A New Measuring Method of Residual Stresses with the Aid of Finite Element Method and Reliability of Estimated Values†

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Abstract

In estimation of residual stresses, the existing methods are mainly based on the idea that variation of strains on the surface of the object is measured by sectioning continuously until any variation of the measured strains corresponding to the residual stresses is not observed. In this kind of methods, some definite mathematical relation between the variation of stresses and the released surface forces is required. This kind of relation was obtained only for the cases where the geometry and the boundary condition of the object and the pattern of residual stress distribution are simple. This restriction is removed when numerical methods, such as the finite element method, etc. are adopted.

In this paper, a general theory is developed based on the finite element method. With this method, three dimensional residual stresses can be measured. Furthermore, reliability of estimated values of residual stresses by this method is mathematically studied when errors are contained in the measured strains.

1. Introduction

It is very important to obtain more accurate information about residual stresses in welded structures for more rational design and safe construction of the structures in relation to weld cracks and strength of the structure.

Although there are strain gauge, X-ray, photo-elastic material etc., as directly measuring devices or instruments of strains (stresses) on the surface of the body, no method has been devised yet to measure strains in the interior. So, with the aid of these instruments, several measuring methods have been already proposed to estimate the interior stresses, for examples, by Sacks, Mather and Rosenthal[1)].

Any of these methods is by means of a relaxation method. By cutting a part of the object, a new surface is exposed and forces acting on the surface before cutting are released. These forces are estimated from the changes of stress measured on the surface. The procedure is repeated until strains do not change and residual stresses are determined by the summation of released stresses at each step. Therefore, the relation between the change of strains measured on the body surface and released surface-forces by sectioning is necessary in advance.

As it is very difficult to obtain such required relation by analytical expressions in any cases that can always express the corresponding released stress distribution to any sectioning and satisfy the boundary conditions of the body, appropriate analytical solutions or approximated expressions are chosen to the specific object in the existing measuring method. As the result, the chosen relationship is only effective within the fairly limited conditions and then in the specified cases, the three dimensional residual stresses can be estimated. Therefore, these methods are not general. If these relations are applied ignoring the restriction, reliability of the estimated value would be very poor. For these reasons, a general theory of measuring method of residual stresses has not been developed yet[2)].

However, once the above relation can be obtained generally, it is naturally possible and very efficient to estimate overall residual stress distribution of a body with a very limited number of measured strains.

As the other aspect of measurement of residual stresses, it should be considered that errors are always contained in measured strains to a certain degree and these influence the accuracy of the result. Accordingly, when the overall distribution is estimated from these data, it is necessary to investigate reliability of the estimate.

Thus, in this paper a new general theory of measuring method of residual stresses with the aid of the finite element method is proposed in order to solve the above mentioned difficulties. Furthermore, the theory is generalized to be able to apply to the cases where errors are contained in the measured strains. Applying this new method, numerical experiments are conducted for several cases taking account of random errors contained in measured strains, in order to varify the validity of the new theory and usefulness of the new method.

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2. Principle of Measuring Method of Residual Stresses

When residual stresses exist in a self-balance object, these should be produced by dislocations, which are generally called residual inherent strains.

Basically, there are two approaches to estimate residual stresses. One is a method which utilize a relation between relaxation of residual stresses at an arbitrary point and released forces on a new surface by sectioning. The other is one which directly estimated the inherent strains. In the former, released surface-forces are a part of the residual stresses which was produced by the inherent strains. Consequently, these methods are of the same nature. In contrast with this, these are so different in procedure that the former is called the released surface-force method, and the latter the inherent strain method in this paper. The theory of each approach is described in the following section.

2.1 Released Surface-Force Method

Elastic strains (ϵ_{ij}) at an arbitrary point of a self-balanced body are generally given by such a function that

$$\epsilon_{ij}\ (x) = R^*_{ij}\ (x;\ e^*,\ V) \qquad (1)$$

where x : vector of position at an interior point of the object
e* : vector of inherent strains
V : vector to experss the body shape.

If an interior plane A of the body is sectioned, the shape changes from V_1 to V_2 producing new free surfaces A_u and A_l with reference to Fig. 1. In this

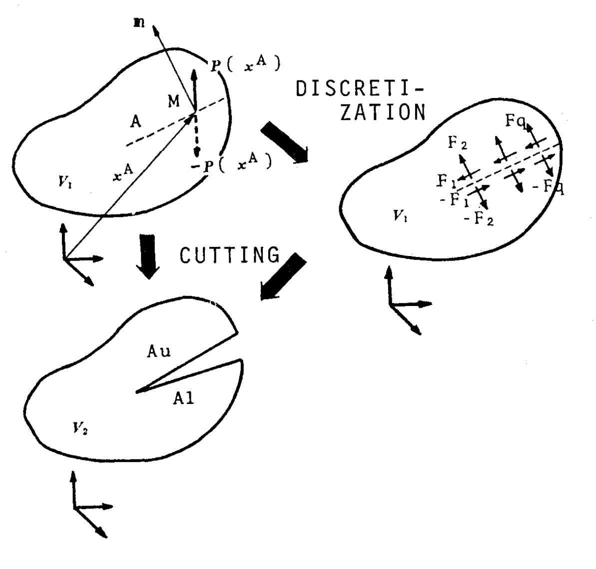


Fig. 1 Released surface-force and its discretization

process, the strain energy stored in the object generally decreases and elastic strains at each point are relaxed partially (there may be some cases where produce new plastic deformations by particular ways of sectioning, but it is assumed that any sectioning is conducted without new plastic zones). Thus,

$$\epsilon^{I}_{ij}\ (x) = R^*_{ij}\ (x;\ e^*,\ V_1) \qquad (2)$$

$$\epsilon^{II}_{ij}\ (x) = R^*_{ij}\ (x;\ e^*,\ V_2) \qquad (3)$$

Relaxed strains $\Delta\epsilon_{ij}$ are expressed,

$$\Delta\epsilon_{ij} = \epsilon^{II}_{ij} - \epsilon^{I}_{ij} = R^*_{ij}\ (x;\ e^*,\ V_2) - R^*_{ij}\ (x;\ e^*,\ V_1) \quad (4)$$

