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A NEW MATRIX THEOREM AND ITS APPLICATION FOR ESTABLISHING INDEPENDENT COORDINATES FOR COMPLEX DYNAMICAL SYSTEMS WITH CONSTRAINTS

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SUMMARY

A new method is presented by which equations of motion of a linear mechanical system can be derived in terms of independent coordinates when the system is described in terms of coordinates which are not independent but instead are governed by linear homogeneous equations of constraint. There is a discussion of the origin in practical vibrations analysis of dynamical systems involving equations of constraint. Methods previously used for handling such systems are discussed and the new method is demonstrated to have the following advantages: (1) For the most general constraint equations, solution of the equations is reduced in substance to computing the eigenvalues and eigenvectors of a symmetric matrix; and (2) the method is applicable when there are redundancies in the equations of constraint.

INTRODUCTION

The purpose of this paper is to present a method by which equations of motion of a linear mechanical system can be derived in terms of independent coordinates when basic information about the system is available in terms of coordinates which are not independent but instead are governed by linear homogeneous equations of constraint. Necessity for this derivation occurs frequently in practical vibration analysis. It arises naturally in studies of the motions of bodies composed of components which have been idealized as separate bodies. Experience in analyses of vibrations of engineering structures has convinced the authors that this method often offers decided advantages in practical computation over methods previously used.

The method is based on a mathematical theorem designated the "zero eigenvalues theorem" which allows the computational procedures to be systematically developed. A search of the mathematical literature has been made and nowhere has this result been found.

The paper begins with a background note on dynamical systems involving constraint equations. A brief discussion of approaches previously taken in treating such systems follows. The zero eigenvalues theorem is proved, and the method of this paper is discussed. There is a development of the relationship between the result obtained by the method of this paper and the result obtained by the method generally taught in engineering textbooks. Two examples of application of the theorem to problems from vibration analysis are presented and the numerical considerations involved in practical computing with the method are discussed.

SYMBOLS

A	an $R \times R$ partitional submatrix of matrix C
B	an $R \times (P - R)$ partitional submatrix of matrix C
C	a constant matrix of order $R \times P$
D	a constant matrix defined by equation (23)
E	a constant matrix defined by equation (14)
F_n	elements of vector Q in first example
G	number of positive finite elements of diagonal matrix λ
H	any nonsingular matrix of order $P - G$
I	identity matrix
K	stiffness matrix referred to coordinates q
$\overline{K}$	stiffness matrix referred to coordinates $\overline{q}$
L	Lagrangian
l	length of cylinder
M	mass matrix referred to coordinates q

$\overline{M}$ mass matrix referred to coordinates $\overline{q}$

m   axial wave number

N   number of elements in vector q

n   subscript denoting general element in vector q

P   number of elements in vector $\overline{q}$

p   subscript denoting general element in vector $\overline{q}$

Q   generalized forces referred to coordinates q

$\overline{Q}$ generalized forces referred to coordinates $\overline{q}$

q   a vector whose elements are independent coordinates

$\overline{q}$ a vector whose elements are dependent coordinates

$\overline{q}^{(a)}$ a partition of $\overline{q}$ containing R elements

$\overline{q}^{(b)}$ a partition of $\overline{q}$ containing those elements not in $\overline{q}^{(a)}$

$\overline{q}_{m,o}$ coordinates associated with axisymmetric circumferential harmonic

$\overline{q}_{m,4}$ coordinate associated with fourth circumferential harmonic

R   number of rows in matrix C

r   radius of cylinder

S   quadratic form defined by equation (19)

T   a constant matrix in equation (33c) relating the coordinates $\overline{q}$ to the coordinates q

$\overline{T}$ matrix defined by equation (35)

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U	modal matrix of matrix E
u	longitudinal displacement of shell
v	an arbitrary vector of order P
$\bar{v}$	a vector defined by equation (22)
W	work of external forces
x_n	elements of vector q for first example
$\bar{z}$	a vector defined by equation (30)
$\bar{z}^{(g)}$	a vector whose elements are first G elements of $\bar{z}$
z_c	displacement of center of disk in ξ direction
α_ξ	rotation about ξ-axis
α_η	rotation about η-axis
β	a constant matrix in equation (9) relating dependent coordinates $\bar{q}$ to independent coordinates $\bar{q}^{(b)}$
δ	variational operator
ξ,η,ζ	coordinates used in second example
λ	a real diagonal matrix
$\bar{\lambda}$	a real diagonal matrix whose elements are positive elements of λ
λ_p	pth diagonal element of matrix λ
(˙)	denotes differentiation with respect to time
( )'	denotes transpose of a matrix

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$\lfloor \ \rfloor$ denotes a row matrix

$\lbrack \ \rbrack$ denotes a rectangular matrix

$\{ \ \}$ denotes a column matrix

BACKGROUND

In conventional analyses of small forced oscillations of mechanical systems, the physical system is idealized so that its configuration at any instant is determined by specification of a finite number of independent coordinates $q_1, q_2, \ldots, q_n, \ldots, q_N$. Then, with approximations allowable because of the assumed smallness of the oscillations, the Lagrangian of the system may be expressed as in reference 1 in the form

$$L = \frac{1}{2} \dot{q}' M \dot{q} - \frac{1}{2} q' K q \qquad (1)$$

where

(1) q is a column matrix the elements of which are the coordinates q_n

(2) M and K are constant symmetric matrices of order N

(3) A prime denotes the transpose of a matrix

(4) A dot denotes differentiation with respect to time.

