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A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative

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Abstract

We present a new method to investigate some fractional integro-differential equations involving the Caputo-Fabrizio derivation and we prove the existence of approximate solutions for these problems. We provide three examples to illustrate our main results. By checking those, one gets the possibility of using some discontinuous mappings as coefficients in the fractional integro-differential equations.

Keywords: approximate solution; Caputo-Fabrizio derivative; fractional integro-differential equation; generalized α-contractive map

1 Introduction

The fractional calculus has an old history and several fractional derivations where defined but the most utilized are Caputo and Riemann-Liouville derivations [1–5]. In 2015, Caputo and Fabrizio defined a new fractional derivation without singular kernel [6]. Immediately, Losada and Nieto wrote a paper about properties of the new fractional derivative [7] and several researchers tried to utilize it for solving different equations (see [8–14] and the references therein).

Let $b > 0$, $u \in H^1(0,b)$ and $\alpha \in (0,1)$. The Caputo-Fabrizio fractional derivative of order α for the function u is defined by ${}^{\mathrm{CF}}D^{\alpha}u(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp(\frac{-\alpha}{1-\alpha}(t-s))u'(s)\,ds$, where $t \geq 0$ and $M(\alpha)$ is normalization constant depending on α such that $M(0) = M(1) = 1$ [6]. Also, Losada and Nieto showed that the fractional integral of order α for the function u is given by ${}^{\mathrm{CF}}I^{\alpha}u(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(s)\,ds$ whenever $0 < \alpha < 1$ [7]. They showed that $M(\alpha) = \frac{2}{2-\alpha}$ for all $0 \leq \alpha \leq 1$ [7]. Thus, the fractional Caputo-Fabrizio derivative of order α for the function u is given by ${}^{\mathrm{CF}}D^{\alpha}u(t) = \frac{1}{1-\alpha} \int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))u'(s)\,ds$, where $t \geq 0$ and $0 < \alpha < 1$ [7]. If $n \geq 1$ and $\alpha \in [0,1]$, then the fractional derivative ${}^{\mathrm{CF}}D^{\alpha+n}$ of order $n+\alpha$ is defined by ${}^{\mathrm{CF}}D^{\alpha+n}u := {}^{\mathrm{CF}}D^{\alpha}(D^n u(t))$ [6]. We need the following results.

Theorem 1.1 ([6]) *Let $u, v \in H^1(0,1)$ and $\alpha \in (0,1)$. If $u'(0) = 0$, then ${}^{\mathrm{CF}}D^{\alpha}({}^{\mathrm{CF}}D^1(u(t))) = {}^{\mathrm{CF}}D^1({}^{\mathrm{CF}}D^{\alpha}(u(t)))$. Also, we have $\lim_{\alpha\to 0} {}^{\mathrm{CF}}D^{\alpha}u(t) = u(t) - u(0)$, $\lim_{\alpha\to 1} {}^{\mathrm{CF}}D^{\alpha}u(t) = u'(t)$ and ${}^{\mathrm{CF}}D^{\alpha}(\lambda u(t) + \gamma v(t)) = \lambda {}^{\mathrm{CF}}D^{\alpha}u(t) + \gamma {}^{\mathrm{CF}}D^{\alpha}v(t)$.*

Lemma 1.2 ([7]) *Let $0 < \alpha < 1$. Then the unique solution for the problem ${}^{\mathrm{CF}}D^{\alpha}u(t) = v(t)$ with boundary condition $u(0) = c$ is given by $u(t) = c + a_{\alpha}(v(t) - v(0)) + b_{\alpha}\int_0^t v(s)\,ds$, where $a_{\alpha} = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} = 1 - \alpha$ and $b_{\alpha} = \frac{2\alpha}{(2-\alpha)M(\alpha)} = \alpha$. Note that $v(0) = 0$ whenever $u(0) = 0$.*

To discuss the existence of solutions for most fractional differential equations in analytic methods, the well-known fixed point results such as the Banach contraction principle is used. In fact, the existence of solutions and the existence of fixed points are equivalent. As is well known, there are many fractional differential equations which have no exact solutions. Thus, the researchers utilize numerical methods usually for obtaining an approximation of the exact solutions. We say that u is an approximate solution for fractional integro-differential equation whenever we could obtain a sequence of functions $\{u_n\}_{n\geq 1}$ with $u_n \to u$. We use this notion when we could not obtain the exact solution u. This appears usually when you want to investigate the fractional integro-differential equation in a non-complete metric space.

In this manuscript, we prove the existence of approximate solutions analytically for some fractional integro-differential equations involving the Caputo-Fabrizio derivative. In fact, the approximate solution of an equation is equivalent to the approximate fixed point of an appropriate operator. This says that by using numerical methods, one can obtain approximations of the unknown exact solution. We will not check the estimates of the exact solution in our examples because our aim is to show the existence of approximate solutions within the analytical method.

Here, we provide some basic needed notions.

Let (X,d) be a metric space, F a selfmap on X, $\alpha : X \times X \to [0,\infty)$ a mapping and ε a positive number. We say that F is α-admissible whenever $\alpha(x,y) \geq 1$ implies $\alpha(Fx,Fy) \geq 1$ [15]. An element $x_0 \in X$ is called ε-fixed point of F whenever $d(Fx_0,x_0) \leq \varepsilon$. We say that F has the approximate fixed point property whenever F has an ε-fixed point for all $\varepsilon > 0$ [15]. Some mappings have approximate fixed points, while they have no fixed points [15]. Denote by $\mathcal{R}$ the set of all continuous mappings $g : [0,\infty)^5 \to [0,\infty)$ satisfying $g(1,1,1,2,0) = g(1,1,1,0,2) := h \in (0,1)$, $g(\mu x_1, \mu x_2, \mu x_3, \mu x_4, \mu x_5) \leq \mu g(x_1,x_2,x_3,x_4,x_5)$ for all $(x_1,x_2,x_3,x_4,x_5) \in [0,\infty)^5$ and $\mu \geq 0$ and also $g(x_1,x_2,x_3,0,x_4) \leq g(y_1,y_2,y_3,0,y_4)$ and $g(x_1,x_2,x_3,x_4,0) \leq g(y_1,y_2,y_3,y_4,0)$ whenever $x_1,\ldots,x_4,y_1,\ldots,y_4 \in [0,\infty)$ with $x_i < y_i$ for $i = 1,2,3,4$ [15]. We say that F is a generalized α-contractive mapping whenever there exists $g \in \mathcal{R}$ such that $\alpha(x,y)d(Fx,Fy) \leq g(d(x,y),d(x,Fx),d(y,Fy),d(x,Fy),d(y,Fx))$ for all $x,y \in X$ ([15]).

Theorem 1.3 ([15]) *Let (X,d) be a metric space, $\alpha : X \times X \to [0,\infty)$ a mapping and F a generalized α-contractive and α-admissible selfmap on X. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,Fx_0) \geq 1$. Then F has an approximate fixed point.*

2 Main results

Now, we are ready to state and prove our main results.

Lemma 2.1 *Suppose that $u,v \in H^1(0,1)$ and there exists a real number K such that*

$$\left|u(t) - v(t)\right| \leq K$$

for all $t \in [0,1]$. Then $|{}^{\mathrm{CF}}D^{\alpha}u(t) - {}^{\mathrm{CF}}D^{\alpha}v(t)| \leq \frac{2-\alpha}{(1-\alpha)^2}K$ for all $t \in [0,1]$.

