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A new method for analysis of complex structures based on FRF's of substructures

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Abstract. A new method is presented for synthesizing the dynamic responses of a complex structure based upon the frequency response functions of the substructures. This method is shown to be superior to traditional methods for several reasons: (i) It can be applied to a generic class of systems. (ii) The analyst is spared the responsibilities of eliminating the coupling forces and rearranging the equations of motion. (iii) The coupling forces and the responses of the total system can be obtained simultaneously and efficiently.

1. Introduction

Structural systems of interests are often complex in nature. Automotive chassis and bodies provide such examples. Since the earlier work of Klosterman [1,2], many authors have attempted to use frequency response functions (FRF's) to represent the substructures in a total system dynamic analysis [3–5], whereby the FRF's of substructures can be obtained by either experimental or analytical methods. This approach is attractive since the dynamic behavior of the total system can be predicted by synthesizing the dynamic behaviors of simpler substructures, and the effects of changes in any substructure on the operating behavior of the total system can be ascertained. Another advantage of this approach is that the order of the final system of equations to be solved is substantially smaller than the number of physical degrees of freedom of the total system.

However, this approach as originally presented requires the elimination of coupling forces and redundant unknown variables by using the constraint equations which represent the coupling conditions of the substructures. This may be difficult in general. For example, if the substructures are connected through bushings (springs and dampers), then the coupling forces and displacements at the coupling points will coexist in the constraint equations [5]. Needless to say that in some cases, the constraint equations may be even more complicated. Moreover, if the coupling forces are of explicit interest, one must use the constraint equations again by back substitution to find a solution. This might be cumbersome and inefficient.

The purpose of this paper is to provide a new method which will circumvent these difficulties. This method is suitable for numerical solutions with computer programs devised for applications to a generic class of systems, without any extraordinary ingenuity from the user, or the burden to rearrange the equations to avoid mathematical difficulties. In addition, this method can determine the coupling forces and the responses of the total system simultaneously and efficiently.

2. Equations of motion

Consider a system consists of N sub-systems. We can set up the dynamic equations of the subsystems as

$$\mathbf{x}_k = \mathbf{H}_k \mathbf{f}_k \quad (k = 1, 2, \ldots, N) \tag{1}$$

where $\mathbf{x}_k$ is the response vector; $\mathbf{f}_k$ is the force vector which includes external forces and coupling forces; $\mathbf{H}_k$

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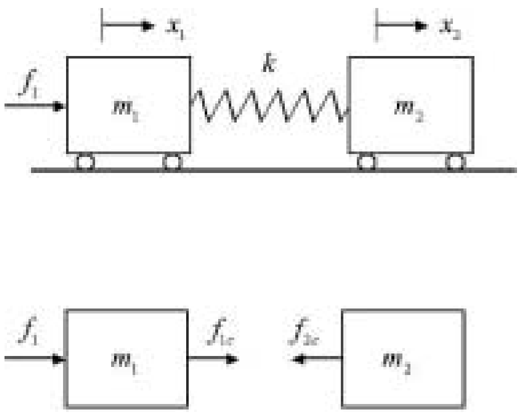

Fig. 1. A mass-spring system and free-body diagram.

is the frequency response function (FRF) matrix. By stacking up the subsystem equations, we obtain the dynamic equations of the total system as

$$\mathbf{x} = \mathbf{Hf} \tag{2}$$

where $\mathbf{x}$ is the total response vector and is of dimension $p \times 1$; $\mathbf{f}$ is the total force vector and is of dimension $q \times 1$, it consists of r coupling forces and $q - r$ external forces; $\mathbf{H}$ is the FRF matrix which is of dimension $p \times q$.

Next, we can write the coupling conditions in a general form as

$$\mathbf{Bx} + \mathbf{Cf} = \mathbf{0} \tag{3}$$

where matrix $\mathbf{B}$ is of dimension $r \times p$, and matrix $\mathbf{C}$ is of dimension $r \times q$. As an example, consider a simple system shown in Fig. 1 composed of masses m_1 and m_2 connected by a spring with stiffness k. A force f_1 is applied on m_1. The coupling conditions in this case are given by

$$k(x_2 - x_1) = f_{1\mathrm{c}} \tag{4a}$$

$$f_{1\mathrm{c}} + f_{2\mathrm{c}} = 0 \tag{4b}$$

where x_1 and x_2 are the displacements of m_1 and m_2, respectively, $f_{1\mathrm{c}}$ and $f_{2\mathrm{c}}$ are the coupling forces at the two ends of the connecting spring. Equations (4a) and (4b) can be written in the form of Eq. (3), that is

$$\begin{bmatrix} -k & k \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = 0 \tag{5}$$

Equations (2) and (3) form a system of $(p+r)$ equations for the p responses and the r coupling forces. We proceed to develop a solution method in the following section.

