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A new localization set for generalized eigenvalues

Jing Gao[1] and Chaoqian Li[2*]

*Correspondence:
lichaoqian@ynu.edu.cn
[2]School of Mathematics and
Statistics, Yunnan University,
Kunming, Yunnan 650091, China
Full list of author information is
available at the end of the article

Abstract

A new localization set for generalized eigenvalues is obtained. It is shown that the new set is tighter than that in (Numer. Linear Algebra Appl. 16:883-898, 2009). Numerical examples are given to verify the corresponding results.

MSC: 15A45; 15A48

Keywords: generalized eigenvalue; inclusion set; matrix pencil

1 Introduction

Let $\mathbb{C}^{n \times n}$ denote the set of all complex matrices of order n. For the matrices $A, B \in \mathbb{C}^{n \times n}$, we call the family of matrices $A - zB$ a matrix pencil, which is parameterized by the complex number z. Next, we regard a matrix pencil $A - zB$ as a matrix pair (A, B) [1]. A matrix pair (A, B) is called regular if $\det(A - zB) \neq 0$, and otherwise singular. A complex number λ is called a generalized eigenvalue of (A, B), if

$$\det(A - \lambda B) = 0.$$

Furthermore, we call a nonzero vector $x \in \mathbb{C}^n$ a generalized eigenvector of (A, B) associated with λ if

$$Ax = \lambda Bx.$$

Let $\sigma(A, B) = \{\lambda \in \mathbb{C} : \det(A - \lambda B) = 0\}$ denote the generalized spectrum of (A, B). Clearly, if B is an identity matrix, then $\sigma(A, B)$ reduces to the spectrum of A, i.e. $\sigma(A, B) = \sigma(A)$. When B is nonsingular, $\sigma(A, B)$ is equivalent to the spectrum of $B^{-1}A$, that is,

$$\sigma(A, B) = \sigma(B^{-1}A).$$

So, in this case, (A, B) has n generalized eigenvalues. Moreover, if B is singular, then the degree of the characteristic polynomial $\det(A - \lambda B)$ is $d < n$, so the number of generalized eigenvalues of the matrix pair (A, B) is d, and, by convention, the remaining $n - d$ eigenvalues are ∞ [1, 2].

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We now list some notation which will be used in the following. Let $N = \{1, 2, \ldots, n\}$. Given two matrices $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, we denote

$$r_i(A) = \sum_{\substack{k \in N, \\ k \neq i}} |a_{ik}|, \qquad r_i^j(A) = \sum_{\substack{k \in N, \\ k \neq i,j}} |a_{ik}|,$$

$$R_i(A, B, z) = \sum_{\substack{k \in N, \\ k \neq i}} |a_{ik} - z b_{ik}|, \qquad R_i^j(A, B, z) = \sum_{\substack{k \in N, \\ k \neq i,j}} |a_{ik} - z b_{ik}|,$$

$$\Gamma_i(A, B) = \left\{ z \in \mathbb{C} : |a_{ii} - z b_{ii}| \leq R_i(A, B, z) \right\},$$

and

$$\Phi_{ij}(A, B) = \left\{ z \in \mathbb{C} : \left| (a_{ii} - z b_{ii})(a_{jj} - z b_{jj}) - (a_{ij} - z b_{ij})(a_{ji} - z b_{ji}) \right| \right.$$
$$\left. \leq |a_{jj} - z b_{jj}| R_i^j(A, B, z) + |a_{ij} - z b_{ij}| R_j^i(A, B, z) \right\}.$$

The generalized eigenvalue problem arises in many scientific applications; see [3–5]. Many researchers are interested in the localization of all generalized eigenvalues of a matrix pair; see [1, 2, 6, 7]. In [1], Kostić *et al.* provided the following Geršgorin-type theorem of the generalized eigenvalue problem.

Theorem 1 ([1], Theorem 7) *Let $A, B \in \mathbb{C}^{n \times n}$, $n \geq 2$ and (A, B) be a regular matrix pair. Then*

$$\sigma(A, B) \subseteq \Gamma(A, B) = \bigcup_{i \in N} \Gamma_i(A, B).$$

Here, $\Gamma(A, B)$ is called the generalized Geršgorin set of a matrix pair (A, B) and $\Gamma_i(A, B)$ the ith generalized Geršgorin set. As showed in [1], $\Gamma(A, B)$ is a compact set in the complex plane if and only if B is strictly diagonally dominant (*SDD*) [8]. When B is not *SDD*, $\Gamma(A, B)$ may be an unbounded set or the entire complex plane (see Theorem 2).

