A New Method for Suboptimal Control of a Class of Nonlinear Systems

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A NEW METHOD FOR SUBOPTIMAL CONTROL OF A CLASS OF NONLINEAR SYSTEMS

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Abstract

In this paper, a new nonlinear control synthesis technique (θ-D approximation) is presented. This approach achieves suboptimal solutions to nonlinear optimal control problems in the sense that it solves the Hamilton-Jacobi- Bellman (HJB) equation approximately by adding perturbations to the cost function. By manipulating the perturbation terms both semi-globally asymptotic stability and suboptimality properties can be obtained. The convergence and stability proofs are given. This method overcomes the large control for large initial states problem that occurs in some other Taylor expansion based methods. It does not need time-consuming online computations like the State Dependent Riccati Equation (SDRE) technique. A vector problem is investigated to demonstrate the effectiveness of this new technique.

1. INTRODUCTION

Numerous techniques exist for the synthesis of control laws for nonlinear systems. Optimal control of nonlinear dynamics with respect to a mathematical index of performance has also been extensively investigated in the last few decades [1-5]. One of the difficulties of controlling a nonlinear system is that optimal feedback control depends on the solution to the Hamilton-Jacobi-Bellman (HJB) equation. The HJB equation is extremely difficult to solve in general rendering optimal control techniques of limited use for nonlinear systems. Therefore, a number of papers investigated methods to find suboptimal solutions to the nonlinear control problem.

Al'Brekht [1] has derived a sufficient condition for obtaining the optimal feedback control of a nonlinear analytic system and developed a formal recursive procedure to construct a suboptimal control in a power series in states. However, a closed form solution for the recursive procedure has not been developed. Garrard et.al.[2] extended the above idea by expanding the optimal cost function as a power series of an artificial variable and utilizing a similar technique to that for the linear systems. The idea simplified the calculations, but this technique can only be applied to some certain class of nonlinear systems in which the nonlinearity can be considered as some small perturbations. Garrard [3] also formulated another approach that expanded both optimal cost and nonlinear dynamics as a power series of the states and used the same idea as before. This applied to more general nonlinear systems. However, finding the coefficients of power series is an involved process.

Another recently emerging technique which systematically solves the nonlinear regulator problem is the State Dependent Riccati Equation (SDRE) method (Cloutier et al.,1996)[4]. By turning the equations of motion into a linear-like structure, this approach permits the designer to employ linear optimal control methods such as the LQR methodology and the H_∞ design technique for the synthesis of nonlinear control systems. The major problem with SDRE is the time-consuming online computation of the Riccati equation. Wernli and Cook [5] proposed a method for solving the state dependent Riccati equation by expanding the solution of SDRE as a power series. But the convergence of this series is not guaranteed and the resulting control law leads to a large control effort or even instability when initial states are large.

In this paper, a new suboptimal nonlinear controller synthesis (θ-D approximation) based on approximate solution to the HJB equation is proposed. By introducing an instrumental variable θ, the co-state λ can be expanded as a power series in terms of θ. The HJB equation is then reduced to a set of recursive algebraic equations. By adding perturbations to the cost function and manipulating these terms appropriately we are able to achieve semi-globally asymptotic stability and overcome the large initial control problem. In Section 2, the formulation of θ-D approximation method will be introduced. Both convergence and stability proofs are given. A two dimensional nonlinear regulator problem is studied and compared with the SDRE technique in Section 3. Conclusions are given in Section 4.

2. SUBOPTIMAL CONTROL OF A CLASS OF NONLINEAR SYSTEMS

In this paper we restrict ourselves to the state feedback control problem for the class of nonlinear time-invariant systems described by

$$\dot{x} = f(x) + gu \tag{1}$$

with the cost function:

$$J = \frac{1}{2}\int_0^\infty [x^T Q x + u^T R u]dt \tag{2}$$

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where $x \in \Omega \subset R^n, f:\Omega \to R^n, g \in R^{n\times m}, u:\Omega \to R^m, Q \in R^{n\times n}, R \in R^{m\times m}$. Q is semi-definite matrix and R is positive definite matrix; g is a constant matrix and f(0)=0.

To ensure that the control problem is well posed we assume that a solution to the optimal control problem (1), (2) exists. We also assume that $f(x)$ is locally Lipschitz in x on a set Ω and zero state observable through Q.