When the Lagrangian has the form shown by equation (1) and the coordinates q_n are independent, Lagrange's equations of motion of the system have the form (see ref. 1):

$$M \ddot{q} + K q = Q \qquad (2)$$

In equation (2) Q is a column matrix with N elements. The elements of Q are usually called generalized forces. The generalized forces are determined by the following requirements: Let δq be an arbitrary infinitesimal variation of the coordinates composing the matrix q. Then the work W done by the forces applied to the system when these forces act through the displacements produced by the variation shall be given by the equation

$$W = Q' \, \delta q \qquad (3)$$

The generalized forces may be functions of the coordinates q_n and/or the time explicitly.

Once the equations of motion are known in the form indicated by equation (2), there is a well-established and very effective body of mathematical theory and computational

technique for determining the behavior of the system. Often, however, it is much easier to express L and W in terms of coordinates which are not independent but which are governed by linear homogeneous equations of constraint. (See ref. 2.) Let $\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_p, \ldots, \bar{q}_P$ represent such a set of coordinates. The constraint equations then take the form

$$C\bar{q} = 0 \tag{4}$$

where $\bar{q}$ is a column matrix the elements of which are the coordinates $\bar{q}_p$, and where C is a constant matrix which has P columns and is, in general, rectangular.

In terms of the dependent coordinates $\bar{q}_p$, the Lagrangian will take the form

$$L = \frac{1}{2} \dot{\bar{q}}' \overline{M} \dot{\bar{q}} - \frac{1}{2} \bar{q}' \overline{K} \bar{q} \tag{5}$$

where $\overline{M}$ and $\overline{K}$ are symmetric matrices of order P. The work W can be found in the form

$$W = \overline{Q}' \, \delta\bar{q} \tag{6}$$

where $\delta\bar{q}$ is an arbitrary variation of $\bar{q}$ compatible with the equations of constraint (eq. (4)) and $\overline{Q}$ is a column matrix with P elements which are functions of the coordinates $\bar{q}_p$ and/or the time explicitly.

It is useful to know a systematic procedure by which equations of motion in terms of independent coordinates, as in equation (2), can be derived by starting with the Lagrangian L and the work W in terms of coordinates governed by homogeneous equations of constraint as in equations (5) and (6). The object of this paper is to set forth such a procedure, but before doing so, it is appropriate to discuss briefly how the problem has been solved previously.

PREVIOUS METHODS

In the past, the equations of motion in terms of independent coordinates have been determined in two ways:

(1) Through consideration of particular physical or geometrical aspects of a problem, the dependent coordinates $\bar{q}_p$ are chosen to impart a very simple form to the equations of constraint, which renders easy and obvious determinations of the independent coordinates.

(2) By using one of many variants of Gauss's classical elimination algorithm, the equations of constraint are solved as simultaneous equations; these solutions lead to the selection of certain of the coordinates as independent coordinates and the expression of the remaining coordinates in terms of those which have been selected to be independent.

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Under the first category of approaches come, for example, those finite-element methods of structural analysis in which the coordinates of a free-body element are displacements and rotations at juncture points among structural elements. In such analyses the equations of constraint are equalities among appropriate displacements and rotations at junctures and equations in which appropriate displacements and rotations are set equal to zero at junctures where there are supposed to be rigid constraints. A set of independent coordinates is determined by the simple expedient of using a single symbol for each set of displacements and rotations which are equated. (See, for example, ref. 3.) This idea is the basis of the now widely used procedure of superimposing stiffness matrices or mass matrices of structural elements to determine a stiffness matrix or a mass matrix of an entire structure composed of the connected elements.

In order to illustrate some advantages of the method to be presented, the method generally taught in engineering textbooks is discussed formally. (See, for example, ref. 4.) This method belongs in the second category of approaches. It is assumed (usually tacitly) that the rank R of the matrix C is equal to the number of rows in C and that, therefore, equation (4) may be written as

$$A\bar{q}^{(a)} + B\bar{q}^{(b)} = 0 \tag{7}$$

where

(1) A is an $R \times R$ nonsingular constant matrix the columns of which are R distinct columns of C

(2) B is an $R \times (P - R)$ constant matrix the columns of which are those columns of C not included in A

(3) $\bar{q}^{(a)}$ and $\bar{q}^{(b)}$ are column matrices the elements of which are elements of $\bar{q}$ corresponding to the columns in A and B, respectively. The goal is to establish the coordinates in $\bar{q}^{(b)}$ as independent coordinates.

By renumbering the coordinates $\bar{q}_p$, it can be arranged that the first R columns of C constitute the matrix A and the last $P - R$ columns of C constitute the matrix B. Correspondingly, the elements of $\bar{q}^{(a)}$ would be the first R elements of $\bar{q}$, and the elements of $\bar{q}^{(b)}$, the last $P - R$ elements of $\bar{q}$. For convenience in the ensuing discussion, it is assumed that such a rearrangement has been made. However, as a practical matter, it is very important to note that in order to actually make a suitable rearrangement, one must be able to identify R linearly independent columns of C. This identification may not be easy.

