A New Look at The Crossed-Product of a C*-algebra by an Endomorphism

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ABSTRACT. We give a new definition for the crossed-product of a C^*-algebra A by a *-endomorphism α, which depends not only on the pair (A, α) but also on the choice of a *transfer operator* (see definition below). With this we generalize some of the earlier constructions in the situations in which they behave best (e.g. for monomorphisms with hereditary range), but we get a different and perhaps more natural outcome in other situations. For example, we show that the Cuntz–Krieger algebra $\mathcal{O}_{\mathcal{A}}$ arises as the result of our construction when applied to the corresponding Markov subshift and a very natural transfer operator.

1. Introduction.

Since the appearance of [**3**], a paper of fundamental importance in which Cuntz showed that $\mathcal{O}_n$ can be regarded as the crossed-product of a UHF algebra by an endomorphism, many authors, notably Paschke [**15**], Cuntz [**4**], Doplicher and Roberts [**8**], Stacey [**21**], and Murphy [**14**], have proposed general theories of crossed-products of C^*-algebras by single as well as semigroups of endomorphisms. Most of these theories are specially well adapted to deal with a monomorphism (injective endomorphism) whose range is sometimes required to be a hereditary subalgebra.

Motivated by Renault's Cuntz groupoid [**19**: 2.1], Deaconu introduced a construction [**6**] that can be applied to some situations in which the endomorphism preserves the unit and therefore usually does not have hereditary range. This construction, and a subsequent generalization in collaboration with Muhly [**7**], is designed to be applied to monomorphisms of commutative C^*-algebras, requiring special topological properties of the map on the spectrum (branched covering) which are difficult to phrase in noncommutative situations.

In this work we propose a new definition of crossed-product that applies to most *-endomorphisms of unital C^*-algebras, generalizing some of the earlier constructions in the situations in which they behave best (e.g. monomorphisms with hereditary range), but giving a different and perhaps more natural outcome in some other situations.

The main novelty is that our construction depends not only on the endomorphism α of the C^*-algebra A but also on the choice a *transfer operator*, i.e. a positive continuous linear map $\mathcal{L} : A \to A$ such that $\mathcal{L}\big(\alpha(a)b\big) = a\mathcal{L}(b)$, for all $a, b \in A$.

Our motivation for considering transfer operators comes from classical dynamics. To be specific let X be a compact topological space and $\sigma : X \to X$ a continuous map, so that the pair (X, σ) is a classical dynamical system. Suppose, in addition, that σ is surjective and a local homeomorphism. Then the inverse image $\sigma^{-1}(\{y\})$ is finite and nonempty for each $y \in X$ and the map $\mathcal{L} : C(X) \to C(X)$ defined by

$$\mathcal{L}(f)\big|_y = \frac{1}{\#\sigma^{-1}(\{y\})} \sum_{\substack{t\in X \\ \sigma(t)=y}} f(t), \qquad \forall f \in C(X),\ y \in X,$$

is often called a transfer operator for the pair (X, σ) and has important uses in classical dynamics ([**20**], [**1**]).

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The realization that a transfer operator is a necessary ingredient for the construction of the crossed-product comes from the fact that in virtually all examples successfully treated so far the crossed-product, say of the C^*-algebra A by the endomorphism α, turns out to be a C^*-algebra B which contains A as well as an extra element S such that

$$S^*AS \subseteq A.$$

Since this takes place inside A, which is an ingredient of the construction rather than the outcome, it is only natural that one should be required to specify in advance what the map

$$\mathcal{L} : a \in A \mapsto S^*aS \in A$$

(which is necessarily positive) should be. Again the examples seem to warrant the requirement that

$$Sa = \alpha(a)S, \quad \forall a \in A,$$

and hence the axiom that $\mathcal{L}\big(\alpha(a)b\big) = a\mathcal{L}(b)$ must be accepted. If we decide to demand no further properties of our extra element S it makes sense to consider, at this point, the universal C^*-algebra generated by a copy of A and an element S subject to the above restrictions. Thanks to Blackadar's theory of C^*-algebras given by generators and relations [**2**] the universal algebra exists. We call it $\mathscr{T}(A, \alpha, \mathcal{L})$.

In case α is an automorphism of A and $\mathcal{L}$ is chosen to be the inverse of α then the fact that $\mathcal{L}(1) = 1$ implies that S must be an isometry and $\mathscr{T}(A, \alpha, \mathcal{L})$ turns out to be the Toeplitz extension of the usual crossed-product considered by Pimsner and Voiculescu in [**17**].

Since we are aiming at the crossed-product rather than the Toeplitz extension, it is clear that the algebra that we seek is not yet $\mathscr{T}(A, \alpha, \mathcal{L})$, but is likely to be a quotient of it. Quotients of universal algebras are usually produced by throwing more relations at it and hence we see that we must go in search of new relations. If we look at the literature for inspiration we will undoubtedly find numerous occurrences of the relation

$$\alpha(a) = SaS^*, \tag{1.1}$$

in which S is moreover required to be an isometry, but this has some serious draw backs, among them that only an endomorphism with hereditary range may be so implemented. Stacey [**21**] suggests

$$\alpha(a) = \sum_{i=1}^{n} S_i a S_i^*,$$

but then one has to choose in advance which n to pick. Stacey shows [**21**: 2.1.b] that n is not intrinsic to the system (A, α) and hence there seems to be no good recipe for the choice of the number of terms in the above sum.

What seems to be the answer is a somewhat subtle point which, in essence, boils down to a very interesting idea of Pimsner, namely the passage from the Toeplitz–Cuntz–Pimsner algebra $\mathscr{T}_E$ to the Cuntz–Pimsner algebra $\mathcal{O}_E$, in the context of Hilbert bimodules (see [**16**] making sure to compare Theorems (3.4) and (3.12)).

In order to describe Pimsner's idea, as adapted to the present context, consider the subspace M of $\mathscr{T}(A, \alpha, \mathcal{L})$ obtained by taking the closure of AS. Using the defining properties of S one easily verifies that

- $MA \subseteq M$,
- $AM \subseteq M$, and
- $M^*M \subseteq A$.

So M may be viewed as a Hilbert bimodule over A and, in particular, $\overline{MM^*}$ is a C^*-algebra. It follows that M is invariant by the left multiplication operators given by elements, either from A or from $\overline{MM^*}$. It may therefore happen that for a certain $a \in A$ and a certain $k \in \overline{MM^*}$ these operators agree. Equivalently

$$abS = kbS, \quad \forall b \in A.$$

In this case we call the pair (a, k) a *redundancy*.

Pimsner's idea [**16**: Theorem 3.12] may be interpreted in the present situation as the need to "eliminate the redundancies", i.e. to mod out the ideal generated by the differences $a - k$. However, due to the fact that many of Pimsner's main hypothesis are lacking here, namely the right module structure needs not be full neither must the left action of A on M be isometric, we find that eliminating all redundancies is a much too drastic approach. Our examples seem to indicate that the redundancies (a, k) which should be modded out are those for which a lies in the closed two-sided ideal generated by the range of α.