The optimal solution of the infinite-horizon nonlinear regulator problem can be obtained by solving the Hamilton-Jacobi-Bellman (HJB) partial differential equation [6]:

$$\frac{\partial V^T}{\partial x} f(x) - \frac{1}{2}\frac{\partial V^T}{\partial x} g R^{-1} g^T \frac{\partial V}{\partial x} + \frac{1}{2} x^T Q x = 0 \qquad (3)$$

where $V(0) = 0$

The optimal control is given by

$$u = -R^{-1} g^T \frac{\partial V}{\partial x} \qquad (4)$$

and V(x) is the optimal cost, i.e.

$$V(x) = \min_u \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u) dt \qquad (5)$$

The HJB equation is extremely difficult to solve in general, rendering optimal control techniques of limited use for nonlinear systems.

Now consider perturbations added to the cost function:

$$J = \frac{1}{2}\int_0^\infty [x^T (Q + \sum_{i=1}^{\infty} D_i \theta^i) x + u^T R u] dt \qquad (6)$$

where θ and D_i are chosen such that $\left\| \sum_{i=1}^{\infty} D_i \theta^i \right\|_2$ is a small number compared to $\|Q\|_2$.

Write the original state equation as:

$$\dot{x} = f(x) + gu = A_0 x + \theta(\frac{A(x)}{\theta}) x + gu \qquad (7)$$

where A_0 is a constant matrix such that (A_0, g) is a stabilizable pair.

Define
$$\lambda = \frac{\partial V}{\partial x} \qquad (8)$$

By using (8) in (3), we have a perturbed HJB equation:

$$\lambda^T f(x) - \frac{1}{2}\lambda^T g R^{-1} g^T \lambda + \frac{1}{2} x^T (Q + \sum_{i=1}^{\infty} D_i \theta^i) x = 0 \qquad (9)$$

Assume a power series expansion of λ as

$$\lambda = \sum_{i=0}^{\infty} T_i \theta^i x \qquad (10)$$

Here T_i s are assumed symmetric and are to be determined.

Substitute Eq. (10) into HJB Eq. (9) and equate the coefficients of powers of θ to zero to get the following equations:

$$T_0 A_0 + A_0^T T_0 - T_0 g R^{-1} g^T T_0 + Q = 0 \qquad (11)$$

$$T_1 (A_0 - g R^{-1} g^T T_0) + (A_0^T - T_0 g R^{-1} g^T) T_1 = -\frac{T_0 A(x)}{\theta} - \frac{A^T(x) T_0}{\theta} - D_1 \qquad (12)$$

$$T_2 (A_0 - g R^{-1} g^T T_0) + (A_0^T - T_0 g R^{-1} g^T) T_2 = -\frac{T_1 A(x)}{\theta} - \frac{A^T(x) T_1}{\theta} + T_1 g R^{-1} g^T T_1 - D_2 \qquad (13)$$

$$\vdots$$

$$T_n (A_0 - g R^{-1} g^T T_0) + (A_0^T - T_0 g R^{-1} g^T) T_n = -\frac{T_{n-1} A(x)}{\theta} - \frac{A^T(x) T_{n-1}}{\theta} - D_n + \sum_{j=1}^{n-1} T_j g R^{-1} g^T T_{n-j} \qquad (14)$$

The expression for control can be obtained in terms of the power series for λ as

$$u = -R^{-1} g^T \lambda = -R^{-1} g^T \sum_{i=0}^{\infty} T_i(x,\theta) \theta^i x \qquad (15)$$

It is easy to find that the equation (11) is an algebraic Riccati equation. All other equations turn out to be simply Lyapunov equations which are *linear* in terms of T_n. In the rest of this paper we will call it the θ-D approximation technique. The algorithm without the D_i term is called the θ approximation. The algorithm in [5] would result in the θ approximation. One of the problems with θ approximation is that large initial conditions may give rise to large control. In order to deal with this problem, we construct the following expression for D_i:

$$D_1 = k_1 e^{-l_1 t}[-\frac{T_0 A(x)}{\theta} - \frac{A^T(x) T_0}{\theta}] \qquad (16)$$

$$D_2 = k_2 e^{-l_2 t}[-\frac{T_1 A(x)}{\theta} - \frac{A^T(x) T_1}{\theta}] \qquad (17)$$

$$\vdots$$

$$D_n = k_n e^{-l_n t}[-\frac{T_{n-1} A(x)}{\theta} - \frac{A^T(x) T_{n-1}}{\theta}] \qquad (18)$$

The idea in constructing D_i in this manner is that large control results from the state dependent term A(x) on the right hand side of the equations (11)-(14). It happens when there are some terms in A(x) which could grow to a high magnitude as x is large. For example, when $A(x)$ includes a cubic term, the higher initial state would result in higher initial T_i and consequently higher initial control. So we choose D_i such that

$$-\frac{T_{i-1} A(x)}{\theta} - \frac{A^T(x) T_{i-1}}{\theta} - D_i = \varepsilon_i[-\frac{T_{i-1} A(x)}{\theta} - \frac{A^T(x) T_{i-1}}{\theta}] \qquad (19)$$

where
$$\varepsilon_i(t) = 1 - k_i e^{-l_i t} \qquad (20)$$

is a small number chosen to satisfy some conditions required in the proof of convergence and stability of the above algorithm. On the other hand, the exponential term $e^{-l_i t}$ is used to let the perturbation terms in the cost function and HJB equation diminish as the time evolves. This will guarantee the HJB equations to be solved asymptotically.

