Geophys. J. R. astr. Soc. (1981) 67, 735–746

A new method of computing geopotential fields

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Received 1981 March 13; in original form 1980 October 7

Summary. A new method is proposed for the geopotential field computation and gravitational attraction modelling. The usual method is to use a uniform density discrete numerical integration to represent either the gravitational potential or the gravitational attraction from a given density configuration. In this paper, an interpolation scheme is explained, using a piecewise continuous basis function to represent the arbitrarily varying density configuration in one, two and three dimensions. This new approach greatly simplifies the potential integrations and, in certain cases where symmetry exists, analytical evaluation of the integrals is also possible. Numerical tests and examples are given for a hypothetical salt dome, a vertical dyke with varying density structure and the hydrostatic ellipticity of earth model 1066B. The numerical error in this method is limited to the analytical approximation and interpolation errors in each case. This new approach can also be used as efficiently for other potential field studies.

1 Introduction

Analytical methods for computing or approximating the Earth's gravity potential have been studied by many people and clearly summarized in Jeffreys (1970). The applied methods of potential field theory are also studied extensively and illustrated well in several applied geophysics textbooks. In most of these studies it is assumed that the density contrast of the model is constant uniformly or constant in a stepwise sense throughout the body. The continuously varying density distribution is in most cases approximated by rectangular blocks of uniform density. In this paper a general formalism is described to include an arbitrarily varying density distribution in the gravitational potential field computation and modelling.

In the following sections, numerical examples are given for a cylindrically symmetric configuration, a general two-dimensional modelling case and a whole Earth problem. The basic formula we have to solve in these examples includes a potential field integral or an equivalent integral with the density of the constituent material in its integrand. In a simple geopotential integral, the integrand of the volume integral is density multiplied by the well-known Green function. When the density is constant throughout the body the integration is very simple. In realistic problems, however, the density varies with coordinate variables and an analytical solution is possible only for extremely simple cases. It has been customary practice to use a discrete numerical method for more complex density distribution.

In this research an interpolation scheme is proposed to represent the density function or integrand itself, in some cases, using polynomial basis functions. The interpolation method using piecewise continuous functions has been a popular tool in many branches of numerical mathematics. It has been particularly effective in the finite element method and variational problem solution in mathematical physics. As demonstrated in the following three examples, if we interpolate the density term using a piecewise continuous basis function, we can evaluate the whole integral analytically between the nodal or data points, and we can increase not only the accuracy of the solution but also the numerical efficiency greatly. Even when the density function has a finite number of jump discontinuities, the accuracy of the solution is in the accuracy of the density interpolation and is higher than any other conventional methods. The basic interpolation scheme is explained in detail by Schultz (1973) and Moon (1979). The piecewise continuous basis functions used in the following examples are cubic Hermite polynomials defined in terms of the input data points.

2 Cylindrically symmetrical bodies

Even until very recently, the attempts to obtain an analytical solution for the gravitational attraction from a cylindrical body have been limited to the cases where the density is uniform throughout the body. The gravitational attraction by a circular lamina or a right cylindrical body has been studied analytically many times in the past, as well as by analytical and numerical methods. The gravitational attractions of salt domes, granitic batholiths, igneous plugs, mine shafts etc. can be approximated by bodies with a cylindrically symmetrical shape and consequently the simplified solution method attracted practical applications. Recently an exact closed form expression for the gravitational attraction of a vertical right cylinder was obtained by Nabihigan (1962) and Singh (1977a, b). However, the density of the model configuration has always been constant.

We will now look at a case where the density of a cylindrical configuration varies with the radius. In fact it can vary with depth as well. The general approach in this example will follow Parasnis (1961) but it will be more general in the scientific context. The anomaly at $\mathbb{P}$ $(\bar{p})$ due to a circular lamina is (see Fig. 1)

$$G\int_0^a\int_0^{2\pi}\frac{\sigma(r)r dr\, d\phi}{S^3}$$

where

$$S=(r^2-2rp\cos\gamma+p^3)^{1/2}.$$

The denominator in this equation, which is Green's function satisfying the Laplace equation, can be written in terms of the Legendre functions (Morse & Feshbach 1953, pp. 589, 748)

$$\frac{1}{S^3}=\frac{1}{p^3}\left[\frac{p}{r}P_0'(\cos\gamma)+P_1'(\cos\gamma)+\frac{r}{p}P_2'(\cos\gamma)+\dots\right]\qquad p\geqslant r$$

or

$$\frac{1}{S^3}=\frac{1}{r^3}\left[\frac{r}{p}P_0'(\cos\gamma)+P_1'(\cos\gamma)+\frac{p}{r}P_2'(\cos\gamma)+\dots\right]\qquad p\leqslant r$$

