STATISTICA, anno LXXIX, n. 4, 2019

A NEW LIFETIME DISTRIBUTION WITH DECREASING AND UPSIDE-DOWN BATHTUB-SHAPED HAZARD RATE FUNCTION

Sanku Dey
Department of Statistics, St. Anthony's College, Shillong-793001, Meghalaya, India
Mazen Nassar
Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah, Kingdom of Saudia Arabia
Devendra Kumar[1]
Department of Statistics, Central University of Haryana, Haryana, India
Ayman Alzaatreh
Department of Mathematics, Nazarbayev University, Astana, Kazakhstan
Muhammad Hussain Tahir
Department of Statistics, The Islamia University of Bahawalpur, Pakistan

1. Introduction

In recent years, researchers proposed various ways of generating new continuous distributions in lifetime data analysis to enhance its capability to fit diverse lifetime data which have a high degree of skewness and kurtosis. These extended distributions provide greater flexibility in modelling certain applications and data in practice. Due to the computational and analytical facilities available in programming softwares such as R, Maple and Mathematica, it is easy to tackle the problems involved in computing special functions in these extended distributions. A detailed survey of methods for generating distributions were discussed by Lee *et al.* (2013) and Jones (2015).

The Lomax (L) distribution (Lomax, 1954), also known as the Pareto Type II distribution (or simply Pareto II), is a heavy-tail probability distribution often used in business, economics and actuarial modeling. It is known as a special form of Pearson type VI distribution and has also considered as a mixture of exponential and gamma distributions. The Lomax distribution has been applied in a variety of contexts ranging from

[1] Corresponding Author. E-mail: devendrastats@cuh.ac.in

modeling the survival times of patients after a heart transplant (see Bain and Engelhardt, 1992) to the sizes of computer files on servers (see Holland *et al.*, 2006). Some authors, such as Bryson (1974), suggest the use of this distribution as an alternative to the exponential distribution when data are heavy-tailed. However, the Lomax distribution does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates, such as the upside-down bathtub failure rates, which are common in reliability and biological studies. For example, the lifetime models that present upside-down bathtub failure rates curves can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually. The need for extended forms of the Lomax model arises in many applied areas. For greater details, readers may refer to Kotz and Nadarajah (2000). It is one of the the most popular distribution in the literature for analyzing lifetime data.

In the recent past, so many generalizations of Lomax distribution have been attempted by researchers to cope with these failure rates. Notable among them Abdul-Moniem and Abdel-Hameed (2012) studied the exponentiated-Lomax (EL), Ghitany *et al.* (2007) introduced the Marshall-Olkin extended Lomax, Lemonte and Cordeiro (2013) investigated the beta-Lomax (BL), Kumaraswamy-Lomax (KuL) and McDonald Lomax and Cordeiro *et al.* (2013) introduced gamma-Lomax (GL) distributions. Tahir *et al.* (2015) introduced the Weibull Lomax distribution and studied its mathematical and statistical properties. Al-Zahrani and Sagor (2014a,b) have introduced the Lomax-logarithmic and Poisson-Lomax distributions. Al-Zahrani (2015) also introduced extended Poisson-Lomax distribution.

Many authors have discussed the situations where the data shows decreasing and the upside-down bathtub (UBT) shape hazard rates. For example, Proschan (1963) found that the air-conditioning systems of planes follows decreasing failure rate. Kus (2007) analyzed earthquakes in the last century in North Anatolia fault zone and found that decreasing failure rate distribution fits well. Efron (1988) analyzed the data set in the context head and neck cancer, in which the hazard rate initially increased, attained a maximum and then decreased before it stabilized owing to a therapy. Bennette (1983) analyzed lung cancer trial data which showed that failure rates were unimodal in nature. Langlands *et al.* (1997) have studied the breast carcinoma data and found that the mortality reached a peak after some finite period, and then declined gradually. A few inverse statistical distributions namely inverse Weibull, inverse Gaussian, inverse Gamma and inverse Lindley etc., are used to model such UBT data in various real life applications.

The aim of this note is to derive a new distribution from the Lomax distribution by alpha-power transformation (APT) as suggested by Mahdavi and Kundu (2017), called APTL distribution which contains special sub-model such as Lomax distribution. This concept of generalization is well established in the statistical literature, see Dey *et al.* (2017a,b). The chief motivation of the generalized distributions for modeling failure time data lies in its flexibility to model both monotonic and non-monotonic failure rates even though the baseline failure rate may be monotonic. The proposed distribution encompasses the behavior of and provides better fits than some well known lifetime distributions, such as L, EL, GL, BL and KuL distributions. We are motivated to introduce the APTL distribution because (i) it contains Lomax lifetime sub model; (ii) it is capable of modeling monotonically decreasing and upside down bathtub shaped hazard rates and thus APTL model is useful where Lomax model is not practically applicable; (iii) it can be viewed as a suitable model for fitting the skewed data which may not be properly fitted by other common distributions and can also be used in a variety of problems in different areas such as industrial reliability and survival analysis; and (iv) two real data applications show that it compares well with other competing lifetime distributions in modeling bladder cancer data and service times of Aircraft Windshield data.

The rest of the paper is organized as follows. In Sections 2 and 3, we introduce the alpha-power transformed Lomax distribution, and discuss some properties of this family of distributions. In Section 4, maximum likelihood estimators of the unknown parameters are obtained. In Section 5, we investigate the maximum likelihood estimation procedure to estimate the model parameters. The analysis of two real data sets have been presented in Section 6. Finally, in Section 7, we conclude the paper.

2. Model description

If $F(x)$ is an absolute continuous distribution function with the probability density function (pdf) $f(x)$, then $F_{APT}(x)$ is also an absolute continuous distribution function with the pdf:

$$f_{APT}(x) = \frac{\log\alpha}{\alpha-1} f(x)\alpha^{F(x)}, \quad \alpha > 0,\ \alpha \neq 1. \tag{1}$$

For more details about APT, see Nassar *et al.* (2017) and Dey *et al.* (2019). Note that $f_{APT}(x)$ is a weighted version of $f(x)$, where the weight function is $w(x) = \alpha^{F(x)}$. Thus, $f_{APT}(x)$ can be written as

$$f_{APT}(x) = \frac{w(x;\alpha) f(x)}{E(w(x;\alpha))}, \tag{2}$$

where $w(x;\alpha)$ is non-negative and

$$E[w(x;\alpha)] = \int_{-\infty}^{\infty} w((x;\alpha) f(x) dx < \infty.$$

Remark 1. *When $\alpha \longrightarrow 1$, $f_{APT}(x)$ reduces to $f(x)$. Therefore, $f_{APT}(x)$ is considered a generalization of the pdf $f(x)$.*

Applications of a weighted distribution to biased samples in various areas including medicine, ecology, reliability, and branching processes can be seen in Rao (1965), Patil and Rao (1978), Patil *et al.* (1986), Gupta and Kirmani (1990), Gupta and Keating (1985), Oluyede (1999), Patil (2002) and Gupta and Kundu (2009) and the references therein. In this case the weight function $w(x;\alpha)$ can be increasing or decreasing depending on whether $\alpha > 1$ or $\alpha < 1$.

3. APTL DISTRIBUTION

Let X be Lomax random variable with parameter $\beta, \lambda > 0$. Recall that the pdf and the cumulative distribution function (cdf) associated to X are, respectively, given by

$$f(x) = \frac{\beta\lambda^{\beta}}{(\lambda + x)^{\beta+1}} \qquad x > 0;\ \beta, \lambda > 0 \tag{3}$$

and

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\beta} \qquad x > 0;\ \beta, \lambda > 0. \tag{4}$$

We now introduce the notion of the APTL distribution.

DEFINITION 2. *A random variable X is said to have APTL distribution if its pdf is of the form*

$$f(x) = \frac{\beta\lambda^{\beta}\log\alpha}{\alpha - 1}(\lambda + x)^{-\beta-1}\alpha^{1-(\frac{\lambda}{\lambda+x})^{\beta}}, \quad x > 0;\ \alpha, \beta, \lambda > 0,\ \alpha \neq 1. \tag{5}$$

The corresponding cdf and hazard rate functions are, respectively, given by

$$F(x) = \frac{\alpha^{1-(\frac{\lambda}{\lambda+x})^{\beta}} - 1}{\alpha - 1}, \quad x > 0;\ \alpha, \beta, \lambda > 0,\ \alpha \neq 1 \tag{6}$$

and

$$h(x) = \beta\lambda^{\beta}\log\alpha(\lambda + x)^{-\beta-1}\left(\alpha^{\left(\frac{\lambda}{\lambda+x}\right)^{\beta}} - 1\right)^{-1}, \quad x > 0;\ \alpha, \beta, \lambda > 0,\ \alpha \neq 1. \tag{7}$$

Hereafter, a random variable X that follows the distribution in (5) is denoted by $X \sim$ APTL(α, β, λ). Note that when $\alpha \longrightarrow 1$, the APTL distribution reduces to the L distribution in (3).