The following theorem will show that the convergence of the series expansion of $\sum_{i=0}^{\infty} T_i(x,\theta)\theta^i$ can be obtained by choosing appropriate D_i matrices.

Theorem 2.1: Assume that the following conditions hold:

(i) $x \in \Omega$, where $\Omega \subset R^n$ is a compact set.

(ii) $f(x)$ is partitioned such that (A_0, g) is controllable.

(iii) A(x) is continuous on Ω.

(iv) D_i are chosen according to equations (16)-(18).

Then there exists a set of perturbation matrices D_i such that series $\sum_{i=0}^{\infty} T_i(x,\theta)\theta^i$ produced by the algorithm in Eqs. (11)-(14) is a pointwise convergent series.

Proof. Considering Eq. (12) and the selection of D_1 in Eq. (16), Eq. (12) can be written as:

$$T_1(A_0-gR^{-1}g^TT_0)+(A_0^T-T_0gR^{-1}g^T)T_1=-\varepsilon_1(T_0A+A^TT_0)\frac{1}{\theta} \quad (21)$$

with $$\varepsilon_1=1-k_1e^{-l_1t} \quad (22)$$

Assume that the solution to the equation

$$\hat{T}_1(A_0-gR^{-1}g^T\hat{T}_0)+(A_0^T-\hat{T}_0gR^{-1}g^T)\hat{T}_1=-\varepsilon_1(\hat{T}_0A+A^T\hat{T}_0) \quad (23)$$

is $\hat{T}_1$, with $$\hat{T}_0=T_0 \quad (24)$$

Then using the linearity property of Lyapunov equation (23), the solution to Eq. (21) is

$$T_1=\frac{1}{\theta}\hat{T}_1 \quad (25)$$

Similarly assume that the solution to the equation

$$\hat{T}_2(A_0-gR^{-1}g^T\hat{T}_0)+(A_0^T-\hat{T}_0gR^{-1}g^T)\hat{T}_2=-\varepsilon_2(\hat{T}_1A+A^T\hat{T}_1)+\hat{T}_1gR^{-1}g^T\hat{T}_1 \quad (26)$$

is $\hat{T}_2$. Then the solution to Eq. (13) is:

$$T_2=\frac{1}{\theta^2}\hat{T}_2 \quad (27)$$

In the same manner, $$T_n=\frac{1}{\theta^n}\hat{T}_n \quad (28)$$

where $\hat{T}_n$ is the solution of

$$\hat{T}_n(A_0-gR^{-1}g^T\hat{T}_0)+(A_0^T-\hat{T}_0gR^{-1}g^T)\hat{T}_n=-\varepsilon_n\left(\hat{T}_{n-1}A(x)+A^T(x)\hat{T}_{n-1}\right)+\sum_{j=1}^{n-1}\hat{T}_jgR^{-1}g^T\hat{T}_{n-j} \quad (29)$$

and

$$\varepsilon_n=1-k_ne^{-l_nt} \quad (30)$$

Therefore, from Eqs. (24), (25), (27) and (28) we note that proving the convergence of $\sum_{i=0}^{\infty}T_i(x,\theta)\theta^i$ is equivalent to proving the convergence of $\sum_{i=0}^{\infty}\hat{T}_i(x)$ where $\hat{T}_i$ satisfy the Eqs. (11), (23), (26) and (29). Here we rewrite them in the following form for the clarity.