When $\alpha(1) = 1$ this ideal is the whole of A and so our approach is indeed to mod out all redundancies. But, in order to make our theory compatible with the successful theories for monomorphisms with hereditary range, we seem to be restricted to modding out only the redundancies indicated above. I must nevertheless admit that, besides the case in which α preserves the unit, I consider this as tentative, especially given the relatively small number of interesting examples so far available.

We thus define $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ to be the quotient of $\mathscr{T}(A, \alpha, \mathcal{L})$ by the closed two-sided ideal generated by the set of differences $a - k$, for all redundancies (a, k) such that $a \in \overline{A\mathcal{R}A}$, where $\mathcal{R}$ denotes the range of α.

It may be argued that our construction is not intrinsic to the C^*-dynamical system (A, α) since one needs to provide a transfer operator, which is often not unique, by hand. However in most of the examples that we have come across there seems to be a transfer operator crying out to be picked. For example, when dealing with a monomorphism α of a C^*-algebra A with hereditary range, the map

$$E : a \in A \mapsto \alpha(1)\, a\, \alpha(1) \in \mathcal{R},$$

is a conditional expectation onto $\mathcal{R}$ and the composition $\mathcal{L} = \alpha^{-1} \circ E$ is a transfer operator for (A, α). Exploring this example further we are able to show that, for any $a \in A$, the pair $\big(\alpha(a), SaS^*\big)$ is a redundancy and hence (1.1) does hold in $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$. Moreover, since $\mathcal{L}(1) = 1$, one has that S an isometry (which is not always the case). We are then able to show that $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is isomorphic to the universal C^*-algebra for an isometry satisfying equation (1.1) and hence our theory is shown to include Murphy's notion of crossed-product in the case of a single monomorphism with hereditary range (see also [**4**]). Likewise we show that Paschke's crossed-product algebras [**15**] may be recovered from our theory.

Leaving the realm of monomorphisms with hereditary range we next discuss the case of a Markov (one-sided) subshift $(\Omega_{\mathcal{A}}, \sigma)$ for a finite transition matrix $\mathcal{A}$. This clearly gives rise to the endomorphism

$$\alpha : f \in C(\Omega_{\mathcal{A}}) \mapsto f \circ \sigma \in C(\Omega_{\mathcal{A}})$$

which is unital and hence usually does not have hereditary range.

Here too one finds a very natural transfer operator and the corresponding crossed-product is shown to be the Cuntz–Krieger algebra $\mathcal{O}_A$.

It is well known that σ is surjective if and only if no column of $\mathcal{A}$ is zero, which in turn is equivalent to α being injective. Markov subshifts whose transition matrix possess trivial columns are admittedly not among the most interesting ones but our result holds just the same (with an Huef and Raeburn's definition of $\mathcal{O}_{\mathcal{A}}$ [**12**]), thus showing that the present theory deals equally well with endomorphisms which are not injective, as long as there is a nontrivial transfer operator.

The main problem left unresolved is the determination of the precise conditions under which the natural map

$$A \to A \rtimes_{\alpha,\mathcal{L}} \mathbb{N} \tag{1.2}$$

is injective, although this is so in all of the examples considered. We prove that the natural map

$$A \to \mathscr{T}(A, \alpha, \mathcal{L})$$

is injective for any transfer operator whatsoever, but it will certainly be necessary to assume special hypotheses in order for (1.2) to be injective. This is false, for instance, if one chooses the zero map to play the role of the transfer operator. But we conjecture[1] that (1.2) is injective whenever $\mathcal{L}$ is a non-degenerate transfer operator (see definition below).

There are certainly numerous other examples where our theory may be tested but in the sake of brevity we will leave these for a later occasion.

2. Transfer operators.

Throughout this section, and most of this work, we will let A be a unital C^*-algebra and $\alpha : A \to A$ be a *-endomorphism. The main concept to be introduced in this section is as follows:

2.1. Definition. A *transfer operator* for the pair (A, α) is a continuous linear map

$$\mathcal{L} : A \to A$$

such that

(i) $\mathcal{L}$ is positive in the sense that $\mathcal{L}(A_+) \subseteq A_+$, and

(ii) $\mathcal{L}\big(\alpha(a)b\big) = a\mathcal{L}(b)$, for all $a, b \in A$.

Let $\mathcal{L}$ be a transfer operator for (A, α). As a consequence of positivity it is clear that $\mathcal{L}$ is self-adjoint and hence the symmetrized version of (2.1.ii), namely $\mathcal{L}\big(a\alpha(b)\big) = \mathcal{L}(a)b$ holds as well. In particular this implies that the range of $\mathcal{L}$ is a two-sided ideal in A. Also

$$a\mathcal{L}(1) = \mathcal{L}(\alpha(a)) = \mathcal{L}(1)a, \tag{2.2}$$

for all $a \in A$, so $\mathcal{L}(1)$ is a positive central element in A. In our examples we will see that quite often, though not always, one has $\mathcal{L}(1) = 1$, in which case $\mathcal{L}$ will necessarily be surjective.

It is interesting to notice that if $a \in \mathrm{Ker}(\alpha)$ (observe that we are not assuming α to be injective), and $b \in A$ we have

$$a\mathcal{L}(b) = \mathcal{L}(\alpha(a)b) = 0,$$

so $\mathrm{Ker}(\alpha)$ is orthogonal to the range of $\mathcal{L}$. Hence, the less injective α is, the less room there is for a nontrivial transfer operator to exist (notice that the zero map is always a transfer operator).

Given the predominant role to be played by the range of α we will adopt the notation

$$\mathcal{R} := \alpha(A).$$

It is well known that $\mathcal{R}$ is a sub-C^*-algebra of A.

Define the map $E : A \to A$ by $E = \alpha \circ \mathcal{L}$, and observe that if $b \in \mathcal{R}$, say $b = \alpha(c)$ with $c \in A$, one has for all $a \in A$ that

$$E(ab) = \alpha(\mathcal{L}(a\alpha(c))) = \alpha(\mathcal{L}(a)c) = E(a)\alpha(c) = E(a)b,$$

and similarly $E(ba) = bE(a)$. Therefore E is linear for the natural $\mathcal{R}$–bimodule structure of A.

Clearly E maps A into $\mathcal{R}$ so one sees that E satisfies almost all of the axioms of a conditional expectation.

2.3. Proposition. *Let $\mathcal{L}$ be a transfer operator for the pair (A, α). Then the following are equivalent:*

(i) *the composition $E = \alpha \circ \mathcal{L}$ is a conditional expectation onto $\mathcal{R}$,*

(ii) $\alpha \circ \mathcal{L} \circ \alpha = \alpha$,

(iii) $\alpha(\mathcal{L}(1)) = \alpha(1)$.