Figures 1a and 1b (see Appendix) show the various curves for the pdf and the hazard rate functions, respectively, of APTL distribution with $\lambda = 1$ and various values of α and β. Figure 1a indicates that the APTL distribution can be uni-modal, reversed J-shaped and positively right skewed. Figure 1b, shows that the hazard function $h(x)$ of APTL distribution can be decreasing or UBT shapes. One of the advantages of the APTL distribution over the L distribution is that the latter cannot model phenomenon showing an UBT shape failure rate.

3.1. Statistical properties

The quantile function $x_p = Q(p) = F^{-1}(p)$, for $0 < p < 1$, of the APTL(α,β,λ) distribution is obtained from (6), it follows that the quantile function $F^{-1}(p)$ is

$$x_p = \lambda\left[\left\{1 - \frac{\log((\alpha-1)p+1)}{\log\alpha}\right\}^{-1/\beta} - 1\right], \quad \alpha \neq 1. \tag{8}$$

The asymptotics of the pdf, cdf and hazard rate function of X are given by

$$\begin{aligned} F(x) &\sim x && \text{as} \quad x \to 0 \\ f(x) &\sim (1+x)^{-\beta-1} && \text{as} \quad x \to \infty, \\ h(x) &\sim (1+x)^{-1} && \text{as} \quad x \to \infty. \end{aligned}$$

REMARK 3. *The $APTL(\alpha,\beta,\lambda)$ distribution has the following mixture representation for $\alpha > 1$. $\frac{\log(\alpha)}{(\alpha-1)}$ is a decreasing function from 1 to 0, as α varies from 1 to ∞. If $X \sim APTL(\alpha,\beta,\lambda)$, then it can be represented as follows:*

$$X = \begin{cases} X_1 & \textit{with probability } \left(\frac{\log\alpha}{\alpha-1}\right) \\ X_2 & \textit{with probability } 1-\left(\frac{\log\alpha}{\alpha-1}\right), \end{cases} \tag{9}$$

where X_1 and X_2 have the following pdfs

$$f_{X_1}(x) = \frac{\beta\lambda^\beta}{(\lambda+x)^{\beta+1}} \tag{10}$$

and

$$f_{X_2}(x) = \frac{\log\alpha}{(\alpha-1-\log\alpha)} \frac{\beta\lambda^\beta}{(\lambda+x)^{\beta+1}}[\alpha^{1-(\frac{\lambda}{\lambda+x})^\beta} - 1], \tag{11}$$

respectively. It is clear from the representation (9) that as α approaches 1, X behaves like L distribution, and as α increases, it behaves like X_2.

The following two theorems discuss the shapes of the hazard and the density functions of the APTL distribution.

THEOREM 4. *The $APTL(\alpha,\beta,\lambda)$ is unimodal. The mode is at $x = 0$ whenever $\alpha \leq e^{1+1/\beta}$ and the mode is at $x = \lambda\left[\left(\frac{\beta\log\alpha}{1+\beta}\right)^{1/b} - 1\right]$ whenever $\alpha > e^{1+1/\beta}$.*

PROOF. Without loss of generality assume $\lambda = 1$. On can easily see that $f'(x) = A[\beta(1+x)^{-\beta}\log\alpha - b - 1]$, where $A > 0$ is a constant. This implies that for $\alpha \leq e^{1+1/\beta}$, $f'(x) \leq 0$ for all $x > 0$. Hence $f(x)$ has a reversed J-shape and the mode is at $x = 0$. Now suppose that $\alpha > e^{1+1/\beta}$, then $f'(x) = 0$ iff $x = x_0 = \left(\frac{\beta\log\alpha}{1+\beta}\right)^{1/b} - 1$. Now since $f''(x) < 0$, $f(0) > 0$ and $\lim_{x\longrightarrow\infty} f(x) = 0$, $f(X)$ has a unique mode at $x = x_0$. □

THEOREM 5. *Let $X \sim APTL(\alpha,\beta,\lambda)$. The hazard rate function of X has the following shapes.*

i. If $\alpha \le e^{1/\beta}$, then the hazard rate function of X has the decreasing failure rate property.

ii. If $\alpha > e^{1/\beta}$ and $\beta \le \frac{\alpha-1}{1-\alpha+\alpha\log\alpha}$, then the hazard rate function of X has the decreasing failure rate property.

iii. If $\alpha > e^{1/\beta}$ and $\beta > \frac{\alpha-1}{1-\alpha+\alpha\log\alpha}$, then the hazard rate function of X has the UBT property.

PROOF. From (7), $h'(x) = \psi(x)k(x)$ where $\psi(x) > 0$ for all $x > 0$ and

$$k(x) = \log\alpha\left(\beta\log(\alpha)(x+1)^{-\beta}\alpha^{(x+1)^{-\beta}} - (\beta+1)(\alpha^{(x+1)^{-\beta}} - 1)\right).$$

Also, $k'(x) = \xi(x)\ell(x)$ with $\xi(x) > 0$ for all $x > 0$ and

$$\ell(x) = 1 - \beta(x+1)^{-\beta}\log\alpha.$$

This implies that the critical value for $k(x)$ is $x = x_1 = (\beta\log\alpha)^{1/b} - 1$. Consider the following two cases.

Case 1: If $\alpha \le e^{1/\beta}$, then x_1 is not defined or $x_1 < 0$ (this is not possible since the support of the distribution is positive). Therefore, $k'(x) \ge 0$ for all $x > 0$. which implies that $k(x)$ is increasing function on $(0,\infty)$. Now the fact $k(x) \longrightarrow 0$ as $x \longrightarrow \infty$ implies that $k(x) \le 0$ and hence, $h(x)$ is decreasing function on $(0,\infty)$. This ends the proof of (i).

Case 2: If $\alpha > e^{1/\beta}$, then $k'(x) < 0$ on $(0,x_1)$ and $k'(x) > 0$ on (x_1,∞). I.e. $k(x)$ is decreasing function on $0 < x < x_1$ and increasing on $x > x_1$. Now, it is not difficult to show that $k(0) < 0$ iff $\beta \le \frac{\alpha-1}{1-\alpha+\alpha\log\alpha}$. Therefore, if $k(0) < 0$ and on using the fact $k(x) \longrightarrow 0$ as $x \longrightarrow \infty$, we conclude that $k(x) < 0$ for all $x > 0$ and hence $h(x)$ is a decreasing function. This ends the proof of (ii). Now if $\beta > \frac{\alpha-1}{1-\alpha+\alpha\log\alpha}$, then $k(0) > 0$. And since $k(x)$ has unique minimum value at $x = x_1$ and $k(x) \longrightarrow 0$ as $x \longrightarrow \infty$, $k(x)$ must have a root at, say, $x = x_2$. Therefore, $h'(x) > 0$ on $(0,x_2)$ and $h'(x) < 0$ on (x_2,∞). Hence, $h(x)$ has UBT shape. This ends the proof of (iii). □

3.1.1. Moments

In this section, we study the existence of the nth moment for the APTL distribution.

THEOREM 6. *Let $X \sim APTL(\alpha,\beta,\lambda)$. Then the nth moment of X exists iff $\beta > n$. Furthermore,*

$$E(X^n) = \frac{n!\,\lambda^n\alpha\beta}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^k(\log\alpha)^{k+1}\Gamma[\beta(k+1)-n]}{k!\Gamma[\beta(k+1)+1]}, \quad \beta > n. \tag{12}$$

PROOF. From (5), $f(x) \sim (1+x)^{-\beta-1}$ as $x \to \infty$. Therefore, $\int_0^\infty x^n f(x)dx$ exists iff $\beta > n$. Now, from (5) it is easy to see that

$$E(X^n) = \frac{\beta\,\alpha}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^k(\log\alpha)^{k+1}\lambda^{\beta(k+1)}}{k!}\int_0^\infty \frac{x^n}{(\lambda+x)^{\beta(k+1)+1}}dx.$$

The result in (12) can be obtained by using equation (3.241.4) in Gradshteyn and Ryzhik (2014). □

REMARK 7. *The central moments of X are* $\mu_n = E(X-\mu)^n = \sum_{k=0}^{n}\binom{n}{k}(-\mu_1')^k\,\mu_{n-k}'$. *The skewness and kurtosis of X can be obtained using the formulas skewness* $(X) = \mu_3/\sigma^3$ *and* $kurtosis(x) = \mu_4/\sigma^4$, *where* $\sigma^2 = Var(X)$.

Using Equations (8), (12) and Remark 4, we obtain the values of mean, median, mode, variance, skewness and kurtosis of the APTL distribution. These values are displayed in Table 1 for $\lambda = 1$ and various values of α and β. It can be noticed from Table 1 that for fixed λ and β the mean, median, mode and the variance of the APTL distribution are increasing functions of α, while the skewness and the kurtosis are decreasing function of α. Also, it is observed that for fixed λ and α, the mean, median, variance, skewness and the kurtosis are decreasing function of β.