3. Solution method

Theoretically, it is possible to rearrange the elements of the force vector $\mathbf{f}$ and the matrix $\mathbf{C}$ so that the last r entries of $\mathbf{f}$ are the coupling forces, and the last r columns of $\mathbf{C}$ are linearly independent. Then one may rewrite Eq. (3) as

$$\mathbf{Bx} + \begin{bmatrix} \mathbf{C}^1_{r\times(q-r)} & \mathbf{C}^2_{r\times r} \end{bmatrix} \begin{Bmatrix} \mathbf{f}_\mathrm{a} \\ \mathbf{f}_\mathrm{c} \end{Bmatrix} = \mathbf{0} \tag{6}$$

where the subscripts on matrices $\mathbf{C}^1$ and $\mathbf{C}^2$ designate the corresponding dimensions. The coupling force vector $\mathbf{f}_\mathrm{c}$ can then be expressed as

$$\mathbf{f}_\mathrm{c} = -[\mathbf{C}^2_{r\times r}]^{-1}[\mathbf{Bx} + \mathbf{C}^1_{r\times(q-r)}\mathbf{f}_a] \tag{7}$$

Hence, by substituting Eqs (7) into (2), one may solve for the dynamic responses in $\mathbf{x}$.

However, the above approach may be tedious and inefficient. We now provide a new method based upon the numerically robust singular value decomposition algorithm (SVD).

The new method under consideration can be explained as follows. First, substituting Eqs (2) into (3), we obtain

$$[\mathbf{BH} + \mathbf{C}]\mathbf{f} = 0 \tag{8}$$

Suppose Φ is an orthogonal complement of $[\mathbf{BH} + \mathbf{C}]^\mathrm{H}$ such that

$$[\mathbf{BH} + \mathbf{C}]\Phi = \mathbf{0} \tag{9}$$

where Φ is of dimension $q \times (q-r)$ with rank $q - r$. The total force vector $\mathbf{f}$ can therefore be expressed as

$$\mathbf{f} = \Phi\eta \tag{10}$$

where the vector η has $q - r$ entries, which may be interpreted as generalized force components. The force vector $\mathbf{f}$ consists of r coupling forces and $q - r$ external forces. Moving the unknown coupling forces to the right side of Eq. (10), we have

$$\left[\Phi \left| \begin{matrix} \mathbf{0}_{(q-r)\times r} \\ -\mathbf{I}_{r\times r} \end{matrix} \right. \right] \begin{Bmatrix} \boldsymbol{\eta} \\ \mathbf{f}_\mathrm{c} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_\mathrm{a} \\ \mathbf{0} \end{Bmatrix} \tag{11}$$

where $\mathbf{I}_{r\times r}$ is an identity matrix with rank r. From Eq. (11) we can solve for $\boldsymbol{\eta}$ and the coupling forces in $\mathbf{f}_\mathrm{c}$. Once this is accomplished, the response $\mathbf{x}$ can be determined as

$$\mathbf{x} = \mathbf{H}\Phi\boldsymbol{\eta} \tag{12}$$

Observe that a key step in this approach is the determination of the orthogonal complement matrix Φ.

There are several ways to accomplish this. See, for example, Walton and Steeves [6], Huston [7], Huston and Liu [8], and Singh and Likins [9]. In this paper, we will use the SVD method. Specifically, let $\mathbf{D} = \mathbf{BH} + \mathbf{C}$, then there are unitary matrices $\mathbf{U}$ and $\mathbf{V}$ of orders $r \times r$ and $q \times q$, respectively, such that

$$\mathbf{U}^{\mathrm{H}}\mathbf{D}\mathbf{V} = [\Sigma \quad \mathbf{0}] \tag{13}$$

in which $\Sigma = diag(\sigma_1, \ldots, \sigma_r)$ and $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_r > 0$. The diagonal non-zero elements of the decomposition are called the singular values of the matrix $\mathbf{D}$. The singular values are unique, but $\mathbf{U}$ and $\mathbf{V}$ are not [10,11].

