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A NEW MODEL FOR ACOUSTIC-STRUCTURE INTERACTION AND ITS EXPONENTIAL STABILITY

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Abstract. A new model for the interaction between the acoustic wave in an enclosed air cavity and the transversal motion of a flexible beam is proposed in this paper. This new boundary condition for the coupled wave and Euler-Bernoulli beam equations introduces sufficient damping of the energy of the system to gain uniform exponential stability. Careful physical justification of the boundary condition is based upon well-established theoretical results in acoustics. The estimate of the energy decay rate is obtained using a multiplier technique.

1. Introduction. In the control design problems for active noise suppression, the interaction between the acoustic field and its surrounding is critically important. In fact, in most recent applications (see for example [13], [14], [7], [12], [2]), control of the acoustic field is provided through the boundary of the acoustic cavity (a notable exception is the problem where the acoustic field is controlled through the interior pointwise controls as formulated in [5] and [21]). This method of active noise control can be viewed in the more general context of boundary control problems for the wave equation, which is the usual model for the acoustic field. The recent efforts in this area (see [14], [15]) have produced valuable theoretical results for solving control problems involving hyperbolic dynamical systems with unbounded control input operators. However, only recently, in the work of Banks et al., [2], the exact nature of the interaction between the acoustic field and its boundary, which is modeled as flexible structures, has been carefully investigated. The contribution made by [2] is significant in that it makes the boundary control of the wave equation much closer to the physical reality. In all previous works, the boundary

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controls are formulated as Dirichlet or Neumann boundary controls which remain as mathematical abstractions yet to be physically realized. In [2], instead, the boundary control is achieved and verified in physical experiments by a piezoceramic actuator which controls the vibration of the flexible boundary. For this reason, the model proposed in [2] has attracted the attention of many mathematical control scientists.

Many important questions about the model in [2] still need careful consideration. In particular, the issues of uniform exponential stability of the homogeneous model equation and stabilizability of the control system need to be addressed. The stabilizability as well as the detectability of a linear control system are not only useful for establishing the existence and the uniqueness of the optimal feedback control for linear quadratic regulator problems (see for example, [8], [6], [15]), they are also needed to ensure the robustness of optimal control laws. In many cases, the uniform exponential stability of the homogeneous model equation allows us to establish stabilizability and detectability for different forms of input and observation operators. Numerical experimental results as well as theoretical results for similar equations suggest that the model in [2] does not have this desired stability.

In this paper, we propose an alternative model for the interaction between the acoustic field and the flexible structure. In particular, we replace the condition of continuity of normal velocity at the boundary of the acoustic cavity by a boundary condition that allows a difference between the normal velocity of the acoustic wave and the transversal velocity of the flexible beam. As in [2], we pay careful attention to the physical interpretation of the model and provide a justification based on well-established acoustics theory (see [18] and [16]). To establish the desired stability result for our model, we use the multiplier technique, which has already been used successfully in establishing exponential decay rates for wave equations with boundary feedback damping (see [7] and [12]). By applying this method we show that the solutions of the model equations for a rectangular acoustic cavity with two sides bounded by flexible beams are uniformly exponentially stable. Our results can easily be generalized to a polygonal air cavity with a subset of its edges bounded by flexible beams that satisfy appropriate geometric conditions similar to those assumed in [7] and [12]. It should be noted that the application of the multiplier method in establishing the decay rate of the solutions of the wave equation is a delicate matter when the boundary of the domain is not C^1 and mixed boundary conditions are imposed on different subsets of the boundary (see [10]). To address similar issues, we use a new approach based on smooth approximations of the solution to the model equations to establish the necessary estimates. We have recently discovered that the exponential stability results for more general geometry similar to the results presented here are proved independently by Avalos ([1]) using different techniques; however, when the part of the boundary of the acoustic cavity corresponding to flexible structure has a shape other than a piecewise straight line, we are unable to determine the physical interpretation of such models. On the other hand, our results can be readily generalized to the cases where the shape of the acoustic cavity is other polygons.

The outline of this paper is as follows. In Sec. 2 we describe the physical interpretation of the model equations. In particular, we discuss the theoretical basis for the new boundary conditions that model the acoustic-structure interaction. In Secs. 3 and 4, we use a variational formulation to establish the well-posedness and regularity of the solutions of the model equations. Section 5 is devoted to the discussions of the decay rate estimate of the solutions of the model equations. We offer our concluding remarks in Sec. 6.

2. Physical model for boundary interaction. In this section, we present arguments to justify our proposed model for the interaction between acoustic waves in an enclosed two-dimensional air cavity and a vibrating flexible beam that forms a portion of the boundary of the air cavity. Even though we are not acoustic scientists, our justifications for the model are based upon widely accepted and commonly cited physical assumptions. We show that our model for acoustic and structure interactions is consistent with the classical acoustic theory. Moreover, we show that the solutions of the system are uniformly exponentially stable, a characteristic that is highly desirable in modeling and control of acoustic-structure interactions. Our model for the most part is similar to the model in [2]. The main difference between the two models is that in our model the condition of continuity of normal velocity at the interface between the air cavity and the beam is replaced by a boundary condition that allows a difference in velocity proportional to the difference in pressure on the two sides of the beam. In the classical acoustic theory, this difference in velocities can arise when the solid structure is made of porous materials (totally or in part); see [18, p. 142]. In that case, air can flow through the pores and thereby make the average velocity of the air on either side of the beam different from the velocity of the beam. In this section, we will show how this physical explanation will lead to our proposed boundary condition.