3.1.2. Moment generating function

Many of the interesting characteristics and features of a distribution can be obtained via its moment generating function and moments. Let X denote a random variable with the pdf (5). By definition of moment generating function of X, we have

$$\begin{aligned} M_x(t) &= E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx \\ &= \int_0^\infty \left(1 + tx + \frac{(t\,x)^2}{2!} + \cdots + \frac{(t\,x)^l}{l!} + \cdots\right) f(x)dx \\ &= \sum_{l=0}^{\infty}\frac{t^l}{l!}\int_0^\infty x^l f(x)dx \\ &= \frac{\alpha}{\alpha-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^k(\lambda t)^l(\log\alpha)^{k+1} l!\,\Gamma[\beta(k+1)-l]}{k!\Gamma[\beta(k+1)+1]}, \quad \beta > l. \qquad (13)\end{aligned}$$

3.2. Conditional moments

THEOREM 8. *Let* $X \sim APTL(\alpha, \beta, \lambda)$. *Then the conditional moment of X exists iff* $\beta > n$. *Furthermore,*

$$E(X^n|X>x)=\frac{\alpha\beta}{(\alpha-1)}\sum_{k=0}^{\infty}\frac{(-1)^k(\log\alpha)^{k+1}\lambda^{\beta(k+1)}\,x^n}{k!(\beta(k+1)-n)[1-V(x)]}$$
$$\times\,{}_2F_1\left(\beta(k+1)+1;\ \beta(k+1)-n;\beta(k+1)-n+1;-\frac{\lambda}{x}\right),\ \beta>n, \tag{14}$$

where $V(x)=\frac{\alpha^{1-\left(\frac{\lambda}{\lambda+x}\right)^{\beta}}-1}{\alpha-1}$ *and* ${}_2F_1(a,b;\ c;\ x)$ *denotes the Gauss hypergeometric function defined by*

$${}_2F_1(a,b;\ c;\ x)=\sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(c)_k}\frac{x^k}{k!},$$

where $(e)_k=e(e+1)\ldots(e+k-1)$ *denotes the ascending fractorial.*

PROOF. From (5), $f(x)\sim(1+x)^{-\beta-1}$ as $x\to\infty$. Therefore, $\int_x^\infty x^n f(x)dx$ exists iff $\beta>n$. The conditional moments, $E(X^n|X>x)$, can be written as

$$E(X^n|X>x)=\frac{1}{S(x)}J_n(x), \tag{15}$$

where

$$\begin{aligned}J_n(x) &= \int_x^\infty y^n f(y)dy\\ &= \frac{\beta\lambda^\beta\log\alpha}{\alpha-1}\int_x^\infty y^n(\lambda+y)^{-(\beta+1)}\alpha^{1-(\frac{\lambda}{\lambda+y})^\beta}dy\\ &= \frac{\alpha\beta}{(\alpha-1)}\sum_{k=0}^{\infty}\frac{(-1)^k(\log\alpha)^{k+1}\lambda^{\beta(k+1)}}{k!}\int_x^\infty\frac{y^n}{(\lambda+y)^{\beta(k+1)+1}}dy\\ &= \frac{\alpha\beta}{(\alpha-1)}\sum_{k=0}^{\infty}\frac{(-1)^k(\log\alpha)^{k+1}\lambda^{\beta(k+1)}\,x^n}{k!\,(\beta(k+1)-n)}\\ &\times\ {}_2F_1\left(\beta(k+1)+1;\ \beta(k+1)-n;\ \beta(k+1)-n+1;\ -\frac{\lambda}{x}\right).\end{aligned} \tag{16}$$

The final step follow by using equation (3.194.2) in Gradshteyn and Ryzhik (2014). The result follows from (15) and (16). □

An application of the conditional moments is the mean residual life (MRL). MRL function is the expected remaining life, $X-x$, given that the item has survived to time x. Thus, in life testing situations, the expected additional lifetime given that a component has survived until time x is called the (MRL). The MRL function in terms of the first conditional moment as

$$m_X(x) = E(X - x|X > x) = \frac{1}{S(x)} J_1(x) - x,$$

where $J_1(x)$ can be obtained from (16) where $n = 1$.

Another application of the conditional moments is the mean deviations about the mean and the median. They are used to measure the dispersion and the spread in a population from the center. If we denote the median by M, then the mean deviations about the mean and the median can be calculated as

$$\delta_\mu = \int_0^\infty |x - \mu| f(x) dx = 2\mu F(\mu) - 2\mu + 2J_1(\mu)$$

and

$$\delta_M = \int_0^\infty |x - M| f(x) dx = 2J_1(M) - \mu,$$

respectively. Where $J_1(\mu)$ and $J_1(M)$ can obtained from (16). Also, $F(\mu)$ and $F(M)$ are easily calculated from (6).

3.3. Mean past lifetime

Assume now that a component with lifetime X has failed at or some time before x, $x \geq 0$. Consider the conditional random variable $x - X|X \leq x$. This conditional random variable shows, in fact, the time elapsed from the failure of the component given that its lifetime is less than or equal to x. Hence, the mean past lifetime of the component can be defined as

$$k(x) = E(X - x|X \leq x) = x - \frac{1}{F(x)} J(x), \tag{17}$$

where

$$\begin{aligned} J(x) &= \int_0^t f(x) dx = \frac{\beta \lambda^\beta \log \alpha}{\alpha - 1} \int_0^t (\lambda + x)^{-(\beta+1)} \alpha^{1-(\frac{\lambda}{\lambda+x})^\beta} dx \\ &= \frac{\alpha\beta}{(\alpha-1)} \sum_{k=0}^\infty \frac{(-1)^k (\log\alpha)^{k+1} \lambda^{\beta(k+1)}}{k!} \int_0^t \frac{1}{(\lambda+x)^{\beta(k+1)+1}} dx \\ &= \frac{\alpha\beta}{(\alpha-1)} \sum_{k=0}^\infty \frac{(-1)^k (\log\alpha)^{k+1} \lambda^{\beta(k+1)}\, t}{k!} {}_2F_1\Big(\beta(k+1)+1;\ 1;\ 2;\ -\frac{x}{\lambda}\Big). \end{aligned}$$

The final step follows by using equation (3.194.1) in Gradshteyn and Ryzhik (2014).

3.3.1. Entropies

An entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution (population). A large value of entropy implies the greater uncertainty in the data. The concept of entropy is important in different areas such as physics, probability and statistics, communication theory, economics, etc. Several measures of entropy have been studied and compared in the literature. The Shannon entropy of a random variable X is defined by $E[-\log(f(X))]$. The Shannon entropy for the APTL distribution can be written as

$$\begin{aligned} E[-\log(f(X))] &= -\log\left(\frac{\beta\lambda^{\beta}\log\alpha}{\alpha-1}\right)+(\beta+1)E[\log(\lambda+x)] \\ &- E\left[\log\, \alpha^{1-\left(\frac{\lambda}{\lambda+x}\right)^{\beta}}\right]. \end{aligned} \tag{18}$$

It is easy to check that

$$\begin{aligned} E[\log(\lambda+x)] &= \frac{\alpha}{\alpha-1}\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^{k+l+1}\, l!\,\lambda^{l}\,(\log\alpha)^{k+1}\,\Gamma[\beta(k+1)-l]}{(k+1)!\, l\;\alpha^{l}\,\Gamma[\beta(k+1)+1]} \\ &+ \log\alpha \end{aligned} \tag{19}$$

and

$$\begin{aligned} E\left[\log\, \alpha^{1-\left(\frac{\lambda}{\lambda+x}\right)^{\beta}}\right] &= \frac{\alpha\beta}{(\alpha-1)}\sum_{k=0}^{\infty}\frac{(-1)^{k}(\log\alpha)^{k+2}}{k!}\Big[B(1,\ \beta(k+1)) \\ &\times\ B(1,\ \beta(k+1))\Big]. \end{aligned} \tag{20}$$

Substituting Equations (19) and (20) into Equation (18), we obtain the Shannon entropy of X.

3.3.2. Bonferroni and Lorenz curve

Boneferroni and Lorenz curves are proposed by Bonferroni (1930). These curves have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. They are defined as

$$B(p)=\frac{1}{p\mu}\int_{0}^{q}xf(x)dx \tag{21}$$

and

$$L(p)=\frac{1}{\mu}\int_{0}^{q}xf(x)dx, \tag{22}$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$. By using (1), one can get

$$\begin{aligned} B(p) &= \frac{\beta\lambda^{\beta}\,\log\alpha}{p\,\mu(\alpha-1)}\int_0^q x(\lambda+x)^{-(\beta+1)}\alpha^{1-(\frac{\lambda}{\lambda+x})^{\beta}}dx \\ &= \frac{\alpha\beta}{p\,\mu\,(\alpha-1)}\sum_{k=0}^{\infty}\frac{(-1)^k(\log\alpha)^{k+1}\lambda^{\beta(k+1)}}{k!}\int_0^q \frac{x}{(\lambda+x)^{\beta(k+1)+1}}dx, \end{aligned} \tag{23}$$

by using equation (3.194.1) in Gradshteyn and Ryzhik (2014) we can calculate the integral in (23). Thus, we get

$$\begin{aligned} B(p) &= \frac{\alpha\beta}{2\,p\,\mu\,\lambda(\alpha-1)}\sum_{k=0}^{\infty}\frac{(-1)^k(\log\alpha)^{k+1}\,q^2}{k!} \\ &\times\ {}_2F_1\Big(\beta(k+1)+1,\ 2;\ 3;\ -\frac{q}{\lambda}\Big), \end{aligned}$$

$$\begin{aligned} L(p) &= \frac{\alpha\beta}{2\,\mu\,\lambda(\alpha-1)}\sum_{k=0}^{\infty}\frac{(-1)^k(\log\alpha)^{k+1}\,q^2}{k!} \\ &\times\ {}_2F_1\Big(\beta(k+1)+1,\ 2;\ 3;\ -\frac{q}{\lambda}\Big), \end{aligned}$$

respectively.

