A new method of determining eigenvalues and eigenfunctions

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An analytical solution of the general eigenequation

$$\mathbf{Y^{(m)}(x) = F(x, Y^{(r)}) \qquad 0 \leqslant r < m}$$

where

$$\mathbf{F(x, Y^{(r)}) = g_0(x, Y^{(r)}) + \lambda g_1(x, Y^{(r)}) + \lambda^2 g_2(x, Y^{(r)}) \dots + \lambda^i g_i(x, Y^{(r)})}$$

is independently linear in the $Y^{(r)}$ and λ^i, and $Y(x)$ is the eigenfunction and λ is the eigenvalue, is developed. The eigenfunction is obtained as an analytic function of x, and the eigenvalue λ is the ratio between successive functions in an expansion. The solution is always convergent, and can be used as a practical method of computing eigenparameters to a high degree of precision. Two examples of computer application are given in which the solutions are obtained as power series.

1. Introduction

In the literature there are numerous methods which describe the solution of finite-difference approximations to eigenequations. In this paper, however, we propose a method of obtaining eigensolutions as analytic functions. The method is basically an analytical solution of the general eigenequation

$$Y^{(m)}(x) = F(x, Y^{(r)}) \qquad 0 \leqslant r < m \tag{1}$$

where

$$F(x, Y^{(r)}) = g_0(x, Y^{(r)}) + \lambda g_1(x, Y^{(r)}) + \lambda^2 g_2(x, Y^{(r)}) \dots + \lambda^i g_i(x, Y^{(r)})$$

is independently linear in the $Y^{(r)}$ and λ^i, and $Y(x)$ is the eigenfunction and λ is the eigenvalue. Throughout, the primes denote differentiation w.r.t. x and each g_i may be a linear combination of $Y(x)$ and its derivatives of the form

$$g_i(x, Y^{(r)}) = \sum_{j=0}^{r} \phi_{ij}(x) Y^{(j)}(x) \quad 0 \leqslant r < m \tag{2}$$

where the ϕ_{ij} are arbitrary real functions of x.

The eigenfunction $Y(x)$ is obtained as an analytic function of x, and the eigenvalue λ is the ratio between successive functions in an expansion. The method appears in principle to be capable of determining all the real solutions of the equation, but it is particularly useful in finding the fundamental eigenparameters. Two examples of computer application are illustrated in which the solution is obtained as a power series in x.

2. The general theory of the solution

Let us consider the equation

$$y^{(m)}(x) = F(x, y^{(r)}) + X^{(m)}(x) \tag{3}$$

which, except for the addition of the arbitrary real function $X^{(m)}(x)$ on the right-hand side, is analogous to equation (1) with the function $Y(x)$ replaced by $y(x)$.

We obtain a solution of equation (3) in the form of the infinite power series

$$y(x) = y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) \dots + \lambda^k y_k(x) \dots \tag{4}$$

where the functions $y_k(x)$ each satisfy identical homogeneous boundary conditions. Substituting for $y(x)$ in equation (3), and remembering that $F(x, y^{(r)})$ is linear in $y^{(r)}$, we have the following equation

$$O = R_0 + \lambda R_1 + \lambda^2 R_2 \dots + \lambda^k R_k \dots \tag{5}$$

where the R_k denote the set of recurrence relations which is expanded below:

$$\begin{aligned}
R_0 &= y_0^{(m)}(x) - g_0(x, y_0^{(r)}) - X^{(m)}(x) \\
R_1 &= y_1^{(m)}(x) - g_0(x, y_1^{(r)}) - g_1(x, y_0^{(r)}) \\
R_2 &= y_2^{(m)}(x) - g_0(x, y_2^{(r)}) - g_1(x, y_1^{(r)}) - g_2(x, y_0^{(r)}) \\
&\dots \\
R_k &= y_k^{(m)}(x) - g_0(x, y_k^{(r)}) - g_1(x, y_{k-1}^{(r)}) \dots \\
&\dots - g_i(x, y_{k-i}^{(r)}).
\end{aligned} \tag{6}$$

Solving equation (5) by setting each

$$R_k = 0 \qquad (0 \leqslant k < \infty) \tag{7}$$

we determine the functions $y_k(x)$, and hence $y(x)$.