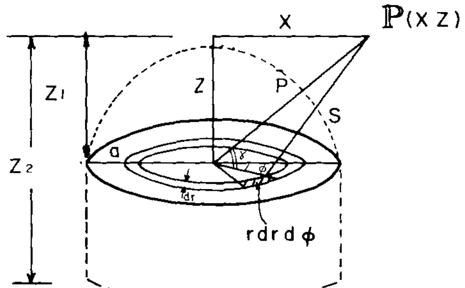


Figure 1. Cylindrical coordinate system for examples of cylindrically symmetrical objects.

where $P_n'(\cos\gamma)$ are the derivatives of the Legendre functions. At this point we can see that the gravitational potential (or attraction) computation has to be treated in two regions: $p \geqslant a$ and $p \leqslant a$ where $r = a = R$ is the radius of the circular lamina. Then the gravitational attraction from the points of $p \geqslant a$ has the general term

$$\Delta g_n = \frac{Gz}{p^{2n+3}} \int_0^a \int_0^{2\pi} r^{2n+1} P'_{2n+1}\, dr\, d\phi. \tag{1}$$

Using the series expansion of the Legendre function

$$P_n'(u) = \sum_{k=0}^{n} (-1)^k \frac{1, 3, \ldots, (4n-2k+1)x^{2n-2k}}{2^k k!\,(2n-2k)!} \cos\frac{2n-2k}{p^{2n-2k}}$$

a general term of gravitational attraction becomes

$$\Delta g_{nk} = \frac{Gzx^{2n-2k}}{p^{4n-2k+1}} (-1)^k \frac{1, 3, \ldots, (4n-2k+1)}{2^k k!\,(2n-2k)!} \int_0^a \sigma(r) r^{2n+1}\, dr \int_0^{2\pi} \cos^{2n-2k}\phi\, d\phi \tag{2}$$

and the total attraction will be

$$\Delta g = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Delta g_{nk}. \tag{3}$$

From equation (2) we see that the density can have any type of functional value. For analytical evaluation $\sigma(r)$ has to be a smooth and simple continuous function. However, as mentioned in Section 2, the density function, $\sigma(r)$, can have any kind of jump discontinuities at $r = r_j$, as long as its gradient is smooth. A piecewise continuous basis function can be used to represent $\sigma(r)$. In that case equation (2) would look like

$$\Delta g_{nk} = \frac{Gzx^{2n-2k}}{p^{4n-2k+3}} (-1)^k \frac{1, 3, \ldots, (4n-2k+1)}{2^k k!(2n-2k)!} \sum_{l=0}^{N-1} \left[\int_{r_l}^{r_{l+1}} \mathrm{SIG}(r)\, dr \right]$$
$$\times \int_0^{2\pi} \cos^{2n-2k}\phi\, d\phi \tag{4}$$

where

$\mathrm{SIG}(r) = \sigma(r) r^{2n+1}$.

In this case SIG(r) is a simple polynomial and the integration along the radius can be analytically evaluated in (r_l, r_{l+1}) and summed over $l = 0, 1, \ldots, N-1$. The integration through azimuthal angle ϕ is a simple one for any value of k.