3.3.3. Stochastic ordering

Stochastic ordering of positive continuous random variable is an important tool for judging the comparative behavior. There are different types of stochastic orderings which are useful in ordering random variables in terms of different properties. Here, we consider four different stochastic orders, namely, the usual stochastic orders, the hazard rate, the mean residual life, and the likelihood ratio order for APTL random variables under a restricted parameter space. If X and Y are independent random variables with cdfs F_X and F_Y respectively, then X is said to be smaller than Y in the

- stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$, for all x;
- hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$, for all x;
- mean residual life order ($X \leq_{mr\ell} Y$) if $m_X(x) \geq m_Y(x)$, for all x;
- likelihood ratio order ($X \leq_{\ell r} Y$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The APTL distribution is ordered with respect to the strongest "likelihood ratio" ordering as shown in the following theorem. It shows the flexibility of three parameter APTL(α,β,λ) distribution.

THEOREM 9. *Let* $X \sim APTL(\alpha_1,\beta_1,\lambda_1)$ *and* $Y \sim APTL(\alpha_2,\beta_2,\lambda_2)$*. If* $\alpha_1=\alpha_2=\alpha$*,* $\beta_1=\beta_2=\beta$ *and* $\lambda_1\geq\lambda_2$ *and if* $\alpha_1=\alpha_2=\alpha$*,* $\lambda_1=\lambda_2=\lambda$ *and* $\beta_1\geq\beta_2$*, then* $X\leq_{lr} Y$*,* $X\leq_{hr} Y$*,* $X\leq_{mrl} Y$ *and* $X\leq_{st} Y$*.*

PROOF. The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)}=\frac{\beta_1\,\lambda_1^{\beta_1}(\alpha_2-1)\,\log\alpha_1\,(\lambda_2+x)^{\beta 2+1}\alpha_1^{1-\left(\frac{\lambda_1}{\lambda_1+x}\right)^{\beta_1}}}{\beta_2\,\lambda_2^{\beta_2}(\alpha_1-1)\,\log\alpha_2\,(\lambda_1+x)^{\beta 1+1}\alpha_2^{1-\left(\frac{\lambda_2}{\lambda_2+x}\right)^{\beta_2}}},$$

thus,

$$\begin{aligned}\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} &= \frac{(\beta_2+1)}{(\lambda_2+x)}+\frac{\lambda_1\,\beta_1\,\log\alpha_1}{(\lambda_1+x)^2}\left(\frac{\lambda_1}{(\lambda_1+x)}\right)^{\beta_1-1}\\ &- \frac{(\beta_1+1)}{(\lambda_1+x)}-\frac{\lambda_2\,\beta_2\,\log\alpha_2}{(\lambda_2+x)^2}\left(\frac{\lambda_2}{(\lambda_2+x)}\right)^{\beta_2-1}.\end{aligned}$$

Case (i): If $\alpha_1=\alpha_2=\alpha$, $\beta_1=\beta_2=\beta$ and $\lambda_1\geq\lambda_2$ then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}\leq 0$, which implies that $X\leq_{lr} Y$ and hence $X\leq_{hr} Y$, $X\leq_{mrl} Y$ and $X\leq_{st} Y$.

Case (ii): If $\alpha_1=\alpha_2=\alpha$, $\lambda_1=\lambda_2=\lambda$ and $\beta_1\geq\beta_2$ then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}\leq 0$, which implies that $X\leq_{lr} Y$ and hence $X\leq_{hr} Y$,$X\leq_{mrl} Y$ and $X\leq_{st} Y$.
Hence from case (i) and case (ii) $X\leq_{lr} Y$ and $X\leq_{hr} Y$, $X\leq_{mrl} Y$ and $X\leq_{st} Y$. □

3.3.4. *Stress strength reliability*

Here, we derive the reliability $R=\Pr(X_2<X_1)$, when $X_1\sim$ APTL$(\alpha_1,\beta_1,\lambda)$. and $X_2\sim$ APTL$(\alpha_2,\beta_2,\lambda)$ are independent random variables. Probabilities of this form have many applications especially in engineering concepts. Thus,

$$\begin{aligned}R &= P(X_2<X_1)\\ &= \int_0^\infty f_1(x)F_2(x)dx\\ &= \frac{\beta_1\,\lambda^{\beta_1}\,\log\alpha_1}{(\alpha_1-1)(\alpha_2-1)}\int_0^\infty(\lambda+x)^{-(\beta_1+1)}\alpha_1^{1-\left(\frac{\lambda}{\lambda+x}\right)^{\beta_1}}\left[\alpha_2^{1-\left(\frac{\lambda}{\lambda+x}\right)^{\beta_2}}-1\right]dx\\ &= \frac{\beta_1\,\lambda^{\beta_1}\,\log\alpha_1}{(\alpha_1-1)(\alpha_2-1)}[I_1-I_2], \qquad (24)\end{aligned}$$

where

$$
\begin{aligned}
I_1 &= \int_0^\infty (\lambda+x)^{-(\beta_1+1)} \alpha_1^{1-\left(\frac{\lambda}{\lambda+x}\right)^{\beta_1}} \alpha_2^{1-\left(\frac{\lambda}{\lambda+x}\right)^{\beta_2}} dx \\
&= \alpha_1\alpha_2 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+l}(\log\alpha_1)^k(\log\alpha_2)^l \ \lambda^{\beta_1 k+\beta_2 l}}{k!\ l!} \int_0^\infty \frac{1}{(\lambda+x)^{\beta_1(k+1)+\beta_2 l+1}} dx \\
&= \alpha_1\alpha_2 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+l}(\log\alpha_1)^k(\log\alpha_2)^l}{k!\ l!\ \lambda^{\beta_1}} B(1,\ \beta_1(k+1)+\beta_2 l),\ \ \lambda_1=\lambda_2=\lambda
\end{aligned}
$$

and

$$
I_2 = \alpha_1 \sum_{k=0}^{\infty} \frac{(-1)^k\ (\log\alpha_1)^k}{k!\ \lambda^{\beta_1}} B(1,\ \beta_1(k+1)l).
$$

Substituting the value of I_1 and I_2 in (24), R reduces to

$$
\begin{aligned}
R &= \frac{\alpha_1\alpha_2\beta_1}{(\alpha_1-1)(\alpha_2-1)} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+l}(\log\alpha_1)^{k+1}(\log\alpha_2)^l}{k!\ l!} B(1,\ \beta_1(k+1)+\beta_2 l) \\
&- \frac{\alpha_1\beta_1}{(\alpha_1-1)(\alpha_2-1)} \sum_{k=0}^{\infty} \frac{(-1)^k\ (\log\alpha_1)^{k+1}}{k!} B(1,\ \beta_1(k+1)l)
\end{aligned}
$$

If $\beta_1=\beta_2=\beta$, then R becomes

$$
\begin{aligned}
R &= \frac{\alpha_1\alpha_2\beta}{(\alpha_1-1)(\alpha_2-1)} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+l}(\log\alpha_1)^{k+1}(\log\alpha_2)^l}{k!\ l!} B(1,\ \beta(k+1+l)) \\
&- \frac{\alpha_1\beta}{(\alpha_1-1)(\alpha_2-1)} \sum_{k=0}^{\infty} \frac{(-1)^k\ (\log\alpha_1)^{k+1}}{k!} B(1,\ \beta(k+1)l).
\end{aligned}
$$

3.3.5. Order statistics

We know that if $X_{(1)} \le \cdots \le X_{(n)}$ denotes the order statistic of a random sample $X_1,\ldots,X_n$ from a continuous population with cdf $G_X(x)$ and pdf $g_X(x)$ then the pdf of $X_{j:n}$ is given by

$$
g_{X_{j:n}}(x) = \frac{n!}{(j-1)!(n-j)!} g_X(x)(G_X(x))^{j-1}(1-G_X(x))^{n-j},
$$

for $j = 1,\ldots,n$. The pdf and cdf of the jth order statistic for a APTL distribution is given by

$$\begin{aligned} g_{X_{j:n}}(x) &= \frac{n!}{(j-1)!\,(n-j)!}\sum_{u=0}^{j-1}(-1)^u \binom{j-1}{u}\left[\frac{\alpha^{1-\left(\frac{\lambda}{\lambda+x}\right)^\beta}-1}{\alpha-1}\right]^{u+j-1} \\ &\times \frac{\beta\lambda^\beta \log\alpha\ \alpha^{1-\left(\frac{\lambda}{\lambda+x}\right)^\beta}}{(\alpha-1)\,(\lambda+x)^{\beta+1}}. \end{aligned}$$