As changes in the stress distributions by sectioning are assumed to be elastic, relaxed stresses $\Delta\sigma_{ij}$ is related with $\Delta\epsilon_{ij}$ by the elasticity matrix [D] (= D_{ijkl})

$$\{\Delta\sigma_{ij}\ (x)\} = [D]\ \{-\Delta\epsilon_{kl}\ (x)\} \qquad (5)$$

Representing the normal vector on the plane A to be sectioned by $\mathbf{n} = \lfloor n_1, n_2, n_3 \rfloor^T$, forces acting on the plane A before sectioning $\mathbf{P} = \lfloor P_1, P_2, P_3 \rfloor^T$ are

$$P_i = \sigma_{ij}\ n_j = D_{ijkl}\epsilon^{I}_{kl} n_j \quad \text{at } M\ (x^A)\ \epsilon A \qquad (6)$$

when M is an arbitrary point on A.
These forces are completely released since A is reduced two free surfaces by sectioning,

$$O = D_{ijkl}\epsilon^{II}_{kl} n_j \qquad \text{at } M\ (x^A)\ \epsilon A \qquad (6)'$$

$$\therefore P_i = -D_{ijkl}\ \Delta\epsilon_{kl} n_j \qquad (7)$$

The relaxed strain $\Delta\epsilon_{ij}$ are produced by relief of the surface-force $P(x^A)$ on the sectioned plane A. So, a relation between the relaxed strain and the released surface-force is given by the following function.

$$\Delta\epsilon_{ij}\ (x) = R_{ij}\ (x;\ \mathbf{P}\ (x^A)) \qquad (8)$$

Although the distribution of released surface-force is generally expressed by a continuous function, this function may be replaced by a finite series (or approximated by discretization) to any desired degree of accuracy, which contain q number of parameters $\lfloor F \rfloor = \lfloor F_1, F_2, \ldots, F_q \rfloor$.

$$\mathbf{P}\ (x^A) = f\ (x^A;\ F_1,\ F_2,\ \cdots,\ F_q) \qquad (9)$$

By substituting Eq. (9) into Eq. (8), $\Delta\epsilon_{ij}$ are of functions of the parameters $\{F\}$ and co-ordinates x . This relation is expressed by the function h_{ij},

$$\Delta\epsilon_{ij}\ (x) = h_{ij}\ (x;\ F_1,\ F_2,\ \cdots,\ F_q) \qquad (10)$$

When the relaxed strains of q number ($\Delta_m\epsilon$) at positions where observation is possible are measured, the simultaneous equations to decide the parameters $\{F\}$ of the released surface-force are constituted as follows;

$$\left.\begin{array}{l}\Delta_m \epsilon_{IJ}(x_1) = h_{IJ}(x_1; F_1, F_2, \cdots, F_q) \\ \Delta_m \epsilon_{IJ}(x_2) = h_{IJ}(x_2; F_1, F_2, \cdots, F_q) \\ \cdots\cdots\cdots\cdots \\ \Delta_m \epsilon_{IJ}(x_q) = h_{IJ}(x_q; F_1, F_2, \cdots, F_q)\end{array}\right\} \quad (11)$$

where the combination of (I,J) represents the particular component of strains at each measuring point.

If Eqs. (11) are composed of independent equations and the inverse functions g_i of h is defined, the parameters of the released surface-forces, $\{F\}$ can be decided by Eq.(12).

$$F_i = g_i(\Delta_m \epsilon_{IJ}(x_1), \Delta_m \epsilon_{IJ}(x_2), \cdots, \Delta_m \epsilon_{IJ}(x_q)) \qquad (i = 1 \sim q) \quad (12)$$

By a combination of Eqs. (10) and (12), the relaxed strains and stresses can be estimated at arbitrary points in the entire body where strains were not measured directly.

2.2 Inherent Strain Method

The above-mentioned method is utilizing elastic response (changes of strains) at arbitrary points, which is induced by changes of the shape of the body, and Eq. (1) is a basic equation. However, this is naturally a function which expresses the relation of inherent strains to the consequent residual strains. In this section, the method by which inherent strains are decided directly and the residual strains are estimated is described.

Inherent strain distribution is replaced by a finite series (or approximated by discretization) with q number of parameters $\lfloor \epsilon^* \rfloor = \lfloor \epsilon^*_1, \epsilon^*_2, \ldots, \epsilon^*_q \rfloor$, which corresponds to the surface-force distribution in the released surface-force method.

$$e^*(x) = f^*(x; \epsilon^*_1, \epsilon^*_2, \cdots, \epsilon^*_q) \quad (13)$$

Substituting Eq. (13) into Eq. (1), it is seen that elastic strains are expressed by a function, h^*_{ij}, of the co-ordinates x, the parameters $\{\epsilon^*\}$, the shape of the object V. That is,

$$\epsilon_{ij}(x) = h^*_{ij}(x; \epsilon^*_1, \epsilon^*_2, \cdots, \epsilon^*_q, V) \quad (14)$$

The above equation is equivalent to Eq. (10). Therefore, if the number of measured strains is q and the inverse function g^*_i of h^*_{ij} can be defined, the parameters $\{\epsilon^*\}$ of the inherent strain distribution are determined as,

$$\epsilon^*_i = g^*_i({}_m\epsilon_{IJ}(x_1), {}_m\epsilon_{IJ}(x_2), \cdots, {}_m\epsilon_{IJ}(x_q); V) \qquad (i = 1 \sim q) \quad \ldots\ldots (15)$$

So, strains or stresses at arbitrary points are estimated by substituting Eq. (15) into Eq. (14).