Now with proper partitioning of matrix $\mathbf{V}$, Eq. (13) can be expressed as

$$\mathbf{D} = \mathbf{U}[\Sigma \quad \mathbf{0}] \begin{bmatrix} \mathbf{V}_1^{\mathrm{H}} \\ \mathbf{V}_2^{\mathrm{H}} \end{bmatrix} = \mathbf{U}\Sigma\mathbf{V}_1^{\mathrm{H}} \tag{14}$$

Because of the fact that $\mathbf{V}$ is unitary, we must have $\mathbf{V}_1^{\mathrm{H}}\mathbf{V}_2 = \mathbf{0}$. Post-multiplying Eq. (14) by $\mathbf{V}_2$, we obtain

$$\mathbf{D}\mathbf{V}_2 = \mathbf{0} \tag{15}$$

which implies that $\mathbf{V}_2$ is indeed an orthogonal complement of $\mathbf{D}^{\mathrm{H}}$.

Now, the following procedural outline may be set forth:

Step 1. Perform SVD of $\mathbf{BH} + \mathbf{C}$.
Step 2. Form an orthogonal complement matrix Φ (the last $q - r$ columns of $\mathbf{V}$);
Step 3. Solve Eq. (11) for the generalized forces and constraint forces;
Step 4. Solve Eq. (12) for the responses of the total system.

4. Example

Figure 2 shows a system composed of two identical steel plates A and B. The points 1 and 2 of plate A are connected to points 1 and 2 of plate B, respectively, through spring-damper systems. In addition, one end of plate A is fixed, and one corner point of plate B is fixed.

Suppose the frequency response function matrices of the two plates have been made available. The objective is to find the acceleration responses at points 1 and 2 of the two plates due to a unit force f_1 applied vertically at point 1 of plate A. Equation (1) in this case becomes

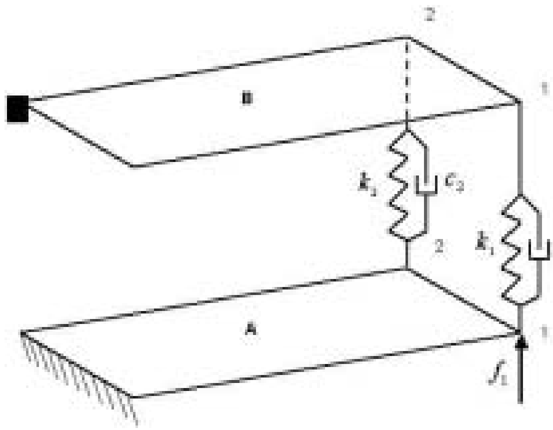

Fig. 2. A system composed of two identical plates connected through two spring-damper systems.

$$\begin{bmatrix} a_1^A \\ a_2^A \end{bmatrix} = \begin{bmatrix} H_{11}^A & H_{11}^A & H_{12}^A \\ H_{12}^A & H_{12}^A & H_{22}^A \end{bmatrix} \begin{bmatrix} f_1 \\ f_{1c}^A \\ f_{2c}^A \end{bmatrix} \tag{16a}$$

and

$$\begin{bmatrix} a_1^B \\ a_2^B \end{bmatrix} = \begin{bmatrix} H_{11}^B & H_{12}^B \\ H_{12}^B & H_{22}^B \end{bmatrix} \begin{bmatrix} f_{1c}^B \\ f_{2c}^B \end{bmatrix} \tag{16b}$$

where f_1 is the applied force, f_{1c}^A and f_{2c}^A are the coupling forces applied respectively at points 1 and 2 of plate A; f_{1c}^B and f_{2c}^B are the coupling forces applied respectively at points 1 and 2 of plate B; $H_{ij}^A(i,j = 1,2)$ are the FRF's of plate A; $H_{ij}^B(i,j = 1,2)$ are the FRF's of plate B.