$$\hat{T}_0A_0+A_0^T\hat{T}_0-\hat{T}_0gR^{-1}g^T\hat{T}_0+Q=0 \quad (31)$$

$$\hat{T}_1(A_0-gR^{-1}g^T\hat{T}_0)+(A_0^T-\hat{T}_0gR^{-1}g^T)\hat{T}_1=-\varepsilon_1(\hat{T}_0A+A^T\hat{T}_0) \quad (23)$$

$$\hat{T}_2(A_0-gR^{-1}g^T\hat{T}_0)+(A_0^T-\hat{T}_0gR^{-1}g^T)\hat{T}_2=-\varepsilon_2(\hat{T}_1A+A^T\hat{T}_1)+\hat{T}_1gR^{-1}g^T\hat{T}_1 \quad (26)$$

$$\vdots$$

$$\hat{T}_n(A_0-gR^{-1}g^T\hat{T}_0)+(A_0^T-\hat{T}_0gR^{-1}g^T)\hat{T}_n=-\varepsilon_n\left(\hat{T}_{n-1}A+A^T\hat{T}_{n-1}\right)+\sum_{j=1}^{n-1}\hat{T}_jgR^{-1}g^T\hat{T}_{n-j} \quad (29)$$

Now we want to find the norm bound for each $\hat{T}_i$ in order to prove the convergence of the series $\sum_{i=1}^{\infty}\hat{T}_i$.

Given a continuous Lyapunov equation

$$A^TP+PA=-Q \quad (32)$$

where $A,P,Q\in R^{n\times n}$. If A is a stable matrix, we have the norm bound for P (Mori [7]).

$$\|P\|\le\frac{\|Q\|}{-\mu(A^T)-\mu(A)} \quad (33)$$

where $$\mu(A)\triangleq\frac{1}{2}\lambda_{\max}(A+A^T) \quad (34)$$

Consider Eq. (31); since (A_0,g) is a controllable pair and R is positive definite and Q is semi-positive definite, we know that Riccati equation (31) has a positive definite solution and $(A_0-gR^{-1}g^T\hat{T}_0)$ is a stable matrix.

Then consider Eq. (23)

$$\|\hat{T}_1\|\le\frac{\left\|\varepsilon_1\left[\hat{T}_0A(x)+A^T(x)\hat{T}_0\right]\right\|}{-\mu\left(A_o-gR^{-1}g^T\hat{T}_0\right)-\mu\left(A_o^T-\hat{T}_0gR^{-1}g\right)} \quad (35)$$

Let $$C=\frac{1}{-\mu\left(A_0-gR^{-1}g^T\hat{T}_0\right)-\mu\left(A_o^T-\hat{T}_0gR^{-1}g^T\right)} \quad (36)$$

Then $$\|\hat{T}_1\|\le C\varepsilon_1\|\hat{T}_0A+A^T\hat{T}_0\|\le C\varepsilon_1\left[\|\hat{T}_0\|\|A+A^T\|\right] \quad (37)$$

Since A(x) is continuous on a compact set Ω, it is bounded on Ω.

Let $$C_A=\max_{x\in\Omega}(\|A+A^T\|) \quad (38)$$

Then we have $$\|\hat{T}_1\|\le C\cdot C_A\cdot\varepsilon_1\cdot\|\hat{T}_0\| \quad (39)$$

Consider Eq. (26); we can obtain a norm-bounded inequality in $\hat{T}_2$ as

$$\begin{aligned}\|\hat{T}_2\|&\le C\varepsilon_2\|\hat{T}_1A+A^T\hat{T}_1\|+C\|\hat{T}_1gR^{-1}g^T\hat{T}_1\| \quad (40)\\ &\le C\cdot\left(\varepsilon_2\|\hat{T}_1\|\cdot\|A+A^T\|+\|\hat{T}_1\|^2\cdot\|gR^{-1}g^T\|\right)\end{aligned}$$

Since $gR^{-1}g^T$ is a constant matrix, let $G=\|gR^{-1}g^T\|$ (41) a constant number.

Then $$\begin{aligned}\|\hat{T}_2\|&\le C\cdot\|\hat{T}_1\|\left(\varepsilon_2\cdot C_A+\|\hat{T}_1\|\cdot G\right)\\ &\le C^2C_A\varepsilon_1\|\hat{T}_0\|\left(\varepsilon_2C_A+CC_A\varepsilon_1\|\hat{T}_0\|\cdot G\right)\end{aligned}$$

(According to Eq. (39))

$$=C^2C_A^2\varepsilon_1\varepsilon_2\|\hat{T}_0\|\left(1+\frac{C\cdot G\cdot\|\hat{T}_0\|\varepsilon_1}{\varepsilon_2}\right) \quad (42)$$

Let $$C_2=\max_{t\in[0,\infty]}\left(1+\frac{C\cdot G\cdot\|\hat{T}_0\|\varepsilon_1}{\varepsilon_2}\right) \quad (43)$$

Then we get $$\|\hat{T}_2\|\le C^2\cdot C_A^2\cdot C_2\varepsilon_1\varepsilon_2\|\hat{T}_0\| \quad (44)$$

In a similar manner we can show for $\hat{T}_n$ that

$$\|\hat{T}_n\|\le(\varepsilon_1\cdots\varepsilon_n)\cdot C^nC_A^nC_2\ldots C_n\|\hat{T}_0\| \quad (45)$$

where $$C_n=\max_{t\in[0,\infty]}\left(1+\frac{2^{n-2}C\cdot G\cdot\|\hat{T}_0\|\varepsilon_1}{\varepsilon_n}\right) \quad (46)$$

Once we obtain the bound for each $\hat{T}_i$, we can prove the convergence of the series $\sum_{i=1}^{\infty}\hat{T}_i$.

