2010; **1**:1–6 *Prepared using* acsauth.cls *[Version: 2002/11/11 v1.00]*

A New Interval-Based Method to Characterize Estimability

O. Reynet[1,*], L. Jaulin[2]

[1] *E^3I^2 laboratory, ENSIETA*

[2] *DTN laboratory, ENSIETA*

SUMMARY

Estimability is a property which states on the accuracy of the parameter estimation in the case of experimental data. This paper defines a new method based on interval analysis and set inversion to characterize estimability in the case of a bounded additive noise. To illustrate this new method, the Time Difference of Arrival (TDOA) passive location estimability is evaluated.

KEY WORDS: Estimability, Bounded-error estimation, Identifiability, Interval Analysis, nonlinear Models, Experimental Design, Worst-case design.

1. Introduction

Estimability is a property which states on the accuracy of the parameter estimation in the case of experimental data [1, 2]. Indeed, a parameter can be identifiable [3, 4] but poorly estimable for a given experiment. Bounded-error estimation has already been used to assess

*Correspondence to: Olivier Reynet (olivier.reynet@ensieta.fr) at E^{3}I^2 laboratory - ENSIETA - 2, rue F. Verny - 29806 Brest Cedex 9 - www.ensieta.fr

the a priori performance of software sensors [5] via Minimax optimization. This paper defines a new method based on interval analysis and set inversion to characterize estimability in the case of a bounded additive noise.

A model of a noisy observation process can be written under the form [6]:

$$\mathbf{y} = \mathbf{f}(\mathbf{p}) + \mathbf{e}, \tag{1}$$

where $\mathbf{e} \in \mathbb{E}$ stands for a bounded additive noise vector, $\mathbb{E}$ stands for an additive noise set with known bounds and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a nonlinear function. Our interval-based estimability approach focuses on $\hat{\mathbf{p}}$ vectors which are estimable from $\mathbf{y}$, i.e. that can lead to the same measurement vector $\mathbf{y}$. We are looking for the set $\mathbb{P}$ defined as follows :

$$\mathbb{P} = \{\hat{\mathbf{p}} \in \mathbb{R}^n \mid \exists(\mathbf{e}_1, \mathbf{e}_2) \in \mathbb{E}^2, f(\hat{\mathbf{p}}) + \mathbf{e}_1 = \mathbf{f}(\mathbf{p}) + \mathbf{e}_2\} \tag{2}$$

This equation can be written :

$$\mathbb{P} = \{\hat{\mathbf{p}} \in \mathbb{R}^n \mid \exists(\mathbf{e}_1, \mathbf{e}_2) \in \mathbb{E}^2, f(\hat{\mathbf{p}}) = \mathbf{f}(\mathbf{p}) + \mathbf{e}_2 - \mathbf{e}_1\} \tag{3}$$

Define the uncertainty set $\mathbb{U} = \{\mathbf{e}_2 - \mathbf{e}_1 | \mathbf{e}_1 \in \mathbb{E}, \mathbf{e}_2 \in \mathbb{E}\}$ and $\mathbb{Y} = \mathbf{f}(\mathbf{p}) + \mathbb{U}$. Then, $\mathbb{P}$ can be written as a set inversion [7]:

$$\mathbb{P} = f^{-1}(\mathbb{Y}). \tag{4}$$

which is typically the problem to be solved in bounded-error parameter estimation [5] . Figure 1 illustrates our estimability approach.

In next section, we define the estimability function ξ_f which characterizes the size of $\mathbb{P}$. Third section shows how interval analysis and set inversion may be used to evaluate of ξ_f. Finally, last section illustrates ξ_f relevance by evaluating the estimability of a nonlinear passive location function.