As described above, it is seen that there are in principle two approaches in measuring residual stresses, which are the released surface-force and inherent strain methods. There are two requirements in the process of the general formulation of each method. One is that continuous functions either P (Eq. (9)) or e* (Eq. (13)) in each method must be replaced by a finite series (or approximated by discretization) to any desired degree of accuracy, which contain parameters $\{F\}$ or $\{\epsilon^*\}$ respectively. And the other is that response functions h or h* must be formulated . It is seen that these functions depend on the shape of a body for both methods are based on Eq. (1). Thus, it is impossible to find general analytical solutions except for special cases. Therefore, it is necessary to apply methods of numerical analysis such as the finite difference method, the finite element method, etc., to satisfy all of the preceding conditions. These numerical analysis are based on discretization of unknown functions and it is very convenient to use the correspondency between the discretization and the finite number of measurement.

In this paper, the finite element method which is capable to satisfy the geometric shape is adopted and the general formulation will be shown in the following chapter.

3. Measuring Methods of Residual Stresses with the Aid of F.E.M.

Since the object is fictitiously divided into finite number of elements in application of the finite element method, the parameters $\{F\}$ of the released surface-forces correspond to forces (nodal forces) acting at the nodal points which are contained in a sectioned surface and the parameters $\{\epsilon^*\}$ of the inherent strains to components of inherent strains imposed in finite elements.

3.1 Released Surface-Force Method (Nodal Force Method)

If nodal forces released in one of sectioned free surfaces Au is expressed as $\lfloor F \rfloor = \{F_1, F_2, \ldots, F_q\}^T$, nodal forces in the other surface A_l is $-\lfloor F \rfloor$. Because the nodal forces are in self-equilibrium with no existance of external forces, nodal displacements $\{\Delta u_r\}$ produced by sectioning is given by the equation.

$$\lfloor O, \lfloor F \rfloor, -\lfloor F \rfloor \rfloor^T = [K]\{-\Delta u_r\} \quad (16)$$

[K] : stiffness matrix

$$\{\Delta u_r\} = -[K]^{-1} \lfloor O, \lfloor F \rfloor, -\lfloor F \rfloor \rfloor^T$$
$$= -[C_o, C_A, C_B] \lfloor O, \lfloor F \rfloor, -\lfloor F \rfloor \rfloor^T$$
$$= -[C_A]\{F\} + [C_B]\{F\}$$
$$= -[C_A - C_B]\{F\} = -[C]\{F\} \quad (16)'$$

Matrix $[T]^e$ is defined such as to transform the nodal displacements, $\{\Delta u_r\}$, of the overall objects to those, $\{\Delta u_r\}^e$, of each element. Thus,

$$\{\Delta u_r\}^e = [T]^e \{\Delta u_r\} \quad (17)$$

where the suffix e indicates an individual element.

The relaxed strains $\{\Delta\epsilon\}^e$ and relaxed stresses $\{\Delta\sigma\}^e$ in an element are represented as,

$$\left.\begin{aligned} \{\Delta\epsilon\}^e &= [B]^e\{\Delta u_r\}^e = -[B]^e[T]^e[C]\{F\} \\ \{\Delta\sigma\}^e &= [D]^e\{-\Delta\epsilon\}^e = [D]^e[B]^e[T]^e[C]\{F\} \end{aligned}\right\} \quad (18)$$

where $[B]^e$: the strain - displacement matrix of an element.

Summarizing Eq. (18) for all finite elements in the object.

$$\{\Delta\epsilon_i\} = [\bar{H}_{ij}]\{F_j\} \quad (i = 1 \sim n, j = 1 \sim q) \quad (19)$$

where n is the total number of the strain components of all elements in the object and q is the total number of the nodal force components acting upon the sectioned surface.

If the number of measured strains $\{\Delta m\epsilon\}$ is m less than n, Eq. (19) is reduced to matrix $[H_{ij}]$ of a size (mxq)

$$\{\Delta_m\epsilon_i\} = [H_{ij}]\{F_j\} \quad (i = 1 \sim m, j = 1 \sim q) \quad (20)$$

If the released surface-forces $\{F\}$ is decided by the above observation equations (20), the magnitudes of the relaxed strains and stresses at any point can be calculated by means of Eq. (18).

Repeating such sectioning procedures until no variation in strains at any point is found and summing up these relaxed strains at each step, the desired residual stresses can be estimated.

3.2 Inherent Strain Method

The inherent strains $\{\epsilon^*\}^e$ imposed in an element produce restraining nodal forces to keep it undeformed, which are usually called as equivalent nodal forces $\{f\}^e$. This relation is shown to be,

$$\{f\}^e = -\int_v [B]^T[D]\{\epsilon^*\}^e \, d(vol) = -[L]^e\{\epsilon^*\}^e \quad (21)$$

By collecting these forces all over the object,

$$\{f\} = \Sigma\{f\}^e = -\Sigma[L]^e\{\epsilon^*\}^e = -[L]\{\epsilon^*\} \quad (22)$$

The nodal displacement $\{u\}$ ($= -\{\Delta u_r\}$) is expressed in the following form, using the stiffness matrix [K],

$$O = [K]\{u\} + \{f\} \quad (23)$$

$$\{u\} = -[K]^{-1}\{f\} = -[C]\{f\} = [C][L]\{\epsilon^*\} \quad \cdots\cdots\cdots (23)$$