Now we seek a scheme summing the series in equations (3) and (4). In this example, the case with $p \geqslant a$ only will be solved but the case with $p \leqslant a$ can also be solved with exactly the same approach. For $k = n$, the gravitational attraction Δg only involves x^0; let Δg_0 represent $\Delta g_0 = \Delta g_{nk}$. Then after the integration with respect to the azimuth and some simplification we have

$$\Delta g_0 = 2\pi G z \sum_{n=0}^{\infty} (-1)^n \frac{1, 3, \ldots, (2n+1)}{2^n n!} \left[\sum_{l=0}^{N-1} \int_{r_l}^{r_{l+1}} \sigma(r) r^{2n+1}\, dr \right] \frac{1}{p^{2n+3}}$$

$$= 2\pi G z \sum_{l=0}^{N-1} \int_{r_l}^{r_{l+1}} \sigma(r) r (p^2 + r^2)^{-3/2}\, dr. \tag{5}$$

This integral is a simple one and can be evaluated analytically (Gradshteyn & Ryzhik 1965). For $k = n - 1$, similarly from equations (2) and (3)

$$\Delta g_2 = \frac{15\pi G z x^2}{2} \sum_{l=0}^{N-1} \int_{r_l}^{r_{l+1}} \sigma(r) r^3 (p^2 + r^2)^{-7/2}\, dr \tag{6}$$

and for $k = n - 2$,

$$\Delta g_4 = \frac{2835\pi G z x^4}{16} \sum_{l=0}^{N-1} \int_{r_l}^{r_{l+1}} \sigma(r) r^5 (p^2 + r^2)^{-11/2}\, dr. \tag{7}$$

In these equations the summation over $k = 0, 1, 2, \ldots, n$ approximates the Legendre polynomial and equations (5), (6) and (7) are only the first three terms for $k = n, n-1, n-2$ from equation (3). The other terms defined in equation (3) for $k = n-3, n-4, \ldots, 0$ can be obtained similarly. The integrals in equation (5), (6) and (7) can be evaluated analytically (Gradshteyn & Ryzhik 1965; 2.263, 2.264, etc.) and summed for the given $N-1$ model intervals (see Appendix).

Now for a cylindrical model, replace $\sigma(r)$ of the lamina by $\rho(r, z)$ and integrate with respect to z for the length of the cylindrical configuration from z_1 to z_2, the vertical distances to the upper and lower faces of the anomalous body. If the density depends only on the radius, $\rho(r, z)$ will be a simple polynomial of r and Δg_0, Δg_2, $\Delta g_4, \ldots$ are all analytically evaluated as before. However, if the density is also a function of depth as well as the radius, $\rho(r, z)$ becomes a polynomial in r and z in the interval $(r_i, r_{i+1}; z_j, z_{j+1})$. Again the integrations involved in Δg_0, Δg_2, $\Delta g_4, \ldots$ can be obtained analytically. The expression for Δg_0, Δg_2, $\Delta g_4, \ldots$ will be, in a general form,

$$\Delta g_0 = 2\pi G \sum_{l=0}^{N-1} \int_{r_l}^{r_{l+1}} \int_{z_1}^{z_2} \rho(r, z) r (p^2 + r^2)^{-3/2} z\, dz\, dr$$

$$\Delta g_2 = \frac{15\pi G z x^2}{2} \sum_{l=0}^{N-1} \int_{r_l}^{r_{l+1}} \int_{z_1}^{z_2} \rho(r, z) r^3 (p^2 + r^2)^{-7/3} z\, dz\, dr$$

$$\Delta g_4 = \frac{2835\pi G x^4}{16} \sum_{l=0}^{N-1} \int_{r_l}^{r_{l+1}} \int_{z_1}^{z_2} \rho(r, z) r^5 (p^2 + r^2)^{-11/2} z\, dz\, dr \tag{8}$$

$$\vdots$$

and so on.