Now, let the response vector be

$$\mathbf{x} = \lfloor a_1^A \quad a_2^A \quad a_1^B \quad a_2^B \rfloor^T \tag{17}$$

and the force vector (including the applied force and the coupling forces) be

$$\mathbf{f} = [f_1 \quad f_{1c}^A \quad f_{1c}^B \quad f_{2c}^A \quad f_{2c}^B]^T \tag{18}$$

Then, the frequency response function matrix $\mathbf{H}$ in Eq. (2) becomes

$$\mathbf{H} = \begin{bmatrix} H_{11}^A & H_{11}^A & 0 & H_{12}^A & 0 \\ H_{12}^A & H_{12}^A & 0 & H_{22}^A & 0 \\ 0 & 0 & H_{11}^B & 0 & H_{12}^B \\ 0 & 0 & H_{12}^B & 0 & H_{22}^B \end{bmatrix} \tag{19}$$

and the matrices $\mathbf{B}$ and $\mathbf{C}$ in Eq. (3) become

$$\mathbf{B} = \tag{20}$$

$$\begin{bmatrix} \frac{k_1+j\omega c_1}{\omega^2} & 0 & -\frac{k_1+j\omega c_1}{\omega^2} & 0 \\ 0 & \frac{k_2+j\omega c_2}{\omega^2} & 0 & -\frac{k_2+j\omega c_2}{\omega^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