2.1. *Basic model for the acoustic waves and the flexible beam.* In describing the interaction between the acoustic wave and the flexible beam, the velocity field $\vec{\kappa}$ of the air mass as well as the air pressure variation in the acoustic cavity are important quantities to model. In the classical acoustic theory, by assuming the air flow is irrotational, i.e., the velocity field $\vec{\kappa}(t,x,y) = (\nu(t,x,y), \mu(t,x,y))$ satisfies

$$\frac{\partial}{\partial y}\nu(t,x,y) - \frac{\partial}{\partial x}\mu(t,x,y) = 0,$$

a scalar-valued function ϕ can be used to model the acoustic wave. More precisely, ϕ is referred to as the *velocity potential* function and it is defined by the equality

$$\nabla\phi(t,x,y) = \begin{pmatrix} \partial_x\phi(t,x,y) \\ \partial_y\phi(t,x,y) \end{pmatrix} = \vec{\kappa}(x,y). \tag{2.1}$$

It can easily be shown that ϕ satisfies a scalar wave equation

$$\partial_t^2\phi(t,x,y) - c^2\Delta\phi(t,x,y) = 0.$$

The function ϕ can also be used to evaluate the acoustic air pressure. In fact, the linearized equation of conservation of momentum (Euler equation) relates the air pressure and the velocity field by

$$\rho\frac{\partial}{\partial t}\vec{\kappa}(t,x,y) = -\nabla p(t,x,y), \tag{2.2}$$

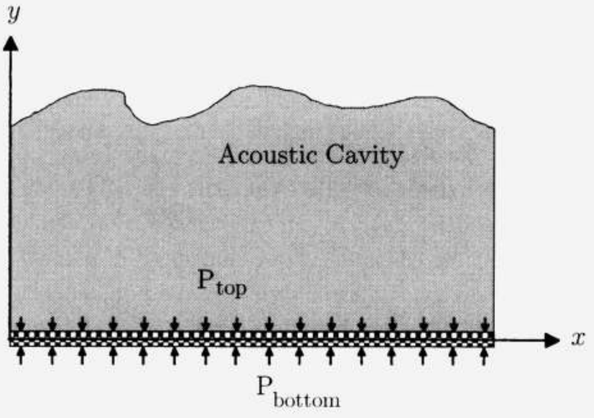

FIG. 2.1. Position of the air cavity and the flexible beam

where ρ is the air density and p is the pressure field. By combining (2.1)–(2.2), it is easy to see that

$$p(t, x, y) = -\rho \partial_t \phi(t, x, y).$$

The flexible beam that constitutes a portion of the boundary of the acoustic cavity is modeled by the Euler-Bernoulli equation:

$$\rho_b \partial_t^2 u(t, x) = -\partial_x^2 M(t, x) + f(t, x),$$

where $u(t, x)$ is the transversal displacement of the beam at position x, and ρ_b is the linear beam density. The term $M(t, x)$ represents the total bending moment of the beam which is given by

$$M(t, x) = EI \partial_x^2 u(t, x) + c_D I \partial_x^2 \partial_t u(t, x).$$

The function f represents the total external loading (force per unit length) on the beam. One can consider this term as the total pressure normalized to the width of the beam. From now on, we will use this notion of pressure in the subsequent discussions.

2.2. *Model of the acoustic-structure interaction.* The interaction between the acoustic field and the flexible beam is characterized by two equations: the force balance equation and the velocity equation. The force balance equation states that the total external loading on the beam is equal to the difference in air pressure on the two sides of the beam. The velocity equation establishes a relationship between the air velocity at the boundary of the acoustic cavity and the velocity of the beam. In the discussion of these two equations, it is important to define the positive direction of the transversal displacement of the beam. We consider the simple case depicted in Fig. 2.1 for our discussion here. The positive direction of the transversal beam deformation is the direction of the y-axis. As a result, the total external loading on the beam is given by

$$f(t, x) = p_{\text{bottom}}(t, x, 0) - p_{\text{top}}(t, x, 0),$$

where p_{bottom} and p_{top} are the pressure field on the bottom side of the beam and the pressure field in the acoustic cavity, respectively. As we have shown in the previous section $p_{\text{top}}(t,x,0) = -\rho\partial_t\phi(t,x,0)$. When we consider the dynamical system that consists of the acoustic field and the flexible beam, we first consider the homogeneous system and its stability. As a result, the dynamics of the flexible beam are given by

$$\rho_b\partial_t^2 u(t,x) = -\partial_x^2 M(t,x) + \rho\partial_t\phi(t,x,0). \tag{2.3}$$

In the usual model of acoustic-structure interaction, it is assumed that the normal components of the fluid velocity on opposite sides of the beam are equal to the transversal velocity of the beam. However, when the beam is made of the porous materials, air flow through the beam can affect the average velocity of the air, and thus the velocity of the beam may be different from the average air velocity on either side of the beam. This difference in velocities gives rise to a frictional force which can be expressed in terms of *flow resistance*. This quantity is the pressure drop required to force a unit flow through the force. In other words, the difference in the air pressure on both sides of the beam is proportional to the difference between the average air velocity and the velocity of the beam, and the constant of proportionality, R, is a measure of flow resistance, similar to the Ohm's law of electric resistance (see [18, p. 146]):

$$\partial_y\phi(t,x,0) - \partial_t u(t,x) = \frac{1}{R}(p_{\text{bottom}}(t,x,0) - p_{\text{top}}(t,x,0)).$$

When the homogeneous system of the acoustic cavity and the flexible beam is considered, the external forcing term p_{bottom} is equal to zero. As a result, we obtain

$$\partial_y\phi(t,x,0) - \partial_t u(t,x) = \alpha\partial_t\phi(t,x,0), \tag{2.4}$$

for a positive constant α. The interaction between the acoustic waves and the flexible beam is completely characterized by Eqs. (2.3) and (2.4). Naturally, when the orientation and the position of the beam are different, appropriate modifications should be made in these equations. We emphasize that the presented justification of the model for acoustic wave and structure interaction may only be appropriate in some special cases where the use of porous materials can be justified. Moreover, in cases where the physical explanation is reasonable, Eqs. (2.3) and (2.4) may provide only a crude approximation to the complex fluid-structure interactions in the physical system. Since it is widely believed in engineering practice that the coupled acoustic structure system is exponentially stable, and the proposed model has the desirable asymptotic properties, alternative physical justification for the above model is the focus of our on-going consultation with experts in acoustic physics. In the remainder of this paper we will show that the solutions of the model equations using (2.3) and (2.4) are uniformly exponentially stable.

3. Well-posedness of the model equations. Based on the discussions in the previous section, we consider the model for the dynamical system consisting of an acoustic cavity interacting with two flexible beams as depicted in Fig. 3.1. The particular geometry of the system is chosen so that the "star-shape" conditions (see [7], [12]) commonly required in the proof of uniform exponential stability are satisfied.