Let $S_n = \varepsilon_1 \cdots \varepsilon_n \cdot C^n C_A^n \cdot C_2 ... C_n \|\hat{T}_0\|$ (47)

Then $$\frac{S_n}{S_{n-1}} = \varepsilon_n \cdot CC_A C_n \quad (48)$$

By choosing a proper ε_n we can make $\frac{S_n}{S_{n-1}} < 1$. That is $\sum_{i=0}^{\infty} S_i$ is a convergent series. Since each $\|\hat{T}_i\| \le S_i$ and according to Weierstrass's Theorem [8], $\sum_{i=0}^{\infty} \hat{T}_i$ is also a convergent series. The choice of ε_n actually depends on the choice of the perturbation terms D_i from Eqs. (16)-(19). If the D_i matrices are chosen such that

$$\varepsilon_n < \frac{1}{CC_A C_n} \quad (49)$$

we can claim that the series $\sum_{i=0}^{\infty} T_i(x,\theta)\theta^i$ is convergent. □

Remark: The above proof shows that θ is just an instrumental variable for the convenience of power series expansion. The value of θ does not matter since it turns out to be cancelled by the choice of D_i matrices (Eqs. (16)-(18)).

Theorem 2.2: Suppose that the conditions in Theorem 2.1 are satisfied and

(v) $f(x)$ is nontrivial.

(vi) $f(x)$ is partitioned such that $[(A_0 + A(x)), g]$ is pointwise controllable.

(vii) D_i are chosen such that $Q + \sum_{i=1}^{\infty} D_i \theta^i$ is semi-positive definite. There exist perturbation matrices D_i such that the closed-loop feedback control system obtained by control law $u = -R^{-1} g^T \sum_{i=0}^{\infty} T_i(x)\theta^i x$ is semi-globally asymptotically stable.

Proof. Let us choose a Lyapunov function

$$V(x) = \frac{1}{2} x^T \sum_{i=0}^{\infty} \hat{T}_i(x) x \quad (50)$$

$\sum_{i=0}^{\infty} \hat{T}_i(x)$ can be obtained as positive definite marix by choosing appropriate ε_i. Please refer to [9]. So we have V(x)>0.

$$\text{Now, } \frac{dV(x)}{dt} = \frac{\partial V^T(x)}{\partial x} \dot{x} = \frac{\partial V^T(x)}{\partial x}[f(x) + gu]$$

$$= \left[x^T \sum_{i=0}^{\infty} \hat{T}_i + \frac{1}{2} x^T \sum_{i=1}^{\infty} \frac{\partial \hat{T}_i}{\partial x} x \right] [f(x) + gu] \quad (51)$$

Since $\frac{\partial V}{\partial x} = \sum_{i=0}^{\infty} \hat{T}_i x$ satisfy the HJB equation

$$\frac{\partial V^T}{\partial x}[f + gu] + \frac{1}{2} u^T R u + \frac{1}{2} x^T (Q + \sum_{i=1}^{\infty} D_i \theta^i) x = 0 \quad (52)$$

We can rearrange (52) to get

$$x^T \sum_{i=0}^{\infty} \hat{T}_i [f + gu] = -\frac{1}{2} x^T Q x - \frac{1}{2} u^T R u - \frac{1}{2} x^T \sum_{i=1}^{\infty} D_i \theta^i x \quad (53)$$

$$\frac{dV(x)}{dt} = -\frac{1}{2} x^T Q x - \frac{1}{2} u^T R u - \frac{1}{2} x^T \sum_{i=1}^{\infty} D_i \theta^i x + \frac{1}{2} x^T \sum_{i=1}^{\infty} \frac{\partial \hat{T}_i}{\partial x} x [f + gu] \quad (54)$$

Since $Q + \sum_{i=1}^{\infty} D_i \theta^i$ is semi-positive definite and R is positive definite, $-\frac{1}{2} x^T Q x - \frac{1}{2} u^T R u - \frac{1}{2} x^T \sum_{i=1}^{\infty} D_i \theta^i x$ is negative definite.