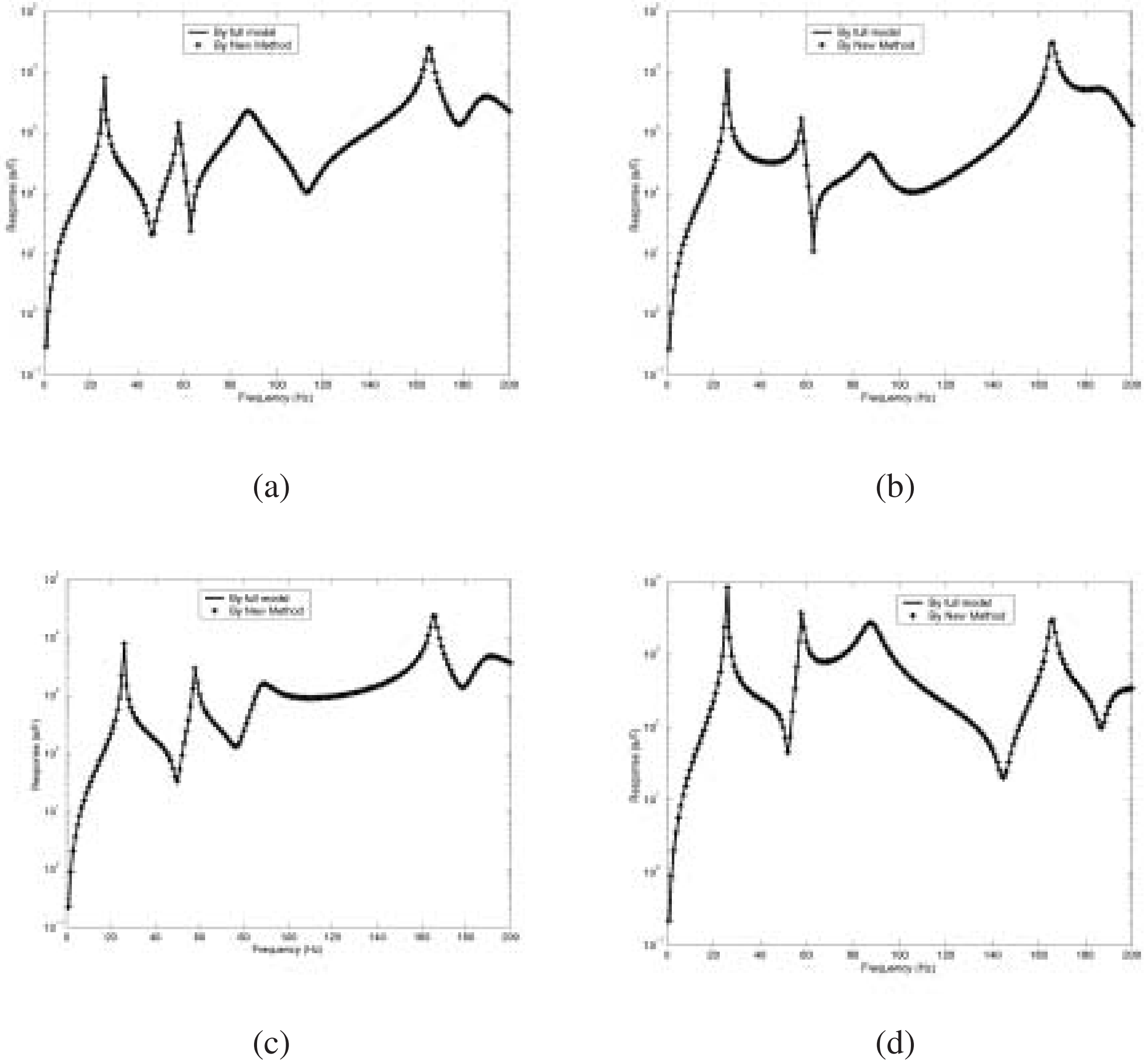

Fig. 3. Case 1 of example problem: FRF's of the system shown in Fig. 2. (a) Response of point 1 of plate A; (b) response of point 1 of plate B; (c) Response of point 2 of plate A; (d) Response of point 2 of plate B.

$$\mathbf{C} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{21}$$

The solution procedure is as follows. At each frequency ω perform SVD

$$\mathbf{BH} + \mathbf{C} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{H}} \tag{22}$$

then the matrix Φ, an orthogonal complement of $(\mathbf{BH}+\mathbf{C})^{\mathrm{H}}$, is of dimension 5×1 and is just the last column of the $\mathbf{V}$. Therefore, Eq. (11) has the form:

$$\begin{bmatrix} V_{15} & 0 & 0 & 0 & 0 \\ V_{25} & -1 & 0 & 0 & 0 \\ V_{35} & 0 & -1 & 0 & 0 \\ V_{45} & 0 & 0 & -1 & 0 \\ V_{55} & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \eta \\ f_{1\mathrm{c}}^{A} \\ f_{1\mathrm{c}}^{B} \\ f_{2\mathrm{c}}^{A} \\ f_{2\mathrm{c}}^{B} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{23}$$

From the above equation we can solve for the generalized force η and the constraint forces $[f_{1\mathrm{c}}^{A} \quad f_{2\mathrm{c}}^{A} \quad f_{1\mathrm{c}}^{B} \quad f_{2\mathrm{c}}^{B}]$. The responses of the total system can be found by solving Eq. (12).

To validate this method we considered a number of cases with various stiffness and damping values of the connecting spring-damper systems. To this end, we