[1] In a paper written and accepted for publication during the refereeing process of the present work we have been able to prove [**10**: Theorem 4.12] the injectivity of this map under natural hypotheses on α and $\mathcal{L}$.

Proof. Assuming (i) one has that $E|_{\mathcal{R}} = id_{\mathcal{R}}$ and hence for all $a \in A$,

$$\alpha(\mathcal{L}(\alpha(a))) = E(\alpha(a)) = \alpha(a),$$

proving (ii). Observe that plugging $a = 1$ in (2.2) we get $\mathcal{L}(1) = \mathcal{L}(\alpha(1))$. Thus, if (ii) is assumed we have

$$\alpha(1) = \alpha(\mathcal{L}(\alpha(1))) = \alpha(\mathcal{L}(1)).$$

Supposing that (iii) holds we have for all $a \in A$ that

$$\alpha(\mathcal{L}(\alpha(a))) = \alpha(\mathcal{L}(\ \alpha(a)1\ )) = \alpha(a\mathcal{L}(1)) = \alpha(a)\alpha(\mathcal{L}(1)) = \alpha(a)\alpha(1) = \alpha(a),$$

and (ii) follows. Finally it is easy to see that (ii) implies that E is the identity on $\mathcal{R}$. From this and from our discussion above we deduce (i). $\square$

This motivates our next:

2.4. Definition. We will say that a transfer operator $\mathcal{L}$ is *non-degenerate* if the equivalent conditions of (2.3) hold.

2.5. Proposition. *Let $\mathcal{L}$ be a non-degenerate transfer operator. Then A may be written as the direct sum of ideals*

$$A = \mathrm{Ker}(\alpha) \oplus \mathrm{Im}(\mathcal{L}).$$

Proof. Regardless of $\mathcal{L}$ being non-degenerate we have seen that $\mathrm{Ker}(\alpha)$ is orthogonal to the range of $\mathcal{L}$. For all $a \in A$ we have that

$$a = \big(a - \mathcal{L}\alpha(a)\big) + \mathcal{L}\alpha(a)$$

and one may easily check that if $\mathcal{L}$ is non-degenerate then $a - \mathcal{L}\alpha(a)$ lies in $\mathrm{Ker}(\alpha)$. $\square$

Let us now discuss a fairly general way to produce non-degenerate transfer operators.

2.6. Proposition. *Suppose that E is a positive conditional expectation from A onto $\mathcal{R}$. Suppose also that we are given an ideal $\mathcal{J}$ in A such that $A = \mathcal{J} \oplus \mathrm{Ker}(\alpha)$ (e.g. $\mathcal{J} = A$ when α is injective). Let $\beta : \mathcal{R} \to \mathcal{J}$ be the inverse of the restriction $\alpha|_{\mathcal{J}} : \mathcal{J} \to \mathcal{R}$. Then the composition $\mathcal{L} = \beta \circ E$ is a non-degenerate transfer operator for (A, α) which moreover satisfies $\alpha \circ \mathcal{L} = E$.*

Proof. It is clear that $\mathcal{L}$ is positive. For $a, b \in A$ we have

$$\mathcal{L}(\alpha(a)b) = \beta(E(\alpha(a)b)) = \beta(\alpha(a)E(b)).$$

We next claim that the last term above coincides with $a\beta(E(b))$. In order to see this observe that these are both in $\mathcal{J}$ where the restriction of α is 1–1. Hence the equality sought is equivalent to

$$\alpha\Big(\ \beta(\alpha(a)E(b))\ \Big) = \alpha\Big(\ a\beta(E(b))\ \Big),$$

which the reader may verify with no difficulty using that $\alpha \circ \beta$ is the identity on $\mathcal{R}$. Finally observe that

$$\alpha \circ \mathcal{L} = \alpha \circ \beta \circ E = E,$$

proving that $\mathcal{L}$ satisfies (2.3.i) and hence is non-degenerate. $\square$

Observe that when the ideal $\mathcal{J}$ mentioned in the statement of (2.6) exists it is necessarily unique because one may easily prove that

$$\mathcal{J} = \big\{a \in A : ab = 0 \text{ for all } b \in \mathrm{Ker}(\alpha)\big\}.$$

We thank the referee for pointing out this observation.

3. The Crossed-Product.

As before we will let A be a C^*-algebra and $\alpha : A \to A$ be a *-endomorphism. From now on we will also fix a transfer operator $\mathcal{L}$ for the pair (A, α). Even though for a degenerate $\mathcal{L}$ (e.g. $\mathcal{L} = 0$) the crossed-product is likely not to be very interesting, it seems that our formalism may be carried out to a certain extent without any further hypothesis on $\mathcal{L}$. In the examples we will explore, however, $\mathcal{L}$ will often be obtained by an application of (2.6) and hence will be non-degenerate.

Our next immediate goal is to introduce a C^*-algebra which will be an extension of the crossed-product algebra to be defined later.

3.1. Definition. Given a *-endomorphism α of a unital C^*-algebra A and a transfer operator $\mathcal{L}$ for (A, α) we let $\mathscr{T}(A, \alpha, \mathcal{L})$ be the universal unital C^*-algebra generated by a copy of A and an element S subject to the relations

(i) $Sa = \alpha(a)S$,

(ii) $S^*aS = \mathcal{L}(a)$,

for every $a \in A$.

For any representation of the above relations one obviously has that $\|S\| = \|\mathcal{L}(1)\|^{1/2}$, so these are *admissible relations* in the sense of [**2**], from where one deduces the existence of $\mathscr{T}(A, \alpha, \mathcal{L})$.

When $\mathcal{L}(1) = 1$ it follows that S is an isometry but, contrary to some existing notions of crossed-products ([**14**], [**15**]), it might well be that S is not even a partial isometry. Another difference with respect to the existing literature is that we do not assume that $SaS^* = \alpha(a)$, or even that $SAS^* \subseteq A$. Such conditions would knock out some of our most interesting examples.

An essential feature of the description of $\mathscr{T}(A, \alpha, \mathcal{L})$ as a universal C^*-algebra is a canonical map

$$A \to \mathscr{T}(A, \alpha, \mathcal{L})$$

(see [**2**]), but there is no a priori guarantee that this will be injective. As is the case with most objects defined by means of generators and relations there may or may not be hidden relations which force nonzero elements of A to be zero in the universal object. It is our next main goal to prove that in the present situation A embeds in $\mathscr{T}(A, \alpha, \mathcal{L})$ injectively.