Since A is nonsingular, an inverse of A exists and is unique. Equation (7) is satisfied therefore if, and only if,

$$\bar{q}^{(a)} = -A^{-1}B\bar{q}^{(b)} \tag{8}$$

where A^{-1} is the inverse of A. It follows that the equations of constraint (eqs. (4)) are satisfied if, and only if,

$$\bar{q} = \beta \bar{q}^{(b)}$$

(9)

where

$$\beta = \begin{bmatrix} -A^{-1}B \\ ----- \\ I \end{bmatrix}$$

(10)

In equation (10) I is an identity matrix of order $P - R$. Thus, the matrix β is a $P \times (P - R)$ matrix.

Substitution of equation (9) into equations (5) and (6) gives an expression for the Lagrangian L and the work W in terms of independent coordinates and in the forms shown by equations (1) and (3), respectively. The components of the expressions are

$$q = \bar{q}^{(b)}$$

(11a)

$$M = \beta' \overline{M} \beta$$

(11b)

$$K = \beta' \overline{K} \beta$$

(11c)

$$Q = \beta' \overline{Q}$$

(12)

where it is to be considered that the substitutions from equation (9) have made the elements of the matrix $\overline{Q}$ functions of the coordinates q_n and/or the time explicitly.

It is noted that the matrices M and K thus derived are symmetric. For emphasis, it is pointed out once more that applicability of this method is restricted to the case where the rank R of the matrix C is equal to the number of rows of C and that as a practical matter in the application, one is required to identify R linearly independent columns of the matrix C.

It is of interest to consider at this point the situation where contrary to the assumption made in the foregoing discussion, the rank R of C is less than the number of rows of C. It is natural for an analyst in idealizing a physical system to try to specify only the minimum number of equations necessary to define the system. In this event the rows of C are linearly independent and consequently the rank of C equals the number of rows. However, in stress and vibration analyses of engineering structures, experience is showing that it is possible to specify inadvertently equations which repeat the content of equations or combinations of equations previously written. In fact, it can be of great convenience to be able to accept redundant equations as may be seen from one of the examples given subsequently. Not much has been written on practical methods for

solving systems with dependent equations. However, reference 5 provides a good illustration of how dependent equations of constraint may arise in practice and also a brief discussion of an elimination approach used to solve them.

ZERO EIGENVALUES THEOREM

The objective is to prove a theorem that is the foundation of the method of this paper. Consider the equation

$$C\bar{q} = 0 \tag{13}$$

where C is a matrix with any number of columns and any number of rows. Let P be the number of columns and R the number of rows.

Let a square matrix E of order P be defined by the equation

$$E = C'C \tag{14}$$

(It may be noted that the determinant of E is the Gramian of the vectors comprising the columns of C. (See ref. 6.)) By transposing both sides of equation (14) and using the familiar rule for transposing products of matrices, it follows that

$$E' = C'(C')' = E \tag{15}$$

Therefore, E, being equal to its own transpose, is symmetric.

It is a well-known property of symmetric matrices that there exist orthogonal matrices U of order P satisfying the following equation:

$$U'EU = \lambda \tag{16}$$

where λ is a real diagonal matrix of order P. By customary usage, an orthogonal matrix having this property is called a modal matrix of E, and the numbers occupying the main diagonal of λ are called eigenvalues of E. To say that U is orthogonal means that

$$U'U = UU' = I \tag{17}$$

where the identity matrix I is of order P.

Let λ_p represent the eigenvalue at the intersection of the pth row and the pth column of λ. If any modal matrix of E is given, it is easy to construct a modal matrix of E so that

$$\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \ldots \geqq \lambda_p \tag{18}$$

since the positions of the eigenvalues can be reordered simply by reordering the columns of the given modal matrix. Henceforth, in this paper when reference is made to a modal matrix, it is to be understood that the columns are ordered so that inequality (18) holds.

The convention of the preceding paragraph being understood, it is well-known that the eigenvalue matrix λ associated with a symmetric matrix E is unique; that is, any modal matrix of E when substituted for U in equation (16) produces the same matrix λ.

Let v be an arbitrary column matrix with P elements, and let a quadratic expression S be defined by the equation

$$S = v'Ev \qquad (19)$$

Note that equation (19) can be written as

$$S = v'C'Cv \qquad (20)$$

or alternatively, by using equations (16) and (17),

$$S = v'UU'EUU'v = \bar{v}'\lambda\bar{v} \qquad (21)$$

where U is any modal matrix of E and where

$$\bar{v} = U'v \qquad (22)$$

Equation (20) shows that S cannot be negative for any non null v. Equation (21) shows that there exist non null forms of $\bar{v}$ which will make S negative if, and only if, at least one of the eigenvalues λ_p is negative. If any choice of $\bar{v}$ is given, then v given by $v = U\bar{v}$ will satisfy equation (22). (See eq. (17).) Therefore, if one or more of the eigenvalues were negative, there would exist forms of v making S negative and this condition would be a contradiction. It follows that the eigenvalues λ_p are each positive or zero.