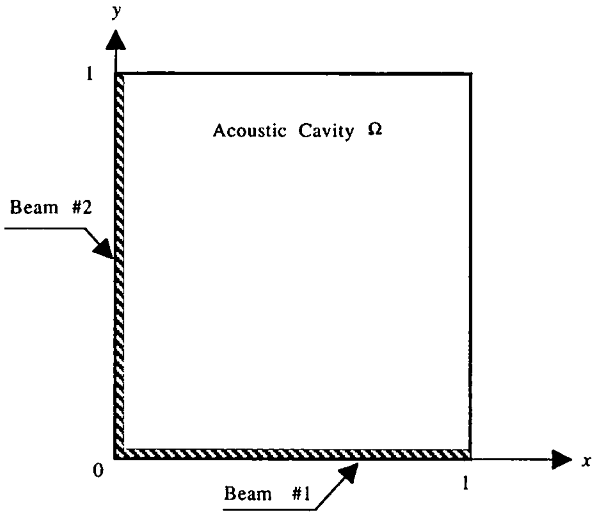

FIG. 3.1. Geometric configuration of the components of the system

Using Newtonian principles of the motion, the system is modeled by the following systems of partial differential equations:

$$\partial_t^2\phi(t,x,y) = \partial_x^2\phi(t,x,y) + \partial_y^2\phi(t,x,y), \quad (x,y) \in (0,1)\times(0,1) = \Omega, \tag{3.1}$$

$$\partial_t^2 u_1(t,x) = -\partial_x^4(Eu_1(t,x) + C\partial_t u_1(t,x)) + \partial_t\phi(t,x,0), \quad x \in (0,1), \tag{3.2}$$

$$\partial_t^2 u_2(t,y) = -\partial_y^4(Eu_2(t,y) + C\partial_t u_2(t,y)) + \partial_t\phi(t,0,y), \quad y \in (0,1), \tag{3.3}$$

where ϕ is the velocity potential function for the acoustic cavity as discussed in the previous section and u_1, u_2 are the transversal displacements of the beams numbered 1 and 2, respectively. Following the discussion in the previous section, the velocity transmission at the interface between the acoustic cavity and the beams is modeled by

$$\partial_y\phi(t,x,0) = \partial_t u_1(t,x) + \alpha\partial_t\phi(t,x,0), \quad x \in (0,1), \tag{3.4}$$

$$\partial_x\phi(t,0,y) = \partial_t u_2(t,y) + \alpha\partial_t\phi(t,0,y), \quad y \in (0,1). \tag{3.5}$$

The coefficient α represents the velocity loss factor. We note that if $\alpha = 0$, Eqs. (3.4) and (3.5) correspond to the continuity of the velocity at the boundaries. In this paper we consider the case of $\alpha > 0$. Additional boundary conditions are given by

$$\phi(t,x,1) = \phi(t,1,y) = 0, \quad x,y \in (0,1), \tag{3.6}$$

$$u_1(t,x) = \partial_x u_1(t,x) = 0, \quad x = 0,1, \tag{3.7}$$

$$u_2(t,y) = \partial_y u_2(t,y) = 0, \quad y = 0,1. \tag{3.8}$$

The results in this paper will still be valid if the boundary condition in (3.6) is changed to the homogeneous Neumann boundary, although minor changes in the norms of the functional spaces may be required.

In this section, we shall establish the well-posedness of the above model equations, that is, the existence and the uniqueness of the solutions of these equations for a given set of initial values. Our approach is based on a variational formulation of the above dynamical system and the theory of the semigroups of bounded linear operators in Hilbert spaces (see for example, [22], [20], [11], [4], [17]). The proposed variational formulation will also be useful in the discussion of the regularity of the solutions in Sec. 4.

The motivation for the proposed variational formulation is by the following observations. Suppose that ϕ, u_1, and u_2 form a classical solution of the model equations (3.1)–(3.8), i.e., these functions are continuously differentiable up to all necessary order and Eqs. (3.1)–(3.8) are satisfied. We can define additional components of the solutions by

$$\theta(t,x,y) = \partial_t\phi(t,x,y), \quad v_1(t,x) = \partial_t u_1(t,x), \quad v_2(t,y) = \partial_t u_2(t,y).$$

Now consider any smooth functions $\tilde{\phi}, \tilde{\theta}$ defined on Ω and arbitrary smooth functions $\tilde{u}_1, \tilde{v}_1, \tilde{u}_2$, and $\tilde{v}_2$ defined on $(0,1)$ such that

$$\tilde{\phi}(x,1) = \tilde{\phi}(1,y) = \tilde{\theta}(x,1) = \tilde{\theta}(1,y) = 0, \quad x,y \in (0,1),$$

and

$$\tilde{u}_i(s) = \tilde{u}_i'(s) = \tilde{v}_i(s) = \tilde{v}_i'(s) = 0, \quad s = 0,1, \quad i = 1,2.$$

We define a function $\lambda(t)$ by

$$\begin{aligned}\lambda(t) = &\int_\Omega (\nabla\phi(t,x,y)\cdot\nabla\tilde{\phi}(x,y) + \theta(t,x,y)\tilde{\theta}(x,y))\,dx\,dy \\ &+ \int_0^1 (E\partial_x^2 u_1(t,x)\tilde{u}_1''(x) + v_1(t,x)\tilde{v}_1(x))\,dx \\ &+ \int_0^1 (E\partial_y^2 u_2(t,x)\tilde{u}_2''(y) + v_2(t,y)\tilde{v}_2(y))\,dy,\end{aligned} \tag{3.9}$$

where ∇ is the gradient operator in the variables x and y. Through the use of integration by parts and the boundary conditions of the functions defined above, it is easy to establish the following equality:

$$\begin{aligned}\frac{d}{dt}\lambda(t) = &\sigma_a(\theta(t,\cdot,\cdot),\tilde{\phi}) - \sigma_a(\phi(t,\cdot,\cdot),\tilde{\theta}) \\ &- \rho_1(v_1(t,\cdot),\tilde{\theta}) - \rho_2(v_2(t,\cdot),\tilde{\theta}) - \gamma_a(\theta(t,\cdot,\cdot),\tilde{\theta}) \\ &+ \sigma_b(v_1(t,\cdot),\tilde{u}_1) - \sigma_b(u_1(t,\cdot),\tilde{v}_1) + \rho_1(\tilde{v}_1,\theta(t,\cdot,\cdot)) - \gamma_b(v_1(t,\cdot),\tilde{v}_1) \\ &+ \sigma_b(v_2(t,\cdot),\tilde{u}_2) - \sigma_b(u_2(t,\cdot),\tilde{v}_2) + \rho_2(\tilde{v}_2,\theta(t,\cdot,\cdot)) - \gamma_b(v_2(t,\cdot),\tilde{v}_2),\end{aligned} \tag{3.10}$$

where the sesquilinear forms are defined by

$$\sigma_a(\hat{\phi},\tilde{\phi}) = \int_\Omega \nabla\hat{\phi}(x,y)\cdot\nabla\tilde{\phi}(x,y)\,dx\,dy,$$

$$\gamma_a(\hat{\phi},\tilde{\phi}) = \int_0^1 \alpha\hat{\phi}(x,0)\tilde{\phi}(x,0)dx + \int_0^1 \alpha\hat{\phi}(0,y)\tilde{\phi}(0,y)\,dy,$$

$$\sigma_b(\hat{u},\tilde{u}) = \int_0^1 E\hat{u}''(s)\tilde{u}''(s)\,ds,$$

$$\gamma_b(\hat{u},\tilde{u}) = \int_0^1 C\hat{u}''(s)\tilde{u}''(s)\,ds,$$

$$\rho_1(\tilde{u},\tilde{\phi}) = \int_0^1 \tilde{u}(x)\tilde{\phi}(x,0)\,dx,$$

$$\rho_2(\tilde{u},\tilde{\phi}) = \int_0^1 \tilde{u}(y)\tilde{\phi}(0,y)\,dy.$$

The variational formulation of the model equations consists of generalizing the notion of the solution for the model equations such that any function $(\phi,\theta,u_1,v_1,u_2,v_2)$ satisfying (3.10) for a selected class of functions $(\tilde{\phi},\tilde{\theta},\tilde{u}_1,\tilde{v}_1,\tilde{u}_2,\tilde{v}_2)$ is called a "weak" solution of the model equation. Moreover, the new formulation of the model allows us to view the model equations as an abstract evolution equation of the form

$$\frac{d}{dt}w(t) = \mathcal{A}w(t),$$

is a Hilbert space $\mathcal{H}$, where $\mathcal{A}$ is typically an unbounded linear operator in $\mathcal{H}$. In the discussions below, we shall specify the definition of the space $\mathcal{H}$ and the operator $\mathcal{A}$.

We define the Hilbert spaces $H_a = L^2(\Omega)$, $H_b = L^2(0,1)$ with the usual L^2-inner product and the Hilbert spaces V_a and V_b by

$$V_a = \{\phi \in H^1(\Omega)\colon \phi(x,1) = \phi(1,y) = 0, \forall x,y \in (0,1)\},$$
$$V_b = \{u \in H^2(0,1)\colon u(s) = u'(s) = 0, s = 0,1\},$$

with inner products defined by

$$\langle \phi,\tilde{\phi}\rangle_{V_a} = \int_\Omega \nabla\phi(x,y)\cdot\nabla\tilde{\phi}(x,y)\,dx\,dy, \qquad \langle u,\tilde{u}\rangle_{V_b} = \int_0^1 Eu''(s)\tilde{u}''(s)\,ds.$$

Let $\mathcal{H} = V_a\times H_a\times V_b\times H_b\times V_b\times H_b$ and let $\mathcal{V} = V_a\times V_a\times V_b\times V_b\times V_b\times V_b$. We define a sesquilinear form in $\mathcal{V}\times\mathcal{V}$ by

$$\begin{aligned}&\sigma((\phi,\theta,u_1,v_1,u_2,v_2),(\tilde{\phi},\tilde{\theta},\tilde{u}_1,\tilde{v}_1,\tilde{u}_2,\tilde{v}_2))\\ &\quad= \sigma_a(\theta,\tilde{\phi}) - \sigma_a(\phi,\tilde{\theta}) - \rho_1(v_1,\tilde{\theta}) - \rho_2(v_2,\tilde{\theta}) - \gamma_a(\theta,\tilde{\theta})\\ &\qquad+ \sigma_b(v_1,\tilde{u}_1) - \sigma_b(u_1,\tilde{v}_1) + \rho_1(\tilde{v}_1,\theta) - \gamma_b(v_1,\tilde{v}_1)\\ &\qquad+ \sigma_b(v_2,\tilde{u}_2) - \sigma_b(u_2,\tilde{v}_2) + \rho_2(\tilde{v}_2,\theta) - \gamma_b(v_2,\tilde{v}_2).\end{aligned}$$

Using these definitions, it is not difficult to see that

$$\lambda(t) = \langle(\phi(t),\theta(t),u_1(t),v_1(t),u_2(t),v_2(t)),(\tilde{\phi},\tilde{\theta},\tilde{u}_1,\tilde{v}_1,\tilde{u}_2,\tilde{v}_2)\rangle_{\mathcal{H}},$$

where the classical solution $(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t))$ is considered as an $\mathcal{H}$-valued function of t. Moreover, a classical solution of the model equation also satisfies

$$\begin{aligned} &\frac{d}{dt}\langle(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)\rangle_{\mathcal{H}} \\ &\qquad = \sigma((\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)), \end{aligned} \tag{3.11}$$

for all $(\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2) \in \mathcal{V}$. As a result, any $\mathcal{H}$-valued function satisfying (3.11) for all $(\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2) \in \mathcal{V}$ is called a weak solution of the model equations (3.1)–(3.6). In the remainder of this section, we shall show that Eq. (3.11) has a unique solution for any given initial value in $\mathcal{H}$.