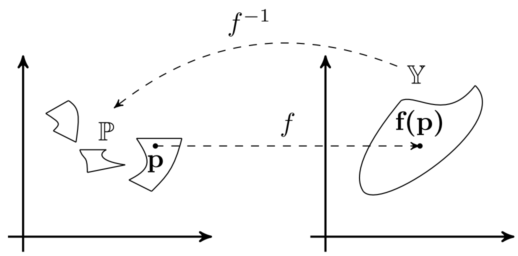


Figure 1. Illustration of our estimability approach: $\mathbb{P}$ constitutes the reciprocal image of $\mathbb{Y} = \mathbf{f(p)} + \mathbb{U}$. Our estimability function ξ_f evaluates the size of $\mathbb{P}$.

2. Estimability Function ξ_f

2.1. Preliminary definition

To define ξ_f, we need a general size function w such as:

$$\begin{array}{rccc} w: & \mathcal{C}(\mathbb{R}^n) & \rightarrow & \mathbb{R}^+ \\ & \mathbb{A} & \rightarrow & w(\mathbb{A}) \end{array} \tag{5}$$

where $\mathcal{C}(\mathbb{R}^n)$ stands for compact sets of $\mathbb{R}^n$. The general size function satisfies two conditions: $w(\mathbb{A})$ always belongs to $\mathbb{R}^+$ and w is monotonic, i.e. $\mathbb{A} \subseteq \mathbb{B} \Rightarrow w(\mathbb{A}) \leq w(\mathbb{B})$. Classically, w is chosen as the largest dimension of the smallest box containing $\mathbb{A}$. Nevertheless, depending on the context and the dimension n, w may account for area, volume or diameter of a compact set [8, 9].

2.2. Estimability Function ξ_f Definition

In the following, $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ stands for a nonlinear function. Then we can define estimability function ξ_f as follow:

$$\begin{array}{rccc} \xi_f : & \mathbb{R}^n & \rightarrow & \mathbb{R}^+ \\ & \mathbf{p} & \rightarrow & w(f^{-1}(\mathbf{f(p)} + \mathbb{U})) \end{array} \tag{6}$$

where $\mathbb{U} = \{\mathbf{e}_2 - \mathbf{e}_1 | \mathbf{e}_1 \in \mathbb{E}, \mathbf{e}_2 \in \mathbb{E}\}$ is the uncertainty set and $\mathbb{E}$ stands for the additive noise set. $\xi_f(\mathbf{p})$ value is the size of the inverted set of $\mathbb{Y} = \mathbf{f(p)} + \mathbb{U}$.

2.3. Illustration of Estimability Function

To illustrate ξ_f concept, let us choose the following one-dimension nonlinear function f:

$$\begin{array}{rccc} f : & [0,6] & \rightarrow & \mathbb{R} \\ & x & \rightarrow & \sqrt{x}\, sin(x) + x. \end{array} \tag{7}$$

This f function is sketched in Fig. 2 and $\xi_f(1)$ evaluation is detailed. We suppose that the additive noise set is $[-\varepsilon/2, \varepsilon/2]$. Then, interval analysis allows us to write :

$$\mathbb{U} = [-\varepsilon/2, \varepsilon/2] - [-\varepsilon/2, \varepsilon/2] = [-\varepsilon, \varepsilon]. \tag{8}$$

In this example, we choose $\varepsilon = 0.7$. Therefore $f^{-1}(f(1) + [-\varepsilon, \varepsilon])$ results in two intervals $\mathbb{A}_1$ and $\mathbb{A}_2$. Let us denote by a_{i_-} and a_{i_+} the $\mathbb{A}_i$ lower and upper bound. w result is the sum of the diameters of these two intervals[†]. That is why:

$$\xi_f(1) = (a_{1_+} - a_{1_-}) + (a_{2_+} - a_{2_-}).$$

$\xi_f(1)$ is found to be about 1.55. It characterizes parameter estimation error due to additive noise and nonlinearity of f near $x = 1$.

The lesser $\xi_f(x)$, the better the accuracy of the parameter estimation. On the contrary, $\xi_f(x) \gg 1$ characterizes the impossibility to properly estimate parameters: it is due to noise, low growing rate or non-injectivity of f [10] .

[†]This definition of w is consistent with the section 2.1 of this paper.