Let an $R \times P$ matrix D be defined by the equation

$$D = CU \qquad (23)$$

Then from equations (14) and (16), it follows that

$$D'D = \lambda \qquad (24)$$

Let G be the number of the eigenvalues λ_p which are positive. Then the last $P - G$ eigenvalues are zero. It follows from equation (24) that D has the partitioned form indicated by the equation

$$D = \left[\bar{D} \vdots 0 \right] \qquad (25)$$

In equation (25) the null matrix 0 is an $R \times (P - G)$ matrix and the matrix $\bar{D}$ is an $R \times G$ matrix with mutually orthogonal columns so that

$$\bar{D}'\bar{D} = \bar{\lambda} \qquad (26)$$

where $\bar{\lambda}$ is a diagonal matrix the diagonal elements of which are the G positive eigenvalues of E as indicated by the equation

$$\bar{\lambda} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \lambda_G \end{bmatrix} \tag{27}$$

By using equation (17), the equations of constraint (eqs. (13)) may be written as

$$CUU'\bar{q} = 0 \tag{28}$$

or alternatively

$$D\bar{z} = 0 \tag{29}$$

where

$$\bar{z} = U'\bar{q} \tag{30}$$

It is clear from equation (25) that equation (29) is satisfied if the first G elements of $\bar{z}$ are zero whatever the last $P - G$ elements of $\bar{z}$ may be.

Premultiplying both sides of equation (29) by $\overline{D}'$ and substitution of equation (25) leads to the equation

$$\bar{\lambda}\bar{z}^{(g)} = 0 \tag{31}$$

where $\bar{z}^{(g)}$ is a column matrix the elements of which are the first G elements of $\bar{z}$. Thus, equation (29) cannot be satisfied unless the first G elements of $\bar{z}$ are zero.

Solving equation (30) for $\bar{q}$ gives the unique solution

$$\bar{q} = U\bar{z} \tag{32}$$

Thus, the following theorem has been proved:

<u>Theorem</u>: Consider any set of linear homogeneous equations

$$C\bar{q} = 0 \tag{33a}$$

and let the symmetric matrix E be defined by

$$E = C'C \tag{33b}$$

The most general solution of the equations may be expressed in the form

$$\bar{q} = Tq \tag{33c}$$

where T is a matrix whose columns are the columns of any modal matrix of E corresponding to eigenvalues of E which have the value zero, and where q is an arbitrary column matrix conformable with T.

COMPUTATIONAL PROCEDURE

With the basic theorem from the preceding section, the following procedural outline may be set forth. It is assumed that the generalized forces in the column matrix $\overline{Q}$ are functions of time alone. (If $\overline{Q}$ is a function of the coordinates $\overline{q}_p$ explicitly, additional substitutions will be required which depend on the functional form of $\overline{Q}$.)

Given:

(1) $\overline{K}$ and $\overline{M}$, both constant symmetric matrices of order P

(2) C, a constant matrix with P columns and any number of rows

(3) $\overline{Q}$, a column matrix with P elements each of which may be a function of time.

Object:

(1) To compute a matrix T so that:

    (a) The transformation $\overline{q} = Tq$ relates the dependent coordinates $\overline{q}$ appearing in equations (5) and (6) to a set of independent coordinates q suitable for use in equation (2)

    (b) The transformation $Q = T'\overline{Q}$ produces a matrix Q suitable for use in equation (2)

(2) To compute matrices K and M suitable for use in equations (1) and (2).

Procedure:

(1) Compute E where $E = C'C$. Then E will be symmetric of order P and positive semidefinite

(2) Compute a modal matrix U and the eigenvalues λ_p of the matrix E (where $p = 1, 2, 3, \ldots, P$). This operation is standard at modern computing installations and, in fact, is one of the most successful applications of digital computers

(3) Identify the columns of U which correspond to zero eigenvalues. This step requires attention because in principle one can fairly question the possibility of a rigorous distinction between finite eigenvalues and eigenvalues having the value zero when, as is normal, there is any roundoff error in the process by which the eigenvalues are computed. This point is discussed in the section "Comments on Numerical Aspects of Computation"

(4) Assemble a matrix the columns of which are the columns of U corresponding to the eigenvalues having the value zero. This matrix is, the required transformation matrix T. Its dimensions are P by $P - G$ where G is the number of positive eigenvalues of E

(5) Compute K and M by the formulas $K = T'\overline{K}T$ and $M = T'\overline{M}T$. Then K and M will be symmetric.

RELATION TO PREVIOUS METHOD

Equation (33c) gives the most general solution to the equations of constraint (eq. (4)). Since the matrix $\bar{q}$ appearing in equation (33c) is completely arbitrary, the solution can just as well be stated in the form

$$\bar{q} = THq = \overline{T}q \tag{34}$$

where H is any nonsingular square matrix of order $P - G$ and where

$$\overline{T} = TH \tag{35}$$

In order that a matrix $\overline{T}$ may be written as in equation (35), it is both necessary and sufficient that the columns of $\overline{T}$ constitute a set of linearly independent eigenvectors of E corresponding to the eigenvalues of E which have the value zero. The eigenvectors in $\overline{T}$ will not, in general, be orthonormal nor even orthogonal. The columns of $\overline{T}$ are orthonormal if, and only if, H is an orthogonal matrix and orthogonal if H is a diagonal matrix. Proof of these statements will not be made as they amount merely to a formal statement of the basic results of that portion of the theory of matrices which deals with repeated eigenvalues of a real symmetric matrix. (See, for example, ref. 7.) A connection may now be made between the method of this paper and the textbook method as given in reference 4.