The kth moments of $X_{j:n}$ can be expressed

$$\begin{aligned} E[X_{j:n}^k] &= \frac{n!}{(j-1)!\,(n-j)!}\int_0^\infty x^k f_{j:n}(x)dx \\ &= \frac{n!}{(j-1)!\,(n-j)!}\sum_{u=0}^{j-1}(-1)^u \binom{j-1}{u}\int_0^\infty x^k \left[\frac{\alpha^{1-\left(\frac{\lambda}{\lambda+x}\right)^\beta}-1}{\alpha-1}\right]^{u+j-1} \\ &\times \frac{\beta\lambda^\beta \log\alpha\ \alpha^{1-\left(\frac{\lambda}{\lambda+x}\right)^\beta}}{(\alpha-1)\,(\lambda+x)^{\beta+1}}dx \\ &= \frac{\beta\ \lambda^k\ k!}{(j-1)!(n-j)!(\alpha-1)}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{n-j}(-1)^{2u+j-1-p+q}\binom{u+j-1}{p} \\ &\times \binom{n-j}{u}\frac{(\log\alpha)^{q+1}\ \alpha^{p+1}\ (p+1)^q\ \Gamma(\beta(q+1)-k)}{(q+1)!\ \Gamma(\beta(q+1)+1)}. \end{aligned}$$

4. Maximum likelihood estimation

The method of maximum likelihood is the most frequently used method of parameter estimation (Casella and Berger, 1990). Its success stems from its many desirable properties including consistency, asymptotic efficiency, invariance property as well as its intuitive appeal. Let $x_1,\cdots,x_n$ be a random sample of size n from (5), then the log-likelihood function of (5) without constant terms is given by

$$\begin{aligned} l(\alpha,\beta,\lambda;x) &= \log L(\alpha,\beta,\lambda;x) = n\log\left(\frac{\log\alpha}{\alpha-1}\right)+n\log\beta+n\beta\log\lambda \\ &- (\beta+1)\sum_{i=1}^{n}\log(\lambda+x_i)+(\log\alpha)\sum_{i=1}^{n}\left[1-\left(\frac{\lambda}{\lambda+x_i}\right)^\beta\right]. \end{aligned} \tag{25}$$

For ease of notation, we will denote the first partial derivatives of (25) by l_α , l_β and l_λ. Now setting $l_\alpha = 0$, $l_\beta = 0$ and $l_\lambda = 0$, we have

$$\frac{n(\alpha - 1 - \alpha \log \alpha)}{\alpha(\alpha - 1)\log \alpha} + \frac{1}{\alpha}\sum_{i=1}^{n}\left[1 - \left(\frac{\lambda}{\lambda + x_i}\right)^{\beta}\right] = 0, \tag{26}$$

$$\frac{n}{\beta} + n\log\lambda - \sum_{i=1}^{n}\log(\lambda + x_i) - (\log\alpha)\sum_{i=1}^{n}\left(\frac{\lambda}{\lambda + x_i}\right)^{\beta}\log\left(\frac{\lambda}{\lambda + x_i}\right) = 0, \tag{27}$$

$$\frac{n\beta}{\lambda} - (\beta + 1)\sum_{i=1}^{n}\frac{1}{\lambda + x_i} - \beta\lambda^{\beta - 1}(\log\alpha)\sum_{i=1}^{n}\frac{x_i}{(\lambda + x_i)^{\beta + 1}} = 0. \tag{28}$$

The maximum likelihood estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ of α, β and λ are obtained by solving these nonlinear system of equations. From Equation (28), and for fixed β and λ, we can obtain $\hat{\alpha}(\beta, \lambda)$ as follows

$$\hat{\alpha}(\beta, \lambda) = \exp\left\{\frac{n/\lambda - (1 + 1/\beta)\sum_{i=1}^{n}(\lambda + x_i)^{-1}}{\lambda^{\beta - 1}\sum_{i=1}^{n} x_i(\lambda + x_i)^{-(\beta + 1)}}\right\} \tag{29}$$

Now, the MLEs of β and λ can be obtained by solving the following two non-linear equations

$$\frac{n(\hat{\alpha}(\beta, \lambda) - 1 - \hat{\alpha}(\beta, \lambda)\log\hat{\alpha}(\beta, \lambda))}{\hat{\alpha}(\beta, \lambda)(\hat{\alpha}(\beta, \lambda) - 1)\log\hat{\alpha}(\beta, \lambda)} + \frac{1}{\hat{\alpha}(\beta, \lambda)}\sum_{i=1}^{n}\left[1 - \left(\frac{\lambda}{\lambda + x_i}\right)^{\beta}\right] = 0, \tag{30}$$

$$\frac{n}{\beta} + n\log\lambda - \sum_{i=1}^{n}\log(\lambda + x_i) - (\log\hat{\alpha}(\beta, \lambda))\sum_{i=1}^{n}\left(\frac{\lambda}{\lambda + x_i}\right)^{\beta}\log\left(\frac{\lambda}{\lambda + x_i}\right) = 0. \tag{31}$$

The MLEs of β and λ can be obtained by means of numerical procedures like the quasi-Newton algorithm. The Mathcad program provides the nonlinear optimization for solving such problems. Once $\hat{\beta}$ and $\hat{\lambda}$ are obtained, then $\hat{\alpha}(\hat{\beta}, \hat{\lambda})$ can be obtained from (29).

5. SIMULATION STUDY

In this section, we use a Monte Carlo simulation to evaluate the performance of the maximum likelihood method for estimating the unknown parameters of the APTL distribution in terms of the sample size n. The variates having the APTL distribution are generated by using (8). We suggest the values $0.5, 1.5$ for parameter λ, $1, 1.5, 2$ for parameter $\beta =$ and $0.5, 1.5, 2.5, 3.5, 5$ for parameter $\alpha =$ and two different sample sizes, $50, 100$

are used. We replicate the process 1000 times. In each setting we obtain the average values of estimates and the corresponding mean squared errors (MSEs). For each sample size, the simulation results for a total of 30 parameter combinations are obtained and reported in Tables 2 and 3. From Tables 2 and 3, it is noticed that the maximum likelihood method works well to estimate the parameters of the APTL distribution. Also, it is observed that, as the sample size increases the MSEs decrease in all cases.

6. APPLICATIONS

In this section, we provide two applications to real data sets to demonstrate the importance and flexibility of the proposed distribution.

Data set 1. The first data set corresponds to remission times (in months) of a random sample of 128 bladder cancer patients, more details about the data set can be found in Lee and Wang (2003). These data were studied by Lemonte and Cordeiro (2013) and Nofal *et al.* (2017). The data are as follows:

0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Data set 2. The second data set consists of service times of 63 Aircraft Windshield. These data were considered by Murthy *et al.* (2004). The data measured in 1000h are as follows:

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

We fitted APTL distribution to the two data sets by using the method of maximum likelihood and the results are compared with the other competitive models namely, L, EL, GL, BL and KuL distributions. The density functions of these models are given below (for $x > 0$) as:

$$\begin{aligned}
\text{EL}: f(x;\alpha,\beta,\lambda) &= \frac{\alpha\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right]^{\alpha-1},\\
\text{GL}: f(x;\alpha,\beta,\lambda) &= \frac{\beta\lambda^{\beta}}{\Gamma(\alpha)(\lambda+x)^{\beta+1}}\left\{-\beta\log\left(\frac{\lambda}{\lambda+x}\right)\right\}^{\alpha-1},\\
\text{BL}: f(x;a,b,\beta,\lambda) &= \frac{\beta}{\lambda B(a,b)}\left(1+\frac{x}{\lambda}\right)^{-(b\beta+1)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right]^{a-1},\\
\text{KuL}: f(x;a,b,\beta,\lambda) &= \frac{ab\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right]^{a-1}\\
&\times \left\{1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right]^{a}\right\}^{b-1}.
\end{aligned}$$

We use the goodness-of-fit statistics: Kolmogorov-Smirnov (K-S) distance and the corresponding p-value. We also considered information theoretic criterion, such as Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*) and Cramér-von Mises (W^*) to compare the above models with the APTL model. The maximum likelihood estimates of the models parameters and the corresponding standard errors (in parentheses) and the approximate confidence intervals (CIs) of the parameters of all the models are displayed in Tables 4 and 6 for data sets 1 and 2, respectively. The statistics $-L$ where (L denotes the log-likelihood function evaluated at the maximum likelihood estimates), AIC, BIC, CAIC, HQIC, A^* and W^* are listed in Tables 5 and 7 for data sets 1 and 2, respectively. Based on the results displayed in Tables 5 and 7, we can see that the APTL distribution has the lowest AIC, BIC, CAIC, HQIC, A^* and W^* values among all other competitive models, and so it could be chosen as the best model. The histogram and the fitted APTL density of data sets 1 and 2, respectively, are displayed in Figures 2(a) and 3(a), while, the plots of the fitted APTL survival and the empirical survival functions for the data sets 1 and 2 are displayed in Figures 2(b) and 3(b), respectively. Also, the Q-Q and P-P plots for the data sets 1 and 2, respectively are presented in Figures 4 and 5.