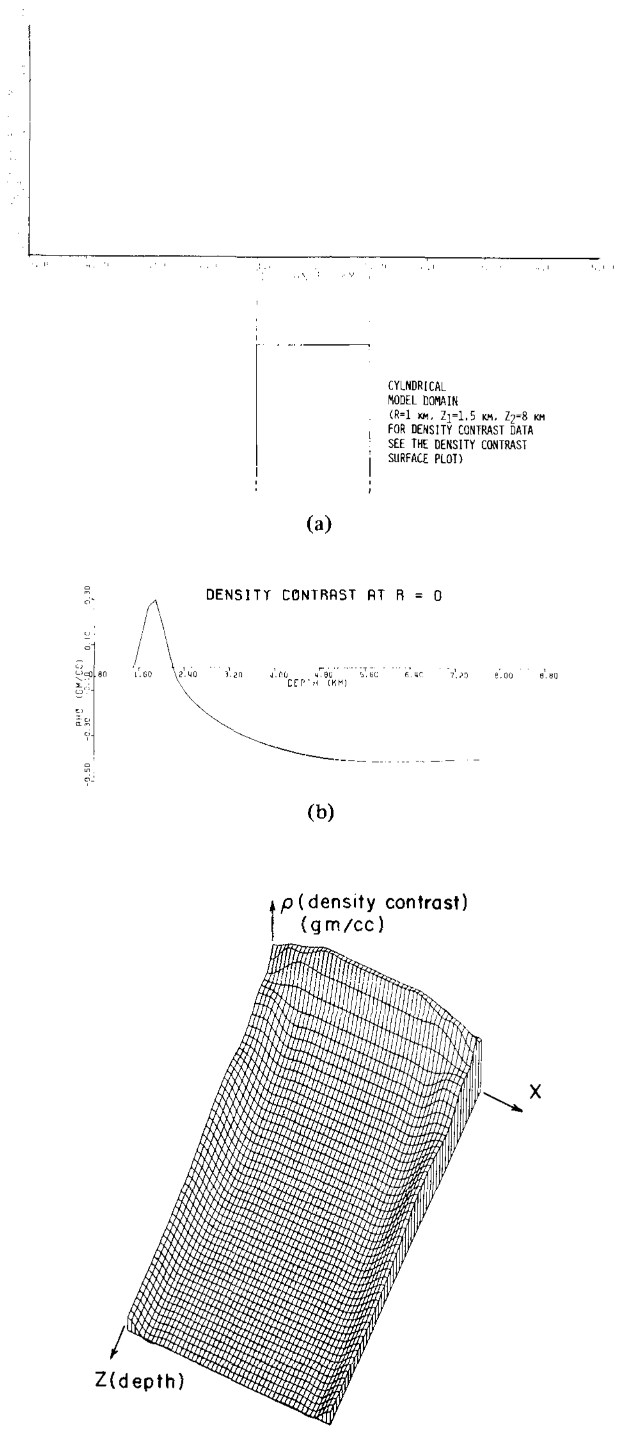


Figure 2. (a) Gravity anomaly produced by a cylindrically symmetrical hypothetical salt dome. (b) Density contrast profile as a function of depth. (c) General shape of hypothetical salt dome and its density contrast surface plot.

A numerical example is shown in Fig. 2. The density model is plotted at the bottom and the gravity anomaly produced from the anomalous body below is plotted in the upper diagram. In this example the density function is interpolated by a cubic spline. The gravitational attraction computed in this example is theoretically exact. However, in practice there are two types of errors involved. The interpolation error for $\rho(r, z)$ is the first one. This error depends on the choice of the piecewise continuous basis function as well as the number of points in the input model. For most practical cases this type of error is very small or can be minimized to the required accuracy in many practical problems. The second type of error is the truncation error. In the above example the infinite series for $n = 0, 1, 2, \ldots,$ is represented as a binomial expansion sum and is exact. But the sum over $k = n, n-1, \ldots, 0$ is truncated at an appropriate length. This truncation error can be reduced simply by adding more terms. Of course this will increase the computing time and the modelling process becomes expensive. However, if we have a close look at the integrands in equation (8) we find that the series of terms Δg_0, Δg_2, $\Delta g_4, \ldots$ converges very fast for $r \leqslant a \leqslant p$ as tested numerically.

3 Two-dimensional body with an arbitrary shape

The computational theory of potential fields from a two-dimensional body is well illustrated by Kellog (1953), and Kogbetliantz (1945) and Hubbert (1948) devised a line integral approach which was used by Talwani, Worzel & Landisman (1959) in their computer algorithm. The method developed by Hubbert (1948) and Talwani *et al.* (1959) is only for a two-dimensional body of an arbitrary shape with uniform density. However, the program is general enough and the density variation may be represented by numerous discrete homogeneous cells which, as a whole, form an n-sided polygon. A similar approach was later proposed by Bhattacharyya & Chan (1977) to use small rectangular blocks of uniform density, but this type of approach is very inefficient in actual modelling process. A very general example of two-dimensional crustal gravity modelling is given below.