By assuming that the column and coordinate rearrangements leading to equation (7) have been carried out, one may write

$$E = C'C = \begin{bmatrix} A' \\ \hline B' \end{bmatrix} \begin{bmatrix} A & \vdots & B \end{bmatrix} = \begin{bmatrix} A'A & \vdots & A'B \\ \hline B'A & \vdots & B'B \end{bmatrix} \tag{36}$$

It follows that

$$
E\beta = \left[\begin{array}{c|c} A'A & A'B \\ \hline B'A & B'B \end{array}\right] \left[\begin{array}{c} -A^{-1}B \\ \hline I \end{array}\right] = \left[\begin{array}{c} 0 \end{array}\right] \tag{37}
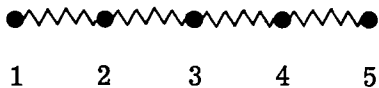$$

where in equation (37) the matrix on the right is a $P \times (P - R)$ null matrix. It is clear from equation (37) that the columns of β are linearly independent eigenvectors of E corresponding to $P - R$ eigenvalues having the value zero. The $R \times R$ submatrix $A'A$ is of rank R. The matrix E is consequently of rank R and possesses no more than $P - R$ eigenvalues with value zero.

Thus, the textbook solution, which is a variant of Gauss's classical elimination algorithm, is seen to be a solution of the form of equation (34).

FIRST EXAMPLE

In the first example, the method of this paper is applied to derive the equations of motion of a simple chain of spring-mass elements. The main intent is to illustrate an application of the method. However, some points of general interest arise.

The system consists of five point masses connected by linear massless springs as shown in sketch (1). Each of the masses and each of the spring constants are assumed to have unit magnitude. The masses may displace only in the horizontal direction and the displacement of the nth mass referred to its undeformed position is denoted by x_n. A positive value of x_n is taken to mean displacement to the right, and a negative value, displacement to the left. A horizontal external force F_n, positive to the right and a function of time only, acts upon the nth mass.

Sketch (1).- Spring-mass system.

The five displacements constitute a set of independent coordinates which determine the configuration of the system at any instant; and in terms of these coordinates, it is easy to write down directly equations of motion of the system in the form

$$
M\ddot{q} + Kq = Q \tag{38}
$$

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where

$$q = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} \tag{39a}$$

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{39b}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{39c}$$

$$Q = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} \tag{39d}$$

Equation (38) has the form of equation (2). Thus, from a practical point of view, the method of this paper is not needed for an analysis of the system since the end result of the method, equations of motion in terms of independent coordinates, is readily obtained by inspection. However, since the object is to illustrate the method, let the system be viewed in a different way as illustrated in sketch (2). There the system of sketch (1)

1   2 3   4 5   6 7   8
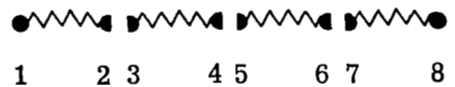
Sketch (2).- Cut system.

is shown figuratively divided into four parts by cuts at the three inner masses to produce an eight-mass system. The half circles represent masses of one-half-unit magnitude. The displacement of the pth mass of this cut system is denoted by $\bar{q}_p$.

It is assumed that three equations of constraint are imposed on the coordinates; namely,

$$\bar{q}_2 = \bar{q}_3 \tag{40a}$$

$$\bar{q}_4 = \bar{q}_5 \tag{40b}$$

$$\bar{q}_6 = \bar{q}_7 \tag{40c}$$

Thus the coordinates $\bar{q}_p$ are not independent, and from the simple geometric considerations involved, it is clear that under these equations of constraint, the systems of sketch (1) and sketch (2) are the same. In terms of the coordinates $\bar{q}_p$, a Lagrangian of the system may be expressed by an equation like equation (5) with

$$\overline{K} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{41a}$$

$$\overline{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{41b}$$

The work W for the system may be expressed by an equation following the form of equation (6) with

$$\overline{Q}' = \lfloor F_1,\, 0,\, F_2,\, 0,\, F_3,\, 0,\, F_4,\, F_5 \rfloor \tag{42}$$

It is noted that the form indicated by equation (42) for the matrix $\overline{Q}'$ is not unique. The following form, for example, will serve equally well:

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$$\overline{Q}' = \left\lfloor F_1, \; (1/2)F_2, \; (1/2)F_2, \; (1/3)F_3, \; (2/3)F_3, \; (1/5)F_4, \; (4/5)F_4, \; F_5 \right\rfloor \qquad (43)$$

All that is required is that $\overline{Q}'$ when introduced in equation (6) should yield the work done during any displacement consistent with the equations of constraint.