7. Conclusion

In this paper, a new three-parameter family of distributions has been proposed, namely the APTL distribution. The proposed APTL distribution has two shape and one scale parameters. The APTL density function can take various forms depending on its shape parameters. Moreover, the APTL distribution failure rate function can have the following two forms: (i) decreasing and (ii) upside down bathtub shaped. Therefore, it can be used quite effectively in analyzing survival and lifetime data. We fit the new model to two real data sets to demonstrate its usefulness in practice. Results indicate that the APTL model is adequate to fit with two, three and four parameters model for remission times bladder cancer patients and failure times of aircraft windshield data sets. We hope that the APTL distribution will attract wider sets of applications in areas such as survival and lifetime data and others.

ACKNOWLEDGEMENTS

The authors would like to thank the reviewers and the editors for their comments that helped improve the article substantially.

APPENDIX

A. TABLES

TABLE 1
Mean, median, mode, variance, skewness, and kurtosis of APTL distribution for various values of α, β and $\lambda = 1$.

β	α	Mean	Median	Mode	Variance	Skewness	Kurtosis
4.5	0.5	0.233	0.126	0	0.113	5.997	177.327
	1.5	0.319	0.194	0	0.167	5.218	137.628
	3.5	0.393	0.261	0.005	0.212	4.827	120.567
	5.5	0.434	0.298	0.077	0.236	4.679	114.568
	10	0.488	0.349	0.151	0.265	4.532	108.941
5	0.5	0.204	0.113	0	0.081	5.116	87.767
	1.5	0.279	0.173	0	0.118	4.427	67.853
	3.5	0.344	0.232	0.009	0.148	4.077	59.228
	5.5	0.377	0.265	0.072	0.164	3.942	56.181
	10	0.424	0.309	0.139	0.183	3.807	53.311
7.5	0.5	0.126	0.074	0	0.025	3.600	29.801
	1.5	0.171	0.112	0	0.0363	3.059	22.786
	3.5	0.208	0.149	0.013	0.044	2.773	19.691
	5.5	0.229	0.169	0.055	0.048	2.659	18.585
	10	0.255	0.197	0.099	0.053	2.544	17.536
10	0.5	0.091	0.055	0	0.0124	3.147	21.402
	1.5	0.123	0.083	0	0.017	2.647	16.282
	3.5	0.149	0.109	0.013	0.021	2.378	14.009
	5.5	0.164	0.125	0.045	0.022	2.270	13.194
	10	0.183	0.144	0.076	0.024	2.158	12.419

TABLE 2
Average value of the estimates of the parameters and the corresponding MSEs (in parentheses) for $n = 50$.

Parameters			MLE		
λ	β	α	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$
0.5	1	0.5	0.558 (0.055)	1.075 (0.108)	0.598 (0.138)
		1.5	0.526 (0.037)	1.029 (0.059)	1.817 (1.097)
		2.5	0.519 (0.035)	1.019 (0.050)	3.024 (2.866)
		3.5	0.516 (0.035)	1.014 (0.046)	3.992 (3.609)
		5	0.514 (0.034)	1.012 (0.043)	5.155 (3.761)
	1.5	0.5	0.597 (0.106)	1.713 (0.534)	0.619 (0.181)
		1.5	0.547 (0.063)	1.594 (0.264)	1.880 (1.351)
		2.5	0.537 (0.057)	1.570 (0.220)	3.068 (2.962)
		3.5	0.533 (0.055)	1.558 (0.201)	4.048 (3.856)
		5	0.529 (0.053)	1.550 (0.185)	5.138 (3.657)
	2	0.5	0.669 (0.251)	2.529 (2.459)	0.641 (0.246)
		1.5	0.583 (0.114)	2.239 (0.956)	1.929 (1.555)
		2.5	0.568 (0.099)	2.186 (0.774)	3.113 (3.198)
		3.5	0.561 (0.093)	2.162 (0.695)	4.086 (4.061)
		5	0.556 (0.089)	2.143 (0.632)	5.145 (3.725)
1.5	1	0.5	1.194 (0.423)	1.713 (0.534)	0.619 (0.181)
		1.5	1.094 (0.251)	1.594 (0.264)	1.880 (1.351)
		2.5	1.074 (0.229)	1.570 (0.220)	3.068 (2.963)
		3.5	1.065 (0.220)	1.559 (0.201)	4.048 (3.856)
		5	1.059 (0.215)	1.550 (0.185)	5.138 (3.657)
	1.5	0.5	1.806 (1.031)	1.725 (0.578)	0.594 (0.142)
		1.5	1.664 (0.586)	1.609 (0.275)	1.795 (1.152)
		2.5	1.638 (0.534)	1.587 (0.228)	2.961 (2.809)
		3.5	1.627 (0.514)	1.577 (0.208)	3.903 (3.529)
		5	1.618 (0.501)	1.569 (0.192)	5.077 (3.921)
	2	0.5	2.052 (2.759)	2.575 (3.014)	0.619 (0.209)
		1.5	1.779 (1.106)	2.268 (1.038)	1.843 (1.391)
		2.5	1.738 (0.962)	2.218 (0.834)	3.001 (3.014)
		3.5	1.719 (0.900)	2.196 (0.747)	3.905 (3.484)
		5	1.706 (0.855)	2.177 (0.677)	5.084 (3.978)

TABLE 3
Average value of the estimates of the parameters and the corresponding MSEs (in parentheses) for $n = 100$.

Parameters			MLE		
λ	β	α	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$
0.5	1	0.5	0.515 (0.018)	1.017 (0.036)	0.535 (0.041)
		1.5	0.504 (0.014)	1.003 (0.023)	1.624 (0.396)
		2.5	0.503 (0.015)	1.001 (0.021)	2.716 (1.049)
		3.5	0.502 (0.015)	1.001 (0.019)	3.805 (2.057)
		5	0.501 (0.015)	0.999 (0.019)	5.249 (3.070)
	1.5	0.5	0.524 (0.027)	1.553 (0.137)	0.541 (0.044)
		1.5	0.510 (0.021)	1.519 (0.090)	1.645 (0.445)
		2.5	0.508 (0.021)	1.514 (0.082)	2.753 (1.155)
		3.5	0.507 (0.022)	1.512 (0.078)	3.847 (2.193)
		5	0.507 (0.022)	1.509 (0.075)	5.271 (3.145)
	2	0.5	0.539 (0.041)	2.121 (0.407)	0.546 (0.047)
		1.5	0.519 (0.032)	2.056 (0.263)	1.654 (0.407)
		2.5	0.517 (0.031)	2.046 (0.243)	2.786 (1.269)
		3.5	0.516 (0.032)	2.042 (0.233)	3.874 (2.270)
		5	0.516 (0.032)	2.038 (0.225)	5.283 (3.198)
1.5	1	0.5	1.543 (0.154)	0.965 (0.031)	0.535 (0.041)
		1.5	1.512 (0.125)	0.953 (0.020)	1.622 (0.392)
		2.5	1.507 (0.126)	0.950 (0.018)	2.712 (1.039)
		3.5	1.505 (0.129)	0.949 (0.017)	3.806 (2.081)
		5	1.503 (0.133)	0.949 (0.016)	5.247 (3.061)
	1.5	0.5	1.575 (0.245)	1.580 (0.145)	0.541 (0.044)
		1.5	1.532 (0.195)	1.546 (0.096)	1.646 (0.447)
		2.5	1.526 (0.197)	1.540 (0.087)	2.755 (1.161)
		3.5	1.523 (0.199)	1.538 (0.083)	3.849 (2.204)
		5	1.522 (0.203)	1.536 (0.080)	5.274 (3.160)
	2	0.5	1.619 (0.379)	2.150 (0.428)	0.546 (0.047)
		1.5	1.561 (0.289)	2.084 (0.276)	1.655 (0.409)
		2.5	1.553 (0.288)	2.073 (0.255)	2.788 (1.275)
		3.5	1.551 (0.289)	2.069 (0.245)	3.877 (2.2795)
		5	1.549 (0.292)	2.065 (0.237)	5.281 (3.187)

TABLE 4
MLEs, standard errors (in parentheses) and the CIs for the fitted distributions based on data set 1.

Model	Estimates			
L(β,λ)	13.94 (15.39)	121.0 (142.7)		
	(-16.2, 44.09)	(-116.2, 358.2)		
APTL(α,β,λ)	26.96 (27.01)	2.907 (0.998)	8.499 (4.975)	
	(-25.9, 79.88)	(1.14, 4.67)	(-1.25, 18.25)	
EL(α,β,λ)	1.586 (0.280)	4.586 (2.227)	24.74 (16.68)	
	(1.037, 2.135)	(0.221, 8.951)	(-7.96, 57.45)	
GL(α,β,λ)	1.586 (0.283)	4.754(2.002)	20.59(14.09)	
	(1.031, 2.141)	(0.831, 8.677)	(-7.05, 48.21)	
BL(β,λ,a,b)	3.919(18.192)	23.93(27.34)	1.585 (0.280)	1.157 (5.024)
	(-31.7, 39.57)	(-29.6, 77.51)	(1.036, 2.134)	(-8.69, 11.01)
KuL(β,λ,a,b)	0.391 (2.386)	12.29 (17.32)	1.516 (0.228)	12.03 (87.14)
	(-4.28, 5.068)	(-21.6, 46.24)	(1.069, 1.963)	(-158.7, 182.8)

TABLE 5
Goodness-of-fit statistics of the fitted distributions for data set 1.