Theorem 2 ([1], Theorem 8) *Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, $n \geq 2$. Then the following statements hold*:

　(i) *Let $i \in N$ be such that, for at least one $j \in N$, $b_{ij} \neq 0$. Then $\Gamma_i(A, B)$ is an unbounded set in the complex plane if and only if $|b_{ii}| \leq r_i(B)$.*

　(ii) *$\Gamma(A, B)$ is a compact set in the complex plane if and only if B is SDD, that is, $|b_{ii}| > r_i(B)$.*

　(iii) *If there is an index $i \in N$ such that both $b_{ii} = 0$ and*

$$|a_{ii}| \leq \sum_{\substack{k \in \beta(i), \\ k \neq i}} |a_{ik}|,$$

where $\beta(i) = \{j \in N : b_{ij} = 0\}$, then $\Gamma_i(A, B)$, and consequently $\Gamma(A, B)$, is the entire complex plane.

Recently, in [2], Nakatsukasa presented a different Geršgorin-type theorem to estimate all generalized eigenvalues of a matrix pair (A, B) for the case that the ith row of either

A (or *B*) is *SDD* for any $i \in N$. Although the set obtained by Nakatsukasa is simpler to compute than that in Theorem 1, the set is not tighter than that in Theorem 1 in general.

In this paper, we research the generalized eigenvalue localization for a regular matrix pair (A, B) without the restrictive assumption that the *i*th row of either *A* (or *B*) is *SDD* for any $i \in N$. By considering $Ax = \lambda Bx$ and using the triangle inequality, we give a new inclusion set for generalized eigenvalues, and then prove that this set is tighter than that in Theorem 1 (Theorem 7 of [1]). Numerical examples are given to verify the corresponding results.

2 Main results

In this section, a set is provided to locate all the generalized eigenvalue of a matrix pair. Next we compare the set obtained with the generalized Geršgorin set in Theorem 1.

2.1 A new generalized eigenvalue localization set

Theorem 3 *Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, with $n \geq 2$ and (A, B) be a regular matrix pair. Then*

$$\sigma(A, B) \subseteq \Phi(A, B) = \bigcup_{\substack{i,j=1, \\ i \neq j}}^{n} \left\{ \Phi_{ij}(A, B) \cap \Phi_{ji}(A, B) \right\}.$$

Proof For any $\lambda \in \sigma(A, B)$, let $0 \neq x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ be an associated generalized eigenvector, *i.e.*,

$$Ax = \lambda Bx. \tag{1}$$

Without loss of generality, let

$$|x_p| \geq |x_q| \geq \max\left\{ |x_i| : i \in N, i \neq p, q \right\}.$$

Then $x_p \neq 0$.

(i) If $x_q \neq 0$, then from Equality (1), we have

$$a_{pp}x_p + a_{pq}x_q + \sum_{\substack{k \in N, \\ k \neq p,q}} a_{pk}x_k = \lambda b_{pp}x_p + \lambda b_{pq}x_q + \lambda \sum_{\substack{k \in N, \\ k \neq p,q}} b_{pk}x_k$$

and

$$a_{qq}x_q + a_{qp}x_p + \sum_{\substack{k \in N, \\ k \neq q,p}} a_{qk}x_k = \lambda b_{qq}x_q + \lambda b_{qp}x_p + \lambda \sum_{\substack{k \in N, \\ k \neq q,p}} b_{qk}x_k,$$

equivalently,

$$(a_{pp} - \lambda b_{pp})x_p + (a_{pq} - \lambda b_{pq})x_q = -\sum_{\substack{k \in N, \\ k \neq p,q}} (a_{pk} - \lambda b_{pk})x_k \tag{2}$$

and

$$(a_{qq} - \lambda b_{qq})x_q + (a_{qp} - \lambda b_{qp})x_p = -\sum_{\substack{k \in N, \\ k \neq q, p}} (a_{qk} - \lambda b_{qk})x_k. \tag{3}$$