Proof Note that

$$\begin{aligned}
&{}^{\mathrm{CF}}D^{\alpha}u(t)\\
&\quad= \frac{1}{1-\alpha}\int_0^t \exp\biggl(-\frac{\alpha}{1-\alpha}(t-s)\biggr)u'(s)\,ds\\
&\quad= \frac{1}{1-\alpha}\exp\biggl(-\frac{\alpha}{1-\alpha}(t-s)\biggr)u(s)\Big|_0^t - \frac{1}{1-\alpha}\int_0^t \frac{\alpha}{1-\alpha}\exp\biggl(-\frac{\alpha}{1-\alpha}(t-s)\biggr)u(s)\,ds\\
&\quad= \frac{1}{1-\alpha}u(t) - \frac{1}{1-\alpha}\exp\biggl(-\frac{\alpha}{1-\alpha}t\biggr)u(0) - \frac{\alpha}{(1-\alpha)^2}\int_0^t \exp\biggl(-\frac{\alpha}{1-\alpha}(t-s)\biggr)u(s)\,ds
\end{aligned}$$

and so

$$\begin{aligned}
{}^{\mathrm{CF}}D^{\alpha}u(t) - {}^{\mathrm{CF}}D^{\alpha}v(t) &\le \frac{1}{1-\alpha}\bigl|\bigl(u(t)-v(t)\bigr)\bigr| + \frac{1}{1-\alpha}\biggl|\exp\biggl(-\frac{\alpha}{1-\alpha}t\biggr)\biggr|\bigl|u(0)-v(0)\bigr|\\
&\quad+ \frac{\alpha}{(1-\alpha)^2}\int_0^t \exp\biggl(-\frac{\alpha}{1-\alpha}(t-s)\biggr)\bigl|\bigl(u(s)-v(s)\bigr)\bigr|\,ds\\
&\le \frac{2}{1-\alpha}K + \frac{\alpha}{(1-\alpha)^2}K = \frac{2-\alpha}{(1-\alpha)^2}K
\end{aligned}$$

for all $t \in [0,1]$. Hence, $|{}^{\mathrm{CF}}D^{\alpha}u(t) - {}^{\mathrm{CF}}D^{\alpha}v(t)| \le (\frac{2-\alpha}{(1-\alpha)^2})K$ for all $t \in [0,1]$. □

If $u \in H^1(0,1)$ and there exists $K \ge 0$ such that $|u(t)| \le K$ for all $t \in [0,1]$, then by using last result we get $|{}^{\mathrm{CF}}D^{\alpha}u(t)| \le (\frac{2-\alpha}{(1-\alpha)^2})K$ for all $t \in [0,1]$. Also by checking the proof of the last result, one can prove the next lemma.

Lemma 2.2 *Suppose that $u, v \in H^1(0,1)$ with $u(0) = v(0)$ and there exists a real number K such that $|u(t) - v(t)| \le K$ for all $t \in [0,1]$. Then $|{}^{\mathrm{CF}}D^{\alpha}u(t) - {}^{\mathrm{CF}}D^{\alpha}v(t)| \le \frac{1}{(1-\alpha)^2}K$ for all $t \in [0,1]$.*

Lemma 2.3 *Suppose that $u, v \in C[0,1]$ and there is $K \ge 0$ such that $|u(t) - v(t)| \le K$ for all $t \in [0,1]$. Then $|{}^{\mathrm{CF}}I^{\alpha}u(t) - {}^{\mathrm{CF}}I^{\alpha}v(t)| \le K$ for all $t \in [0,1]$.*

Proof Note that for each $t \in [0,1]$ we have

$${}^{\mathrm{CF}}I^{\alpha}u(t) - {}^{\mathrm{CF}}I^{\alpha}v(t) = a_{\alpha}\bigl(u(t)-v(t)\bigr) + b_{\alpha}\int_0^t \bigl(u(s)-v(s)\bigr)\,ds \le a_{\alpha}K + b_{\alpha}K = K,$$

where a_{α} and b_{α} are given in Lemma 1.2. This completes the proof. □

If u is an element of $C[0,1]$ such that $|u(t)| \le K$ for some $K \ge 0$ and all $t \in [0,1]$, then the last result implies that $|{}^{\mathrm{CF}}I^{\alpha}u(t)| \le K$ for all $t \in [0,1]$.

Lemma 2.4 *Let $b > 0$ be given and $0 \le \alpha \le 1$. If u is an element of $H^1(0,b)$ such that $u(0) = 0$, $u'(0) = 0$, $u' \in H^1(0,b)$ and ${}^{\mathrm{CF}}D^{\alpha}u \in H^1(0,b)$, then ${}^{\mathrm{CF}}D^{1}({}^{\mathrm{CF}}I^{\alpha}u(t)) = {}^{\mathrm{CF}}I^{\alpha}({}^{\mathrm{CF}}D^{1}u(t)) = a_{\alpha}u'(t) + b_{\alpha}u(t)$ and $({}^{\mathrm{CF}}D^{\alpha}u(t))' = {}^{\mathrm{CF}}D^{\alpha}u'(t)$ for all $t \ge 0$. If $u''(t) \ge 0$ for all $t \ge 0$, then ${}^{\mathrm{CF}}D^{\alpha}u$ is increasing on $[0,b]$. Also, ${}^{\mathrm{CF}}D^{\alpha}u$ is decreasing on $[0,b]$ whenever $u''(t) \le 0$ for all $t \ge 0$.*

Proof Note that ${}^{\mathrm{CF}}I^{\alpha}({}^{\mathrm{CF}}D^{1}u(t)) = a_{\alpha}u'(t) + b_{\alpha}\int_0^t u'(s)\,ds = a_{\alpha}u'(t) + b_{\alpha}u(t)$ and

$$\begin{aligned} {}^{\mathrm{CF}}D^{1}\bigl({}^{\mathrm{CF}}I^{\alpha}u(t)\bigr) &= {}^{\mathrm{CF}}D^{1}\left(a_{\alpha}u(t) + b_{\alpha}\int_0^t u(s)\,ds\right) \\ &= a_{\alpha}u'(t) + b_{\alpha}\left(\int_0^t u(s)\,ds\right)' \\ &= a_{\alpha}u'(t) + b_{\alpha}u(t) \end{aligned}$$

for all $t \geq 0$. Also, $({}^{\mathrm{CF}}D^{\alpha}u(t))' = {}^{\mathrm{CF}}D^{1}({}^{\mathrm{CF}}D^{\alpha}u(t)) = {}^{\mathrm{CF}}D^{\alpha}({}^{\mathrm{CF}}D^{1}u(t)) = {}^{\mathrm{CF}}D^{\alpha}u'(t)$ for all $t \geq 0$. Since $({}^{\mathrm{CF}}D^{\alpha}u(t))' = {}^{\mathrm{CF}}D^{\alpha}u'(t) = \frac{1}{1-\alpha}\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))u''(s)\,ds$ for all $t \geq 0$, we see that ${}^{\mathrm{CF}}D^{\alpha}u$ is increasing on $[0,b]$ whenever $u''(t) \geq 0$ for all $t \in [0,b]$. Also, ${}^{\mathrm{CF}}D^{\alpha}u$ is decreasing on $[0,b]$ whenever $u''(t) \leq 0$ for all $t \in [0,b]$. □