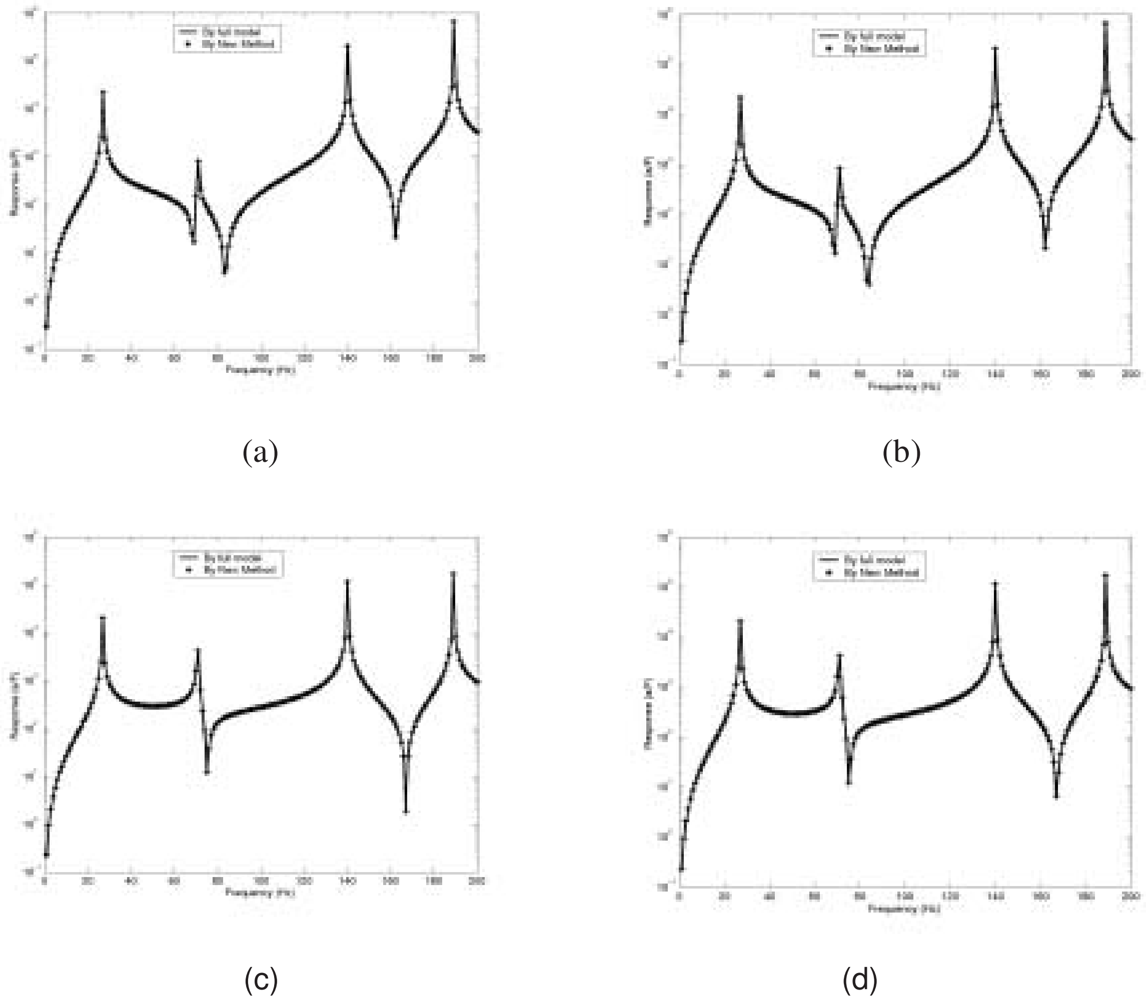

Fig. 4. Case 2 of example problem: FRF's of the system shown in Fig. 2. (a) Response of point 1 of plate A; (b) response of point 1 of plate B; (c) Response of point 2 of plate A; (d) Response of point 2 of plate B.

used these typical values for both plates: dimensions were 500.0 mm $\times$ 250.0 mm $\times$ 10.0 mm, Young's modulus was 250000 N/mm^2, Poisson's ratio was 0.3, and mass density was 7.850×10^{-6} Kg/mm^3.

Case 1: Moderate stiffness and damping values

First, consider a case in which the spring stiffness values are $k_1 = 195.0$ N/mm and $k_2 = 205.0$ N/mm, and the damping coefficients are $c_1 = c_2 = 0.05$N $-$ sec/mm. Figure 3 shows the FRF's obtained by employing the proposed method as compared to results from full model finite element analysis. The results agree to seven significant figures.

Case 2: Very large stiffness and damping values

Next, consider the case in which the stiffness and damping values are larger than those in case 1 by three orders of magnitude. That is, $k_1 = 195000.0$ N/mm, $k_2 = 205000.0$ N/mm, $c_1 = c_2 = 50.0$N $-$ sec/mm. Figure 4 shows the comparison between the FRF's obtained by the proposed method and by full model finite element analysis. Again, the results agree to seven significant figures. As expected, the responses of the points 1 on the two plates are the same (see Figs 4(a) and (b)) because the stiffness value of the connecting spring is extremely large, thus the two points will move together. The same is true for the points 2 of the two plates (see Figs 4(c) and (d)).