Substituting $u = -R^{-1} g^T \sum_{i=0}^{\infty} \hat{T}_i x$ into (54) we get

$$-\frac{1}{2} x^T Q x - \frac{1}{2} u^T R u - \frac{1}{2} x^T \sum_{i=1}^{\infty} D_i \theta^i x = -\frac{1}{2} x^T \left[Q + \sum_{i=0}^{\infty} \hat{T}_i g R^{-1} g^T \sum_{i=0}^{\infty} \hat{T}_i + \sum_{i=1}^{\infty} D_i \theta^i \right] x \quad (55)$$

According to Courant-Fischer theorem [8],

$$-\frac{1}{2} x^T \left[Q + \sum_{i=0}^{\infty} \hat{T}_i g R^{-1} g^T \sum_{i=0}^{\infty} \hat{T}_i + \sum_{i=1}^{\infty} D_i \theta^i \right] x$$

$$\le -\frac{1}{2} \lambda_{\min} \left[Q + \sum_{i=0}^{\infty} \hat{T}_i g R^{-1} g^T \sum_{i=0}^{\infty} \hat{T}_i + \sum_{i=1}^{\infty} D_i \theta^i \right] \|x\|_2^2 \quad (56)$$

Then we get

$$\frac{dV(x)}{dt} \le -\frac{1}{2} \lambda_{\min} \left[Q + \sum_{i=0}^{\infty} \hat{T}_i g R^{-1} g^T \sum_{i=0}^{\infty} \hat{T}_i + \sum_{i=1}^{\infty} D_i \theta^i \right] \|x\|_2^2 + \frac{1}{2} x^T \sum_{i=1}^{\infty} \frac{\partial \hat{T}_i}{\partial x} x [f + gu]$$

$$\le -\frac{1}{2} C_\lambda \|x\|_2^2 + \frac{1}{2} \|x\|_2^2 \left\| \sum_{i=1}^{\infty} \frac{\partial \hat{T}_i}{\partial x} x \right\|_2 \left\| A_o + A - g R^{-1} g^T \sum_{i=0}^{\infty} \hat{T}_i \right\|_2 \quad (57)$$

$$\text{where } C_\lambda = \lambda_{\min} \left[Q + \sum_{i=0}^{\infty} \hat{T}_i g R^{-1} g^T \sum_{i=0}^{\infty} \hat{T}_i + \sum_{i=1}^{\infty} D_i \theta^i \right] > 0 \quad (58)$$

$$\frac{dV(x)}{dt} \le -\frac{1}{2} \|x\|_2^2 \left[C_\lambda - \left\| \sum_{i=1}^{\infty} \frac{\partial \hat{T}_i}{\partial x} x \right\|_2 \left\| A_o + A - g R^{-1} g^T \sum_{i=0}^{\infty} \hat{T}_i \right\|_2 \right] \quad (59)$$

According to the linearity property of Eqs. (23), (26) and (29), we know that $\hat{T}_n$ can always be written in the form of

$$\hat{T}_n = \varepsilon_1 \cdots \varepsilon_n \overline{T}_n(x) \quad (60)$$

$$\text{and} \quad \frac{\partial \hat{T}_n}{\partial x} x = \varepsilon_1 \cdots \varepsilon_n \frac{\partial \overline{T}_n}{\partial x} x \quad (61)$$

Thus by choosing $\varepsilon_1 \cdots \varepsilon_n$ properly we can always make

$$C_\lambda > \left\| \sum_{i=1}^{\infty} \frac{\partial \hat{T}_i}{\partial x} x \right\|_2 \left\| A_o + A - g R^{-1} g^T \sum_{i=0}^{\infty} \hat{T}_i \right\|_2 \quad (62)$$

Then $\frac{dV(x)}{dt} < 0$. Here we do not limit the proof in a specified region for x. As long as x lies in a compact set with A(x) bounded, we can always choose a set of ε_i such that $\frac{dV(x)}{dt} < 0$. Therefore the underlying system is semi-globally asymptotically stable. □

3. ILLUSTRATIVE EXAMPLE [4]

Find control u to minimize the cost function:

$$J = \frac{1}{2}\int_0^\infty x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^T \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} u dt \quad (63)$$

with a system defined by

$$\begin{cases} \dot{x}_1 = x_1 - x_1^3 + x_2 + u_1 \\ \dot{x}_2 = x_1 + x_1^2 x_2 - x_2 + u_2 \end{cases} \quad (64)$$

We reorganize f(x) as

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad A = \begin{bmatrix} -x_1^2 & 0 \\ 0 & x_1^2 \end{bmatrix} \quad (65)$$

Figure 1 shows the state and control response when the initial condition is 1 (using only θ approximation). Figure 2 shows the results when the initial states are [10,10]. We find that the initial control effort increases to -1.859×10^5. Please note that the control response in Figure 2 is a zoomed plot for comparison with the SDRE results.