Note that the conditions $u' \in H^1(0,b)$ and ${}^{\mathrm{CF}}D^{\alpha}u \in H^1(0,b)$ in Lemma 2.4 just impose a unique condition on u. Let $\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$ be two continuous maps such that $\sup_{t\in I}|\int_0^t \lambda(t,s)\,ds| < \infty$ and $\sup_{t\in I}|\int_0^t \gamma(t,s)\,ds| < \infty$. Consider the maps ϕ and φ defined by $(\phi u)(t) = \int_0^t \gamma(t,s)u(s)\,ds$ and $(\varphi u)(t) = \int_0^t \lambda(t,s)u(s)\,ds$. Throughout this paper, we put $\gamma_0 = \sup_{t\in I}|\int_0^t \gamma(t,s)\,ds|$, $\lambda_0 = \sup_{t\in I}|\int_0^t \lambda(t,s)\,ds|$ and $\eta(t) \in L^{\infty}(I)$ with $\eta^* = \sup_{t\in I}|\eta(t)|$. Here, we investigate the fractional integro-differential problem

$$ {}^{\mathrm{CF}}D^{\alpha}u(t) = f\bigl(t, u(t), (\phi u)(t), (\varphi u)(t)\bigr) \tag{1}$$

with boundary condition $u(0) = 0$, where $\alpha \in (0,1)$.

Theorem 2.5 *Let $\eta(t) \in L^{\infty}(I)$ and $f : I \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function such that*

$$\bigl|f(t,x,y,w) - f\bigl(t,x',y',w'\bigr)\bigr| \leq \eta(t)\bigl(\bigl|x-x'\bigr| + \bigl|y-y'\bigr| + \bigl|w-w'\bigr|\bigr)$$

for all $t \in I$ and $x,y,w,x',y',w' \in \mathbb{R}$. Then the problem (1) *with the boundary condition has an approximate solution whenever $\Delta_1 = \eta^*(1+\gamma_0+\lambda_0) < 1$.*

Proof Consider the space H^1 endowed with the metric $d(u,v) = \|u-v\|$, where $\|u\| = \sup_{t\in I}|u(t)|$. Now, define the selfmap $F : H^1 \to H^1$ by

$$(Fu)(t) = a_{\alpha}f\bigl(t,u(t),(\phi u)(t),(\varphi u)(t)\bigr) + b_{\alpha}\int_0^t f\bigl(s,u(s),(\phi u)(s),(\varphi u)(s)\bigr)\,ds,$$

where a_{α} and b_{α} are given in Lemma 1.2. Note that

$$\begin{aligned} (Fu)(t) &= a_{\alpha}{}^{\mathrm{CF}}D^{\alpha}u(t) + b_{\alpha}\int_0^t f\bigl(s,u(s),(\phi u)(s),(\varphi u)(s)\bigr)\,ds \\ &= a_{\alpha}\frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)u'(s)\,ds \\ &\quad + b_{\alpha}\int_0^t f\bigl(s,u(s),(\phi u)(s),(\varphi u)(s)\bigr)\,ds \end{aligned}$$

for all t. This shows that F maps H^1 into H^1. Thus, we have

$$\begin{aligned}
&\big|(Fu)(t) - (Fv)(t)\big| \\
&\quad\leq a_\alpha \big|f\big(t,u(t),(\phi u)(t),(\varphi u)(t)\big) - f\big(t,v(t),(\phi v)(t),(\varphi v)(t)\big)\big| \\
&\qquad + b_\alpha \int_0^t \big|f\big(s,u(s),(\phi u)(s),(\varphi u)(s)\big) - f\big(s,v(s),(\phi v)(s),(\varphi v)(s)\big)\big|\,ds \\
&\quad\leq a_\alpha \big|\eta(t)\big|\big(\big|u(t)-v(t)\big| + \big|(\phi u)(t)-(\phi v)(t)\big| + \big|(\varphi u)(t)-(\varphi v)(t)\big|\big) \\
&\qquad + b_\alpha \int_0^t \big(\big|u(s)-v(s)\big| + \big|(\phi u)(s)-(\phi v)(s)\big| + \big|(\varphi u)(s)-(\varphi v)(s)\big|\big)\big|\eta(s)\big|\,ds \\
&\quad\leq \eta^*(1+\gamma_0+\lambda_0)[a_\alpha+b_\alpha]\|u-v\| = \eta^*(1+\gamma_0+\lambda_0)\|u-v\|
\end{aligned}$$

for all $t \in I$ and $u,v \in H^1$. Now, consider $g : [0,\infty)^5 \to [0,\infty)$ and $\alpha : H^1 \times H^1 \to [0,\infty)$ defined by $g(t_1,t_2,t_3,t_4,t_5) = \Delta_1 t_1$ and $\alpha(x,y) = 1$ for all $x,y \in H^1$. One can easily check that $g \in \mathcal{R}$ and F is a generalized α-contraction. By using Theorem 1.3, F has an approximate fixed point which is an approximate solution for the problem (1). $\square$

Note that H^1 with the sup norm is not Banach. Thus, we used a new method for investigation of the problem. Now, we investigate the fractional integro-differential problem

$$\begin{aligned}
{}^{\mathrm{CF}}D^\alpha u(t) &= \mu\big({}^{\mathrm{CF}}D^\beta u(t) + {}^{\mathrm{CF}}D^\gamma u(t)\big) \\
&\quad + f\big(t,u(t),(\phi u)(t),(\varphi u)(t),{}^{\mathrm{CF}}I^\theta u(t),{}^{\mathrm{CF}}D^\delta u(t)\big)
\end{aligned}\tag{2}$$

with boundary condition $u(0) = c$, where $\mu \geq 0$ and $\alpha,\beta,\gamma,\theta,\delta \in (0,1)$ and $c \in \mathbb{R}$.

Theorem 2.6 *Let $\eta(t) \in L^\infty(I)$ and $f : [0,1] \times \mathbb{R}^5 \to \mathbb{R}$ be a continuous function such that*

$$\begin{aligned}
&\big|f(t,x,y,w,u_1,u_2) - f\big(t,x',y',w',v_1,v_2\big)\big| \\
&\quad\leq \eta(t)\big(\big|x-x'\big| + \big|y-y'\big| + \big|w-w'\big| + |u_1-v_1| + |u_2-v_2|\big)
\end{aligned}$$

for all $t \in I$ and $x,y,w,x',y',w',u_1u_2,v_1,v_2 \in \mathbb{R}$. Then the problem (2) *with the boundary condition has an approximate solution whenever $\Delta_2 < 1$, where*

$$\Delta_2 = \eta^*\left(2+\gamma_0+\lambda_0+\frac{1}{(1-\delta)^2}\right) + \mu\left(\frac{1}{(1-\beta)^2}+\frac{1}{(1-\gamma)^2}\right).$$