The notable feature of this set of equations is that

R_0 contains only functions of $y_0(x)$

R_1 contains only functions of $y_0(x)$ and $y_1(x)$

R_2 contains only functions of $y_0(x)$, $y_1(x)$ and $y_2(x)$, etc.

Hence each y_k may be determined successively by applying m homogeneous boundary conditions to the

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integration of an ordinary and linear differential equation of the form

$$y_k^{(m)}(x) - g_0(x, y_k^{(r)}) = \chi(x)$$

where

$$\chi(x) = g_1(x, y_{k-1}^{(r)}) \ldots + g_i(x, y_{k-i}^{(r)})$$

is a known function of x involving only previously determined y_{k-i}. Provided that the expansion (4) is convergent, therefore, we may calculate $y(x)$ to any desired degree of accuracy.

Our task now is to relate the function $y(x)$ to an eigenfunction $Y(x)$ of equation (1).

3. The eigensolution

Let us consider the special cases of equation (3) in which $|\lambda|$ is equal to the radius of convergence $|\lambda_0|$ of the power series (4). Now suppose that the type of convergence is such that as $k \to \infty$, the $y_k(x)$ become similar for all x, and λ_0 is defined by the limit relation,

$$\lim_{k\to\infty}\left(\frac{y_{k-1}(x)}{y_k(x)}\right) = \lambda_0 \tag{8}$$

for all x. Then from the set of equations (6) we have,

$$y_k^{(m)}(x) = g_0(x, y_k^{(r)}) + g_1(x, y_{k-1}^{(r)}) + \ldots g_i(x, y_{k-i}^{(r)}). \tag{9}$$

Hence considering equation (9) in the limit $k \to \infty$ and substituting equation (8) we obtain,

$$\begin{aligned}\lim_{k\to\infty} y_k^{(m)}(x) &= g_0\left(x, \lim_{k\to\infty} y_k^{(r)}\right) + \lambda_0 g_1\left(x, \lim_{k\to\infty} y_k^{(r)}\right)\\ &\quad + \ldots \lambda_0^i g_i\left(x, \lim_{k\to\infty} y_k^{(r)}\right)\\ &= F\left(x, \lim_{k\to\infty} y_k^{(r)}\right) \end{aligned} \tag{10}$$

But equation (10) is identical in form with our eigenequation (1), therefore

$$Y(x) = \lim_{k\to\infty} y_k(x) \tag{11}$$

is an eigenfunction, and λ_0 is the corresponding eigenvalue.

The existence of a limit of the form of equation (8) therefore implies that equation (1) has a real eigensolution. In practice the fundamental eigensolution is usually obtained, but in special cases the $y_k(x)$ may converge to a harmonic (cf. Example 1). Any reasonable choice of $X(x)$* may be made, and the convergence to the limit has been found to be rapid.

On the other hand, if the convergence is of a different kind to that described at the beginning of the section, i.e. as $k \to \infty$, the $y_k(x)$ do not become similar, equation (1) may not have a real eigensolution, although "special" values of λ can exist. It is hoped that the solution of an equation of this type will be described in a later paper.

* If $X(x) \neq 0$, but $X^m(x) = 0$, it should be noted that the solution $y(x)$ may be interpreted as having homogeneous boundary conditions on all $y_k(x)$ except $y_0(x)$.

4. Computational techniques

A powerful computational procedure is to express all the functions, $\phi_{ij}(x)$, of the eigenequation as power series. It then follows that the eigenfunction is also obtained as a power series. The higher the approximation (k) of equation (11), the higher the order, which increases by at least m for each approximation. Further if

$$g_0(x, Y^{(r)}) = 0$$

the solution is obtained simply by term-by-term integration of power series, since each recurrence equation (cf. Section 2) is of the form

$$y_k^{(m)}(x) = \text{power series } (x)$$

and hence

$$y_k(x) = \int \overset{(m)}{\ldots} \int (\text{power series } (x))dx \overset{(m)}{\ldots} dx. \tag{12}$$

Integration (12) yields m boundary constants, which are satisfied by the homogeneous boundary conditions, and is particularly simple to program for a computer. The solution may easily be monitored as obtained by evaluating each $y_k(x)$ at a set of values of x.

5. Examples of solution

The potential value of the general solution is illustrated below in two examples. The first example is a very simple eigenequation of which the closed analytical solution is known. The second example has many of the properties of the first example. However, its analytical solution has not previously been studied. In this case, we compare our results with finite-difference estimates.