The two-dimensional potential may be written as

$$U(\bar{p}) = 2G \int_S \rho(\bar{p}_0) \ln |\bar{p} - \bar{p}_0| \, dS \tag{9}$$

and the vertical gravitational attraction would be

$$\Delta g(x, z) = 2G \frac{\partial}{\partial z} \int_S \rho(x, z) \ln R \, dx \, dz \tag{10}$$

where

$$R = \sqrt{(x - x_0)^2 + (z - z_0)^2}.$$

In this example, the body is infinitely long in the y-axis. As mentioned above, the density function $\rho(x, z)$ can first be interpolated and then integrated to obtain the gravity effect. However, in some cases, the integrand is evaluated at each model point and the integration is performed using bicubic basis functions, in which case the integration inside each rectangular cell $(x_i, x_{i+1}; z_j, z_{j+1})$ is evaluated analytically. In such cases, the potential function integrand rather than the density function is smoothly interpolated. In any case, the density function $\rho(x, z)$ or the potential function integrand can be interpolated smoothly in any cross-section x–z plane and it may even have discontinuities of simple nature. The example given in Fig. 3 shows two-dimensional modelling of a simple geological structure and its computed gravity effects for the uniform and varying density contrast cases. For the given

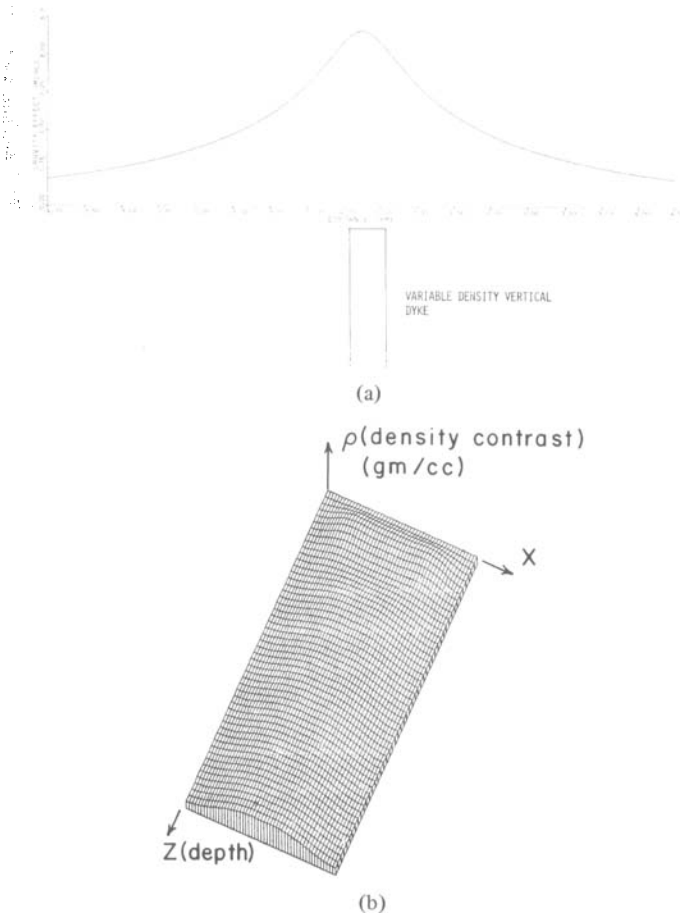


Figure 3. (a) Gravitational attraction from a hypothetical vertical dyke with varying density contrast as shown in (b). (b) The density contrast plot of a hypothetical vertical dyke.

model, the accuracy of the method is tested using a conventional scheme and the accuracy for the uniform density model is better than 0.1 per cent at all points. For the continuously varying density model, the accuracy of the method is tested using the available block model method and it is again better than 0.1 per cent at all points. Besides the above-mentioned accuracy of the method, this new method is much more efficient computationally and simple to use. However, if the anomalous body becomes exposed near the surface one has to be careful. When the top of the anomalous body comes close to the surface, the integrand of the potential field integral or the gravitational attraction integral may become nearly singular and consequently some of the gravitational attractions computed may become very erratic. In such cases, extra care has to be taken in the numerical procedure to accommodate the situation.