The relation between displacement and strain is given by Eq. (24) because the total strain of an element is the summation of elastic and inherent strains.

$$\begin{aligned} \{\epsilon\}^e + \{\epsilon^*\}^e &= [B]^e\{u\}^e = [B]^e[T]^e\{u\} \\ &= [B]^e[T]^e[C][L]\{\epsilon^*\} \end{aligned} \quad (24)$$

Matrix $[U]^e$ is defined as one to transform the inherent strains $\{\epsilon^*\}$ over the object into those $\{\epsilon^*\}^e$ of an element. Thus,

$$\{\epsilon^*\}^e = [U]^e\{\epsilon^*\} \quad (25)$$

Therefore, Eq. (24) is transformed into the form,

$$\begin{aligned} \{\epsilon\}^e &= [B]^e[T]^e[C][L]\{\epsilon^*\} - [U]^e\{\epsilon^*\} \\ &= ([B]^e[T]^e[C][L] - [U]^e)\{\epsilon^*\} \\ &\equiv [H^*]^e\{\epsilon^*\} \end{aligned} \quad (26)$$

And stresses are evaluated as

$$\{\sigma\}^e = [D]^e\{\epsilon\}^e = [D]^e[H^*]^e\{\epsilon^*\}^e \quad (27)$$

The elastic strains $\{\epsilon\}_A^e$, $\{\epsilon\}_B^e$, . . ., the stresses $\{\sigma\}^e{}_A$, $\{\sigma\}^e{}_B$, . . . of elements A, B, . . ., respectively are summarized over the object in the following forms,

$$\{\epsilon\} = \lfloor \lfloor \epsilon \rfloor^e{}_A, \lfloor \epsilon \rfloor^e{}_B, \cdots \rfloor^T = [H^{*\prime}]\{\epsilon^*\} \quad (28)$$

$$\{\sigma\} = \lfloor \lfloor \sigma \rfloor^e{}_A, \lfloor \sigma \rfloor^e{}_B, \cdots \rfloor^T = [M']\{\epsilon^*\} \quad (29)$$

Generally speaking, when the total number, q, of inherent strain components is equal to the total number, n, of the elastic strain components, the sizes of matrices $[H^{*\prime}]$, and $[M']$ are (nxn).

If a special attension is paid to welding residual stresses, the portion where the inherent strains exist is limited in the vicinity of welded lines because the inherent strains are originated by dislocations due to plastic deformation. Therefore, it is not difficult to presume such elements that apparently contain no inherent strains. Representing the inherent strain vector by the same notation, $\{\epsilon^*\}$ which consist of only non-zero inherent strain components q ($<$n), the above mentioned matrices are reduced to matrices $[\bar{H}^*]$, [M] of the sizes (nxq).

$\{\epsilon_i\} = [\bar{H}^*_{ij}]\ \{\epsilon^*_j\} \qquad (i = 1 \sim n, j = 1 \sim q) \qquad (28)'$

$\{\sigma_i\} = [M_{ij}]\ \{\epsilon^*_j\} \qquad (i = 1 \sim n, j = 1 \sim q) \qquad (29)'$

For example, if m number of measurements of elastic strains can be done, the matrix $[\bar{H}^*] = (nxq)$ is reduced to $[H^*] = (mxq)$ and the observation equations are constituted as,

$$\{{}_m\epsilon_i\} = [H^*_{ij}]\ \{\epsilon^*_j\} \qquad (i = 1 \sim m, j = 1 \sim q) \qquad (30)$$

If the unknown inherent strains, $\{\epsilon^*\}$ can be decided by these measured strains, $\{{}_m\epsilon_i\}$, the residual strain and stress distributions can be calculated over the entire object.

As seen in the above, both observation equations appeared in the released surface-force and the inherent strain methods are given in the same form, and the necessary condition to determine the parameters $\{F\}$, $\{\epsilon^*\}$ which residual stress distribution is estimated by is to satisfy the inequality $m \geqq q$. For example, if $m = q$, and $[H]^{-1}$, $[H^*]^{-1}$ exist, the resulting stress distribution can be obtained uniquely. In contrast with this, if the number of Eqs. (30) is not sufficient, that is $m < q$, the inherent strains $\{\epsilon^*\}$ can not be determined. In this case, it is required to increase such relations as those in Eqs. (30). These relations can be obtained simply by adding measuring points, if not, by producing a new self-balanced state by sectioning.

4. Most Probable Value and Its Confidence Interval

In the case of $m > q$ in the observation equations, the number of equations is greater than the number of unknown parameters and consequently, there should be $(m - q)$ number of dependent equations. But these dependent equations do not exist apparently because errors contained in measured values change the dependency. So, for such cases where $m > q$ and errors are contained in the measured values, the theory of statistics[3)] is introduced into the above mentioned methods in order to decide the most probable values and these confidence intervals with the observation equations.

Here, the errors used are accidental ones and satisfy the three axioms of errors.

In this paper, the process of evaluation of the most probable values and these confidence intervals are discussed only when the inherent strain method is applied, for the observation equations of the released surface-force method is of the same form as the inherent strain method.