For convenience, we will not give the actual coefficients of the power series representing the functions y_k but only the evaluations at a series of values of η

$$0 < \eta < 1$$

normalized w.r.t. the maximum $y_k(\eta)$.

Example 1

We are required to determine the eigenvalues and eigenfunctions of the equation

$$Y''(x) = \lambda\phi_{10}(x)Y(x) \tag{13}$$

where $\quad \phi_{10}(x) = -1$

with the boundary conditions

$$Y(0) = 0$$

$$Y(1) = 0.$$

Table 1

Numerical solutions of Example 1

POWER SERIES EIGENSOLUTIONS (EXAMPLE 1)

	$X = 1$					$X = 1 - 2x$				
k \ $\eta \to$	$y_k(\eta)$ 0·111	0·222	0·333	0·5	$\frac{y_{k-1}(0 \cdot 5)}{y_k(0 \cdot 5)}$	$y_k(\eta)$ 0·111	0·222	0·333	0·444	$\frac{y_{k-1}(0 \cdot 222)}{y_k(0 \cdot 222)}$
0	1	1	1	1	—	1	0·714	0·428	0·143	—
1	0·395	0·691	0·888	1	8·00	0·88	1	0·772	0·286	34·75
2	0·347	0·648	0·869	1	9·60	0·682	1	0·846	0·328	39·51
3	0·343	0·643	0·866	1	9·84	0·659	1	0·870	0·342	39·66
4	0·342	0·643	0·866	1	9·87	0·654	1	0·877	0·346	39·54
5	0·342	0·643	0·866	1	9·87	0·653	1	0·879	0·347	39·50
6	0·342	0·643	0·866	1	9·87	0·653	1	0·879	0·348	39·48

Substituting in our general equations, we have

$$g_0(x, y^{(r)}) = 0$$
$$g_1(x, y^{(r)}) = \phi_{10}(x)y(x)$$
$$g_i(x, y^{(r)}) = 0, \quad (i \geqslant 2)$$

and

$$R_0 = y_0'' - X'' = 0$$
$$R_1 = y_1'' - \phi_{10}y_0 = 0$$
$$\dots$$
$$R_k = y_k'' - \phi_{10}y_{k-1} = 0. \tag{14}$$

By means of a computer program incorporating sequence (14) results were obtained for two choices of X, and evaluated at $\eta = n/18$ (where n is an integer, $0 \leqslant n \leqslant 18$). They are recorded in **Table 1,** and may be summarized as follows:

$$\text{(i) } X = 1, \qquad \lambda = \pi^2, \quad Y = \sin \pi x$$
$$\text{(ii) } X = 1 - 2x, \quad \lambda = 4\pi^2, \quad Y = \sin 2\pi x. \tag{15}$$

These two determinations are the fundamental and the first harmonic of the eigenequation (13).

In solution (15 (i)),

$$\lim_{k \to \infty} y_k = a \sin \pi x$$

and

$$\lim_{k \to \infty} y_{k-1} = a\, \pi^2 \sin x$$

where a is an arbitrary multiplier.

Similarly for solution (15 (ii)).

For these two examples, we may also obtain closed analytical solutions (the expansions of which using the series for sine and cosine functions exactly reproduce the power series solutions), as follows:

$$\text{(i) } y = \cos \sqrt{\lambda}x + \left(\frac{1 - \cos \sqrt{\lambda}}{\sin \sqrt{\lambda}}\right) \sin \sqrt{\lambda}x$$

$$\text{(ii) } y = \cos \sqrt{\lambda}x + \left(\frac{-1 - \cos \sqrt{\lambda}}{\sin \sqrt{\lambda}}\right) \sin \sqrt{\lambda}x. \tag{16}$$

We see immediately that at $\lambda = \lambda_0$ there is a singularity in the solutions at which $y \to \infty$, in agreement with the behaviour of the power series.

The results obtained above are in full agreement with the solutions of equation (13), known by other methods:

$$Y = \sin n\pi x$$
$$\lambda = n^2\pi^2$$

where n is an integer.