Proof Consider the space H^1 endowed with the metric $d(u,v) = \|u-v\|$, where $\|u\| = \sup_{t\in I}|u(t)|$. Define the map $F : H^1 \to H^1$ by

$$\begin{aligned}
(Fu)(t) &= u(0) + a_\alpha\big[\mu\big({}^{\mathrm{CF}}D^\beta u(t) + {}^{\mathrm{CF}}D^\gamma u(t)\big) \\
&\quad + f\big(t,u(t),(\phi u)(t),(\varphi u)(t),{}^{\mathrm{CF}}I^\theta u(t),{}^{\mathrm{CF}}D^\delta u(t)\big) - \mu\big({}^{\mathrm{CF}}D^\beta u(0) + {}^{\mathrm{CF}}D^\gamma u(0)\big) \\
&\quad - f\big(0,u(0),(\phi u)(0),(\varphi u)(0),{}^{\mathrm{CF}}I^\theta u(0),{}^{\mathrm{CF}}D^\delta u(0)\big)\big] \\
&\quad + b_\alpha \int_0^t \big[\mu\big({}^{\mathrm{CF}}D^\beta u(s) + {}^{\mathrm{CF}}D^\gamma u(s)\big) \\
&\quad + f\big(s,u(s),(\phi u)(s),(\varphi u)(s),{}^{\mathrm{CF}}I^\theta u(t),{}^{\mathrm{CF}}D^\delta u(s)\big)\big]\,ds,
\end{aligned}$$

where a_α and b_α are given in Lemma 1.2. By using Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}
\bigl|(Fu)(t)-(Fv)(t)\bigr| &\le \bigl|u(0)-v(0)\bigr| + a_\alpha\bigl[\mu\bigl|{}^{\mathrm{CF}}D^\beta u(t)-{}^{\mathrm{CF}}D^\beta v(t)\bigr| \\
&\quad + \mu\bigl|{}^{\mathrm{CF}}D^\gamma u(t)-{}^{\mathrm{CF}}D^\gamma v(t)\bigr| + \mu\bigl|{}^{\mathrm{CF}}D^\beta u(0)-{}^{\mathrm{CF}}D^\beta v(0)\bigr| \\
&\quad + \mu\bigl|{}^{\mathrm{CF}}D^\gamma u(0)-{}^{\mathrm{CF}}D^\gamma v(0)\bigr| \\
&\quad + \bigl|f\bigl(t,u(t),(\phi u)(t),(\varphi u)(t),{}^{\mathrm{CF}}I^\theta u(t),{}^{\mathrm{CF}}D^\delta u(t)\bigr) \\
&\quad - f\bigl(t,v(t),(\phi v)(t),(\varphi v)(t),{}^{\mathrm{CF}}I^\theta v(t),{}^{\mathrm{CF}}D^\delta v(t)\bigr)\bigr| \\
&\quad + \bigl|f\bigl(0,u(0),(\phi u)(0),(\varphi u)(0),{}^{\mathrm{CF}}I^\theta u(0),{}^{\mathrm{CF}}D^\delta u(0)\bigr) \\
&\quad - f\bigl(0,v(0),(\phi v)(0),(\varphi v)(0),{}^{\mathrm{CF}}I^\theta v(0),{}^{\mathrm{CF}}D^\delta v(0)\bigr)\bigr|\bigr] \\
&\quad + b_\alpha\int_0^t\bigl[\mu\bigl|{}^{\mathrm{CF}}D^\beta u(s)-{}^{\mathrm{CF}}D^\beta v(s)\bigr| + \mu\bigl|{}^{\mathrm{CF}}D^\gamma u(s)-{}^{\mathrm{CF}}D^\gamma v(s)\bigr| \\
&\quad + \mu\bigl|{}^{\mathrm{CF}}D^\beta u(0)-{}^{\mathrm{CF}}D^\beta v(0)\bigr| + \mu\bigl|{}^{\mathrm{CF}}D^\gamma u(0)-{}^{\mathrm{CF}}D^\gamma v(0)\bigr| \\
&\quad + \bigl|f\bigl(s,u(t),(\phi u)(s),(\varphi u)(s),{}^{\mathrm{CF}}I^\theta u(s),{}^{\mathrm{CF}}D^\delta u(s)\bigr) \\
&\quad - f\bigl(s,v(s),(\phi v)(s),(\varphi v)(s),{}^{\mathrm{CF}}I^\theta v(s),{}^{\mathrm{CF}}D^\delta v(s)\bigr)\bigr|\bigr]\,ds \\
&\le a_\alpha\bigl[\mu\bigl|{}^{\mathrm{CF}}D^\beta u(t)-{}^{\mathrm{CF}}D^\beta v(t)\bigr| + \mu\bigl|{}^{\mathrm{CF}}D^\gamma u(t)-{}^{\mathrm{CF}}D^\gamma v(t)\bigr| \\
&\quad + \bigl|\eta(t)\bigr|\bigl(\bigl|u(t)-v(t)\bigr| + \bigl|(\phi u)(t)-(\phi v)(t)\bigr| + \bigl|(\varphi u)(t)-(\varphi v)(t)\bigr| \\
&\quad + \bigl|{}^{\mathrm{CF}}I^\theta u(t)-{}^{\mathrm{CF}}I^\theta v(t)\bigr| + \bigl|{}^{\mathrm{CF}}D^\delta u(t)-{}^{\mathrm{CF}}D^\delta v(t)\bigr|\bigr)\bigr] \\
&\quad + b_\alpha\int_0^t\bigl[\mu\bigl|{}^{\mathrm{CF}}D^\beta u(s)-{}^{\mathrm{CF}}D^\beta v(s)\bigr| + \mu\bigl|{}^{\mathrm{CF}}D^\gamma u(s)-{}^{\mathrm{CF}}D^\gamma v(s)\bigr| \\
&\quad + \bigl|\eta(s)\bigr|\bigl(\bigl|u(s)-v(s)\bigr| + \bigl|(\phi u)(s)-(\phi v)(s)\bigr| + \bigl|(\varphi u)(s)-(\varphi v)(s)\bigr| \\
&\quad + \bigl|{}^{\mathrm{CF}}I^\theta u(s)-{}^{\mathrm{CF}}I^\theta v(s)\bigr| + \bigl|{}^{\mathrm{CF}}D^\delta u(s)-{}^{\mathrm{CF}}D^\delta v(s)\bigr|\bigr)\bigr]\,ds \\
&\le \eta^*\left(2+\gamma_0+\lambda_0+\frac{1}{(1-\delta)^2}\right)+\mu\left(\frac{1}{(1-\beta)^2}+\frac{1}{(1-\gamma)^2}\right)\|u-v\|
\end{aligned}$$

for all $u,v\in H^1$ and $t\in I$. Hence,

$$\begin{aligned}
\|Fu-Fv\| &\le \eta^*\left(2+\gamma_0+\lambda_0+\frac{1}{(1-\delta)^2}\right)+\mu\left(\frac{1}{(1-\beta)^2}+\frac{1}{(1-\gamma)^2}\right)\|u-v\| \\
&= \Delta_2\|u-v\|
\end{aligned}$$

for $u,v\in H^1$. Now, consider the maps $g:[0,\infty)^5\to[0,\infty)$ and $\alpha:H^1\times H^1\to[0,\infty)$ defined by $g(t_1,t_2,t_3,t_4,t_5)=\frac{\Delta_2}{3}(t_1+2t_2)$ and $\alpha(x,y)=1$ for all $x,y\in H^1$. One can easily see that $g\in\mathcal{R}$ and F is a generalized α-contractive map. By using Theorem 1.3, F has an approximate fixed point which is an approximate solution for the problem (2). □