Solving for x_p and x_q in (2) and (3), we obtain

$$\begin{aligned}
&\big((a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big)x_p \\
&= -(a_{qq} - \lambda b_{qq}) \sum_{\substack{k \in N, \\ k \neq p, q}} (a_{pk} - \lambda b_{pk})x_k + (a_{pq} - \lambda b_{pq}) \sum_{\substack{k \in N, \\ k \neq q, p}} (a_{qk} - \lambda b_{qk})x_k
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\big((a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big)x_q \\
&= -(a_{pp} - \lambda b_{pp}) \sum_{\substack{k \in N, \\ k \neq q, p}} (a_{qk} - \lambda b_{qk})x_k + (a_{qp} - \lambda b_{qp}) \sum_{\substack{k \in N, \\ k \neq p, q}} (a_{pk} - \lambda b_{pk})x_k.
\end{aligned} \tag{5}$$

Taking absolute values of (4) and (5) and using the triangle inequality yield

$$\begin{aligned}
&\big|(a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big||x_p| \\
&\leq |a_{qq} - \lambda b_{qq}| \sum_{\substack{k \in N, \\ k \neq p, q}} |a_{pk} - \lambda b_{pk}||x_k| + |a_{pq} - \lambda b_{pq}| \sum_{\substack{k \in N, \\ k \neq q, p}} |a_{qk} - \lambda b_{qk}||x_k|
\end{aligned}$$

and

$$\begin{aligned}
&\big|(a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big||x_q| \\
&\leq |a_{pp} - \lambda b_{pp}| \sum_{\substack{k \in N, \\ k \neq q, p}} |a_{qk} - \lambda b_{qk}||x_k| + |a_{qp} - \lambda b_{qp}| \sum_{\substack{k \in N, \\ k \neq p, q}} |a_{pk} - \lambda b_{pk}||x_k|.
\end{aligned}$$

Since $x_p \neq 0$ and $x_q \neq 0$ are, in absolute value, the largest and second largest components of x, respectively, we divide through by their absolute values to obtain

$$\begin{aligned}
&\big|(a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big| \\
&\leq |a_{qq} - \lambda b_{qq}|R_p^q(A, B, \lambda) + |a_{pq} - \lambda b_{pq}|R_p^q(A, B, \lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\big|(a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big| \\
&\leq |a_{pp} - \lambda b_{pp}|R_q^p(A, B, \lambda) + |a_{qp} - \lambda b_{qp}|R_p^q(A, B, \lambda).
\end{aligned}$$

Hence,

$$\lambda \in \Big(\Phi_{pq}(A, B) \cap \Phi_{qp}(A, B)\Big) \subseteq \Phi(A, B).$$

(ii) If $x_q = 0$, then x_p is the only nonzero entry of x. From equality (1), we have

$$
A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_p \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1p}x_p \\ \vdots \\ a_{p-1,p}x_p \\ a_{pp}x_p \\ a_{p+1,p}x_p \\ \vdots \\ a_{np}x_p \end{pmatrix} = \lambda \begin{pmatrix} b_{1p}x_p \\ \vdots \\ b_{p-1,p}x_p \\ b_{pp}x_p \\ b_{p+1,p}x_p \\ \vdots \\ b_{np}x_p \end{pmatrix},
$$

which implies that, for any $i \in N$, $a_{ip} = \lambda b_{ip}$, i.e., $a_{ip} - \lambda b_{ip} = 0$. Hence for any $i \in N$, $i \neq p$,

$$
\lambda \in \left(\Phi_{pi}(A,B) \cap \Phi_{ip}(A,B) \right) \subseteq \Phi(A,B).
$$

From (i) and (ii), $\sigma(A,B) \subseteq \Phi(A,B)$. The proof is completed. □

Since the matrix pairs (A,B) and (A^T, B^T) have the same generalized eigenvalues, we can obtain a theorem by applying Theorem 3 to (A^T, B^T).

Theorem 4 *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, with $n \geq 2$, and (A^T, B^T) be a regular matrix pair. Then*

$$
\sigma(A,B) \subseteq \Phi\left(A^T, B^T\right).
$$

Remark 1 If B is an identity matrix, then Theorems 3 and 4 reduce to the corresponding results of [9].