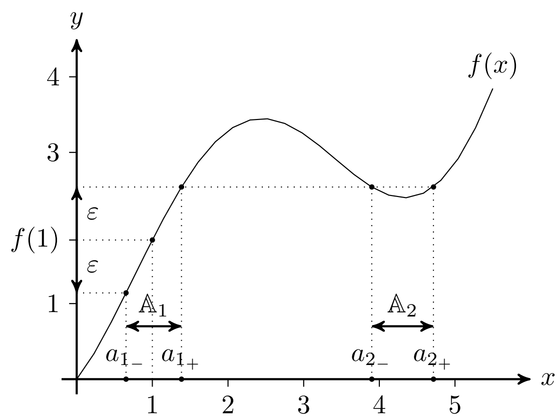


Figure 2. One dimension $f(x) = \sqrt{x}\, sin(x) + x$ function and ξ_f concept: $\varepsilon/2 = 0.35$, $x = 1$. In this example, $f^{-1}(f(1) + [-\varepsilon, \varepsilon])$ results in two intervals $\mathbb{A}_1$ and $\mathbb{A}_2$.$\xi_f(1) = (a_{1_+} - a_{1_-}) + (a_{2_+} - a_{2_-})$.

3. Estimability Evaluation

3.1. Methodology

To evaluate ξ_f, four steps are required: first, $\mathbb{U}$ must be deduced from $\mathbb{E}$. Second, $\mathbf{f(p)} + \mathbb{U}$ of (6) is evaluated. Third, $f^{-1}(\mathbb{Y})$ is characterized by using set inversion [11]. Finally, w computes the sum of the sizes of the resulting sets.

Powerful set methods exist to address set inversion problems [7]. In this paper, we are using Quimper, a high-level language for QUick Interval Modeling and Programming in a bounded-ERror context[‡]. Quimper uses interval analysis and constraint propagation [12] to solve equations. It guarantees that the computed intervals enclose all solutions for given initial intervals. In addition, it provides built-in contractors which speed up computation. Details about Quimper and contractor programming can be found in [13]. But let us now illustrate

[‡]See Ibex/Quimper site at `http://ibex-lib.org/`

the estimability function ξ_f on a one dimension example.

3.2. 1-D Estimability Evaluation

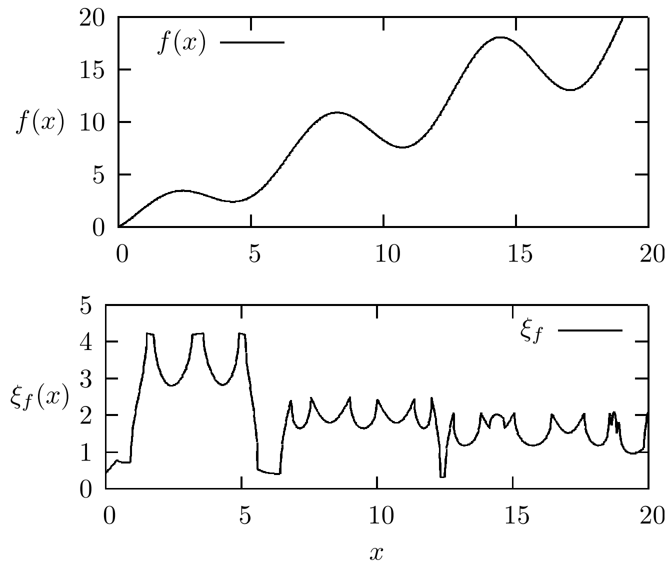
ξ_f and f of (7) are drawn for $x \in [0, 20]$ and $\mathbb{E} = [-0.35, 0.35]$ in Fig. 3. Each point of $\xi_f(x)$ has been evaluated using Quimper software. For $x \in [0, 20]$ and $\mathbb{U} = [-0.7, 0.7]$, the following algorithm is applied :

1. compute $\mathbb{Y} = f(x) + [-0.7, 0.7]$,
2. apply the natural contractor derived from $\sqrt{x}\, sin(x) + x$ on $\mathbb{Y}$,
3. apply the thickness contractor [13],
4. compute the size of $\mathbb{Y}^{-1}$.