We first define an unbounded linear operator $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \mapsto \mathcal{H}$ as follows:

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{r} (\phi, \theta, u_1, v_1, u_2, v_2) \in \mathcal{V}\colon \exists(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2) \in \mathcal{H}, \text{ such that} \\ \sigma((\phi, \theta, u_1, v_1, u_2, v_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)) \\ = \langle(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)\rangle_{\mathcal{H}}, \\ \text{for all } (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2) \in \mathcal{V}. \end{array} \right\},$$

and for all $(\phi, \theta, u_1, v_1, u_2, v_2) \in \mathcal{D}(\mathcal{A})$,

$$\mathcal{A}(\phi, \theta, u_1, v_1, u_2, v_2) = (\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2),$$

where $(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2)$ satisfies

$$\sigma((\phi, \theta, u_1, v_1, u_2, v_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)) = \langle(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)\rangle_{\mathcal{H}},$$

for all $(\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2) \in \mathcal{V}$.

The main result of this section is the following theorem.

THEOREM 3.1. The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions denoted by $e^{\mathcal{A}t}$ in $\mathcal{H}$ and for any $(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0}) \in \mathcal{D}(\mathcal{A})$, the function

$$(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)) = e^{\mathcal{A}t}(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0})$$

is the unique weak solution of (3.11) with initial value $(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0})$.

The proof of the above theorem is based on the Lumer-Phillips theorem. The verification of the conditions in the Lumer-Phillips theorem is established through the following series of technical lemmas.

LEMMA 3.1. The operator $\mathcal{A}$ is dissipative in $\mathcal{H}$.

Proof. Let $(\phi, \theta, u_1, v_1, u_2, v_2) \in \mathcal{D}(\mathcal{A})$. Then

$$\begin{aligned} &\langle\mathcal{A}(\phi, \theta, u_1, v_1, u_2, v_2), (\phi, \theta, u_1, v_1, u_2, v_2)\rangle_{\mathcal{H}} \\ &\qquad = \sigma((\phi, \theta, u_1, v_1, u_2, v_2), (\phi, \theta, u_1, v_1, u_2, v_2)) \\ &\qquad = -\gamma_a(\theta, \theta) - \gamma_b(v_1, v_1) - \gamma_b(v_2, v_2). \end{aligned}$$

Since γ_a and γ_b are positive definite, we have

$$\langle\mathcal{A}(\phi, \theta, u_1, v_1, u_2, v_2), (\phi, \theta, u_1, v_1, u_2, v_2)\rangle_{\mathcal{H}} \leq 0,$$

for all $(\phi, \theta, u_1, v_1, u_2, v_2) \in \mathcal{D}(\mathcal{A})$. □

LEMMA 3.2. For all real numbers $\lambda > 0$, the operator $\mathcal{A} - \lambda I\colon \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \mapsto \mathcal{H}$ is onto.

Proof. Consider an arbitrary element $(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2) \in \mathcal{H}$. If there exists $(\phi, \theta, u_1, v_1, u_2, v_2) \in \mathcal{D}(\mathcal{A})$ such that

$$(\mathcal{A} - \lambda I)(\phi, \theta, u_1, v_1, u_2, v_2) = (\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2),$$

then

$$\mathcal{A}(\phi, \theta, u_1, v_1, u_2, v_2) = (\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2) + \lambda(\phi, \theta, u_1, v_1, u_2, v_2).$$

Using the definition of the operator $\mathcal{A}$, this implies

$$\begin{aligned}&\sigma((\phi, \theta, u_1, v_1, u_2, v_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2))\\ &\quad = \langle(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2) + \lambda(\phi, \theta, u_1, v_1, u_2, v_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)\rangle_{\mathcal{H}},\end{aligned}$$

for all $(\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2) \in \mathcal{V}$. By taking $\tilde{\theta} = 0$ and $\tilde{v}_1 = \tilde{v}_2 = 0$, the above equality implies

$$\hat{\phi} + \lambda\phi = \theta, \quad \hat{u}_1 + \lambda u_1 = v_1, \quad \hat{u}_2 + \lambda u_2 = v_2.$$

Now let $\tilde{\phi} = 0$ and $\tilde{u}_1 = \tilde{u}_2 = 0$ and let $\tilde{\theta}, \tilde{v}_1, \tilde{v}_2$ be arbitrary elements in V_a and V_b, respectively. We have

$$\begin{aligned}&\sigma((\phi, \theta, u_1, v_1, u_2, v_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2))\\ &\quad = -\sigma_a(\phi, \tilde{\theta}) - \rho_1(\hat{u}_1 + \lambda u_1, \tilde{\theta}) - \rho_2(\hat{u}_2 + \lambda u_2, \tilde{\theta}) - \gamma_a(\hat{\phi} + \lambda\phi, \tilde{\theta})\\ &\qquad - \sigma_b(u_1, \tilde{v}_1) + \rho_1(\tilde{v}_1, \hat{\phi} + \lambda\phi) - \gamma_b(\hat{u}_1 + \lambda u_1, \tilde{v}_1)\\ &\qquad - \sigma_b(u_2, \tilde{v}_2) + \rho_2(\tilde{v}_2, \hat{\phi} + \lambda\phi) - \gamma_b(\hat{u}_2 + \lambda u_2, \tilde{v}_2)\\ &\quad = \langle\hat{\theta} + \lambda\hat{\phi} + \lambda^2\phi, \tilde{\theta}\rangle_{H_a} + \langle\hat{v}_1 + \lambda\hat{u}_1 + \lambda^2 u_1, \tilde{v}_1\rangle_{H_b} + \langle\hat{v}_2 + \lambda\hat{u}_2 + \lambda^2 u_2, \tilde{v}_2\rangle_{H_b}.\end{aligned}$$