Figure 3 demonstrates the effect of the D_i terms (θ-D approximation). The parameters in this case are:

$$D_1 = diag\{1.00001e^{-21t}[-\frac{T_0 A(x)}{\theta} - \frac{A^T(x)T_0}{\theta}], 1.00001e^{-14t}[-\frac{T_1 A(x)}{\theta} - \frac{A^T(x)T_1}{\theta}] \quad (66)$$

$$D_2 = diag\{1.00001e^{-40t}[-\frac{T_1 A(x)}{\theta} - \frac{A^T(x)T_1}{\theta}], 1.00001e^{-8t}[-\frac{T_1 A(x)}{\theta} - \frac{A^T(x)T_1}{\theta}] \quad (67)$$

We only pick the first three terms, e.g. up to T_2 terms in the λ expansion. Usually three terms are enough for a good approximation. More terms could be added if needed. In Figure 3, the initial control magnitude is reduced to [-95, -22]. The SDRE solution produces [-75, -17]. Although SDRE achieves less effort on the control, it needs **on line** solution of Riccati Equation. In addition, our method results in a small cost compared to the SDRE. The cost of the $\theta - D$ method is about 400 while the cost of the SDRE is over 800. This pattern of lower cost for the $\theta - D$ method was observed in all the cases we tried.

When implementing the $\theta - D$ method, we need it to be insensitive to the initial states. Notice that initial states come into play when t=0 and we can take care of large initial control problem by manipulating the D matrix. Since large control is always due to the $T_i A(x) + A^T(x) T_i$ term and $\varepsilon_i(t=0) = 1 - k_i$, we pick $k_i = 1 - \frac{1}{\|T_0 A(x_0) + A^T(x_0) T_0\|_2}$ such that k_i is a function of the initial states to help alleviate the large initial control problem.

Figure 4-5 show the responses under different large initial states. We can see that they are not sensitive to the variation of x_0. Compared to SDRE method, our approach can keep low cost and small initial control.

Another advantage of this method is that once we find one set of appropriate parameters of the exponential terms in D matrix that give satisfactory transient performance, they are also not sensitive to the variation of the initial states. In Figure 4-5, we keep the same parameters in (66) and (67). These parameters in the D matrix can be adjusted *off-line* to achieve the desired performance.

As for the implementation, θ-D algorithm needs a matrix inverse operation only one time *offline* when solving the linear Lyapunov equations (12)-(14) and solution to the first algebraic Riccati equation (11) only one time, *offline*. That is to say, when solving (12)-(14), we only need to rearrange the left hand side of the equations such that they form a linear matrix equation: $\hat{A}_0 T_n = Q_n(x,t)$ and then $T_n = \hat{A}_0^{-1} Q_n(x,t)$ where $\hat{A}_0$ is a constant matrix and $Q_n(x,t)$ is the right hand side of (12)-(14). When implemented online, this method involves only two 2×2 matrix multiplications and three 2×2 matrix additions if we take three terms. However, in comparison, SDRE needs computation of the 2×2 algebraic Riccati equation at *each* sample time. The number of computations will become significant if we want to solve higher order problems.

4. CONCLUSIONS

In this paper, a new suboptimal nonlinear control synthesis technique was proposed. This method can solve the HJB equation asymptotically by adding some perturbations to the cost function. The recursive solution for control does not need complex on-line computations compared to the SDRE technique. In addition, the large initial control problem encountered in [5] was overcome by manipulating the perturbation terms appropriately. An illustrative example demonstrated the good results and comparisons with other methods. This technique can be applied to a broad class of nonlinear control problems.