The natural contractor eliminates all the intervals which do not satisfy (7). The thickness contractor is a special contractor to fix the bisection limits. It collects all the intervals whose maximum size is 0.01. Therefore the intervals that are not solutions and the intervals that are indiscernible are included by this contractor[§]. Computation takes about 0.002s per point[¶].

ξ_f is not monotonic over $[0, 20]$. Structural identifiability [10] tells us that it is due to variation of the cardinality of $f^{-1}(\mathbb{Y})$. ξ_f can take high values because of non-injectivity. On the contrary, if the injective part of f is considered and if the growing rate of f is high, then ξ_f tends to 0.

[§]See Quimper manual for examples.
[¶]on an Intel Core 2 Duo CPU at 2.00GHz



Prepared using acsauth.cls

Figure 3. One dimension $f(x) = \sqrt{x}sin(x) + x$ function and ξ_f over x-range $[0, 20]$ for $\varepsilon = 0.7$.

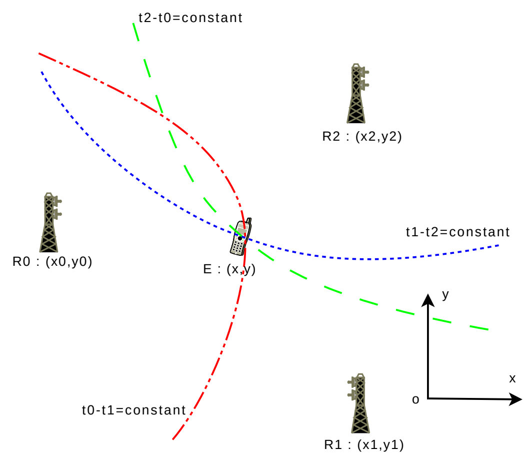


Figure 4. Wireless network using TDOA

4. Application to Passive Location

4.1. TDOA Hyperbolic Equations

Let (x, y) be the unknown location of the emitter, and (x_i, y_i) the location of the receivers. Distance from emitter to receiver i is:

$$D_i = \sqrt{(x - x_i)^2 + (y - y_i)^2} \tag{9}$$

Let t_{ij} be the measured[‖] Time Difference Of Arrival (TDOA) of the signal between receiver i and j. As $D_i - D_j = ct_{ij}$, hyperbolic TDOA equations are:

$$\sqrt{(x - x_i)^2 + (y - y_i)^2} - \sqrt{(x - x_j)^2 + (y - y_j)^2} = ct_{ij} \tag{10}$$

where c is the speed of the signal and $(i, j) \in \{(0, 1), (1, 2), (2, 0)\}$, as sketched in Fig. 4.

Solving these nonlinear equations for (x, y) is not a trivial problem [16, 17, 18], especially

[‖]See [14] and [15] for correlation techniques used to measure TDOA.

when time measurements are noisy. However, we have shown in [19] that our approach based on interval analysis, constraint propagation and contractor programming allows us to avoid any approximations and naturally results in bounded-error estimation.

4.2. TDOA Estimability

Consider the following function:

$$\begin{aligned} f: \quad \mathbb{R}^2 &\rightarrow \mathbb{R}^3 \\ (x, y) &\rightarrow (t_{01}, t_{12}, t_{20}). \end{aligned} \tag{11}$$

where t_{ij} is defined by (10). The estimability of this function allows us to refine our TDOA approach: for a given time additive noise and a special receivers configuration, we can now easily build a map which states on the TDOA passive location error.

In this example, receivers are located at R0 (-1000, 0) m, R1 (0,1000) m and R2 (1000,0) m. We choose to define w as area operator : it means that w computes the sum of the areas of all solution boxes. Therefore, ξ_f unit is km^2. In this simulation, $\mathbb{E} = [-\varepsilon/2, \varepsilon/2] \times [-\varepsilon/2, \varepsilon/2] \times [-\varepsilon/2, \varepsilon/2]$ and $\varepsilon/2 = 15$ns. This time error** corresponds to an analog to digital converter with a good precision and a basic signal correlation.