4.1 Most Probable Value

The relation between the true value $\{\epsilon\}$ of the elastic strain and the true value $\{\epsilon^*\}$ of the inherent strain is expressed by Eqs. (30). Substituting measured values $\{{}_m\epsilon\}$ of strains into Eqs. (30) in place of $\{\epsilon\}$, the errors $\{x\}$ are obtained in the following form.

$$\{{}_m\epsilon\} - [H^*]\ \{\epsilon^*\} = \{x\} \qquad (31)$$

Furthermore, replacing $\{\epsilon^*\}$ by the most probable value $\{\hat{\epsilon}^*\}$ in Eq. (31), residuals $\{v\}$ are given as,

$$\{{}_m\epsilon\} - [H^*]\ \{\hat{\epsilon}^*\} = \{v\} \qquad (32)$$

In the case where each measured value of strains is of the same precision, the sum of squares of the residuals S is

$$S = \lfloor v \rfloor \cdot \{v\} \qquad (33)$$

According to the method of least squares, the most probable values $\{\hat{\epsilon}^*\}$ are decided so as to minimize the sum of squares of the residuals S. Thus, from the condition $\partial S/\partial \{\hat{\epsilon}^*\} = 0$,

$$[H^*]^T\ \{{}_m\epsilon\} = [H^*]^T [H^*]\ \{\hat{\epsilon}^*\} \qquad (34)$$

The above equation which was normalized is called normal equations and $[H^*]^T\ [H^*]$ is a square matrix of a size (qxq). So, if the square matrix is regular, its inverse matrix can be obtained and the most probable values are given as follows,

$$\{\hat{\epsilon}^*\} = ([H^*]^T [H^*])^{-1} [H^*]^T \{{}_m\epsilon\} \equiv [G^*]\ \{{}_m\epsilon\} \qquad (35)$$

By using this result and Eq. (29)′, the residual stress distribution over the object is estimated.

$$\{\hat{\sigma}\} = [M]\ \{\hat{\epsilon}^*\} = [M][G^*]\ \{{}_m\epsilon\} \equiv [N]\ \{{}_m\epsilon\} \qquad (36)$$

4.2 Confidence Interval

4.2.1 Accuracy of Most Probable Value

The relation among measurement variance s^2 of unit weight, inherent strain variances $\{s^{*2}\}$ and these weights $\{p^*\}$ is represented as follows.

$$p^*_i\, s^*_i{}^2 = s^2 \qquad (i = 1 \sim q) \qquad (37)$$

The unbiased estimate $\hat{s}^2$ of the measurement variance is given as,

$$\hat{s}^2 = \lfloor v \rfloor \{v\}\ / (m - q) = S / (m - q) \qquad (38)$$

And with components g^*_{ij} of the matrix $[G^*]$, the weights of Eq. (37) can be evaluated.

$$p^*_i = 1 / \sum_{j=1}^{m} g^*_{ij}{}^2 \qquad (i = 1 \sim q) \qquad (39)$$

By substituting Eqs. (38) and (39) into Eq. (37), the unbiased estimate of population variance of the most probable value $\{\hat{\epsilon}^*\}$ can be determined.

Otherwise, the unbiased variance $\hat{s}_{\sigma i}^2$ of the most probable value $\{\hat{\sigma}\}$ of residual stresses is expressed in the following equation if the components of the matrix [N] are n_{ij}.

$$\hat{s}_{\sigma i}^2 = \left(\sum_{j=1}^{m} n_{ij}^2 \right) \cdot \hat{s}^2 \qquad (i = 1 \sim n) \qquad (40)$$

4.2.2 Confidence Interval

Stochastic variable $(\hat{\epsilon}^*_i - \epsilon^*_i)/s^*_i$ which is dimensionlessly normalized $\{\epsilon^*\}$ of the inherent strains obeys the standard normal distribution N(0, 1). A variable S/s^2 obeys χ^2 - distribution with degree of freedom $\phi = m - q$. So, a variable t in the following expression depends upon Student's t-distribution with ϕ degree of freedom.

$$\begin{aligned} t &= (\hat{\epsilon}^*_i - \epsilon^*_i) / s^*_i / \sqrt{S / s^2 / \phi} \\ &= (\hat{\epsilon}^*_i - \epsilon^*_i) / \hat{s}^*_i \end{aligned} \qquad (41)$$

So, the relation between confidence coefficient $(1-\alpha)$ and t-value is represented as follows.

$$1 - \alpha = 2 \int_0^t \frac{\Gamma\{(\phi+1)/2\}}{\sqrt{\phi\pi}\ \Gamma(\phi/2)} \left(1 + \frac{x^2}{\phi}\right)^{-\frac{\phi+1}{2}} dx \qquad (42)$$

where Γ : gamma function

If the confidence coefficient is given, t-value is decided by Eq. (42) and the confidence intervals are obtained in the following forms.

$$\left. \begin{aligned} \hat{\epsilon}^*_i - t\hat{s}^*_i \leqq \epsilon^*_i \leqq \hat{\epsilon}^*_i + t\hat{s}^*_i \\ \hat{\sigma}_i - t\hat{s}_{\sigma i} \leqq \sigma_i \leqq \hat{\sigma}_i + t\hat{s}_{\sigma i} \end{aligned} \right\} \text{(confidence coef. } 1 - \alpha) \qquad (43)$$

These intervals do not contain errors in the process of the discretization, which are inevitable in the finite element method but contain round-off errors in the calculation.

5. Examples Analyzed by the Present Theory

5.1 Measurement of Three Dimensional Residual Stresses (without Error in Measurement)

First, in order to confirm the validity of the new general theory of measuring methods of residual stresses with the aid of the finite element method, a few experiments are conducted in the following.

As indicated in Fig. 2, a thick plate with a slit weld is considered and constant inherent strains were imposed along the slit. The resulting stress distribution is analyzed by means of the convensional finite element technique (ref. Figs. 7(a), (b), (c)). This distribution is called the true value in the following discussion. The aim is to reproduce this stress distribution with the aid of the inherent strain method. In this analysis, only three components $(\epsilon_x^*, \epsilon_y^*, \epsilon_z^*)$ of unknown inherent strains in an element are assumed. As the number of elements where the inherent strains exist is 9, taking account of three axes of symmetry, the total number of the unknowns (q) equals to 3 x 9 = 27.