Example 2

We are required to determine the eigenvalues and eigenfunctions of the equation

$$Y''''(x) = \lambda(\phi_{10}(x)Y(x) + \phi_{12}(x)Y''(x))$$

where $\quad \phi_{10}(x) = 3x^2 - 4x + 1$

and $\quad \phi_{12}(x) = 6 \qquad (17)$

with the boundary conditions

$$Y(0) = 0$$
$$Y''(0) = 0$$
$$Y(1) = 0$$
$$Y'(1) = 0.$$

Substituting in our general equations we have

$$g_0(x, y^{(r)}) = 0$$

Table 2

Numerical solutions of Example 2

POWER SERIES EIGENSOLUTION (EXAMPLE 2) $X = x^3 - 2x^2 + x$						FINITE-DIFFERENCE EIGENSOLUTION (EXAMPLE 2)					
$k \downarrow$ η	$y_k(\eta)$ 0·2	0·4	0·6	0·8	$\frac{y_{k-1}(0 \cdot 4)}{y_k(0 \cdot 4)}$	η	$y_k(\eta)$ 0·2	0·4	0·6	0·8	λ_k
0	0·889	1	0·667	0·222	—		—	—	—	—	—
1	−0·751	−1	−0·751	−0·276	−29·597	*	—	—	—	—	−24
2	0·732	1	0·765	0·285	−38·040	*	—	—	—	—	−28·5
3	−0·728	−1	−0·767	−0·287	−39·520	*	—	—	—	—	−31·8
4	0·728	1	0·768	0·287	−39·753		0·73	1	0·79	0·33	−34·09
5	−0·728	−1	−0·768	−0·287	−39·788	†	—	—	—	—	—
6	0·728	1	0·768	0·287	−39·792		0·73	1	0·79	0·31	−36·60
7	−0·728	−1	−0·768	−0·287	−39·793		0·72	1	0·77	0·30	−37·29
18	0·728	1	0·768	0·287	−39·793		0·73	1	0·77	0·29	−39·33

* Significant estimates of the finite-difference eigenfunction (from graphs plotted for the network points) cannot be made for coarse networks.
† No eigensolution was found.

$$g_1(x, y^{(r)}) = \phi_{10}(x)y(x) + \phi_{12}(x)y''(x)$$
$$g_i(x, y^{(r)}) = 0 \quad (i \geqslant 2)$$

and

$$R_0 = y_0'''' - X'''' = 0$$
$$R_1 = y_1'''' - \phi_{10}y_0 - \phi_{12}y_0'' = 0$$
$$R_2 = y_1'''' - \phi_{10}y_1 - \phi_{12}y_1'' = 0$$
$$\ldots$$
$$R_k = y_k'''' - \phi_{10}y_{k-1} - \phi_{12}y_{k-1}'' = 0^* \qquad (18)$$

By means of a computer program incorporating sequence (18), we obtained the fundamental eigensolution, which was evaluated at $\eta = n/10$, where n is an integer $0 \leqslant n \leqslant 10$ (cf. **Table 2**).

The fundamental eigensolution of this example has also been studied in detail by finite-difference techniques. We approximated the differentials of the eigenequation as first-order finite-difference formulae for a set of k grid points between $x = 0$ and 1.

The resultant set of equations was then expressed as the matrix equation

$$A = \lambda B$$

where A and B are $k \times k$ matrices, and solved by a standard procedure.

* As λ is the highest power of the eigenvalue in equation (17), the number of terms in R_k does not increase for $k > 1$. The maximum number of terms in R_k is equal to the number of terms in the expanded eigenequation.

The results are also shown in Table 2. If we compare the estimate of λ obtained with k grid points, to that obtained with a polynomial of order (k), it is notable that our present method converges much more rapidly.

We may interpret the solution in a rather interesting way.

In an analogous manner to the sine function which preserves its form after being integrated twice, the functions (Y) which are solutions of this example preserve their form after the linear combination

$$(\phi_{10}Y + \phi_{12}Y'')$$

has been integrated four times.

We thus see in the present type of solution a method of obtaining generalized functions with specified integral properties.

6. Conclusion

We have developed an analytical solution of a very general class of eigenequation. The examples of solution illustrate the precision of the method compared with finite-difference techniques.

The convergence process normally leads to the fundamental eigenparameters; however, it is hoped that it can be extended in a straightforward manner to give also the harmonics. The significance of the solution for those equations which have no real eigensolution, in which case the convergence is of a rather different nature, is also being investigated.