Equations (40), the equations of constraint, may be put in the form of equation (4) with

$$C = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \qquad (44)$$

The first step in the application of the method is to compute the matrix E defined by equation (14). This computation yields

$$C'C = E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (45)$$

The matrix U which follows is a modal matrix of the matrix E, as may be easily verified by substitution of the matrix into equations (16) and (17).

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & -1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (46)$$

The matrix λ containing the eigenvalues associated with the modal matrix is given by

$$\lambda = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{47}$$

The first three eigenvalues are finite and the last five have the value zero. Therefore the last five columns of U constitute a suitable transformation matrix T. It follows that acceptable independent coordinates for describing any configuration of the system consistent with the equations of constraint are five coordinates q_n related to the displacements $\bar{q}_p$ by the equation

$$\begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \\ \bar{q}_4 \\ \bar{q}_5 \\ \bar{q}_6 \\ \bar{q}_7 \\ \bar{q}_8 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix} \tag{48}$$

From the equation $Q = T'\overline{Q}$ it follows also by using either equation (42) or equation (43) that generalized forces suitable for use with the coordinates q_n are given by

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ \dfrac{1}{\sqrt{2}} F_2 \\ \dfrac{1}{\sqrt{2}} F_3 \\ \dfrac{1}{\sqrt{2}} F_4 \\ F_5 \end{Bmatrix} \tag{49}$$

Completing the steps in the method gives

$$M = T'\overline{M}T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{50a}$$

$$K = T'\overline{K}T = \begin{bmatrix} 1 & -1/\sqrt{2} & 0 & 0 & 0 \\ -1/\sqrt{2} & 1 & -1/2 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -1/\sqrt{2} \\ 0 & 0 & 0 & -1/\sqrt{2} & 1 \end{bmatrix} \tag{50b}$$

Equations (49), (50a), and (50b) give all the quantities necessary for writing the equations of motion for the spring-mass system in the form of equation (2). By use of equation (48), solutions of the equations giving time histories of the coordinates q_n can be transformed into time histories of the original coordinates $\bar{q}_p$. If initial conditions consistent with the equations of constraint are given in terms of the coordinates $\bar{q}_p$, the equations

$$q = T'\bar{q} \tag{51}$$

and

$$\dot{q} = T'\dot{\bar{q}} \tag{52}$$

may be used to convert them into initial conditions on the coordinates q_n.

It may be noted that the matrices K, M, and Q given in equations (50b), (50a), and (49) are not identical to the corresponding matrices in equations (39) which were written down directly from simple physical considerations. Either set of matrices forms a valid basis for equations of motion of the system of sketch (1). The difference between the matrices arises from the fact that the coordinates q_n determined by the method of this paper are not related to the coordinates $\bar{q}_p$ in the same way as are the displacement coordinates x_n. Equation (48) shows the relationship between the coordinates q_n and $\bar{q}_p$ whereas coordinates x_n and $\bar{q}_p$ are related by the equation

$$\begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \\ \bar{q}_4 \\ \bar{q}_5 \\ \bar{q}_6 \\ \bar{q}_7 \\ \bar{q}_8 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} \tag{53}$$

Equation (53) may be written

$$\bar{q} = \overline{T}x \tag{54}$$

where

$$\overline{T} = TH \tag{55}$$

in which T is the transformation matrix in equation (48), derived by the method of this paper, and

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{56}$$

Thus the coordinates x_n which represent displacements of masses are in the category of coordinates discussed in connection with equation (34). The foregoing discussion illustrates a feature of the method of this paper which should be recognized by anyone using the method; that is, the coordinates q_n produced by the method are generally abstract in character and do not lend themselves to simple physical interpretations.

It is instructive to reexamine the matrix C in equation (44) and to think about the decisions involved in applying the textbook method. Consider the three pairs of displacements $(\bar{q}_2, \bar{q}_3)$, $(\bar{q}_4, \bar{q}_5)$, and $(\bar{q}_6, \bar{q}_7)$ straddling the cuts in sketch (2). Let triplets of displacements be formed by taking one and only one displacement from each pair, for example, $(\bar{q}_2, \bar{q}_5, \bar{q}_7)$. If the displacements in any such triplet are taken to make up the elements of the column $\tilde{q}^{(a)}$ in equation (7), the matrix A formed from the corresponding columns will be nonsingular and the textbook method will succeed. If the elements of $\bar{q}^{(a)}$ are chosen from among the eight coordinates $\bar{q}$ in any other way, the

matrix A will be singular. In applying the textbook method to this simple problem, recognition of the combinations of coordinates suitable to form $\bar{q}^{(a)}$ must come about either from physical insight or from understanding of linear dependence among the columns of C. In applying the method of this paper, it is not necessary to think directly about the physics involved or about the linear dependence. Instead, the problem becomes one of finding a modal matrix of E and identifying the columns associated with eigenvalues having the value zero. Because of the block-diagonal form of E in this case, it was possible by inspection to put down exactly a modal matrix and the eigenvalues of E. Therefore, all decisions in the application of the method of this paper could be made easily on a purely mathematical basis.