Since the vector $(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2)$ is a given vector in $\mathcal{H}$, we need to show that there exist vectors $\phi \in V_a$ and $u_1, u_2 \in V_b$ such that the above equation holds for all $\tilde{\theta} \in V_a$ and $\tilde{v}_1, \tilde{v}_2 \in V_b$. By rearranging the terms, we have

$$\begin{aligned}&\sigma_a(\phi, \tilde{\theta}) + \lambda\rho_1(u_1, \tilde{\theta}) + \lambda\rho_2(u_2, \tilde{\theta}) + \lambda\gamma_a(\phi, \tilde{\theta}) + \lambda^2\langle\phi, \tilde{\theta}\rangle_{H_a}\\ &\quad + \sigma_b(u_1, \tilde{v}_1) - \lambda\rho_1(\tilde{v}_1, \phi) + \lambda\gamma_b(u_1, \tilde{v}_1) + \lambda^2\langle u_1, \tilde{v}_1\rangle_{H_b}\\ &\quad + \sigma_b(u_2, \tilde{v}_2) - \lambda\rho_2(\tilde{v}_2, \phi) + \lambda\gamma_b(u_2, \tilde{v}_2) + \lambda^2\langle u_2, \tilde{v}_2\rangle_{H_b}\\ &= -\langle\hat{\theta} + \lambda\hat{\phi}, \tilde{\theta}\rangle_{H_a} - \rho_1(\hat{u}_1, \tilde{\theta}) - \rho_2(\hat{u}_2, \tilde{\theta}) - \gamma_a(\hat{\phi}, \tilde{\theta})\\ &\quad - \langle\hat{v}_1 + \lambda\hat{u}_1, \tilde{v}_1\rangle_{H_b} + \rho(\tilde{v}_1, \hat{\phi}) - \gamma_b(\hat{u}_1, \tilde{v}_1)\\ &\quad - \langle\hat{v}_2 + \lambda\hat{u}_2, \tilde{v}_2\rangle_{H_b} + \rho_2(\tilde{v}_2, \hat{\phi}) - \gamma_b(\hat{u}_2, \tilde{v}_2).\end{aligned} \tag{3.12}$$

Consider the space $\mathcal{U} = V_a \times V_b \times V_b$. We define a sesquilinear form τ on $\mathcal{U} \times \mathcal{U}$ by

$$\begin{aligned}&\tau((\phi, u_1, u_2), (\tilde{\phi}, \tilde{u}_1, \tilde{u}_2))\\ &\quad = \sigma_a(\phi, \tilde{\phi}) + \lambda\rho_1(u_1, \tilde{\phi}) + \lambda\rho_2(u_2, \tilde{\phi}) + \lambda\gamma_a(\phi, \tilde{\phi}) + \lambda^2\langle\phi, \tilde{\phi}\rangle_{H_a}\\ &\qquad + \sigma_b(u_1, \tilde{u}_1) - \lambda\rho_1(\tilde{u}_1, \phi) + \lambda\gamma_b(u_1, \tilde{u}_1) + \lambda^2\langle u_1, \tilde{u}_1\rangle_{H_b}\\ &\qquad + \sigma_b(u_2, \tilde{u}_2) - \lambda\rho_2(\tilde{u}_2, \phi) + \lambda\gamma_b(u_2, \tilde{u}_2) + \lambda^2\langle u_2, \tilde{u}_2\rangle_{H_b}.\end{aligned}$$

Similarly, we can define the continuous linear functional f on $\mathcal{U}$ by

$$\begin{aligned} f(\phi, u_1, u_2) = &-\langle \hat{\theta} + \lambda\hat{\phi}, \phi\rangle_{H_a} - \rho_1(\hat{u}_1, \phi) - \rho_2(\hat{u}_2, \phi) - \gamma_a(\hat{\phi}, \phi) \\ &- \langle \hat{v}_1 + \lambda\hat{u}_1, u_1\rangle_{H_b} + \rho_1(u_1, \hat{\phi}) - \gamma_b(\hat{u}_1, u_1) \\ &- \langle \hat{v}_2 + \lambda\hat{u}_2, u_2\rangle_{H_b} + \rho_2(u_2, \hat{\phi}) - \gamma_b(\hat{u}_2, u_2). \end{aligned}$$

Let $\mathcal{Z} = H_a \times H_b \times H_b$. It is easy to see that $\mathcal{U}$ is densely continuously embedded in $\mathcal{Z}$. Moreover, we have

$$\begin{aligned} |\tau((\phi, u_1, u_2), (\tilde{\phi}, \tilde{u}_1, \tilde{u}_2))| &\le c_0\|(\phi, u_1, u_2)\|_{\mathcal{U}} \cdot \|(\tilde{\phi}, \tilde{u}_1, \tilde{u}_2)\|_{\mathcal{U}}, \\ \tau((\phi, u_1, u_2)(\phi, u_1, u_2)) &\ge c_1\|(\phi, u_1, u_2)\|_{\mathcal{U}}^2, \end{aligned}$$

for some positive constants c_0, c_1. By the Lax-Milgram theorem [23, p. 82] (also see [11, pp. 98–105]), for any element f in the dual space of $\mathcal{U}$, there exists a unique element $(\phi, u_1, u_2) \in \mathcal{U}$ such that

$$f(\tilde{\phi}, \tilde{u}_1, \tilde{u}_2) = \tau((\phi, u_1, u_2), (\tilde{\phi}, \tilde{u}_1, \tilde{u}_2)),$$

for all $(\tilde{\phi}, \tilde{u}_1, \tilde{u}_2) \in \mathcal{U}$. As a result, by combining the above results, we conclude that there exists a unique element $(\phi, \theta, u_1, v_1, u_2, v_2) \in \mathcal{V}$ such that

$$\begin{aligned} &\sigma((\phi, \theta, u_1, v_1, u_2, v_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)) \\ &\quad = \langle(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2) + \lambda(\phi, \theta, u_1, v_1, u_2, v_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)\rangle_{\mathcal{H}}, \end{aligned}$$

for all $(\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2) \in \mathcal{V}$ and $\lambda \ge 0$. Hence, $\mathcal{A} - \lambda I$ is onto for all $\lambda \ge 0$. □

Proof of Theorem 3.1. Using Lemmas 3.1–3.2, we conclude that the operator $\mathcal{A}$ is a dissipative linear operator and for all $\lambda \ge 0$ the operator $\mathcal{A} - \lambda I$ is onto. By the Lumer-Phillips theorem [17, p. 16, Theorem 4.6] $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contraction operators on $\mathcal{H}$. Moreover, if $(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0}) \in \mathcal{D}(\mathcal{A})$, the function

$$(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)) = e^{\mathcal{A}t}(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0})$$

is strongly differentiable and

$$\frac{d}{dt}(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)) = \mathcal{A}(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)).$$

Using the continuity of the $\mathcal{H}$-inner product, we obtain that $(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t))$ is the unique weak solution of the model equations. □

4. Regularity of the weak solutions. In this section, we consider the regularity of the weak solution of the model equations given by

$$(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)) = e^{\mathcal{A}t}(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0}),$$

for some initial vector $(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0}) \in \mathcal{D}(\mathcal{A})$. Our definition of the operator $\mathcal{A}$ offers very little explicit statement on the smoothness of elements in $\mathcal{D}(\mathcal{A})$. Therefore, the results in the previous section give very little explicit information on the regularity of the weak solutions of the model equation. On the other hand, the proof of the exponential stability of the semigroup $e^{\mathcal{A}t}$ in Sec. 5 requires specific regularity results for the weak solutions. In this section, we will investigate the smoothness of the elements in $\mathcal{D}(\mathcal{A})$ and a few related issues.