Let $A_{\mathcal{L}}$ be a copy of the underlying vector space of A. Define the structure of a right A–module on $A_{\mathcal{L}}$ by

$$m \cdot a = m\alpha(a), \quad \forall m \in A_{\mathcal{L}},\ a \in A.$$

Also define an A-valued (possibly degenerate) inner-product $\langle \cdot\,, \cdot \rangle_{\mathcal{L}}$ on $A_{\mathcal{L}}$ by

$$\langle m, n \rangle_{\mathcal{L}} = \mathcal{L}(m^*n), \quad \forall m, n \in A_{\mathcal{L}}.$$

Upon modding out vectors of norm zero and completing we get a right Hilbert A–module which we denote by $M_{\mathcal{L}}$. For every a in A and m in $A_{\mathcal{L}}$ we have

$$\| \langle am, am \rangle \| = \|\mathcal{L}(m^*a^*am)\| \leq \|a\|^2 \|\mathcal{L}(m^*m)\| = \|a\|^2 \| \langle m, m \rangle \|,$$

so we see that left multiplication by a on $A_{\mathcal{L}}$ extends to a bounded operator $\phi(a) : M_{\mathcal{L}} \to M_{\mathcal{L}}$, which may be easily proven to be adjointable (recall that a map $T : M \to N$ between Hilbert modules M and N is said to be *adjointable* when there exists a map $S : N \to M$ such that $\langle T(x), y \rangle = \langle x, T(y) \rangle$ for all $x \in M$ and $y \in N$. See [**13**: 1.1.7] for more details). This in turn defines a *-homomorphism

$$\phi : A \to \mathscr{L}(M_{\mathcal{L}}),$$

thus making $M_{\mathcal{L}}$ a Hilbert bimodule over A (in the sense of [**16**]).

3.2. Lemma. *If $\mathcal{L}$ is a transfer operator for (A,α) and $n \in \mathbb{N}\cup\{0\}$ then $\mathcal{L}^n$ is a transfer operator for (A,α^n). Moreover the map*

$$\gamma_n : x \in A_{\mathcal{L}^n} \longmapsto \alpha(x) \in A_{\mathcal{L}^{n+1}}$$

extends to an adjointable linear map

$$\gamma_n : M_{\mathcal{L}^n} \to M_{\mathcal{L}^{n+1}},$$

with $\|\gamma_n\| \le \|\mathcal{L}(1)\|^{1/2}$, whose adjoint is given by

$$\gamma_n^*(x) = \mathcal{L}(x), \quad \forall x \in A_{\mathcal{L}^{n+1}}.$$

Proof. Clearly $\mathcal{L}^n$ is positive. With respect to (2.1.ii) let $a,b \in A$. By induction we have

$$\mathcal{L}^n\big(\alpha^n(a)b\big) = \mathcal{L}^{n-1}\Big(\mathcal{L}\big(\alpha(\alpha^{n-1}(a))b\big)\Big) = \mathcal{L}^{n-1}\big(\alpha^{n-1}(a)\mathcal{L}(b)\big) = a\mathcal{L}^{n-1}\big(\mathcal{L}(b)\big) = a\mathcal{L}^n(b),$$

proving that $\mathcal{L}^n$ is in fact a transfer operator for (A,α^n). Next observe that for all $x \in A_{\mathcal{L}^n}$ one has

$$\langle \alpha(x),\alpha(x)\rangle_{\mathcal{L}^{n+1}} = \mathcal{L}^n\big(\mathcal{L}(\alpha(x^*)\alpha(x))\big) = \mathcal{L}^n(x^*\mathcal{L}(1)x) \le$$

$$\le \|\mathcal{L}(1)\|\ \mathcal{L}^n(x^*x) = \|\mathcal{L}(1)\|\ \langle x,x\rangle_{\mathcal{L}^n}.$$

It follows that γ_n is bounded on $A_{\mathcal{L}^n}$ and hence extends to $M_{\mathcal{L}^n}$ with $\|\gamma_n\| \le \|\mathcal{L}(1)\|^{1/2}$.

Let $x \in A_{\mathcal{L}^n}$ and $y \in A_{\mathcal{L}^{n+1}}$. Then

$$\langle \gamma_n(x), y\rangle_{\mathcal{L}^{n+1}} = \mathcal{L}^n\big(\mathcal{L}(\alpha(x^*)y)\big) = \mathcal{L}^n\big(x^*\mathcal{L}(y)\big) = \langle x, \mathcal{L}(y)\rangle_{\mathcal{L}^n}.$$

We therefore have that

$$\|\mathcal{L}(y)\|^2_{\mathcal{L}^n} = \|\langle \mathcal{L}(y),\mathcal{L}(y)\rangle_{\mathcal{L}^n}\| = \|\langle \gamma_n(\mathcal{L}(y)), y\rangle_{\mathcal{L}^{n+1}}\| \le \|\gamma_n\|\,\|\mathcal{L}(y)\|_{\mathcal{L}^n}\|y\|_{\mathcal{L}^{n+1}},$$

which implies that $\|\mathcal{L}(y)\|_{\mathcal{L}^n} \le \|\gamma_n\|\,\|y\|_{\mathcal{L}^{n+1}}$, and hence the correspondence $y \to \mathcal{L}(y)$ extends continuously to a map from $A_{\mathcal{L}^{n+1}}$ to $A_{\mathcal{L}^n}$ which may now easily be shown to be the adjoint of γ_n. ☐

The following should now be evident:

3.3. Proposition. *Consider the Hilbert bimodule M_∞ over A given by*

$$M_\infty = \bigoplus_{n=0}^{\infty} M_{\mathcal{L}^n},$$

and define $S : M_\infty \to M_\infty$ by

$$S(x_0,x_1,\ldots) = \big(0,\gamma_0(x_0),\gamma_1(x_1),\ldots\big).$$

Then S is an adjointable map on M_∞ and its adjoint is given by

$$S^*(x_0,x_1,x_2,\ldots) = \big(\mathcal{L}(x_1),\mathcal{L}(x_2),\ldots\big),$$

whenever $x = (x_0,x_1,\ldots) \in M_\infty$ is such that $x_n \in A_{\mathcal{L}^n}$ for every n.

Proof. Left to the reader. ☐

We now come to a main technical result:

3.4. Theorem. *Let α be a *-endomorphism of a unital C^*-algebra A and let $\mathcal{L}$ be a transfer operator for (A, α). Then there exists a faithful representation ρ of A on a Hilbert space H and an operator $S \in \mathscr{B}(H)$ such that*

(i) $S\rho(a) = \rho(\alpha(a))S$,

(ii) $S^*\rho(a)S = \rho(\mathcal{L}(a))$,

for every $a \in A$.