Let k and h be bounded functions on $I=[0,1]$ and s an integrable bounded function on I with $M_1=\sup_{t\in I}|k(t)|$, $M_2=\sup_{t\in I}|s(t)|<\infty$ and $M_3=\sup_{t\in I}|h(t)|<\infty$. Now, we investigate the fractional integro-differential problem

$$\begin{aligned} {}^{\mathrm{CF}}D^{\alpha}u(t) &= \mu_1 k(t)\,{}^{\mathrm{CF}}D^{\beta}u(t) + \mu_2(\varphi s)(t)\,{}^{\mathrm{CF}}D^{\theta}\big({}^{\mathrm{CF}}D^{\rho}u(t)\big) \\ &\quad + f\big(t, u(t), (\phi u)(t), h(t)\,{}^{\mathrm{CF}}D^{\nu}u(t)\big) \end{aligned} \tag{3}$$

with boundary condition $u(0) = 0$, where $\mu_1, \mu_1 \geq 0$ and $\alpha, \beta, \theta, \rho, \nu \in (0,1)$. Note that the functions k, s and h are not necessarily continuous. Since the left side of equation (3) is continuous, the right side so is as the problem (3) is a well-defined equation. For this reason, we supposed continuity of the function f in Theorems 2.5 and 2.6 where equations (1) and (2) are well defined.

Theorem 2.7 *Let $\eta(t) \in L^{\infty}(I)$ and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be a function such that*

$$\big|f(t,x,y,w) - f\big(t,x',y',w'\big)\big| \leq \eta(t)\big(\big|x-x'\big| + \big|y-y'\big| + \big|w-w'\big|\big)$$

for all $t \in I$ and $x, y, w, v, x', y', w', v' \in \mathbb{R}$. Then the problem (3) *has an approximate solution whenever $\Delta_3 < 1$, where*

$$\Delta_3 = \left(1 + \frac{1}{(1-\beta)^2} + \frac{1}{(1-\theta)^2(1-\rho)^2} + \frac{1}{(1-\nu)^2}\right)\big[\mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3)\big].$$

Proof Consider the space H^1 endowed with the metric $d(u,v) = \|u - v\|$, where

$$\|u\| = \max_{t\in I}\big|u(t)\big| + \max_{t\in I}\big|{}^{\mathrm{CF}}D^{\beta}u(t)\big| + \max_{t\in I}\big|{}^{\mathrm{CF}}D^{\theta}\big({}^{\mathrm{CF}}D^{\rho}u(t)\big)\big| + \max_{t\in I}\big|{}^{\mathrm{CF}}D^{\nu}u(t)\big|.$$

Define the map $F : H^1 \to H^1$ by

$$\begin{aligned} (Fu)(t) &= a_{\alpha}\big[\mu_1 k(t)\,{}^{\mathrm{CF}}D^{\gamma}\big({}^{\mathrm{CF}}D^{\beta}u(t)\big) + \mu_2(\varphi s)(t)\,{}^{\mathrm{CF}}D^{\theta}\big({}^{\mathrm{CF}}D^{\rho}u(t)\big) \\ &\quad + f\big(t, u(t), (\phi u)(t), h(t)\,{}^{\mathrm{CF}}D^{\nu}u(t)\big)\big] \\ &\quad + \int_0^t \big[\mu_1 k(s)\,{}^{\mathrm{CF}}D^{\gamma}\big({}^{\mathrm{CF}}D^{\beta}u(s)\big) + \mu_2(\varphi s)(t)\,{}^{\mathrm{CF}}D^{\theta}\big({}^{\mathrm{CF}}D^{\rho}u(s)\big) \\ &\quad + f\big(t, u(s), (\phi u)(s), h(s)\,{}^{\mathrm{CF}}D^{\nu}u(s)\big)\big]\,ds \end{aligned}$$

for all $t \in I$, where a_{α} and b_{α} introduced in Lemma 1.2. By using Lemma 2.2, we get

$$\begin{aligned} &\big[\mu_1 k(t)\,{}^{\mathrm{CF}}D^{\gamma}\big({}^{\mathrm{CF}}D^{\beta}u(t)\big) + \mu_2(\varphi s)(t)\,{}^{\mathrm{CF}}D^{\theta}\big({}^{\mathrm{CF}}D^{\rho}u(t)\big) + f\big(t, u(t), (\phi u)(t), h(t)\,{}^{\mathrm{CF}}D^{\nu}u(t)\big)\big] \\ &\quad - \big[\mu_1 k(t)\,{}^{\mathrm{CF}}D^{\gamma}\big({}^{\mathrm{CF}}D^{\beta}v(t)\big) + \mu_2(\varphi s)(t)\,{}^{\mathrm{CF}}D^{\theta}\big({}^{\mathrm{CF}}D^{\rho}v(t)\big) \\ &\quad + f\big(t, v(t), (\phi v)(t), h(t)\,{}^{\mathrm{CF}}D^{\nu}v(t)\big)\big] \\ &\leq \mu_1\big|k(t)\big|\big|{}^{\mathrm{CF}}D^{\gamma}\big({}^{\mathrm{CF}}D^{\beta}\big(u(t) - v(t)\big)\big)\big| + \mu_2\big|(\varphi s)(t)\big|\big|{}^{\mathrm{CF}}D^{\theta}\big({}^{\mathrm{CF}}D^{\rho}\big(u(t) - v(t)\big)\big)\big| \\ &\quad + \big|f\big(t, u(t), (\phi u)(t), h(t)\,{}^{\mathrm{CF}}D^{\nu}u(t)\big) - f\big(t, v(t), (\phi v)(t), h(t)\,{}^{\mathrm{CF}}D^{\nu}v(t)\big)\big| \\ &\leq \mu_1 M_1\|u - v\| + \mu_2\lambda_0 M_2\|u - v\| + \eta^*\big(\|u - v\| + \gamma_0\|u - v\| + M_3\|u - v\|\big) \\ &= \big[\mu_1 M_1 + \mu_2\lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3)\big]\|u - v\| \end{aligned}$$

for all $t \in I$ and $u, v \in H^1$. Hence

$$\big|(Fu)(t) - (Fv)(t)\big| \leq \big[\mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3)\big]\|u - v\| = \Delta_3 \|u - v\|$$

for all $t \in I$ and $u, v \in H^1$. Also, we have

$$\begin{aligned} &\big|{}^{\mathrm{CF}}D^{\beta}(Fu - Fv)(t)\big| \leq \frac{1}{(1-\beta)^2}\big[\mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3)\big]\|u - v\|, \\ &\big|{}^{\mathrm{CF}}D^{\theta}\big({}^{\mathrm{CF}}D^{\rho}(Fu - Fv)(t)\big)\big| \\ &\quad \leq \frac{1}{(1-\theta)^2(1-\rho)^2}\big[\mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3)\big]\|u - v\|, \end{aligned}$$

and $|{}^{\mathrm{CF}}D^{\nu}(Fu - Fv)(t)| \leq \frac{1}{(1-\nu)^2}[\mu_1 M_1 + \mu_2 \lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3)]\|u - v\|$ for all $u, v \in H^1$ and $t \in I$. Hence, $\|Fu - Fv\| \leq \Delta \|u - v\|$ for all $u, v \in H^1$. Now, consider the maps $g : [0,\infty)^5 \to [0,\infty)$ and $\alpha : H^1 \times H^1 \to [0,\infty)$ defined by $g(t_1, t_2, t_3, t_4, t_5) = \Delta_3 \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\}$ and $\alpha(x, y) = 1$ for all $x, y \in H^1$. One can check that $g \in \mathcal{R}$ and F is a generalized α-contraction. By using Theorem 1.3, F has an approximate fixed point which is an approximate solution for the problem (3). □