4 First-order hydrostatic ellipticity

The computation of first-order hydrostatic ellipticity of the Earth is a classical exercise in geophysics. However, as the earthquake seismology and whole Earth geophysics require a reasonably accurate approximation of the Earth's ellipticity, it is frequently desired to compute the first-order hydrostatic ellipticity for a given earth model. The potential at an internal point of a massive body may be further developed to obtain the famous Clairaut differential equation (Jeffreys 1970).

$$\bar{\rho}\left(\frac{d^2\epsilon_n}{dr^2}-\frac{n(n+1)\epsilon_n}{r^2}\right)+\frac{6\rho}{r}\left(\frac{d\epsilon_n}{dr}+\frac{\epsilon_n}{r}\right)=0. \tag{11}$$

After a Darwin–de Sitter–Radau transformation, we obtain, to a first order

$$\sqrt{1+\eta}=\frac{5}{\bar{\rho}r'^5}\int_0^{r'}\bar{\rho}r^4F(\eta)\,dr \tag{12}$$

where r' is the mean spherical radius, $\bar{\rho}$ the mean density inside the radius r' and

$$F(\eta)=(1+\eta)^2\left[1+\frac{1}{2}\eta-\frac{1}{10}\eta^2+\ldots\right].$$

Then the first-order ellipticity may be obtained by solving

$$\eta=r'\frac{1}{\epsilon}\frac{d\epsilon}{dr'}. \tag{13}$$

The value of $F(\eta)$ is very nearly constant around 1.0 and has a minimum for $\eta=0$ and a maximum for $\eta=1/3$. Bullard (1948) and Jeffreys (1963) numerically integrated the above relations to obtain the approximate values of ellipticity inside a given earth model. Jeffreys (1963) also computed correction terms for his results. In this example, the interpolation scheme described in Moon (1979) is used for earth model 1066B. Then the mean density

$$\bar{\rho}=\frac{3}{r'^3}\int_0^{r'}\rho(r)\,r'^2\,dr \tag{14}$$

is set up to be computed by the method described in Section 3. Equation (12) can be written as

$$\eta(r')=\frac{25}{4}\left[1-\left\{\int_0^{r'}\rho(r)r^4\,dr\right\}\left\{r^2\int_0^{r'}\rho(r)r^2\,dr\right\}^{-1}\right]^2-1 \tag{15}$$

and the ellipticity term, from equation (13), is

$$\epsilon(r')=\epsilon(R)\exp\left[-\int_{r'}^{R}\frac{\eta(r)}{r}\,dr\right] \tag{16}$$

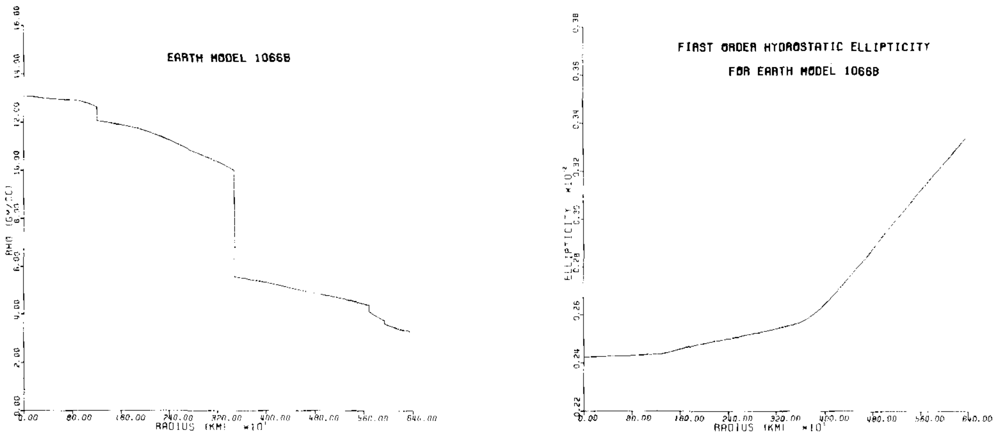
where the surface ellipticity is

$$\epsilon(R)=\frac{5m}{2[\eta(R)+2]}$$

and

$$m=\frac{3}{4}\frac{\Omega^2}{\pi G\bar{\rho}}.$$

As described in previous sections, the integrals involved in equations (14), (15) and (16) can be analytically evaluated in any interval (r_i, r_{i+1}) of a given earth model. The resulting