Case 3: Very small stiffness and damping values

Finally, consider the case in which the stiffness and damping values are smaller than those in case 1 by three orders of magnitude. That is, $k_1 = 0.195$ N/mm, $k_1 = 0.205$ N/mm, $c_1 = c_2 = 0.005$ N $-$ sec/mm. Figure 5 shows the comparison between the FRF's ob-

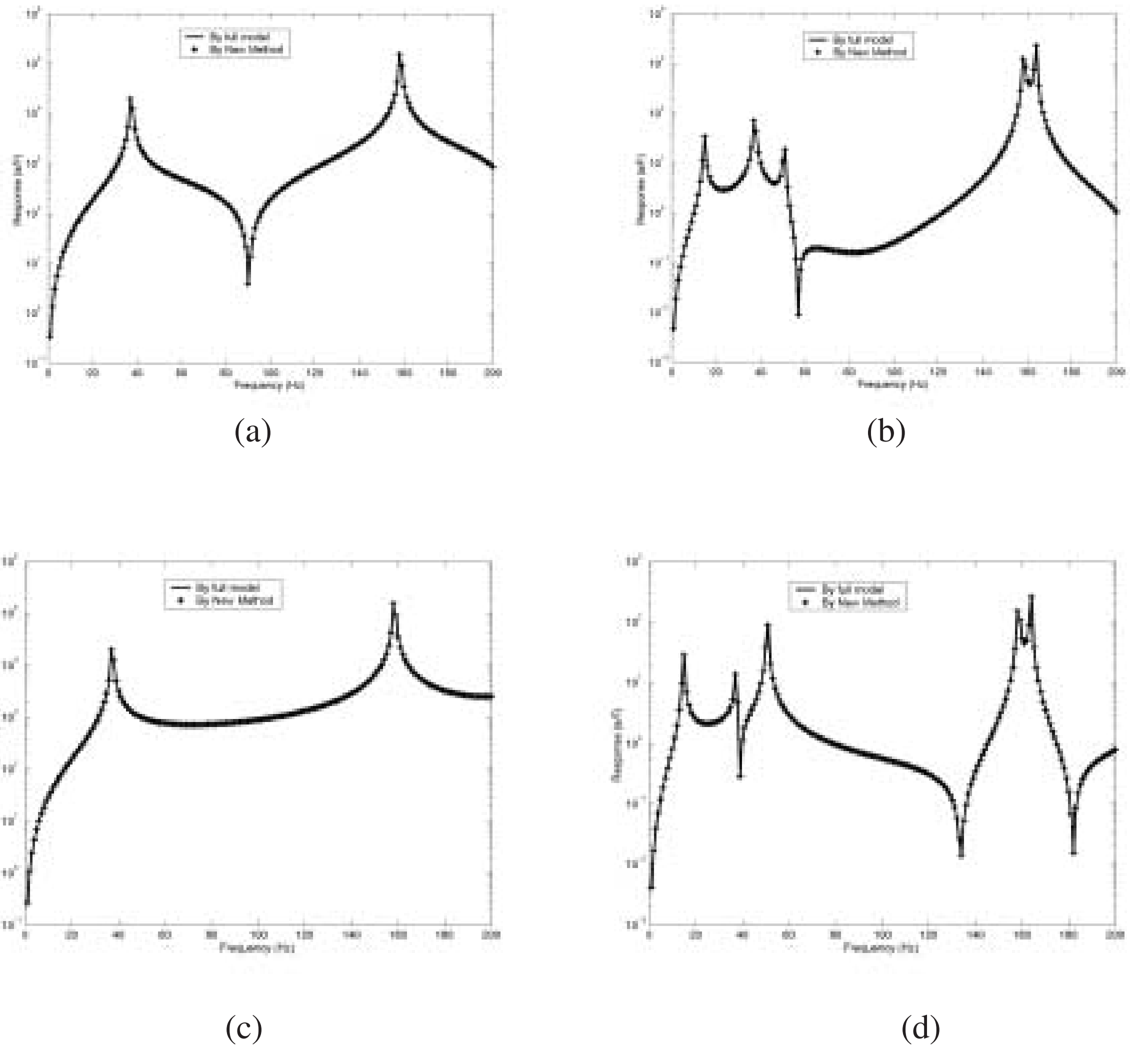

Fig. 5. Case 3 of example problem: FRF's of the system shown in Fig. 2. (a) Response of point 1 of plate A; (b) response of point 1 of plate B; (c) Response of point 2 of plate A; (d) Response of point 2 of plate B.

tained by the proposed method and by full model finite element analysis. Once again, the results agree to seven significant figures.