Let k, s, h, g and q be bounded functions on $[0,1]$ with $M_1 = \sup_{t\in I} |k(t)| < \infty$, $M_2 = \sup_{t\in I} |s(t)| < \infty$, $M_3 = \sup_{t\in I} |h(t)| < \infty$, $M_4 = \sup_{t\in I} |g(t)| < \infty$, and $M_5 = \sup_{t\in I} |q(t)| < \infty$. Here, we investigate the fractional integro-differential problem

$$\begin{aligned} {}^{\mathrm{CF}}D^{\alpha}u(t) = {}& \lambda k(t){}^{\mathrm{CF}}D^{\beta}u(t) + \mu s(t){}^{\mathrm{CF}}I^{\rho}u(t) \\ & + f_1\big(t, u(t), (\phi u)(t), h(t){}^{\mathrm{CF}}I^{\nu}u(t), g(t){}^{\mathrm{CF}}D^{\delta}u(t)\big) \\ & + \int_0^t f_2\big(s, u(s), (\varphi u)(s), q(t){}^{\mathrm{CF}}D^{\gamma}u(s)\big)\,ds \end{aligned} \tag{4}$$

with boundary condition $u(0) = 0$, where $\lambda, \mu \geq 0$ and $\alpha, \beta, \rho, \nu, \delta, \gamma \in (0,1)$. Note that the maps k, s, h, g and q should be chosen such that the right side of equation (4) is continuous.

Theorem 2.8 *Let ξ_1, ξ_2, ξ_3, ξ_4, ξ_1', ξ_2', and ξ_3' be nonnegative real numbers. Suppose that $f_1 : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ and $f_2 : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are integrable functions such that*

$$\big|f_1(t, x, y, w, v) - f_1\big(t, x', y', w', v'\big)\big| \leq \xi_1\big|x - x'\big| + \xi_2\big|y - y'\big| + \xi_3\big|w - w'\big| + \xi_4\big|v - v'\big|$$

and $|f_2(t, x, y, w) - f_2(t, x', y', w')| \leq \xi_1'|x - x'| + \xi_2'|y - y'| + \xi_3'|w - w'|$ for all real numbers x, y, w, v, x', y' and w' and $t \in I$. If $\Delta_4 < 1$, then the problem (4) has an approximate solution, where $\Delta_4 := \max\{\frac{1}{(1-\beta)^2}, \frac{1}{(1-\delta)^2}, \frac{1}{(1-\gamma)^2}\}[\lambda \frac{M_1}{(1-\beta)^2} + \mu M_2 + \xi_1 + \xi_2\gamma_0 + \xi_3 M_3 + \xi_4 \frac{M_4}{(1-\delta)^2} + \xi_1' + \xi_2'\lambda_0 + \xi_3' \frac{M_5}{(1-\gamma)^2}]$.

Proof Consider the space H^1 endowed with the metric $d(u, v) = \|u - v\|$, where

$$\begin{aligned} \|u\| = \max\Big\{ & \sup_{t\in I}\big|u(t)\big|, \sup_{t\in I}\big|{}^{\mathrm{CF}}D^{\beta}u(t)\big|, \sup_{t\in I}\big|{}^{\mathrm{CF}}I^{\rho}u(t)\big|, \sup_{t\in I}\big|{}^{\mathrm{CF}}I^{\nu}u(t)\big|, \\ & \sup_{t\in I}\big|{}^{\mathrm{CF}}D^{\delta}u(t)\big|, \sup_{t\in I}\big|{}^{\mathrm{CF}}D^{\gamma}u(t)\big|\Big\}. \end{aligned}$$

Define the map $F : H^1 \to H^1$ by

$$
\begin{aligned}
(Fu)(t) = a_\alpha \Big[& \lambda k(t)^{\mathrm{CF}}D^\beta u(t) + \mu s(t)^{\mathrm{CF}}I^\rho u(t) \\
& + f_1\big(t, u(t), (\phi u)(t), h(t)^{\mathrm{CF}}I^\nu u(t), g(t)^{\mathrm{CF}}D^\delta u(t)\big) \\
& + \int_0^t f_2\big(s, u(s), (\varphi u)(s), q(t)^{\mathrm{CF}}D^\gamma u(s)\big)\, ds \Big] \\
& + b_\alpha \Big[\int_0^t \lambda k(s)^{\mathrm{CF}}D^\beta u(s) + \mu s(s)^{\mathrm{CF}}I^\rho u(s) \\
& + f_1\big(s, u(s), (\phi u)(s), h(s)^{\mathrm{CF}}I^\nu u(s), g(s)^{\mathrm{CF}}D^\delta u(s)\big) \\
& + \int_0^t \int_0^s f_2\big(r, u(r), (\varphi u)(r), q(r)^{\mathrm{CF}}D^\gamma u(r)\big)\, dr\, ds \Big],
\end{aligned}
$$

where a_α and b_α are introduced in Lemma 1.2. By using Lemmas 2.2 and 2.3, we obtain

$$
\begin{aligned}
& \Big\| \Big[\lambda k(t)^{\mathrm{CF}}D^\beta u(t) + \mu s(t)^{\mathrm{CF}}I^\rho u(t) + f_1\big(t, u(t), (\phi u)(t), h(t)^{\mathrm{CF}}I^\nu u(t), g(t)^{\mathrm{CF}}D^\delta u(t)\big) \\
& \quad + \int_0^t f_2\big(s, u(s), (\varphi u)(s), q(t)^{\mathrm{CF}}D^\gamma u(s)\big)\, ds \Big] \\
& \quad - \Big[\lambda k(t)^{\mathrm{CF}}D^\beta v(t) + \mu s(t)^{\mathrm{CF}}I^\rho v(t) + f_1\big(t, v(t), (\phi v)(t), h(t)^{\mathrm{CF}}I^\nu v(t), g(t)^{\mathrm{CF}}D^\delta v(t)\big) \\
& \quad + \int_0^t f_2\big(s, v(s), (\varphi v)(s), q(t)^{\mathrm{CF}}D^\gamma v(s)\big)\, ds \Big] \Big\| \\
& \leq \lambda \big|k(t)\big| \big|{}^{\mathrm{CF}}D^\beta \big(u(t) - v(t)\big)\big| + \mu \big|s(t)\big| \big|{}^{\mathrm{CF}}I^\rho \big(u(t) - v(t)\big)\big| \\
& \quad + \big| f_1\big(t, u(t), (\phi u)(t), h(t)^{\mathrm{CF}}I^\nu u(t), g(t)^{\mathrm{CF}}D^\delta u(t)\big) \\
& \quad - f_1\big(t, v(t), (\phi v)(t), h(t)^{\mathrm{CF}}I^\nu v(t), g(t)^{\mathrm{CF}}D^\delta v(t)\big) \big| \\
& \quad + \int_0^t \big| f_2\big(s, u(s), (\varphi u)(s), q(s)^{\mathrm{CF}}D^\gamma u(s)\big) - f_2\big(s, v(s), (\varphi v)(s), q(s)^{\mathrm{CF}}D^\gamma v(s)\big) \big|\, ds \\
& \leq \Big[\lambda \frac{M_1}{(1-\beta)^2} \|u - v\| + \mu M_2 \|u - v\| + \xi_1 \|u - v\| + \xi_2 \gamma_0 \|u - v\| + \xi_3 M_3 \|u - v\| \\
& \quad + \xi_4 \frac{M_4}{(1-\delta)^2} \|u - v\| + \xi_1' \|u - v\| + \xi_2' \lambda_0 \|u - v\| + \xi_3' \frac{M_5}{(1-\gamma)^2} \Big] \|u - v\| \\
& = \Big[\lambda \frac{M_1}{(1-\beta)^2} + \mu M_2 + \xi_1 + \xi_2 \gamma_0 + \xi_3 M_3 + \xi_4 \frac{M_4}{(1-\delta)^2} + \xi_1' + \xi_2' \lambda_0 + \xi_3' \frac{M_5}{(1-\gamma)^2} \Big] \\
& \quad \times \|u - v\|
\end{aligned}
$$