Remark 2 When all entries of the ith and jth rows of the matrix B are zero, then

$$
\Phi_{ij}(A,B) = \left\{ z \in \mathbb{C} : |a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{jj}|r_i^j(A) + |a_{ij}|r_j^i(A) \right\}
$$

and

$$
\Phi_{ji}(A,B) = \left\{ z \in \mathbb{C} : |a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{ii}|r_j^i(A) + |a_{ji}|r_i^j(A) \right\}.
$$

Hence, if

$$
|a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{jj}|r_i^j(A) + |a_{ij}|r_j^i(A) \tag{6}
$$

and

$$
|a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{ii}|r_j^i(A) + |a_{ji}|r_i^j(A), \tag{7}
$$

then

$$
\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B) = \mathbb{C},
$$

otherwise,

$$\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B) = \emptyset.$$

Moreover, when inequalities (6) and (7) hold, the matrix B is singular, and $\det(A - zB)$ has degree less than n. As we are considering regular matrix pairs, the degree of the polynomial $\det(A - zB)$ has to be at least one; thus, at least one of the sets $\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B)$ has to be nonempty, implying that the set $\Phi(A,B)$ of a regular matrix pair is always nonempty.

We now establish the following properties of the set $\Phi(A,B)$.

Theorem 5 *Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, with $n \geq 2$ and (A,B) be a regular matrix pair. Then the set $\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B)$ contains zero if and only if inequalities (6) and (7) hold.*

Proof The conclusion follows directly from putting $z = 0$ in the inequalities of $\Phi_{ij}(A,B)$ and $\Phi_{ji}(A,B)$. □

Theorem 6 *Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, with $n \geq 2$ and (A,B) be a regular matrix pair. If there exist $i,j \in N$, $i \neq j$, such that*

$$b_{ii} = b_{jj} = b_{ij} = b_{ji} = 0,$$

$$|a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{jj}| \sum_{\substack{k \in \beta(i), \\ k \neq i,j}} |a_{ik}| + |a_{ij}| \sum_{\substack{k \in \beta(j), \\ k \neq j,i}} |a_{jk}|,$$

and

$$|a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{ii}| \sum_{\substack{k \in \beta(j), \\ k \neq j,i}} |a_{jk}| + |a_{ji}| \sum_{\substack{k \in \beta(i), \\ k \neq i,j}} |a_{ik}|,$$

where $\beta(i) = \{k \in N : b_{ik} = 0\}$, then $\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B)$, and consequently $\Phi(A,B)$ is the entire complex plane.

Proof The conclusion follows directly from the definitions of $\Phi_{ij}(A,B)$ and $\Phi_{ji}(A,B)$. □

2.2 Comparison with the generalized Geršgorin set

We now compare the set in Theorem 3 with the generalized Geršgorin set in Theorem 1. First, we observe two examples in which the generalized Geršgorin set is an unbounded set or the entire complex plane.

Example 1 Let

$$A = (a_{ij}) = \begin{pmatrix} -1 & 1 & 0 & 0.2 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0 & i & 1 \\ 0.2 & 0 & 0 & -i \end{pmatrix}, \qquad B = (b_{ij}) = \begin{pmatrix} 0.3 & 0.1 & 0.1 & 0.1 \\ 0 & -1 & 0.1 & 0.1 \\ 0 & 0 & i & 0.1 \\ 0.1 & 0 & 0 & -0.2i \end{pmatrix}.$$
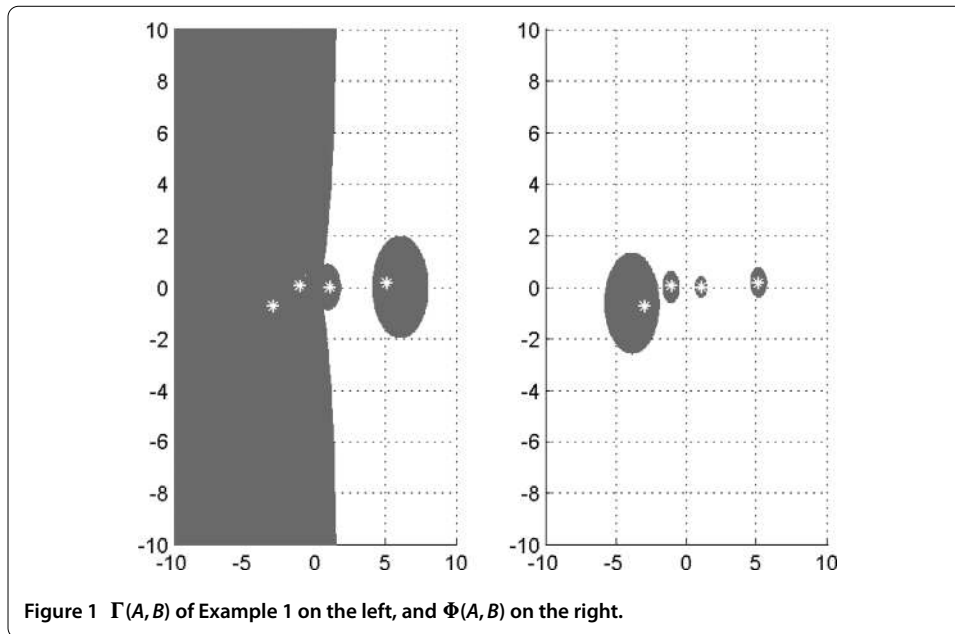
Figure 1 $\Gamma(A, B)$ of Example 1 on the left, and $\Phi(A, B)$ on the right.