In our simulation, the x-range is $[-5000, 5000]$ m and the y-range is $[-5000, 5000]$ m. A 100x100 sampling grid has been chosen. For each point (x, y), a Quimper file similar to listing 1 is computed. The area corresponding to $\xi_f(x, y)$ is extracted from Quimper results. Figure 5 shows ξ_f computation. Each point takes about 0.02 s to compute.

Listing 1. Example of Quimper script for TDOA set inversion

**Different $\varepsilon/2$ could have been chosen for each t_{ij}. They also could have been chosen randomly.

```
Constants
x0=0.0; y0=-1000.0;
x1=0.0; y1=1000.0;
x2=1000.0; y2=0.0;
ct01 in [-592.12,-577.12];
ct12 in [-588.53,-573.53];
ct20 in [1158.15,1173.19];
Variables
x in [-5000,5000];
y in [-5000,5000];
constraint-list dtoaeq
sqrt((x-x0)^2+(y-y0)^2)-sqrt((x-x1)^2+(y-y1)^2) in ct01;
sqrt((x-x1)^2+(y-y1)^2)-sqrt((x-x2)^2+(y-y2)^2) in ct12;
sqrt((x-x2)^2+(y-y2)^2)-sqrt((x-x0)^2+(y-y0)^2) in ct20;
end
contractor-list pinter
  for i=1:3;
    dtoaeq(i)
  end
end
contractor propInter
 propag(pinter)
end
contractor isThick
  maxdiamGT(20)
end
```

To our knowledge, it is the first time that such a function is evaluated. This map highlights the emitter positions for which the TDOA passive location error is the most important. These

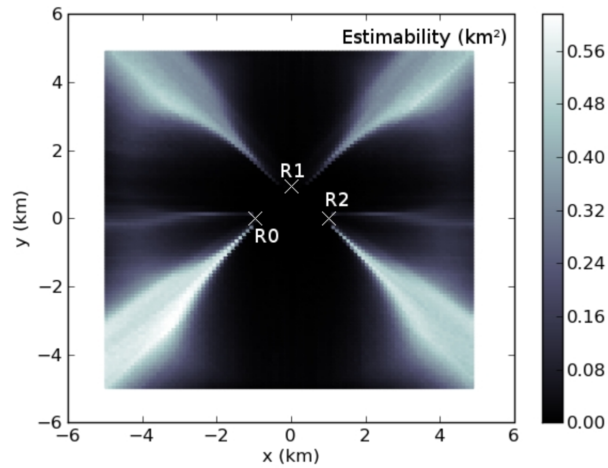


Figure 5. TDOA estimability: receivers are sketched with white crosses: R0 (-1000, 0) m, R1 (0,1000) m and R2 (1000,0) m.

emitter's positions are shown to be located over complex regions really difficult to predict because of the nonlinear hyperbolic equations. These intrinsic properties of f are very useful to properly design passive location systems.

5. Conclusion

We have introduced a new interval-based method to evaluate the estimability and shown that it is possible to predict the accuracy of the parameter estimation of a nonlinear model in the case of noisy data. Our method also differs from the Cramer-Rao Lower Bound (CRLB) approach, because we have not built a statistics-based estimator. Unlike CRLB, no special assumption is required on the bias or the linearity of the model, neither on the additive noise.

Our approach is not another sensitivity analysis to study the influence of the variation of the parameters on the function's result: ξ_f directly evaluates the error of parameter estimation from f and additive noise set $\mathbb{E}$. Estimability function ξ_f does not require global identifiability. Besides, its use is not restricted to small additive noise. This is due to evaluation method based on interval analysis and set inversion. Application to passive location illustrates the relevance of our approach. We are certain that numerous experimental design problems can be solved thanks to ξ_f. Therefore, we are now working at extending these results to a more general case where estimability is guaranteed and no more gridded.