Without any error in measured strains, the true distribution can be obtained if m = 27 and the matrix [H*] is regular. In the process of this calculation, the relation between the inherent and the measured elastic strains is used. As the result, when this relation is very insensitive, some of the relation loosen their independency numerically even if the matrix [H*] is regular. In such a case, the round-off errors in the numerical calculation influence the accuracy of the result. Therefore, in order to improve these unfavorable relations, another measurement is conducted on a piece which is cut out so as to include the slit weld from the original object as shown in Fig. 3. Points and directions of measurement of strains are indicated in Fig. 4. For the new piece, the residual stresses are calculated under the same inherent strains as the original. These values on the surfaces will be used as measured strains on the piece for the further analysis. Based on these data, the inherent strain method predicts the desired inherent strains. Using these inherent strains, a residual stress distribution to the original object (Fig. 2) is computed. The resulting residual stresses agree precisely with the true values. The inherent strains were predicted within an error less than $\pm 1\mu$ for true values, -4000μ.

Although the released surface-force method is not applied in this chapter, its validity has been already confirmed in reference 4).

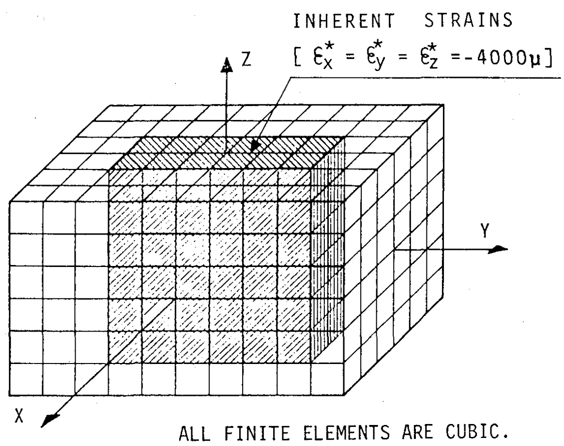


Fig. 2 Model for analysis

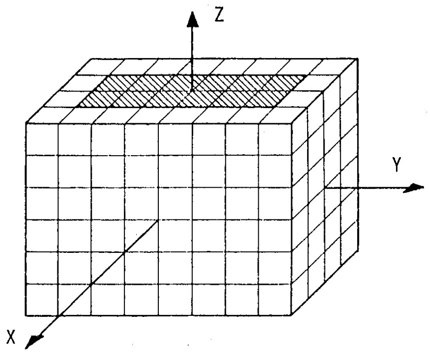


Fig. 3 Model peeled off one outer layer around the core of inherent strains

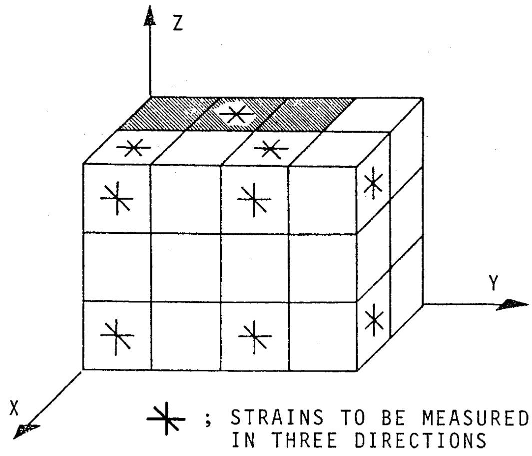


Fig. 4 Location of measuring points and directions of strains

5.2 Measurement of Three Dimensional Residual Stresses (with Random Errors in Measurement)

When errors are contained in measured strains on the same problem as dealt above, the present method can be also applied. Measurement of strains are conducted on the same surface of the piece as shown in Fig. 3. The absolute value of the specified errors is equal to 25μ. Fig. 5 represents points and directions of the measurement (m = 78) and signs of the errors given in accordance with random number. In this case, t-distribution almost coincides with the standard normal one for $\phi = m - q = 51$.

Most probable values and these confidence intervals of inherent strains which are estimated by the measured values with these errors are represented in Fig. 6. And a relation between the true values and the most probable values of the residual stresses to the original object are shown in Figs. 7(a), (b) and (c).

Estimated 68%–confidence intervals of the inherent strains contain the true values except several points and these are almost coincided with the original. Though the standard deviation of the estimated residual stresses in the portion containing the inherent strains are locally great (these values are less than 10 kg/mm^2) and the confidence intervals become wide, the estimates of these deviation are small in the other part.

Based on the above discussions, it is concluded that residual stresses even in three dimensional state can be accurately estimated by the new method, except some locally disturbed portion.

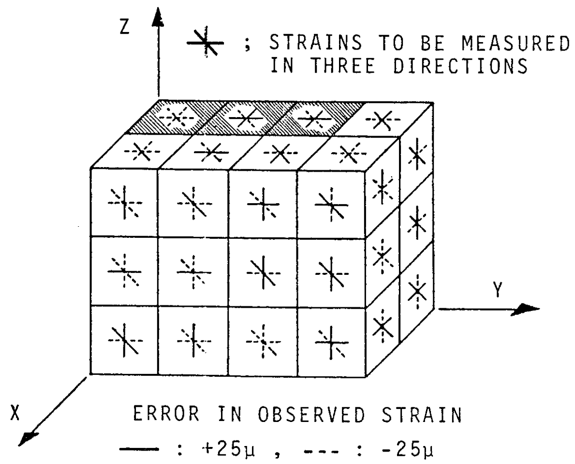
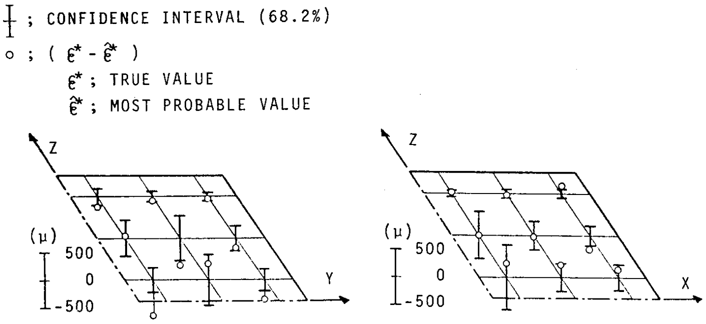
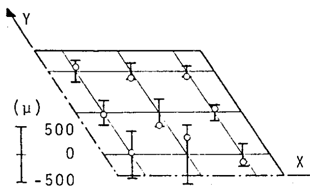