It is easy to see that $b_{12} = 0.1 > 0$ and

$$|b_{11}| = \sum_{k=2,3,4} |b_{1k}| = 0.3.$$

Hence, from the part (i) of Theorem 2, we see that $\Gamma(A, B)$ is unbounded. However, the set $\Phi(A, B)$ in Theorem 3 is compact. These sets are given by Figure 1, where the actual generalized eigenvalues are plotted with asterisks. Clearly, $\Phi(A, B) \subset \Gamma(A, B)$.

Example 2 Let

$$A = (a_{ij}) = \begin{pmatrix} -1 & 1 & 0 & 0.2 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0 & i & 1 \\ 0.2 & 0 & 0 & -i \end{pmatrix}, \qquad B = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0.1 & 0.1 \\ 0 & -1 & 0.1 & 0.1 \\ 0 & 0 & i & 0.1 \\ 0.1 & 0 & 0 & -0.2i \end{pmatrix}.$$

It is easy to see that $b_{11} = 0$, $\beta(1) = \{2\}$ and

$$|a_{11}| = \sum_{\substack{k \in \beta(1), \\ k \neq 1}} |a_{1k}| = |a_{12}| = 1.$$

Hence, from the part (iii) of Theorem 2, we see that $\Gamma(A, B)$ is the entire complex plane, but the set $\Phi(A, B)$ in Theorem 3 is not. $\Phi(A, B)$ is given by Figure 2, where the actual generalized eigenvalues are plotted with asterisks.

We establish their comparison in the following.

Figure 2 $\Phi(A, B)$ of Example 2.
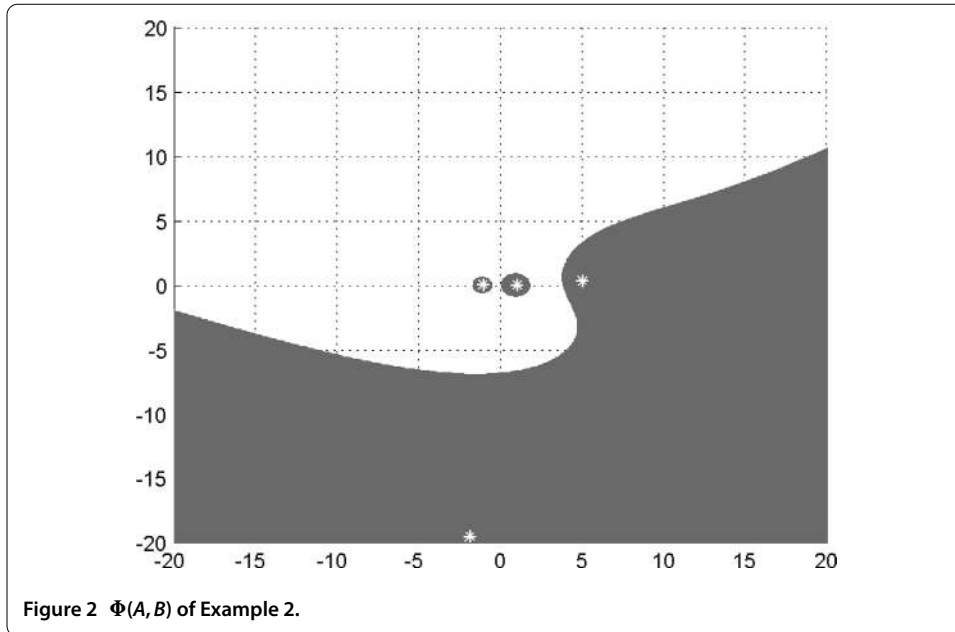
Theorem 7 *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, with $n \geq 2$ and (A, B) be a regular matrix pair. Then*