Model	$-L$	AIC	BIC	$CAIC$	$HQIC$	A^*	W^*	K-S	p-value
L	413.8	831.7	837.4	831.8	833.9	0.487	0.080	0.089	0.265
APTL	409.4	824.8	833.4	824.9	828.3	0.086	0.014	0.029	1.000
EL	410.1	826.1	834.7	826.3	829.6	0.190	0.028	0.034	0.999
GL	410.1	826.2	834.7	826.4	829.7	0.203	0.031	0.034	0.998
BL	410.1	828.1	839.5	828.5	832.8	0.190	0.028	0.034	0.998
KuL	409.9	827.8	839.3	828.2	832.5	0.173	0.026	0.033	0.998

TABLE 6
MLEs, standard errors (in parentheses) and the CIs for the fitted distributions based on data set 2.

Model	Estimates			
L(β,λ)	99279.8 (11863.5)	207019.4 (301.24)		
	(76017.3, 122522.3)	(206428.9, 207609.8)		
APTL(α,β,λ)	29.04 (23.22)	89.69 (163.9)	100.8 (188.2)	
	(-16.5 74.56)	(-231.6, 410.9)	(-267.9, 469.6)	
EL(α,β,λ)	1.915 (0.35)	22971.1 (3209.52)	32881.99 (162.23)	
	(1.23, 2.59)	(16680.4, 29261.8)	(32564.0, 33199.9)	
GL(α,β,λ)	1.91 (0.32)	35842.4 (6945.1)	39197.6 (151.6)	
	(1.28, 2.54)	(22230.1, 49454.8)	(38900.3, 39494.8)	
BL(β,λ,a,b)	4.97 (50.53)	169.57 (339.2)	1.93 (0.32)	31.26 (316.84)
	(-94.07, 104.01)	(-495.27, 834.42)	(1.29, 2.55)	(-589.75, 652.27)
KuL(β,λ,a,b)	2.57 (4.76)	65.06 (177.60)	1.67 (0.26)	60.57 (86.01)
	(-6.76, 11.89)	(-283.02, 413.14)	(1.17, 2.17)	(-108.02, 229.15)

TABLE 7
Goodness-of-fit statistics of the fitted distributions for data set 2.

Model	$-L$	AIC	BIC	$CAIC$	$HQIC$	A^*	W^*	K-S	p-value
L	109.3	222.6	226.9	222.8	224.3	1.126	0.186	0.192	0.019
APTL	100.6	207.1	213.5	207.5	209.6	0.626	0.102	0.091	0.671
EL	103.6	213.1	219.6	213.6	215.7	1.233	0.204	0.129	0.239
GL	102.8	211.7	218.1	212.1	214.2	1.112	0.183	0.123	0.299
BL	102.9	213.9	222.5	214.6	217.4	1.134	0.187	0.124	0.290
KuL	100.9	209.7	218.3	210.4	213.1	0.739	0.122	0.099	0.571

B. FIGURES

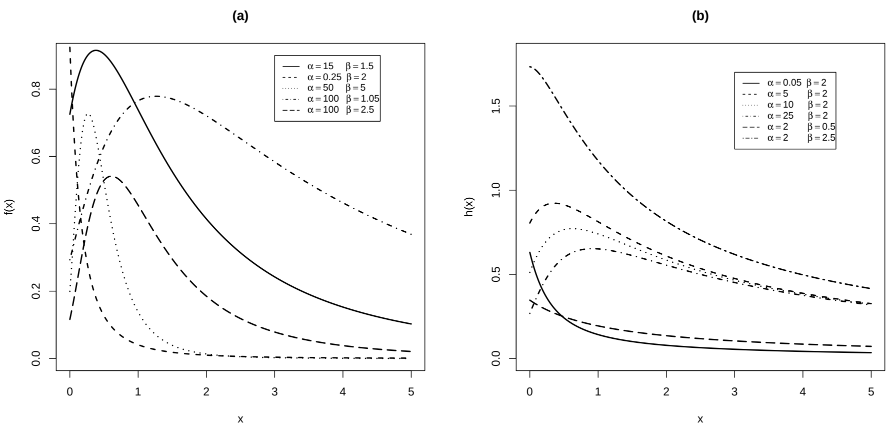

Figure 1 – Plots for density and hazard rate functions of APTL distribution with $\lambda = 1$.

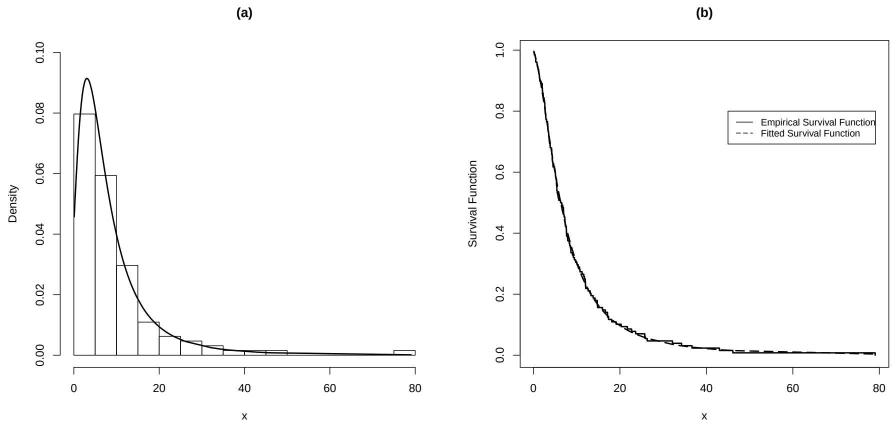

Figure 2 – Graphs of the fitted APTL density and survival functions for data set 1.

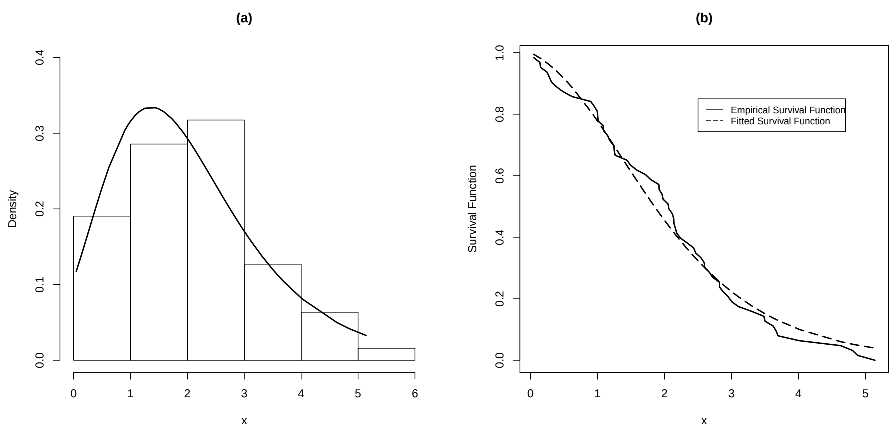

Figure 3 – Graphs of the fitted APTL density and survival functions for data set 2.

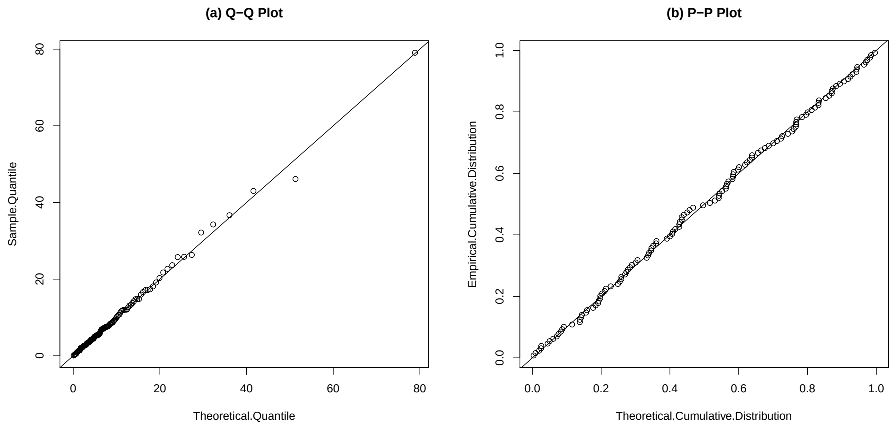

Figure 4 – P-P and Q-Q plots for the APTL distribution for data set 1.