REFERENCES

1. Jacquez JA, Greif P. Numerical parameter identifiability and estimability: Integrating identifiability, estimability, and optimal sampling design. *MATH. BIOSCI.* 1985; **77**(1-2):201–227. URL `http://www.scopus.com/scopus/record/display.url?view=extended\&origin=resultslist\&eid=2-s2.0-0022361161`.
2. Jayasankar B, Ben-Zvi A, Huang B. Identifiability and estimability study for a dynamic solid oxide fuel cell model. *Computers & Chemical Engineering* 2009; **33**(2):484 – 492, doi:DOI:10.1016/j.compchemeng.2008.11.005. URL `http://www.sciencedirect.com/science/article/B6TFT-4TYYT2C-5/2/7c75e2bb6ab80db40b6666e8ea155c9b`.
3. Ljung L, Glad T. On global identifiability of arbitrary model parametrizations. *Automatica* 1994; **30**:265–276.
4. Berthier F, Diard JP, Pronzato L, Walter E. Identifiability and distinguishability in electrochemistry. *Automatica* 1996; **32**:973–984.
5. Braems I, Kieffer M, Walter E. Prior characterization of the performance of software sensors. *Proceedings of the 13th IFAC Symposium on System Identification, SYSID*, 2003.
6. Walter E, Pronzato L. *Identification of Parametric Models From Experimental Data.* Springer: London, 1997.
7. Jaulin L, Kieffer M, Didrit O, Walter E. *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics.* Springer-Verlag: London, 2001.

8. Fogel E, Huang YF. On the value of information in system identification: Bounded noise case. *Automatica* 1982; **18**(2):229–238.
9. Durieu C, Walter E, Polyak B. Set-membership estimation with the trace criterion made simpler than with the determinant criterion. *12th IFAC Symposium on System Identification Sysid'2000, Cd-Rom, Santa Barbara, Californie, USA*, 2000.
10. Lagrange S, Delanoue N, Jaulin L. Injectivity analysis using interval analysis: Application to structural identifiability. *Automatica* 2008; **44**(11):2959 – 2962, doi:DOI:10.1016/j.automatica.2008.04.018. URL http://www.sciencedirect.com/science/article/B6V21-4TKXDBK-2/2/c69f4e542bdbf6c5cc0b6e1c4dec3c59.
11. Jaulin L, Walter E. Set inversion via interval analysis for nonlinear bounded-error estimation. *Automatica* 1993; **29**(4):1053–1064.
12. Jaulin L, Kieffer M, Braems I, Walter E. Guaranteed nonlinear estimation using constraint propagation on sets. *International Journal of Control* 2001; **74**(18):1772–1782.
13. Chabert G, Jaulin L. Contractor programming. *Artificial Intelligence* 2009; **173**(11):1079 – 1100, doi:DOI:10.1016/j.artint.2009.03.002. URL http://www.sciencedirect.com/science/article/B6TYF-4VVXSXC-1/2/3219355b6db4f86ba25cc09646f6899e.
14. Knapp C, Carter G. The generalized correlation method for estimation of time delay. *Acoustics, Speech and Signal Processing, IEEE Transactions on* Aug 1976; **24**(4):320–327.
15. Carter G. Time delay estimation for passive sonar signal processing. *Acoustics, Speech and Signal Processing, IEEE Transactions on* Jun 1981; **29**(3):463–470.
16. Chan YT, Ho KC. A simple and efficient estimator for hyperbolic location. *IEEE Transactions on Signal Processing* Aug 1994; **42**(8):1905–1915, doi:10.1109/78.301830.
17. Foy WH. Position-location solutions by Taylor-series estimation. *IEEE Transactions on Aerospace and Electronic Systems* Mar 1976; **12**(2):187–194, doi:10.1109/TAES.1976.308294.
18. Torrieri DJ. Statistical theory of passive location systems. *IEEE Transactions on Aerospace and Electronic Systems* Mar 1984; **20**(2):183–198, doi:10.1109/TAES.1984.310439.
19. Reynet O, Jaulin L, Chabert G. Robust TDOA passive location using interval analysis and contractor programming. *Radar 2009 - International Radar Conference, Bordeaux, France*, 2009.