(a) (b)

Figure 4. (a) Earth model 1066B. (b) First-order hydrostatic ellipticity of the Earth as shown above.

ellipticity is absolutely accurate except for the theoretical approximations in the derivation up to equation (16) and the interpolation error included in $\rho(r)$. Earth model 1066B and the first-order hydrostatic ellipticity are shown in Fig. 4. A further computer algorithm is briefly explained in Moon (1980). The theoretical and analytical correction terms are explained in Jeffreys (1963).

5 Conclusion

The geopotential field theory and its applications in geophysics have been important for our understanding of our whole Earth as well as near surface small-scale geological effects. In this paper, a general scheme is proposed to compute geopotentials and attractions of an object with continuously varying density distributions. Jump discontinuities such as the core–mantle boundary in an earth model or a fault plane in the Earth's crust can also be included adequately in this approach. As shown in the detailed examples, the interpolation of the density function, with an appropriate piecewise continuous polynomial basis function, enables us to simplify the field integrals, in most cases, and we can accommodate a continuous density variation inside the given object as well as reasonable jump discontinuities associated within the body. This often makes it possible to evaluate the field integrals analytically and to increase the numerical efficiency. The basis functions of interpolation schemes, explained and used in the above sections and examples, are all cubic polynomials. However, in simple examples, a linear relation may be adequate in which case the problem will be even further simplified. In fact, for the problems with a higher density gradient than normal examples, a linear basis function is preferred to avoid overshooting of the cubic functions. Of course this problem may also be avoided easily with cubic basis functions by allocating data points at appropriate points in the respective models. This scheme can also be applied to magnetic potential field theory as well (Moon & Hall 1981). As shown in the examples, this method can be applied in exploration geophysics, crustal geophysics and more complicated whole Earth problems.

Acknowledgments

Some of the original ideas in this paper were conceived while the author was working with Dr C. H. Chapman at the University of Toronto. Professors W. C. Brisbin and D. H. Hall kindly provided informative discussions. Patrick Lui tested the computer programs. This research is supported by the University of Manitoba research grant and NSERC of Canada operating grants A-7400 and A-9130.

The University of Manitoba, Centre for Precambrian Studies Publication No. 55.

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Appendix

The integrations given in equations (5), (6) and (7) are integrated (Gradshteyn & Ryzhik 1965) and, after some algebra, can be written as

$$\Delta g_0 = 2\pi G z \sum_{l=0}^{N-1} \left[A\left\{ \frac{1}{z} r\sqrt{p^2+r^2} + \frac{p^2 r}{\sqrt{p^2+r^2}} - \frac{3}{2} p^2 \ln\left(r+\sqrt{p^2+r^2}\right) \right\} + B\left\{ \sqrt{p^2+r^2} + \frac{p^2}{\sqrt{p^2+r^2}} \right\} + C\left\{ -\frac{r}{\sqrt{p^2+r^2}} + \ln\left(r+\sqrt{p^2+r^2}\right) \right\} + D\left\{ -\frac{1}{\sqrt{p^2+r^2}} \right\} \right]_{r_{l+1}}^{r_{l+1}} \tag{A1}$$

$$\Delta g_2 = \frac{15\pi Gzx^2}{2} \sum_{l=0}^{N-1} \left[A \left\{ -\frac{23}{15} \frac{r^5}{(p^2+r^2)^{5/2}} - \frac{7}{3} \frac{p^2 r^3}{(p^2+r^2)^{5/2}} - \frac{p^4 r}{(p^2+r^2)^{5/4}} \right.\right.$$

$$\left. + \ln\left(r + \sqrt{p^2+r^2}\right) \right\} + B \left\{ -\frac{1}{\sqrt{p^2+r^2}} + \frac{2}{3} \frac{p^2}{(p^2+r^2)^{3/2}} \right.$$