<div align="center">SECOND EXAMPLE</div>

The purpose of this second example is to show a condition in which redundancies in the equations of constraint arise in a natural way. The mechanical system is shown in sketch (3). A cylindrical elastic shell is fixed at one end to an immovable base. At the other end, a thin massive rigid disk is attached to the wall of the shell by four pins placed at 90° intervals around the circumference. Points in the shell wall are assumed to displace only longitudinally.

Sketch (3).- Shell with attached disk.

By adopting an approximation common in practical vibration analysis, the longitudinal displacement u of a general point in the shell wall is expressed as a linear combination of a finite number of displacement functions. The expansion assumed is

$$u = \sum \left(\bar{q}_{m,o} + \bar{q}_{m,4} \cos 4\theta \right) \sin m \frac{\pi \zeta}{2l} \tag{57}$$

where m takes on positive integral values and the summation sign indicates summation of the terms corresponding to some finite number of selected values of m. The coefficients $\bar{q}_{m,o}$ and $\bar{q}_{m,4}$ are functions of time alone and serve as coordinates which describe the instantaneous configuration of the shell.

By assuming small displacements, the instantaneous position of the disk is determined by specification of three coordinates z_c, α_ξ, and α_η defined as follows:

(1) z_c is the displacement of the center of the disk parallel to the longitudinal axis of the shell

(2) α_ξ and α_η are small rotations about axis ξ and η, respectively, as shown in sketch (3).

Equating the displacements of the disk to the displacements of the shell at each of the four pins gives

$$\left.\begin{aligned}
z_c - r\alpha_\xi &= \sum \left(q_{m,o} + q_{m,4} \right) \sin \frac{m\pi}{2} \\[2mm]
z_c + r\alpha_\eta &= \sum \left(q_{m,o} + q_{m,4} \right) \sin \frac{m\pi}{2} \\[2mm]
z_c + r\alpha_\xi &= \sum \left(q_{m,o} + q_{m,4} \right) \sin \frac{m\pi}{2} \\[2mm]
z_c - r\alpha_\eta &= \sum \left(q_{m,o} + q_{m,4} \right) \sin \frac{m\pi}{2}
\end{aligned}\right\} \tag{58}$$

where r is the radius of the cylinder. If, in the summation on the right, only the terms corresponding to $m = 1$ are retained, the equations may be put in the form

$$C\bar{q} = 0 \tag{59}$$

where

$$C = \begin{bmatrix}
1 & 0 & -1 & -1 & -1 \\
1 & 1 & 0 & -1 & -1 \\
1 & 0 & 1 & -1 & -1 \\
1 & -1 & 0 & -1 & -1
\end{bmatrix} \tag{60}$$

and

$$\bar{q} = \begin{Bmatrix} z_c \\ r\alpha_\eta \\ r\alpha_\xi \\ q_{1,0} \\ q_{1,4} \end{Bmatrix} \tag{61}$$

It will be clear on inspection that an attempt to determine independent coordinates for this system by a straightforward application of the textbook method must fail because any choice of the matrix A will lead to a matrix which has at least two proportional columns and which is therefore singular. This difficulty stems from the fact that the system of equations is redundant; the redundancy may be demonstrated by adding rows 1 and 3 of matrix C and subtracting row 2 from the result to produce row 4.

One way to determine independent coordinates would be to discard the fourth equation from the system and apply the textbook method to the first three equations. However, this approach requires, in general, the following:

(1) Recognition in the first place that the system is redundant

(2) Identification of dependent equations

(3) Identification of a nonsingular submatrix A after redundant equations are discarded.

For the example problem under consideration, the required understanding of the structure of the equations may be gained by inspection. In practical work, however, there may be many equations of constraint involving many unknowns, and the coefficients making up the matrix C will usually not be small integers. Generally, in such situations, little of use can be deduced about the system merely by inspection of the matrix of coefficients. Also, one cannot always rely on physical insight to detect and understand redundancies. Furthermore, there are considerable theoretical and practical difficulties in making computational tests for redundancy when there is error, such as roundoff error, in the process by which the coefficients of the equations of constraint are generated. (See ref. 5.)

Proceeding now to apply the method of this paper yields the matrix E as

$$E = C'C = \begin{bmatrix} 4 & 0 & 0 & -4 & -4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ -4 & 0 & 0 & 4 & 4 \\ -4 & 0 & 0 & 4 & 4 \end{bmatrix} \tag{62}$$

The eigenvalues of E are

$$
\left.\begin{array}{l}
\lambda_1 = 12 \\
\lambda_2 = 2 \\
\lambda_3 = 2 \\
\lambda_4 = 0 \\
\lambda_5 = 0
\end{array}\right\} \tag{63}
$$

It may be easily verified that the two columns of the matrix T which follow are orthonormal eigenvectors of E corresponding to the two eigenvalues λ_4 and λ_5 which have the value zero.

$$
T = \begin{bmatrix}
1/\sqrt{6} & 1/\sqrt{2} \\
0 & 0 \\
0 & 0 \\
-1/\sqrt{6} & 1/\sqrt{2} \\
2/\sqrt{6} & 0
\end{bmatrix} \tag{64}
$$

Therefore the system may be described by two independent coordinates q_1 and q_2 related to the coordinates in $\bar{q}$ by the equation

$$
\bar{q} = Tq \tag{65}
$$

As can be seen, direct concern with the number and nature of redundancies in the equations of constraint is unnecessary when the method of this paper is used. The problem reduces in substance to that of determining a modal matrix of E and identifying the columns which correspond to eigenvalues with the value of zero.