for all $u, v \in H^1$ and $t \in I$. Hence,

$$
\begin{aligned}
& \big|(Fu)(t) - (Fv)(t)\big| \\
& \leq a_\alpha \Big| \lambda k(t)^{\mathrm{CF}}D^\beta \big(u(t) - v(t)\big) + \mu s(t)^{\mathrm{CF}}I^\rho \big(u(t) - v(t)\big) \\
& \quad + f_1\big(t, u(t), (\phi u)(t), h(t)^{\mathrm{CF}}I^\nu u(t), g(t)^{\mathrm{CF}}D^\delta u(t)\big)
\end{aligned}
$$

$$
\begin{aligned}
&-f_1\big(t,v(t),(\phi v)(t),h(t)^{\mathrm{CF}}I^{\nu}v(t),g(t)^{\mathrm{CF}}D^{\delta}v(t)\big)\\
&+\int_0^t \big(f_2\big(s,u(s),(\varphi u)(s),q(s)^{\mathrm{CF}}D^{\gamma}u(s)\big)-f_2\big(s,v(s),(\varphi v)(s),q(s)^{\mathrm{CF}}D^{\gamma}v(s)\big)\big)\,ds\bigg|\\
&+b_\alpha\bigg[\int_0^t\bigg|\lambda k(s)^{\mathrm{CF}}D^{\beta}\big(u(s)-v(s)\big)+\mu s(s)^{\mathrm{CF}}I^{\rho}\big(u(s)-v(s)\big)\\
&+f_1\big(s,u(s),(\phi u)(s),h(s)^{\mathrm{CF}}I^{\nu}u(s),g(s)^{\mathrm{CF}}D^{\delta}u(s)\big)\\
&-f_1\big(s,v(s),(\phi v)(s),h(s)^{\mathrm{CF}}I^{\nu}v(s),g(s)^{\mathrm{CF}}D^{\delta}v(s)\big)\\
&+\int_0^s\big(f_2\big(r,u(r),(\varphi u)(r),q(r)^{\mathrm{CF}}D^{\gamma}u(r)\big)\big)-f_2\big(r,v(r),(\varphi v)(r),q(r)^{\mathrm{CF}}D^{\gamma}v(r)\big)\,dr\bigg|\,ds\bigg]\\
\le{}& a_\alpha\bigg[\lambda\frac{M_1}{(1-\beta)^2}+\mu M_2+\xi_1+\xi_2\gamma_0+\xi_3M_3+\xi_4\frac{M_4}{(1-\delta)^2}+\xi_1'+\xi_2'\lambda_0+\xi_3'\frac{M_5}{(1-\gamma)^2}\bigg]\\
&\times\|u-v\|\\
&+b_\alpha\int_0^t\bigg[\lambda\frac{M_1}{(1-\beta)^2}+\mu M_2+\xi_1+\xi_2\gamma_0+\xi_3M_3+\xi_4\frac{M_4}{(1-\delta)^2}+\xi_1'+\xi_2'\lambda_0\\
&+\xi_3'\frac{M_5}{(1-\gamma)^2}\bigg]\|u-v\|\,ds\\
\le{}&[a_\alpha+b_\alpha]\bigg[\lambda\frac{M_1}{(1-\beta)^2}+\mu M_2+\xi_1+\xi_2\gamma_0+\xi_3M_3+\xi_4\frac{M_4}{(1-\delta)^2}+\xi_1'+\xi_2'\lambda_0\\
&+\xi_3'\frac{M_5}{(1-\gamma)^2}\bigg]\|u-v\|:=\Delta'\|u-v\|
\end{aligned}
$$

for all $u,v\in H^1$. Also by using Lemmas 2.2 and 2.3, we get

$$\big|{}^{\mathrm{CF}}D^{\beta}Fu(t)-{}^{\mathrm{CF}}D^{\beta}Fv(t)\big|\le\frac{1}{(1-\beta)^2}\Delta'\|u-v\|,$$

$|{}^{\mathrm{CF}}D^{\delta}Fu(t)-{}^{\mathrm{CF}}D^{\delta}Fv(t)|\le\frac{1}{(1-\delta)^2}\Delta'\|u-v\|$, $|{}^{\mathrm{CF}}D^{\gamma}Fu(t)-{}^{\mathrm{CF}}D^{\gamma}Fv(t)|\le\frac{1}{(1-\gamma)^2}\Delta'\|u-v\|$, $|{}^{\mathrm{CF}}I^{\rho}Fu(t)-{}^{\mathrm{CF}}D^{\rho}Fv(t)|\le\Delta'\|u-v\|$ and $|{}^{\mathrm{CF}}I^{\nu}Fu(t)-{}^{\mathrm{CF}}D^{\nu}Fv(t)|\le\Delta'\|u-v\|$ for all $u,v\in H^1$ and $t\in I$. Hence, we obtain

$$\|Fu-Fv\|\le\max\bigg\{\frac{1}{(1-\beta)^2},\frac{1}{(1-\delta)^2},\frac{1}{(1-\gamma)^2}\bigg\}\Delta'\|u-v\|=\Delta_4\|u-v\|$$

for all $u,v\in H^1$. Consider the maps $g:[0,\infty)^5\to[0,\infty)$ and $\alpha:H^1\times H^1\to[0,\infty)$ defined by $g(t_1,t_2,t_3,t_4,t_5)=\frac{\Delta_4}{9}(3t_1+2t_2+4t_3)$ and $\alpha(x,y)=1$ for all $x,y\in H^1$. One can check that $g\in\mathcal{R}$ and F is a generalized α-contraction. By using Theorem 1.3, F has an approximate fixed point which is an approximate solution for the problem (4). □

Here, we provide three examples to illustrate our some main results.