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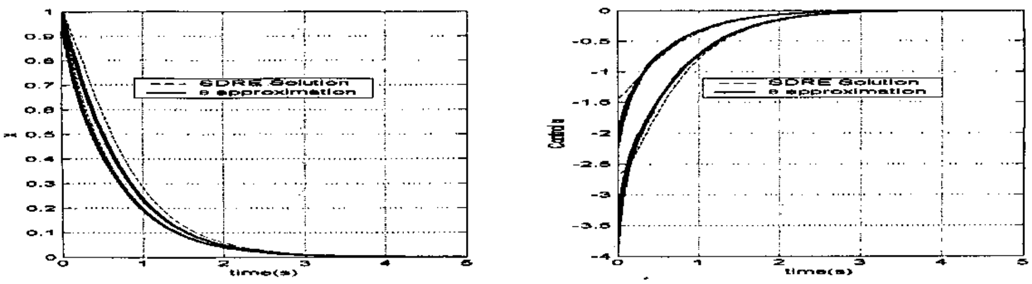

Figure 1: x_0=[1,1] without D term

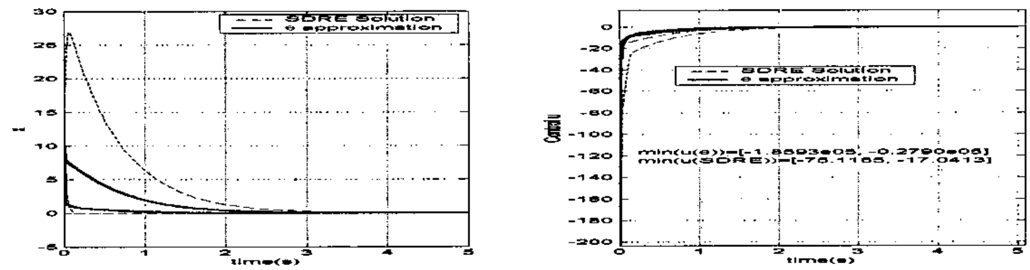

Figure 2: x_0=[10,10] without D term (control plot is zoomed)

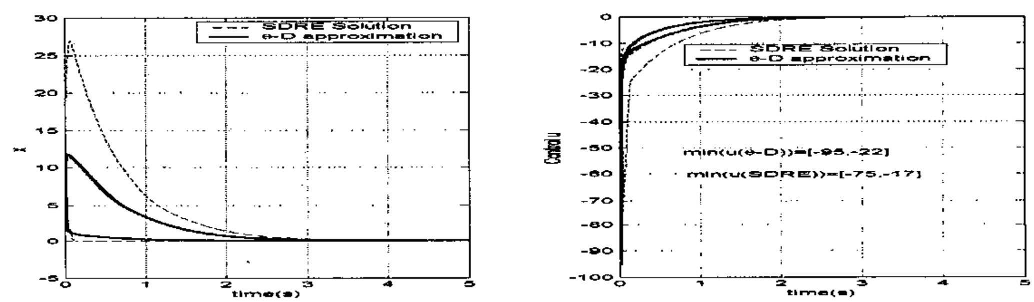

Figure 3: x_0=[10,10], with D term added

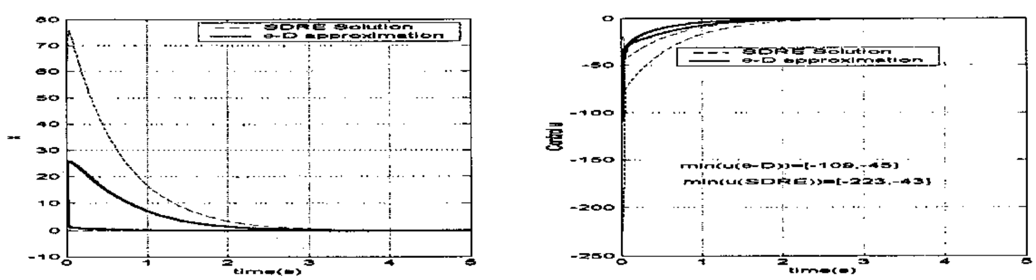

Figure 4: x_0=[20,20], with D term added

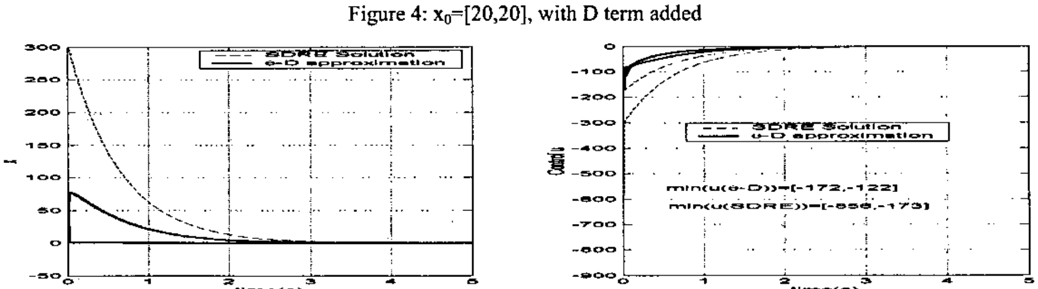

Figure 5: x_0=[50,50], with D term added