Proof. Consider the C^*-algebra $\mathscr{L}(M_\infty)$ and view the left A–module structure of M_∞ as *-homomorphism $\rho : A \to \mathscr{L}(M_\infty)$. Observe that each $M_{\mathcal{L}^n}$ is invariant under ρ and that the restriction of ρ to $M_{\mathcal{L}^0}$ is faithful. It follows that ρ itself is faithful. Considering the operator S defined in (3.3) one may now easily prove that

$$S\rho(a) = \rho(\alpha(a))S, \quad \text{and} \quad S^*\rho(a)S = \rho(\mathcal{L}(a)), \quad \forall a \in A.$$

It now suffices to compose ρ with any faithful Hilbert space representation of $\mathscr{L}(M_\infty)$. □

As an immediate consequence we have:

3.5. Corollary. *Let α be a *-endomorphism of a unital C^*-algebra A and let $\mathcal{L}$ be a transfer operator for (A, α). Then the canonical map $A \to \mathscr{T}(A, \alpha, \mathcal{L})$ is injective.*

From now on we will view A as a sub-C^*-algebra of $\mathscr{T}(A, \alpha, \mathcal{L})$, by (3.5). The canonical representation of S within $\mathscr{T}(A, \alpha, \mathcal{L})$ will be denoted simply by S.

3.6. Definition. (Compare [**16**: Theorem 3.12]). By a *redundancy* we will mean a pair $(a, k) \in A \times \overline{ASS^*A}$ such that

$$abS = kbS,$$

for all $b \in A$.

We may now introduce our main concept. In what follows we will be mostly interested in the redundancies (a, k) for $a \in \overline{A\mathcal{R}A}$ (the closed two-sided ideal generated by $\mathcal{R} = \alpha(A)$).

3.7. Definition. The crossed-product of the unital C^*-algebra A by the *-endomorphism $\alpha : A \to A$ relative to the choice of the transfer operator $\mathcal{L}$, which we denote by $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$, or simply by $A \rtimes \mathbb{N}$ when α and $\mathcal{L}$ understood, is defined to be the quotient of $\mathscr{T}(A, \alpha, \mathcal{L})$ by the closed two-sided ideal $\mathcal{I}$ generated by the set of differences $a - k$, for all redundancies (a, k) such that $a \in \overline{A\mathcal{R}A}$.

Unfortunately we have not been able to show that the natural inclusion of A in $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is injective. We will see, however, that this is the case in a series of examples which we will now discuss. Please see also the footnote at the end of Section (1).

4. Monomorphisms with hereditary range.

We now wish to study the crossed-product under special hypotheses which have been considered in [**14**]. Let A be a unital C^*-algebra and let $\alpha : A \to A$ be a *-endomorphism. Denote by P the idempotent $\alpha(1)$ and notice that the range of α, which we keep denoting by $\mathcal{R}$, is contained in the hereditary subalgebra PAP.

4.1. Proposition. *If $\mathcal{R}$ is a hereditary subalgebra of A then $\mathcal{R} = PAP$.*

Proof. As noticed above $\mathcal{R} \subseteq PAP$. Given $a \in PAP$ with $a \geq 0$ we have that

$$0 \leq a = PaP \leq \|a\|P \in \mathcal{R},$$

which implies that $a \in \mathcal{R}$. □

Throughout this section we will suppose that $\mathcal{R}$ is hereditary and hence that $\mathcal{R} = PAP$. It follows that the map

$$E : a \in A \mapsto PaP \in \mathcal{R}$$

is a conditional expectation onto $\mathcal{R}$. We will assume moreover that α is injective so that we are under the hypothesis of (2.6) (with $\mathcal{J} = A$) and hence the composition

$$\mathcal{L} = \alpha^{-1} \circ E \tag{4.2}$$

defines a non-degenerate transfer operator for (A, α).

4.3. Proposition. *Suppose that α is injective, that $\mathcal{R}$ is hereditary, and that $\mathcal{L}$ is given as above. Then:*

(i) *The canonical element $S \in \mathscr{T}(A, \alpha, \mathcal{L})$ is an isometry, and hence also its image $\dot{S} \in A \rtimes \mathbb{N}$.*

(ii) *For every $a \in A$ one has that $\big(\alpha(a), SaS^*\big)$ is a redundancy.*

Proof. With respect to (i) we have by (3.1.ii) that

$$S^*S = \mathcal{L}(1) = \alpha^{-1}(E(1)) = \alpha^{-1}(P) = \alpha^{-1}(\alpha(1)) = 1.$$

In order to prove (ii) let $b \in A$ and notice that

$$SaS^*bS = Sa\alpha^{-1}(E(b)) = \alpha(a)E(b)S = \alpha(a)PbPS = \alpha(a)bS,$$

where we have used that $PS = \alpha(1)S = S1 = S$. It remains to notice that $SaS^* = \alpha(a)SS^* \in \overline{ASS^*A}$. □

4.4. Definition. We will denote by $\mathscr{U}(A, \alpha)$ the universal unital C^*-algebra generated by A and an isometry T subject to the relation $\alpha(a) = TaT^*$, for every $a \in A$.

$\mathscr{U}(A, \alpha)$ has been proposed ([**4**], [**21**], [**14**]) as the definition for the crossed-product of A by α. It is easy to see that, under the present hypothesis that α is a monomorphism, A embeds in $\mathscr{U}(A, \alpha)$ injectively. In fact, using the terminology of [**4**: 6.1] (where α is denoted by φ), consider a faithful representation of $A^\infty \rtimes_{\alpha^\infty} \mathbb{Z}$ on a Hilbert space H, and let U be the unitary implementing the automorphism α^∞. Then, denoting the unit of A by 1_A, one may easily prove that $T := \alpha(1_A)U1_A$ is an isometry on 1_AH, and that $\alpha(a) = TaT^*$, for all a in A. By universality there exists a canonical representation of $\mathscr{U}(A, \alpha)$ on $\mathscr{L}(1_AH)$ which is the identity on A and hence the canonical map $A \to \mathscr{U}(A, \alpha)$ must be injective (see also [**21**: Section 2] and [**14**: Section 2]).

We will therefore view A as a subalgebra of $\mathscr{U}(A, \alpha)$. As a consequence of (4.3) we have:

4.5. Corollary. *Under the hypotheses of (4.3) there exists a unique *-epimorphism*

$$\phi : \mathscr{U}(A, \alpha) \to A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$$

such that $\phi(T) = \dot{S}$, and $\phi(a) = \dot{a}$, for all $a \in A$, where $\dot{S}$ and $\dot{a}$ are the canonical images of S and a in $A \rtimes \mathbb{N}$, respectively.

Proof. By (4.3.i) $\dot{S}$ is an isometry and by (4.3.ii) we have that $\alpha(\dot{a}) = \dot{S}\dot{a}\dot{S}^*$. Hence the conclusion follows from the universal property of $\mathscr{U}(A, \alpha)$. □

On the other hand we have:

4.6. Proposition. *Under the hypotheses of (4.3) there exists a unique *-epimorphism*

$$\psi : \mathscr{T}(A, \alpha, \mathcal{L}) \to \mathscr{U}(A, \alpha)$$

such that $\psi(S) = T$, and $\psi(a) = a$, for all $a \in A$.