$$\Phi(A, B) \subseteq \Gamma(A, B).$$

Proof Let $z \in \Phi(A, B)$. Then there are $i, j \in N$, $i \neq j$ such that

$$z \in \left(\Phi_{ij}(A, B) \cap \Phi_{ji}(A, B) \right).$$

Next, we prove that

$$\Phi_{ij}(A, B) \subseteq \left(\Gamma_i(A, B) \cup \Gamma_j(A, B) \right) \tag{8}$$

and

$$\Phi_{ji}(A, B) \subseteq \left(\Gamma_i(A, B) \cup \Gamma_j(A, B) \right). \tag{9}$$

(i) For $z \in \Phi_{ij}(A, B)$, then $z \in \Gamma_i(A, B)$ or $z \notin \Gamma_i(A, B)$. If $z \in \Gamma_i(A, B)$, then (8) holds. If $z \notin \Gamma_i(A, B)$, that is,

$$|a_{ii} - zb_{ii}| > R_i(A, B, z), \tag{10}$$

then

$$|a_{jj} - zb_{jj}|R_i^j(A, B, z) + |a_{ij} - zb_{ij}|R_i^j(A, B, z)$$

$$\geq \left| (a_{ii} - zb_{ii})(a_{jj} - zb_{jj}) - (a_{ij} - zb_{ij})(a_{ji} - zb_{ji}) \right|$$

$$\geq |a_{ii} - zb_{ii}||a_{jj} - zb_{jj}| - |a_{ij} - zb_{ij}||a_{ji} - zb_{ji}|. \tag{11}$$

Note that $R_i^j(A,B,z) = R_i(A,B,z) - |a_{ij} - zb_{ij}|$ and $R_j^i(A,B,z) = R_j(A,B,z) - |a_{ji} - zb_{ji}|$. Then from inequalities (10) and (11), we have

$$|a_{jj} - zb_{jj}|\big(R_i(A,B,z) - |a_{ij} - zb_{ij}|\big) + |a_{ij} - zb_{ij}|\big(R_j(A,B,z) - |a_{ji} - zb_{ji}|\big)$$
$$\geq |a_{jj} - zb_{jj}|R_i(A,B,z) - |a_{ij} - zb_{ij}||a_{ji} - zb_{ji}|,$$

which implies that

$$|a_{ij} - zb_{ij}|R_j(A,B,z) \geq |a_{ij} - zb_{ij}||a_{jj} - zb_{jj}|. \tag{12}$$

If $a_{ij} = zb_{ij}$, then from $z \in \Phi_{ij}(A,B)$, we have

$$|a_{ii} - zb_{ii}||a_{jj} - zb_{jj}| \leq |a_{jj} - zb_{jj}|R_i^j(A,B,z) \leq |a_{jj} - zb_{jj}|R_i(A,B,z).$$

Moreover, from inequality (10), we obtain $|a_{jj} - zb_{jj}| = 0$. It is obvious that

$$z \in \Gamma_j(A,B) \subseteq \big(\Gamma_i(A,B) \cup \Gamma_j(A,B)\big).$$

If $a_{ij} \neq zb_{ij}$, then from inequality (12), we have

$$|a_{jj} - zb_{jj}| \leq R_j(A,B,z),$$

that is,

$$z \in \Gamma_j(A,B) \subseteq \big(\Gamma_i(A,B) \cup \Gamma_j(A,B)\big).$$

Hence, (8) holds.

(ii) Similar to the proof of (i), we also see that, for $z \in \Phi_{ji}(A,B)$, (9) holds.

The conclusion follows from (i) and (ii). □

Since the matrix pairs (A,B) and (A^T, B^T) have the same generalized eigenvalues, we can obtain a theorem by applying Theorem 7 to (A^T, B^T).