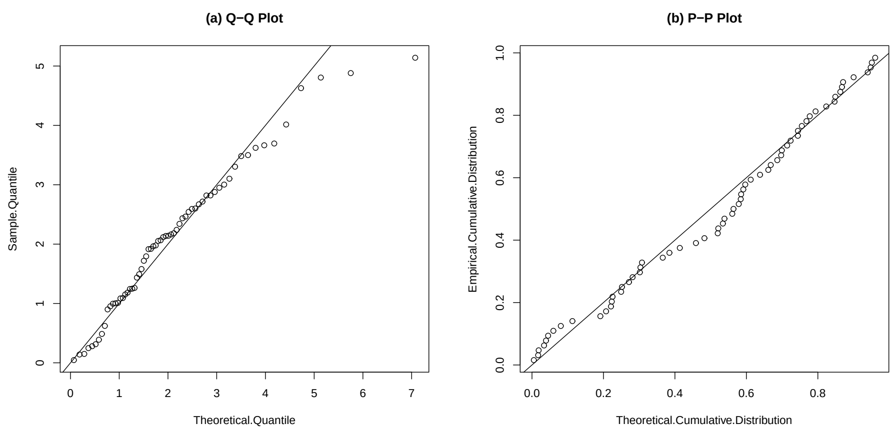

Figure 5 – P-P and Q-Q plots for the APTL distribution for data set 2.

REFERENCES

I. B. ABDUL-MONIEM, H. F. ABDEL-HAMEED (2012). *On exponentiated Lomax distribution*. International Journal of Mathematical Archive, 3, pp. 2144–2150.

B. AL-ZAHRANI (2015). *An extended Poisson-Lomax distribution*. Advances in Mathematics: Scientific Journal, 4, pp. 79–89.

B. AL-ZAHRANI, H. SAGOR (2014a). *The Poisson-Lomax distribution*. Colombian Journal of Statistics, 37, pp. 223–243.

B. AL-ZAHRANI, H. SAGOR (2014b). *Statistical analysis of the Lomax-logarithmic distribution*. Journal of Statistical Computation and Simulation, 85, pp. 1883–1901.

L. BAIN, M. ENGELHARDT (1992). *Introduction to Probability and Mathematical Statistics*. Duxbury Press, London.

S. BENNETTE (1983). *Log-logistic regression models for survival data*. Applied Statistics, 32, pp. 165–171.

C. E. BONFERRONI (1930). *Elmenti di Statistica Generale*. Libreria Seber, Firenze.

M. C. BRYSON (1974). *Heavy-tailed distributions: Properties and tests*. Technometrics, 16, pp. 61–68.

G. CASELLA, R. L. BERGER (1990). *Statistical Inference*. Duxbury Press, Belmont, CA.

G. M. CORDEIRO, E. M. M. ORTEGA, C. C. DANIEL (2013). *The exponentiated generalized class of distributions*. Journal of Data Science, 11, pp. 1–27.

S. DEY, A. ALZAATREH, C. ZHANG, D. KUMAR (2017a). *A new extension of generalized exponential distribution with application to ozone data*. OZONE: Science and Engineering, 39, pp. 273–285.

S. DEY, M. NASSAR, D. KUMAR (2017b). *Alpha logarithmic transformed family of distributions with application*. Annals of Data Sciences, 4, no. 4, pp. 457–482.

S. DEY, M. NASSAR, D. KUMAR (2019). *Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function*. Journal of Computational and Applied Mathematics, 348, pp. 130–145.

B. EFRON (1988). *Logistic regression, survival analysis, and the Kaplan-Meier curve*. Journal of the American Statistical Association, 83, pp. 414–425.

M. E. GHITANY, F. A. AL-AWADHI, L. A. ALKHALFAN (2007). *Marshall-Olkin extended Lomax distribution and its application to censored data.* Communication in Statistics-Theory and Methods, 36, pp. 1855–1866.

I. S. GRADSHTEYN, I. M. RYZHIK (2014). *Table of Integrals, Series, and Products*. Academic Press, San Diego, CA, sixth ed.

R. C. GUPTA, J. KEATING (1985). *Relations for reliability measures under length biased sampling*. Scandinavian Journal of Statistics, 13, pp. 49–56.

R. C. GUPTA, S. KIRMANI (1990). *The role of weighted distribution in stochastic modeling*. Communication in Statistics-Theory and Methods, 19, pp. 3147–3162.

R. D. GUPTA, D. KUNDU (2009). *A new class of weighted exponential distributions*. Statistics: A Journal of Theoretical and Applied Statistics, 43, pp. 621–634.

O. HOLLAND, A. GOLAUP, A. H. AGHVAMI (2006). *Traffic characteristics of aggregated module downloads for mobile terminal reconfiguration*. IEE Proceedings - Communications, 153, pp. 683–690.

M. C. JONES (2015). *On families of distributions with shape parameters*. International Statistical Review, 83, pp. 175–192.

S. KOTZ, S. NADARAJAH (2000). *Extreme Value Distributions: Theory and Applications*. Imperial College Press, London.

C. KUS (2007). *A new lifetime distribution*. Computational Statistics and Data Analysis, 51, pp. 4497–4509.

A. LANGLANDS, S. POCOCK, G. KERR, S. GORE (1997). *Long-term survival of patients with breast cancer: A study of the curability of the disease*. British Medical Journal, 2, pp. 1247–1251.

C. LEE, F. FAMOYE, A. ALZAATREH (2013). *Methods for generating families of continuous distribution in the recent decades*. Wiley Interdisciplinary Reviews: Computational Statistics, 5, pp. 219–238.

E. LEE, J. W. WANG (2003). *Statistical Methods for Survival Data Analysis*. John Wiley, New York, 3rd ed.

A. J. LEMONTE, G. M. CORDEIRO (2013). *An extended Lomax distribution*. Statistics: A Journal of Theoretical and Applied Statistics, 47, pp. 800–816.

K. C. LOMAX (1954). *Business failures: Another example of the analysis of failure data*. Journal of the American Statistical Association, 49, pp. 847–852.

A. MAHDAVI, D. KUNDU (2017). *A new method for generating distributions with an application to exponential distribution*. Communications in Statistics-Theory and Methods, 46, pp. 6543–6557.

D. N. P. MURTHY, M. XIE, R. JIANG (2004). *Weibull Models*. John Wiley, New York.

M. NASSAR, A. ALZAATREH, M. MEAD, O. ABO-KASEM (2017). *Alpha power Weibull distribution: Properties and applications*. Communications in Statistics-Theory and Methods, 46, pp. 10236–10252.

Z. M. NOFAL, A. Z. AFIFY, H. YOUSOF, G. M. CORDEIRO (2017). *The generalized transmuted-G family of distributions*. Communications in Statistics-Theory and Methods, 46, no. 8, pp. 4119–4136.

B. OLUYEDE (1999). *On inequalities and selection of experiments for length-biased distributions*. Probability in the Engineering and Informational Sciences, 13, pp. 169–185.

G. PATIL (2002). *Weighted distributions*. Encyclopedia of Environmetics, 4, pp. 2369–2377.

G. PATIL, C. RAO (1978). *Weighted distributions and size-biased sampling with application to wildlife populations and human families*. Biometrics, 34, pp. 179–189.

G. PATIL, C. RAO, M. RATNAPARKHI (1986). *On discrete weighted distributions and their use in model choice for observed data*. Communications in Statistics-Theory and Methods, 15, pp. 907–918.

F. PROSCHAN (1963). *Theoretical explanation of observed decreasing failure rate*. Technometrics, 5, pp. 375–383.

C. R. RAO (1965). *On discrete distributions arising out of methods of ascertainment*. In G. PATIL (ed.), *Classical and Contagious Discrete Distributions*, Statistical Publishing Society, Calcutta, pp. 320–333.

M. H. TAHIR, G. M. CORDEIRO, M. MANSOOR, M. ZUBAIR (2015). *The Weibull-Lomax distribution: Properties and applications*. Hacettepe Journal of Mathematics and Statistics, 44, p. 461 – 480.

SUMMARY

We introduce a new lifetime distribution, called the alpha-power transformed Lomax (APTL) distribution which generalizes the Lomax distribution to provide better fits than the Lomax distribution and some of its known generalizations. Various properties of the proposed distribution, including explicit expressions for the quantiles, mode, moments, conditional moments, mean residual lifetime, stochastic ordering, Bonferroni and Lorenz curve, stress-strength reliability and order statistics are derived. The new distribution can have a decreasing and upside-down bathtub failure rate function depending on its parameters. The maximum likelihood estimators of the three unknown parameters of APTL are obtained. A simulation study is carried out to examine the performances of the maximum likelihood estimates in terms of their mean squared error using simulated samples. Finally, the potentiality of the distribution is analyzed by means of two real data sets. For the real data sets, this distribution is found to be superior in its ability to sufficiently model both the data sets as compared to the Lomax (L) distribution, exponentiated-Lomax (EL) distribution, gamma-Lomax (GL) distribution, beta-Lomax (BL) distribution and Kumaraswamy-Lomax (KuL) distribution.

Keywords: Lomax distribution; Hazard rate function; Maximum likelihood estimation; Survival function.