COMMENTS ON NUMERICAL ASPECTS OF COMPUTATION

In the examples it was possible to put down exactly the matrix C, to carry out exactly the multiplication $C'C$ to produce the matrix E, and to determine exactly the eigenvalues of E and orthonormal eigenvectors corresponding to the eigenvalues with value zero. In practical work, however, numerical error due to roundoff and/or truncation may be introduced at any of these three stages of calculation. The extreme effect of such errors would, of course, be complete loss of numerical significance in the digits representing the eigenvalues of E and the elements of the eigenvectors of E. In the event the computation is subject to serious loss of significance, the matrix C is said to be "ill-conditioned" with respect to the computing process used. The best indication of ill-conditioning is sensitivity of final results to small changes in the elements of C. The authors have applied the method of this paper a number of times in practical vibration

analysis and have not encountered a situation in which the matrix C is ill-conditioned. From general experience, however, the possibility of ill-conditioning must be anticipated whenever simultaneous equations are solved numerically, and the method of this paper presents no exception to this statement. When an ill-conditioned system arises, the recourse most often open is to increase the number of digits carried in the computation. If this procedure is attempted in connection with the method of this paper, it should be recognized that it may be necessary to increase the carried significant figures in the stage of the calculation in which the elements of C are generated as well as in the implementation of the multiplication $C'C$ and in the calculation of the eigenvalues and eigenvectors of E.

Another consequence of numerical error is that finite numbers may be generated for eigenvalues of E which would be precisely zero if there were no error in the computing process. Thus, the question is raised, in principle at least, of the possibility of rigorous distinction between finite numbers representing finite eigenvalues of E and finite numbers representing eigenvalues of E which are, in fact, zero. In the authors' experience this possibility has not proved to be a problem in practice. The authors use the threshold Jacobi method (ref. 8) to compute the eigenvalues and a modal matrix of E. Approximately 15 significant figures are carried throughout the calculation. With this procedure, inspection of the eigenvalues computed for E has always revealed two clearly distinguishable sets of numbers, the numbers in one set being many orders of magnitude smaller than the numbers in the other set. The set of numbers with relatively large magnitudes are regarded as finite eigenvalues, and the remaining numbers are considered to be eigenvalues with value zero.

It is not difficult to show that the number of finite eigenvalues of E is equal to R, the rank of C. Frequently, R is known from physical or geometric considerations. In particular, it often occurs that one knows that the equations of constraint are linearly independent in which case the rank R of C is equal to the number of rows of C. Such advance information, of course, enhances confidence in the identification of zero eigenvalues.

RÉSUMÉ

The purpose of this paper is to present a new method by which equations of motion of a linear mechanical system can be derived in terms of independent coordinates when the Lagrangian of the system and the generalized forces are expressed with reference to coordinates which are not independent but instead are governed by linear homogeneous equations of constraint.

As background, there are recalled well-known mathematical forms of the Lagrangian and the work statement associated with small oscillations of mechanical systems. When the coordinates utilized to develop these expressions are independent, Lagrange's method may be applied to determine differential equations of motion of the system in a form which has been well studied and is subject to powerful methods of solution. However, as a matter of convenience, practical analysis frequently starts with coordinates which are dependent by virtue of the imposition of linear homogeneous equations of constraint. Thus, it is useful to know a relationship by which the dependent coordinates can be transformed into a set of independent coordinates.

Next, there is a discussion of methods previously used for constructing a transformation from dependent to independent coordinates. In one category of analysis, the dependent coordinates are chosen to impart a very simple form to the equations of constraint so that the transformation may be written from inspection. In a second category of approaches based on Gaussian elimination, some of the original dependent coordinates are selected to be independent coordinates and those of the original coordinates remaining are related to those selected to be independent. This procedure has the drawback that it requires in effect the identification of a square nonsingular submatrix in the matrix of the coefficients of the equations of constraint.

The third part of the paper is devoted to the basic result, a theorem which serves as a foundation for a new method for constructing the transformation from independent to dependent coordinates.

> Theorem: Let a real symmetric matrix be constructed by multiplying the matrix of coefficients in the equations of constraint by the transpose of the same matrix. Consider any modal matrix of the symmetric matrix so defined and select from the modal matrix the columns corresponding to eigenvalues with value zero. Then the matrix of these columns is a legitimate transformation relating the original dependent coordinates to a set of independent coordinates.

The advantages of constructing the transformation matrix by this method are (1) computation is reduced essentially to generating a modal matrix and the eigenvalues of a real symmetric matrix and (2) the method is applicable to systems where there are redundant equations among the equations of constraint.

Computing procedures for applying the method are given in outline form, the method is applied to two simple physical problems, and numerical considerations in the application of the method are discussed. Also a connection is made between the method of this paper and the method generally given in engineering textbooks. In addition to illustrating

the method, the examples bring out the abstract nature of the coordinates produced by the method and indicate how redundancies in the equations of constraint may arise in practical vibration analyses.

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