$$\left. - \frac{1}{5} \frac{p^4}{(p^2+r^2)^{5/2}} \right\}$$

$$\left. + C \left\{ \frac{1}{5} \frac{r^5}{p^2(r^2+p^2)^{5/2}} \right\} + D \left\{ -\frac{1}{3} \frac{1}{(p^2+r^2)^{3/2}} + \frac{1}{5} \frac{p^2}{(p^2+r^2)^{5/2}} \right\} \right]_{r_l}^{r_{l+1}} \tag{A2}$$

$$\Delta g_4 = \frac{2835\pi Gzx^4}{16} \sum_{l=0}^{N-1} \left[A \left\{ -\frac{1}{2} \frac{r^7}{(p^2+r^2)^{9/2}} + \frac{1}{2} \frac{r^7}{p^2(p^2+r^2)^{7/2}} \right.\right.$$

$$\left. - \frac{7}{18} \frac{r^9}{p^2(p^2+r^2)^{9/2}} \right\}$$

$$+ B \left\{ -\frac{1}{3} \frac{1}{(p^2+r^2)^{3/2}} + \frac{3p^2}{5(p^2+r^2)^{5/2}} - \frac{3p^4}{7(p^2+r^2)^{7/2}} \right.$$

$$\left. + \frac{p^6}{9(p^2+r^2)^{9/2}} \right\}$$

$$+ C \left\{ \frac{1}{7} \frac{r^7}{p^4(p^2+r^2)^{7/2}} - \frac{1}{9} \frac{r^9}{p^4(p^2+r^2)^{9/2}} \right\}$$

$$\left. + D \left\{ -\frac{1}{5} \frac{1}{(p^2+r^2)^{5/2}} + \frac{2}{7} \frac{p^2}{(p^2+r^2)^{7/2}} - \frac{1}{9} \frac{p^4}{(p^2+r^2)^{9/2}} \right\} \right]_{r_l}^{r_{l+1}} \tag{A3}$$

where A, B, C and D are the interpolation coefficients for $\sigma(r)$ in each interval (r_l, r_{l+1}). When these equations for a circular lamina are extended for a right vertical cylinder, the expansions for Δg_0, Δg_2, $\Delta g_4, \ldots$ will be in exactly the same form as above except that there will be an integration from z_1, the top of the cylinder, to z_2, the bottom interface. However, if the density also depends on z the interpolation constants A, B, C and D will be polynomials of z

$$A = a_1 z^3 + b_1 z^2 + c_1 z + d_1$$

$$B = a_2 z^3 + b_2 z^2 + c_2 z + d_1$$

for any type of bicubic interpolation. Then we have

$$\Delta g_0 = 2\pi G \int_{z_1}^{z_2} \sum_{l=0}^{N-1} [A\{\ \}_{A_0} + B\{\ \}_{B_0} + C\{\ \}_{C_0} + D\{\ \}_{D_0}]_{r_l}^{r_{l+1}}\, z\, dz$$

$$\Delta g_2 = \frac{15\pi G x^2}{2} \int_{z_1}^{z_2} \sum_{l=0}^{N-1} [A\{\ \}_{A_2} + B\{\ \}_{B_2} + C\{\ \}_{C_2} + D\{\ \}_{D_2}]_{r_l}^{r_{l+1}}\, z\, dz \tag{A5}$$

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and the final form will be of form

$$\begin{aligned}\Delta g_0 \simeq 2\pi G \int_{z_1}^{z_2} \sum_{l=0}^{N-1} \Bigg[& A\left(\{\}_{A_0} + \frac{15x^2}{4}\{\}_{A_2} + \frac{2835x^4}{32}\{\}_{A_4} + \dots\right) \\ & + B\left(\{\}_{B_0} + \frac{15x^2}{4}\{\}_{B_2} + \frac{2835x^4}{32}\{\}_{B_4} + \dots\right) \\ & + C\left(\{\}_{C_0} + \frac{15x^2}{4}\{\}_{C_2} + \frac{2835x^4}{32}\{\}_{C_4} + \dots\right) \\ & + D\left(\{\}_{D_0} + \frac{15x^2}{4}\{\}_{D_2} + \frac{2835x^4}{32}\{\}_{D_4} + \dots\right)\Bigg]_{r_l}^{r_{l+1}} z\, dz. \end{aligned} \tag{A6}$$

This integration is, of course, an elementary one.