LEMMA 4.1. The following properties hold for any element $(\phi, \theta, u_1, v_1, u_2, v_2) \in \mathcal{D}(\mathcal{A})$:
(i) $D^4(Eu_i + Cv_i) \in L^2(0,1)$, for $i = 1, 2$.
(ii) $\Delta\phi \in L^2(\Omega)$ and the trace of the function $\partial_y \phi(x, y)$ exists on the boundary $y = 0$ as a function in $H^{1/2}(0,1)$. Similarly, the trace of the function $\partial_x \phi(x, y)$ exists on the boundary $x = 0$ as a function in $H^{1/2}(0,1)$. Moreover, we have

$$\partial_y \phi(x, 0) = v_1(x) + \alpha\theta(x, 0), \quad \partial_x \phi(0, y) = v_2(y) + \alpha\theta(0, y).$$

It may be necessary for us to be more precise about the use of the notation Δ and the term trace of $\partial_y \phi(x, y)$. The following definition states the exact meaning of these terms in this paper.

DEFINITION 4.1. Consider a vector $\phi \in V_a$. We say that $\Delta\phi \in L^2(\Omega)$ if and only if there exists a function $\eta \in L^2(\Omega)$ such that

$$\int_\Omega \phi(x, y)(\partial_x^2 \xi(x, y) + \partial_y^2 \xi(x, y))\, dx\, dy = \int_\Omega \eta(x, y)\xi(x, y)\, dx\, dy,$$

for all $\xi \in C_0^\infty(\Omega)$. In this case, we also say $\Delta\phi = \eta$. Moreover, if $\Delta\phi \in L^2(\Omega)$ and there exist functions $\mu_1, \mu_2 \in H^{1/2}(0,1)$ such that

$$\int_\Omega \nabla\phi(x, y) \cdot \nabla\xi(x, y)\, dx\, dy + \int_\Omega \Delta\phi(x, y)\xi(x, y)\, dx\, dy$$
$$= -\int_0^1 \mu_1(x)\xi(x, 0) dx - \int_0^1 \mu_2(y)\xi(0, y) dy,$$

for all $\xi \in V_a$, then μ_1 and μ_2 are called the trace of $\partial_y \phi$ and $\partial_x \phi$ on the boundary $y = 0$ and $x = 0$, respectively.

We note that if $\phi \in H^2(\Omega)$, our definition above is simply the usual definition of the Laplace operator in the distributional sense and the trace operator in the usual sense [19, pp. 34–36].

Proof of Lemma 4.1. Let $(\phi, \theta, u_1, v_1, u_2, v_2) \in \mathcal{D}(\mathcal{A})$. Based on the definition of $\mathcal{D}(\mathcal{A})$, there exists $(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2) \in \mathcal{H}$ such that

$$\sigma((\phi, \theta, u_1, v_1, u_2, v_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)) = \langle(\hat{\phi}, \hat{\theta}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2), (\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)\rangle_{\mathcal{H}},$$

for all $(\tilde{\phi}, \tilde{\theta}, \tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2) \in \mathcal{V}$. By taking $\tilde{\phi} = 0$ and $\tilde{u}_1 = \tilde{u}_2 = 0$ and by taking $\tilde{\theta} \in C_0^\infty(\Omega)$ and $\tilde{v}_1, \tilde{v}_2 \in C_0^\infty(0,1)$, we obtain

$$\int_\Omega \hat{\theta}(x, y)\tilde{\theta}(x, y)\, dx\, dy = \int \nabla\phi(x, y) \cdot \nabla\tilde{\theta}(x, y)\, dx\, dy,$$
$$\int_0^1 \hat{v}_1(x)\tilde{v}_1(x)\, dx = \int_0^1 \theta(x, 0)\tilde{v}_1(x)\, dx - \int_0^1 (Eu_1''(x) + Cv_1''(x))\tilde{v}_1''(x)\, dx,$$
$$\int_0^1 \hat{v}_2(y)\tilde{v}_2(y)\, dy = \int_0^1 \theta(0, y)\tilde{v}_2(y)\, dy - \int_0^1 (Eu_2''(y) + Cv_2''(y))\tilde{v}_2''(y)\, dy.$$

By integration by parts on the right-hand sides of the above equations and using the fact that $\hat{\theta} \in L^2(\Omega)$, $\theta \in V_a$ and $\hat{v}_1, \hat{v}_2 \in L^2(0,1)$, and the fact that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$

we conclude that $\Delta\phi \in L^2(\Omega)$, $D^4(Eu_1+Cv_1) \in L^2(0,1)$, and $D^4(Eu_2+Cv_2) \in L^2(0,1)$. Then, by taking $\tilde{\theta} \in V_a$ and $\tilde{v}_1 = \tilde{v}_2 = 0$, we obtain

$$\int_\Omega \nabla\phi(x,y)\nabla\tilde{\theta}(x,y)\,dx\,dy + \int_\Omega \hat{\theta}(x,y)\tilde{\theta}(x,y)\,dx\,dy$$
$$= -\int_0^1 (v_1(x) + \alpha\theta(x,0))\tilde{\theta}(x,0)\,dx - \int_0^1 (v_2(y) + \alpha\theta(0,y))\tilde{\theta}(0,y)\,dy.$$