Fig. 5 Location of measuring points and directions of strains

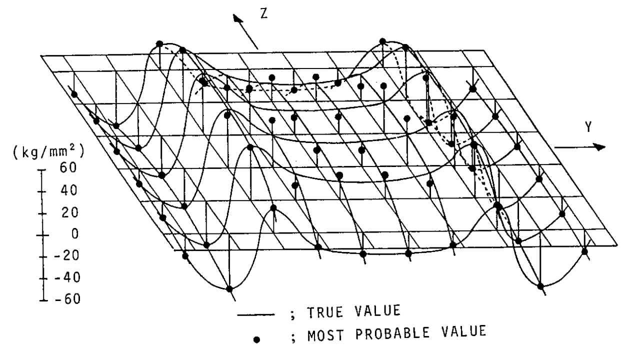


(a) Inherent strains in X-direciton (b) Inherent strains in Y-direction

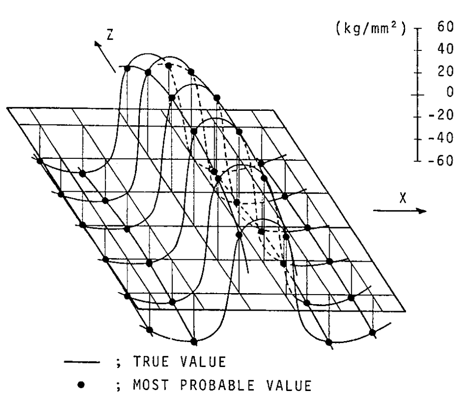


(c) Inherent strains in Z-direction

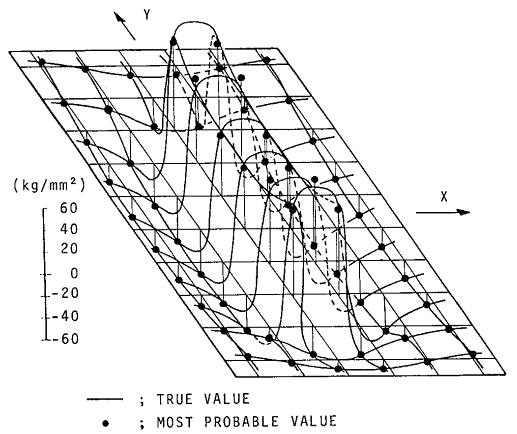
Fig. 6 Relations of true values to confidence intervals of inherent strains



(a) Stresses in X-direction on the plane, X = 0



(b) Stresses in Y-direction on the plane, Y = 0



(c) Stresses in Z-direction on the plane, Z = 0

Fig. 7 True and estimated values of residual stresses

5.3 Measurement of Two Dimensional Residual Stresses (with Random Errors in Measurement)

There are many cases in which residual stress distribution is regarded as two dimensional state. So, as such typical example, welding residual stresses induced in a butt welded joint of plates as shown in Fig. 8 are analyzed. The original residual stress distribution is produced by a uniform inherent strain $\epsilon_x{}^* = -4000\mu$ in each element located along the center line of the plate.

The number of unknown values is q = 11 ($\epsilon_x{}^*$ of 11 elements) and the number of measurements is m = 22. Location of measuring points are along the line $Y = Y_0$ shown in Fig. 8 (analysis is conducted in each case of $Y_0 = 2.5$, 7.5 and 12.5 mm). Residual stresses are estimated by the method, following the same procedure and providing the same error as used in Sec. 5.2. Fig. 9 represents the estimated inherent strains and their accuracy (the original inherent strain is -4000μ). In the cases of measuring lines $Y_0 = 2.5$ and 7.5 mm, the estimated inherent strains are almost agreeable to the original and then the accuracy is high and of same order in both cases. On the other hand, it is seen that the accuracy is abruptly worse in the case of $Y_0 = 12.5$ mm since the relation between inherent and elastic strains is insensitive. Furthermore, residual stresses are estimated in Fig. 10 and t-value corresponds to 95% – confidence coefficient is equal to 2.2 for $\phi = m - q = 11$. As the estimates of the standard deviation are 1 kg/mm^2 at the largest in the cases of $Y_0 = 2.5$ and 7.5 mm, there are confidences of 95% if these bands of 2 kg/mm^2, to the most probable values of about 75 kg/mm^2 are adopted.

Therefore, as seen above, if points of measurement are located close to the welded line, very accurate estimate of stress distribution all over the plate can be expected by a few points of measurement.