Proof. Observe that, for all $a \in A$,

$$Ta = TaT^*T = \alpha(a)T,$$

and

$$T^*aT = T^*TT^*aTT^*T = T^*PaPT = T^*E(a)T = T^*\alpha(\alpha^{-1}(E(a)))T =$$

$$= T^*T\mathcal{L}(a)T^*T = \mathcal{L}(a).$$

In other words, relations (3.1.i–ii) hold for A and T within $\mathscr{U}(A,\alpha)$ and hence the conclusion follows from the universal property of $\mathscr{T}(A,\alpha,\mathcal{L})$. ◻

Putting together the conclusions of (4.5) and (4.6) we see that the quotient map from $\mathscr{T}(A,\alpha,\mathcal{L})$ to $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ factors through $\mathscr{U}(A,\alpha)$ yielding the commutative diagram

$$\begin{tikzcd} \mathscr{T}(A,\alpha,\mathcal{L}) \arrow[rr] \arrow[dr, "\psi"'] & & A \rtimes_{\alpha,\mathcal{L}} \mathbb{N} \\ & \mathscr{U}(A,\alpha) \arrow[ur, "\phi"'] & \end{tikzcd}$$

in which the horizontal arrow is the quotient map. In the main result of this section we will show that ϕ is in fact an isomorphism.

4.7. Theorem. *Let A be a unital C^*-algebra and let $\alpha : A \to A$ be an injective *-endomorphism whose range is a hereditary subalgebra of A. Consider the transfer operator $\mathcal{L}$ given by*

$$\mathcal{L}(a) = \alpha^{-1}(PaP), \quad \forall a \in A,$$

*where $P = \alpha(1)$. Then the map ϕ of (4.5) is a *-isomorphism between $\mathscr{U}(A,\alpha)$ and $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.*

Proof. We begin by claiming that the map ψ of (4.6) vanishes on the ideal $\mathcal{I}$ mentioned in (3.7). In order to prove this let $(a,k) \in A \times \overline{ASS^*A}$ be a redundancy with $a \in \overline{A\mathcal{R}A}$. Therefore for all $b \in A$ one has $abS = kbS$. Applying ψ to both sides of this equation gives

$$abT = \psi(k)bT.$$

Observing that $P = \alpha(1) = TT^*$ we have for all $b, c \in A$ that

$$abPc = abTT^*c = \psi(k)bTT^*c = \psi(k)bPc.$$

It follows that $ax = \psi(k)x$ for all $x \in \overline{APA}$. Since $k \in \overline{ASS^*A}$ we have that $\psi(k) \in \overline{ATT^*A} = \overline{APA}$. Finally we have that

$$a \in \overline{A\mathcal{R}A} = \overline{APAPA} = \overline{APA}.$$

Therefore we must have that $a = \psi(k)$. It follows that $\psi(a-k) = 0$ and hence that ψ vanishes on $\mathcal{I}$ as claimed. By passage to the quotient we get a map

$$\widetilde{\psi} : A \rtimes \mathbb{N} \to \mathscr{U}(A,\alpha),$$

which one may now easily prove to be the inverse of the map ϕ of (4.5). ◻

5. Paschke's Crossed-Product.

Let us now comment on the relationship between Paschke's notion of crossed-product [**15**] and the one introduced in this work. Following [**15**] we will fix, throughout this section, a unital C^*-algebra A acting on a Hilbert space H, and an isometry T such that both TAT^* and T^*AT are contained in A. Following Paschke's notation we let $C^*(A,T)$ be the C^*-algebra of operators on H generated by A and T. One may then define an endomorphism

$$\alpha : A \to A$$

by the formula $\alpha(a) = TaT^*$. Because $T^*\alpha(a)T = a$ we have that α is necessarily injective. Let

$$P := \alpha(1) = TT^*,$$

and observe that the range of α is precisely the hereditary subalgebra $\mathcal{R} = PAP$ of A. This suggests choosing the conditional expectation

$$E : a \in A \mapsto PaP \in \mathcal{R},$$

and the transfer operator $\mathcal{L} = \alpha^{-1} \circ E$, as in (4.2).

5.1. Theorem. *Let A and T be as above and suppose that the hypothesis of* [**15**: Theorem 1] *are satisfied (that is, A is strongly amenable, T is not unitary, and there is no nontrivial ideal J of A such that $TJT^* \subseteq J$). Then $C^*(A,T)$ is canonically isomorphic to $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.*

Proof. By (4.7) it follows that there exists a *-epimorphism

$$\phi : A \rtimes_{\alpha,\mathcal{L}} \mathbb{N} \to C^*(A,T),$$

such that $\phi(\dot{a}) = a$, for all $a \in A$, and $\phi(\dot{S}) = T$. We will now use [**15**: Theorem 1] in order to prove that $A \rtimes \mathbb{N}$ is simple. By (4.3) we have that $\dot{S}$ is an isometry and that $\dot{S}A\dot{S}^* = \alpha(A) \subseteq A$, while the fact that $\dot{S}^*A\dot{S} = \mathcal{L}(A) \subseteq A$ follows from (3.1.ii).

Given that T is not unitary and that $\phi(\dot{S}) = T$, it is clear that $\dot{S}$ is not unitary either. For any ideal J of A we have that $\dot{S}J\dot{S}^* = \alpha(J) = TJT^*$, and hence there is no nontrivial ideal J of A such that $\dot{S}J\dot{S}^* \subseteq J$. By Paschke's Theorem [**15**: Theorem 1] we have that $C^*(A,\dot{S}) = A \rtimes \mathbb{N}$ is simple and hence that ϕ is an isomorphism. □

6. Cuntz–Krieger algebras as Crossed-Products.

Throughout this section we will let $\mathcal{A}$ be an $n \times n$ matrix with $\mathcal{A}(i,j) \in \{0,1\}$ for all i and j, and such that no row of $\mathcal{A}$ is identically zero. Our main goal is to show that the Cuntz–Krieger algebra $\mathcal{O}_{\mathcal{A}}$ (see [5], [**12**]) may be characterized as the crossed-product of a commutative C^*-algebra by an endomorphism arising from Markov sub-shifts.