Theorem 8 *Let* $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $B = (b_{ij}) \in \mathbb{C}^{n \times n}$, *with* $n \geq 2$ *and* (A^T, B^T) *be a regular matrix pair. Then*

$$\Phi\big(A^T, B^T\big) \subseteq \Gamma\big(A^T, B^T\big).$$

Example 3 ([1], Example 1) Let

$$A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 0 & 0.2 \\ 0 & -1 & 0.4 & 0 \\ 0 & 0 & i & 1 \\ 0.2 & 0 & 0 & -i \end{pmatrix}, \qquad B = (b_{ij}) = \begin{pmatrix} 0.5 & 0.1 & 0.1 & 0.1 \\ 0 & -1 & 0.1 & 0.1 \\ 0 & 0 & i & 0.1 \\ 0.1 & 0 & 0 & -0.5i \end{pmatrix}.$$

It is easy to see that B is *SDD*. Hence, from the part (ii) of Theorem 2, we see that $\Gamma(A,B)$ is compact. $\Gamma(A,B)$ and $\Phi(A,B)$ are given by Figure 3, where the exact generalized eigenvalues are plotted with asterisks. Clearly, $\Phi(A,B) \subset \Gamma(A,B)$.

Figure 3 $\Gamma(A, B)$ of Example 3 on the left, and $\Phi(A, B)$ on the right.
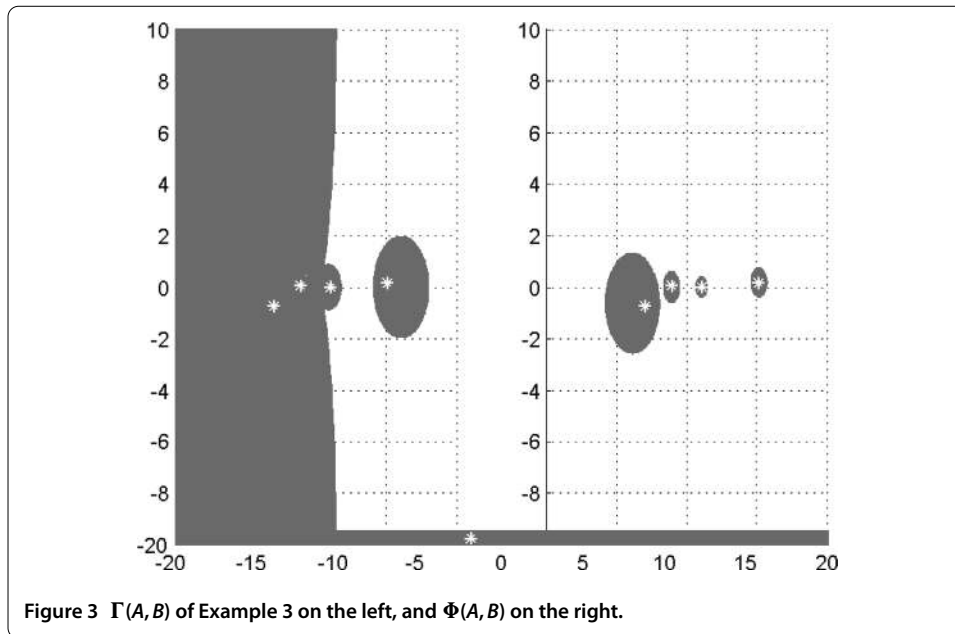
Remark 3 From Examples 1, 2 and 3, we see that the set in Theorem 3 is tighter than that in Theorem 1 (Theorem 7 of [1]). In addition, note that A and B in Example 1 satisfy

$$|a_{11}| = 1 < \sum_{k=2,3,4} |a_{1k}| = 1.2$$

and

$$|b_{11}| = \sum_{k=2,3,4} |b_{1k}| = 0.3,$$

respectively. Hence, we cannot use the method in [2] to estimate the generalized eigenvalues of the matrix pair (A,B). However, the set we obtain is very compact.

3 Conclusions

In this paper, we present a new generalized eigenvalue localization set $\Phi(A, B)$, and we establish the comparison of the sets $\Phi(A, B)$ and $\Gamma(A, B)$ in Theorem 7 of [1], that is, $\Phi(A, B)$ captures all generalized eigenvalues more precisely than $\Gamma(A, B)$, which is shown by three numerical examples.

Author details
[1]Department of Mathematics, Guangzhou Vocational College of Technology & Business, Guangzhou, Guangdong 510000, China. [2]School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, China.

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