	SYMBOL	Y_0 (mm)
CASE 1	○	2.5
CASE 2	●	7.5
CASE 3	◆	12.5

TOTAL NUMBER OF OBSERVED STRAIN COMPONENTS : m = 22
TOTAL NUMBER OF INHERENT STRAIN COMPONENTS : q = 11
DEGREE OF FREEDOM : ϕ = 11

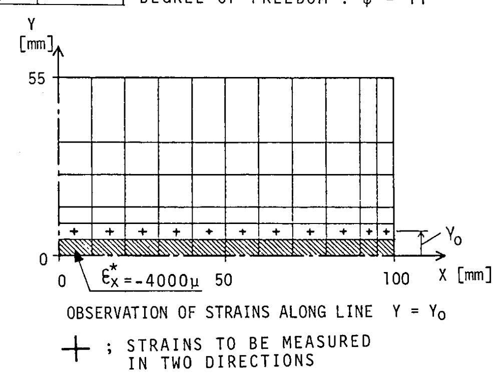


Fig. 8 Location of measuring points

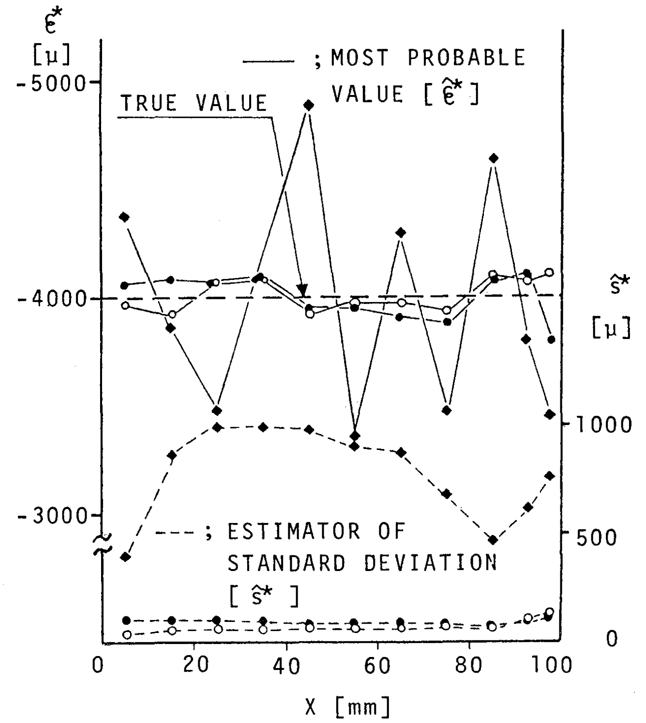


Fig. 9 Estimated values and their accuracy of inherent strains

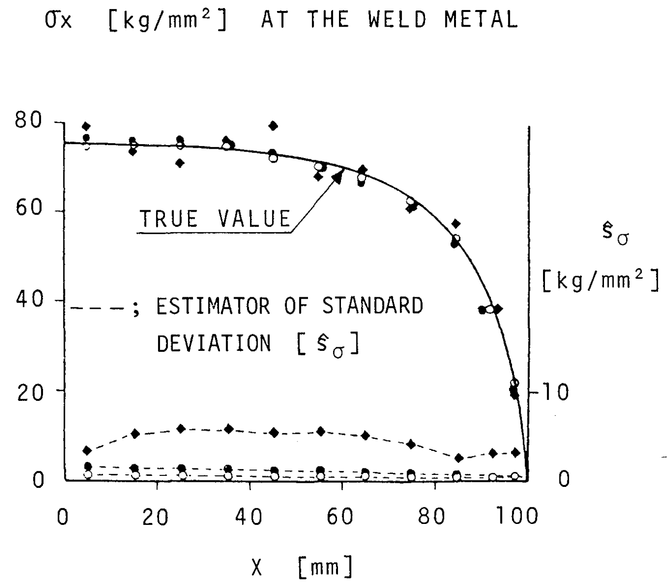


Fig. 10 Estimated values and their accuracy of residual stresses at the weld metal

6. Conclusion

In this paper, the principle of measuring residual stresses is presented, and the release surface-force and the inherent strain methods are introduced as basic approaches for the application of the principle. A new general theory of measuring method of residual stresses is proposed by formulating each method with the aid of the finite element method.

Furthermore, this theory is generalized by statistic approach in order to be applied in the cases where measured values contain random errors.

Several numerical experiments are conducted by using the present method. The following information is obtained.

(1) The new general theory of measuring method of residual stresses has been developed.

(2) Measurement of three dimensional residual stresses in general cases, which is unable by the existing methods, is shown to be possible by this theory.

(3) In the case where measured strains contain errors, residual stresses in three dimensional stress state, can be estimated accurately over the whole object but there are still some portions where those are predicted without good accuracy.

(4) In the case of butt welded joint in two dimensional stress state, extremely accurate estimates can be obtained only with a few measured strain if points of measurement are located close to welded line.

(5) Furthermore, it is possible to predict such points of measurement by the present method as to obtain the most accurate result.

Reference

1) For example, S. Watanabe and K. Satoh, "Welding Mechanics and its Application", 3rd Ed., Asakurashoten, Ltd., (1971), p.332-364 (in Japanese)

2) Y. Ueda, K. Fukuda and S. Endo, "A Study on the Accuracy of Estimated Residual Stresses by Existing Measuring Methods", Trans. of JWRI (Welding Research Institute of Osaka University, Japan) Vol.4 No.2 (1975) (to be published).

3) J. Honma and S. Kasugaya, "Dimensional Analysis, Method of Least Squares and Empirical Formula", 13th Ed., Corona Publishing Company, Ltd., (1968), p.61-304 (in Japanese)

4) Y. Ueda, K. Fukuda, K. Nakacho and S. Endo, "A New Theory of Measuring Method of Residual Stresses with the Aid of Finite Element Method and Reliability of Estimated Values", Proc. of National Symp. of Methods of Structural Analysis and Design, The Society of Steel Construction of Japan (1975), pp.465-470 (in Japanese) and J. of Naval Architects, Japan, Vol.138 (1975), pp. 525–533 (in Japanese).