For convenience we let

$$\mathcal{G} = \{1, 2, \ldots, n\}.$$

Considering the Cantor space $\mathcal{G}^{\mathbb{N}}$ (we let $\mathbb{N} = \{1, 2, 3, \ldots\}$ by convention) let $\Omega_{\mathcal{A}}$ be the compact subspace of $\mathcal{G}^{\mathbb{N}}$ given by

$$\Omega_{\mathcal{A}} = \{\xi = (\xi_i)_{i\in\mathbb{N}} \in \mathcal{G}^{\mathbb{N}} : \mathcal{A}(\xi_i, \xi_{i+1}) = 1, \ \forall i \in \mathbb{N}\}.$$

Let $\sigma : \Omega_{\mathcal{A}} \to \Omega_{\mathcal{A}}$ be given by

$$\sigma(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots),$$

so that $(\Omega_{\mathcal{A}}, \sigma)$ is the well known dynamical system usually referred to as the Markov sub-shift. Given $\xi \in \Omega_{\mathcal{A}}$ observe that

$$\sigma^{-1}(\{\xi\}) = \big\{(x, \xi_1, \xi_2, \ldots) : x \in \mathcal{G}, \ \mathcal{A}(x, \xi_1) = 1\big\}.$$

In particular the number of preimages of ξ is exactly the number of elements $x \in \mathcal{G}$ such that $\mathcal{A}(x, \xi_1) = 1$. In other words

$$\#\sigma^{-1}(\{\xi\}) = \sum_{x\in\mathcal{G}} \mathcal{A}(x, \xi_1).$$

From now on we will let Q be the integer-valued function on $\Omega_{\mathcal{A}}$ defined by

$$Q(\xi) = \sum_{x\in\mathcal{G}} \mathcal{A}(x, \xi_1).$$

Since Q depends only on the first component ξ_1 of ξ we have that Q is a continuous function on $\Omega_{\mathcal{A}}$. Observe also that $\sigma(\Omega_{\mathcal{A}}) = \{\xi \in \Omega_{\mathcal{A}} : Q(\xi) > 0\}$, whence $\sigma(\Omega_{\mathcal{A}})$ is a clopen set. In particular σ is surjective if and only if $Q > 0$, which in turn is equivalent to the fact that no column of $\mathcal{A}$ is zero.

Consider the endomorphism of the commutative C^*-algebra $C(\Omega_{\mathcal{A}})$ given by

$$\alpha(f) = f \circ \sigma, \quad \forall f \in C(\Omega_{\mathcal{A}}).$$

Clearly α is injective if and only if σ is surjective, which we have seen to be equivalent to the absence of trivial columns. Markov sub-shifts with a transition matrix $\mathcal{A}$ possessing identically zero columns are admittedly not among the most interesting examples. Nevertheless we will allow zero columns here mostly to illustrate that our theory deals equally well with endomorphisms which are not injective.

The kernel of α therefore consists of the ideal $C_0\left(\Omega_{\mathcal{A}} \setminus \sigma(\Omega_{\mathcal{A}})\right)$ formed by the functions f vanishing on $\sigma(\Omega_{\mathcal{A}})$. Since the latter is a clopen set we have that $C(\Omega_{\mathcal{A}})$ may be written as the direct sum of ideals

$$C_0\left(\sigma(\Omega_{\mathcal{A}})\right) \oplus \mathrm{Ker}(\alpha),$$

as required by (2.6). Accordingly we will let $\mathcal{J} = C_0\left(\sigma(\Omega_{\mathcal{A}})\right)$ and we will denote by β the inverse of the map $\alpha|_{\mathcal{J}} : \mathcal{J} \to \mathcal{R}$, where $\mathcal{R} = \alpha(C(\Omega_{\mathcal{A}}))$, as usual.

Given $f \in C(\Omega_{\mathcal{A}})$ let $E(f)$ be the function on $\Omega_{\mathcal{A}}$ defined by

$$E(f)\big|_{\xi} = \frac{1}{\#\sigma^{-1}(\{\sigma(\xi)\})} \sum_{\substack{\eta\in\Omega_{\mathcal{A}}\\ \sigma(\eta)=\sigma(\xi)}} f(\eta), \quad \forall \xi \in \Omega_{\mathcal{A}},$$

so that $E(f)\big|_{\xi}$ is the average of the values of f on the elements of $\Omega_{\mathcal{A}}$ which have the same image as ξ under σ. We leave it for the reader to prove:

6.1. Proposition. *For every $f \in C(\Omega_{\mathcal{A}})$ one has that $E(f) \in C(\Omega_{\mathcal{A}})$. Moreover E is a positive conditional expectation from $C(\Omega_{\mathcal{A}})$ to $\mathcal{R}$.*

Following (2.6) we therefore have that $\mathcal{L} = \beta \circ E$ is a transfer operator for $(C(\Omega_{\mathcal{A}}), \alpha)$. One may alternatively define $\mathcal{L}$ directly as

$$\mathcal{L}(f)\big|_{\xi} = \begin{cases} \dfrac{1}{\#\sigma^{-1}(\{\xi\})} \displaystyle\sum_{\substack{\eta\in\Omega_{\mathcal{A}}\\ \sigma(\eta)=\xi}} f(\eta), & \text{if } \xi \in \sigma(\Omega_{\mathcal{A}}), \\ \\ 0, & \text{if } \xi \notin \sigma(\Omega_{\mathcal{A}}). \end{cases}$$

It is the goal of this section to prove:

6.2. Theorem. *For every $n \times n$ matrix $\mathcal{A}$ with no zero rows one has that the Cuntz–Krieger C^*-algebra $\mathcal{O}_{\mathcal{A}}$ is isomorphic to $C(\Omega_{\mathcal{A}}) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.*

Proof. The proof will be accomplished in several steps. We begin by introducing some notation. Given $x \in \mathcal{G}$ we let Q_x and P_x be the continuous functions defined for all $\xi \in \Omega_{\mathcal{A}}$ by

$$Q_x(\xi) = \mathcal{A}(x, \xi_1), \quad \text{and} \quad P_x(\xi) = [\xi_1 = x],$$

where the brackets correspond to the obvious boolean valued function. Observe that the function Q defined above may then be written as $Q = \sum_{x\in\mathcal{G}} Q_x$. Observe also that for all $x \in \mathcal{G}$ one has that

$$Q_x = \sum_{y\in\mathcal{G}} \mathcal{A}(x, y)P_y.$$

Working within $\mathscr{T}(C(\Omega_{\mathcal{A}}), \alpha, \mathcal{L})$ let, for each $x \in \mathcal{G}$,

$$S_x = P_x\alpha(Q)^{1/2}S,$$

and observe that

$$S_x^*S_x = S^*\alpha(Q)P_xS = \mathcal{L}\left(\alpha(Q)P_x\right) = Q_x,$$

where the last step should be verified by direct computation. Since Q_x is a projection we have that S_x is a partial isometry. We now claim that for all $x \in \mathcal{G}$,

$$S_x^*S_x \equiv \sum_{y\in\mathcal{G}} \mathcal{A}(x, y)S_yS_y^* \tag{†}$$

modulo the ideal $\mathcal{I}$ of (3.7). In order to see this observe that, for $y \in \mathcal{G}$ and all $b \in C(\Omega_{\mathcal{A}})$, one has that

$$S_yS_y^*bS = P_y\alpha(Q)^{1/2}SS^*\alpha(Q)^{1/2}P_ybS = P_y\alpha(Q)^{1/2}S\mathcal{L}\left(\alpha(Q)^{1/2}P_yb\right) =$$

$$= P_y\alpha(Q)^{1/2}E\left(\alpha(Q)^{1/2}P_yb\right)S = P_y\alpha(Q)E(P_yb)S = P_ybS,$$

where the last step follows from the fact that $P_y\alpha(Q)E(P_yb) = P_yb$, which the reader may once again prove by direct computation. It follows that

$$\sum_{y\in\mathcal{G}} \mathcal{A}(x, y)S_yS_y^*bS = \sum_{y\in\mathcal{G}} \mathcal{A}(x, y)P_ybS = Q_xbS.$$