5. Concluding remarks

The method presented in this paper is useful for the dynamic analysis of complex mechanical systems composed of several substructures. This method not only solves the dynamic responses of the total systems, but also determines the interactions between the substructures. It requires only the frequency response functions at the points of interests and the coupling points, therefore, the order of the mathematical model to be solved is significantly reduced with respect to the physical degrees of freedom of the total system. The method is ideally suitable for the development of a general purpose computer program. A key step in the method is the determination of the orthogonal complement matrix of $(\mathbf{BH}+\mathbf{C})^{\mathrm{H}}$, which can be achieved easily by using the numerically robust singular value decomposition algorithm.

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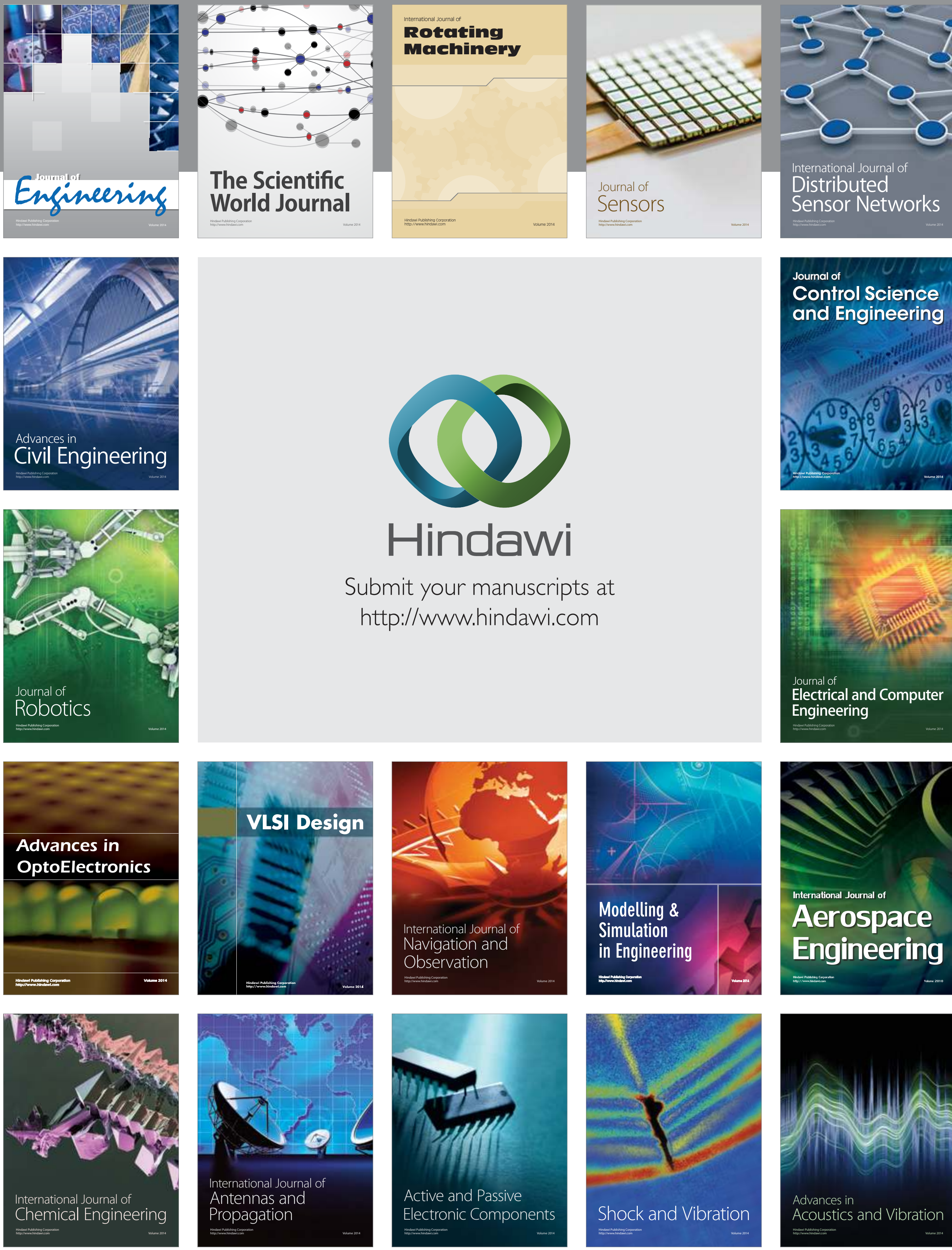
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