Example 2.1 Define the functions $\eta\in L^{\infty}([0,1])$ and $\gamma,\lambda:[0,1]\times[0,1]\to[0,\infty)$ by $\eta(t)=e^{-(\pi t+6)}$, $\gamma(t,s)=\sin(1)$ and $\lambda(t,s)=e^{t-s}$. Then $\eta^*=\frac{1}{e^6}$, $\gamma_0=\sin(1)$ and $\lambda_0\le e$. Put $\alpha=\frac{1}{3}$. Consider the problem

$$ {}^{\mathrm{CF}}D^{\frac{1}{3}}u(t)=e^{-(\pi t+6)}\bigg[2t+u(t)+\frac{1}{20}\int_0^t\sin(1)u(s)\,ds+\frac{1}{6}\int_0^t e^{s}u(s)\,ds\bigg] \tag{5}$$

with boundary condition $u(0) = 0$ and the function $f(t,x,y,w) = e^{-(\pi t+6)}(2t + x + \frac{1}{20}y + \frac{1}{6}w)$. Note that $\Delta_1 = \eta^*(1 + \gamma_0 + \lambda_0) < 0.00926 < 1$. By utilizing Theorem 2.5, (5) has an approximate solution.

Example 2.2 Define the functions $\eta \in L^\infty([0,1])$ and $\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$ by $\eta(t) = \frac{\pi}{e^{(t+16)}}$, $\gamma(t,s) = e^{t-s}$ and $\lambda(t,s) = \log e^{\sin(\ln(\pi|t-s|+1))}$. Then $\eta^* = \frac{\pi}{e^{16}}$, $\gamma_0 \leq e$ and $\lambda_0 \leq \log e$. Put $\alpha = \frac{1}{2}$, $\mu_1 = \frac{1}{120}$, $\mu_2 = \frac{1}{28}$, $\beta = \frac{2}{3}$, $\theta = \frac{1}{3}$, $\rho = \frac{1}{2}$ and $\nu = \frac{1}{2}$. Consider the functions $k(t) = \sin t$, $h(t) = \tan^{-1}(t)$, and $s(t) = \frac{1}{n}$ whenever $x = \frac{m}{n} \in Q \cap [0,1]$ with $(m,n) = 1$ and $s(t) = 0$ whenever $x \in Q^c \cap [0,1]$ or $x = 0$. Then $M_1 = \sup_{t\in[0,1]} |k(t)| = 1$, $M_2 = \sup_{t\in[0,1]} |s(t)| = 1$ and $M_3 = \sup_{t\in[0,1]} |h(t)| = \frac{\pi}{2}$. Consider the fractional integro-differential problem

$$\begin{aligned}
{}^{\mathrm{CF}}D^{\frac{1}{2}} u(t) &= \frac{1}{120}\sin(t)^{\mathrm{CF}}D^{\frac{2}{3}} u(t) \\
&\quad + \frac{1}{28}\left(\int_0^t \log\left(e^{\sin(\ln(\pi|t-s|+1))}\right)s(s)\,ds\right)^{\mathrm{CF}}D^{\frac{1}{3}}\left({}^{\mathrm{CF}}D^{\frac{1}{2}} u(t)\right) \\
&\quad + \frac{\pi}{e^{(t+16)}}\left[t + u(t) + \int_0^t e^{t-s}u(s)\,ds + \tan^{-1}(t)^{\mathrm{CF}}D^{\frac{1}{2}} u(t)\right] \qquad (6)
\end{aligned}$$

with boundary condition $u(0) = 0$. Put $f(t,x,y,w,v) = \frac{\pi}{e^{(t+16)}}(t + x + y + w)$. Note that $\Delta_3 = (1 + \frac{1}{(1-\beta)^2} + \frac{1}{(1-\theta)^2(1-\rho)^2} + \frac{1}{(1-\nu)^2})[\mu_1 M_1 + \mu_2\lambda_0 M_2 + \eta^*(1 + \gamma_0 + M_3)] < 0.5485 < 1$. Then by using Theorem 2.7, it implies that (6) has an approximate solution.

Example 2.3 Define the functions $\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$ by $\lambda(t,s) = \frac{e^{2t-s}}{e}$ and $\gamma(t,s) = 0$. Then $\gamma_0 = 0$ and $\lambda_0 \leq e$. Put $\alpha = \frac{1}{4}$, $\beta = \frac{1}{4}$, $\nu = \frac{1}{2}$, $\delta = \frac{1}{4}$, $\gamma = \frac{1}{2}$, $\lambda = \frac{1}{200}$, $\mu = 0$, $\xi_1 = \frac{2}{41}$, $\xi_3 = \frac{1}{48}$, $\xi_4 = \frac{1}{400}$, $\xi_1' = \frac{1}{320}$, $\xi_2' = \frac{1}{40}$, and $\xi_3' = \frac{1}{119}$. Let s be an arbitrary bounded map, $q(t) = \tan^{-1}(t)$, $h(t) = \sin(t)$ for all $t \in I$, $k(t) = 1$ whenever $x \in Q \cap [0,1]$ and $k(t) = 0$ whenever $x \in Q^c \cap [0,1]$ and $g(t) = 0$ whenever $x \in Q \cap [0,1]$ and $g(t) = 2$ whenever $x \in Q^c \cap [0,1]$. Then $M_1 = \sup_{t\in[0,1]} |k(t)| = 1$, M_2 is a real number, $M_3 = \sup_{t\in[0,1]} |h(t)| = 1$, $M_4 = \sup_{t\in[0,1]} |g(t)| = 2$ and $M_5 = \sup_{t\in[0,1]} |q(t)| = \frac{\pi}{2}$. Now, consider the well-defined fractional integro-differential problem

$$\begin{aligned}
&{}^{\mathrm{CF}}D^{\frac{1}{4}} u(t) \\
&\quad = \frac{1}{200}k(t)^{\mathrm{CF}}D^{\frac{1}{4}} u(t) + \frac{3}{40}t + \frac{2}{41}u(t) + \frac{1}{48}\sin(t)^{\mathrm{CF}}I^{\frac{1}{2}} u(t) + \frac{1}{400}g(t)^{\mathrm{CF}}D^{\frac{1}{4}} u(t) \\
&\qquad + \int_0^t \left[\frac{2}{56}s + \frac{1}{320}u(s) + \frac{1}{40}\int_0^s \frac{e^{2s-r}}{e}u(r)\,dr + \frac{1}{119}\tan^{-1}{}^{\mathrm{CF}}D^{\frac{1}{2}} u(s)\right]ds \qquad (7)
\end{aligned}$$

with boundary condition $u(0) = 0$. Put $f_1(t,x,y,w,v) = \frac{3}{40}t + \xi_1 x + \xi_2 y + \xi_3 w + \xi_4 v$ and $f_2(t,x,y,w) = \frac{2}{56}t + \xi_1' x + \xi_2' y + \xi_3' w$ for all $t \in I$ and $x, y, w, v \in \mathbb{R}$. Note that

$$\begin{aligned}
\Delta_4 &= \max\left\{\frac{1}{(1-\beta)^2}, \frac{1}{(1-\delta)^2}, \frac{1}{(1-\gamma)^2}\right\} \\
&\quad \times \left[\lambda\frac{M_1}{(1-\beta)^2} + \mu M_2 + \xi_1 + \xi_2\gamma_0 + \xi_3 M_3 + \xi_4\frac{M_4}{(1-\delta)^2} + \xi_1' + \xi_2'\lambda_0 + \xi_3'\frac{M_5}{(1-\gamma)^2}\right] \\
&< 0.8451 < 1.
\end{aligned}$$

Thus, taking into account Theorem 2.8 we conclude that the problem (7) has an approximate solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this study equally and approved the final version of the manuscript.

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