We then conclude that the pair

$$\left(Q_x, \sum_{y\in\mathcal{G}} \mathcal{A}(x, y)S_yS_y^*\right)$$

is a redundancy and, observing that $\alpha(1) = 1$ and hence the ideal generated by $\mathcal{R}$ is the whole of $C(\Omega_{\mathcal{A}})$, we have that

$$S_x^*S_x = Q_x \equiv \sum_{y\in\mathcal{G}} \mathcal{A}(x, y)S_yS_y^*$$

modulo $\mathcal{I}$, as claimed. It follows from the universal property of $\mathcal{O}_{\mathcal{A}}$ [**12**] that there exists a unique *-homomorphism

$$\phi : \mathcal{O}_{\mathcal{A}} \to C(\Omega_{\mathcal{A}}) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$

sending each canonical partial isometry generating $\mathcal{O}_{\mathcal{A}}$, which we denote by s_x, to the class of S_x in $C(\Omega_{\mathcal{A}}) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.

We now set out to define a map in the reverse direction. In order to do this recall from [**18**] (see also [**9**] and [**11**]) that $\mathcal{O}_{\mathcal{A}}$ is graded over the free group $\mathbb{F} = \mathbb{F}(\mathcal{G})$, and that the fiber over the identity group element may be naturally identified with $C(\Omega_{\mathcal{A}})$. Making this identification we have that $s_x^*s_x = Q_x$ and $s_xs_x^* = P_x$. Defining

$$s = \alpha(Q)^{-1/2}\sum_{x\in\mathcal{G}} s_x,$$

we claim that s satisfies (3.1.i–ii) with respect to $C(\Omega_{\mathcal{A}})$. Speaking of (3.1.i) one may use the partial crossed-product [**11**] structure of $\mathcal{O}_{\mathcal{A}}$ to show that $s_xf = \alpha(f)s_x$, for all $x \in \mathcal{G}$ and $f \in C(\Omega_{\mathcal{A}})$ so that also $sf = \alpha(f)s$. In order to verify (3.1.ii) observe that for $f \in C(\Omega_{\mathcal{A}})$ we have

$$s^*fs = \sum_{x,y\in\mathcal{G}} s_x^*\alpha(Q)^{-1}fs_y = \sum_{x\in\mathcal{G}} s_x^*\alpha(Q)^{-1}fs_x = \mathcal{L}(f),$$

where the last step may also be proven using the partial crossed-product structure of $\mathcal{O}_{\mathcal{A}}$. By the universal property of $\mathscr{T}(C(\Omega_{\mathcal{A}}),\alpha,\mathcal{L})$ there exists a unique *-homomorphism

$$\psi : \mathscr{T}(C(\Omega_{\mathcal{A}}),\alpha,\mathcal{L}) \to \mathcal{O}_{\mathcal{A}}$$

which is the identity on $C(\Omega_{\mathcal{A}})$ and satisfies $\psi(S) = s$.

We claim that ψ vanishes on the ideal $\mathcal{I}$ of (3.7). In fact let (a,k) be a redundancy and notice that since $k \in \overline{ASS^*A}$ (from now on we will denote $C(\Omega_{\mathcal{A}})$ simply by A), one has that $\psi(k) \in \overline{Ass^*A}$. Observe however that

$$Ass^*A \subseteq \sum_{x,y\in\mathcal{G}} As_xs_y^*A \subseteq \sum_{x,y\in\mathcal{G}} As_xs_y^*.$$

The subspace $As_xs_y^*$ happens to be precisely the fiber over xy^{-1} when $x \neq y$, while $\bigoplus_{x\in\mathcal{G}} As_xs_x^* = A$. It follows that $\sum_{x,y\in\mathcal{G}} As_xs_y^*$ is closed and hence $\psi(k)$ must have the form

$$\psi(k) = \sum_{x,y\in\mathcal{G}} k_{xy}s_xs_y^*,$$

where the coefficients k_{xy} are in A. By hypothesis we have that $abS = kbS$ and hence

$$abs = \psi(k)bs,$$

for all $b \in A$, which translates into

$$ab\alpha(Q)^{-1/2}\sum_{x\in\mathcal{G}} s_x = \sum_{x,y,z\in\mathcal{G}} k_{xy}s_xs_y^*b\alpha(Q)^{-1/2}s_z.$$

Substituting b for $b\alpha(Q)^{-1/2}$, observing that $s_y^*bs_z = 0$ when $y \neq z$, and projecting on the fiber over x, we have that

$$abs_x = \sum_{y\in\mathcal{G}} k_{xy}s_xs_y^*bs_y, \tag{†}$$

for all $x \in \mathcal{G}$. Taking $b = P_y$, with $y \in \mathcal{G}$, we get

$$aP_ys_x = k_{xy}s_xs_y^*P_ys_y = k_{xy}s_xQ_y.$$

When $x \neq y$ the left hand side vanishes and hence

$$k_{xy}s_xs_y^* = k_{xy}s_xQ_ys_y^* = 0.$$

It follows that

$$\psi(k) = \sum_{x\in\mathcal{G}} k_{xx}s_xs_x^* = \sum_{x\in\mathcal{G}} k_{xx}P_x,$$

and also that (†) reduces to

$$abs_x = k_{xx}s_x s_x^* b s_x.$$

Multiplying this on the right by s_x^* leads to

$$abP_x = k_{xx}s_x s_x^* b s_x s_x^* = k_{xx}bP_x.$$

With $b = 1$ we get

$$a = \sum_{x\in\mathcal{G}} aP_x = \sum_{x\in\mathcal{G}} k_{xx}P_x = \psi(k).$$

Therefore $\psi(a-k) = 0$ and hence ψ vanishes on $\mathcal{I}$ as claimed and hence factors through the quotient yielding a *-homomorphism

$$\widetilde{\psi} : C(\Omega_{\mathcal{A}}) \rtimes_{\alpha,\mathcal{L}} \mathbb{N} \to \mathcal{O}_{\mathcal{A}}$$

which may now be proven to be the inverse of ϕ. This concludes the proof. □

It should be remarked that in the last part of the proof above we have *proved* that $\psi(a-k) = 0$ for *all* redundancies. This is perhaps an indication that one should in fact mod out all redundancies in Definition (3.7) when $\alpha(1) = 1$.

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