Using the fact that $\hat{\theta} = \Delta\phi$, we conclude that

$$\partial_y\phi(x,0) = v_1(x) + \alpha\theta(x,0), \quad \partial_x\phi(0,y) = v_2(y) + \alpha\theta(0,y).$$

Since $v_1, v_2 \in H^2(0,1)$ and $\theta \in V_a$, and therefore, $\theta(\cdot,0) \in H^{1/2}(0,1)$ and $\theta(0,\cdot) \in H^{1/2}(0,1)$, we have $\partial_y\phi(\cdot,0) \in H^{1/2}(0,1)$ and $\partial_x\phi(0,\cdot) \in H^{1/2}(0,1)$. □

In the discussion of the exponential decay of the solutions of the model equations, the regularity of the first component ϕ of the solution is an important issue. In fact, this is a commonly known problem for the solution of the wave equation on a bounded domain with mixed boundary conditions and non-smooth boundary [9], [10]. The main problem is that the function $\phi(t,\cdot,\cdot)$ is not necessarily a function in $H^2(\Omega)$. However, our regularity results in this section will show that $\phi(t,\cdot,\cdot)$ has many useful properties of an $H^2(\Omega)$ function. In the remainder of this section, we will focus our attention on the approximation of $\phi(t,\cdot,\cdot)$ with a sequence of $H^2(\Omega)$ functions in a special way in order to allow us to obtain the desired estimate despite the lack of regularity of the function $\phi(t,\cdot,\cdot)$.

Consider the following subset of V_a defined by

$$U_a = \{\phi \in V_a, \Delta\phi \in L^2(\Omega), \partial_y\phi(\cdot,0) \in H^{1/2}(0,1), \partial_x\phi(0,\cdot) \in H^{1/2}(0,1)\},$$

with the inner product defined by

$$\langle\phi,\tilde{\phi}\rangle_{U_a} = \int_\Omega \nabla\phi(x,y)\nabla\tilde{\phi}(x,y)\,dx\,dy + \int_\Omega \Delta\phi(x,y)\Delta\tilde{\phi}(x,y)\,dx\,dy$$
$$+ \langle\partial_y\phi(\cdot,0), \partial_y\phi(\cdot,0)\rangle_{H^{1/2}(0,1)} + \langle\partial_x\phi(0,\cdot), \partial_x\phi(0,\cdot)\rangle_{H^{1/2}(0,1)}.$$

It is easy to see that the first component ϕ of any vector $(\phi,\theta,u_1,v_1,u_2,v_2) \in \mathcal{D}(\mathcal{A})$ belongs to U_a. Also, U_a is clearly a pre-Hilbert space.

LEMMA 4.2. The space U_a is a Hilbert space.

Proof. Consider any Cauchy sequence $\{\phi_n\}$ in U_a. By the completeness of V_a, $L^2(\Omega)$ ad $H^{1/2}(\Omega)$, there exist functions $\phi \in V_a$, $\eta \in L^2(\Omega)$, and $\xi, \zeta \in H^{1/2}(0,1)$ such that

$$\lim_{n\to\infty} \phi_n = \phi, \text{ in } V_a, \qquad \lim_{n\to\infty} \Delta\phi_n = \eta, \text{ in } L^2(\Omega),$$
$$\lim_{n\to\infty} \partial_y\phi_n(\cdot,0) = \xi, \text{ in } H^{1/2}(0,1), \quad \lim_{n\to\infty} \partial_x\phi_n(0,\cdot) = \zeta, \text{ in } H^{1/2}(0,1).$$

We need to show that $\Delta\phi = \eta$ and $\partial_y\phi(\cdot,0) = \xi$, $\partial_x\phi(0,\cdot) = \zeta$. Let $\theta \in C_0^\infty(\Omega)$. We have

$$\int_\Omega \phi_n(x,y)\Delta\theta(x,y)\,dx\,dy = \int_\Omega \Delta\phi_n(x,y)\theta(x,y)\,dx\,dy,$$

for all n. By continuity of the $L^2(\Omega)$-inner product, we have

$$\int_\Omega \phi(x,y)\Delta\theta(x,y)\,dx\,dy = \int_\Omega \eta(x,y)\theta(x,y)\,dx\,dy.$$

Moreover, for all $\theta \in V_a$ we have

$$\int_\Omega \nabla\phi_n(x,y)\cdot\nabla\theta(x,y)\,dx\,dy + \int_\Omega \Delta\phi_n(x,y)\theta(x,y)\,dx\,dy \\ = -\int_0^1 \partial_y\phi_n(x,0)\theta(x,0)\,dx - \int_0^1 \partial_x\phi_n(0,y)\theta(0,y)\,dy.$$

By taking the limit as n tends to infinity, we obtain

$$\int_\Omega \nabla\phi(x,y)\cdot\nabla\theta(x,y)\,dx\,dy + \int_\Omega \Delta\phi(x,y)\theta(x,y)\,dx\,dy \\ = -\int_0^1 \partial_y\phi(x,0)\theta(x,0)\,dx - \int_0^1 \partial_x\phi(0,y)\theta(0,y)\,dy.$$

Using Definition 4.1, we have

$$\Delta\phi = \eta, \quad \partial_y\phi(\cdot,0) = \xi, \quad \partial_x\phi(0,\cdot) = \zeta. \quad \square$$

It is also clear that $H^2(\Omega)\cap V_a$ is a subset of U_a. We will show that every element of U_a can be approximated by a sequence in $H^2(\Omega)\cap V_a$ in U_a.

LEMMA 4.3. The subspace $H^2(\Omega)\cap V_a$ is a dense subset of U_a.

Proof. Let $\phi \in U_a$ be different from zero such that $\langle\phi,\tilde\phi\rangle_{U_a} = 0$ for all $\tilde\phi \in H^2(\Omega)\cap V_a$. By integration by parts, we have

$$0 = \langle\Delta\phi, -\tilde\phi + \Delta\tilde\phi\rangle_{L^2(\Omega)} - \int_0^1 \partial_y\phi(x,0)\tilde\phi(x,0)\,dx - \int_0^1 \partial_x\phi(0,y)\tilde\phi(0,y)\,dy \\ + \langle\partial_y\phi(\cdot,0), \partial_y\tilde\phi(\cdot,0)\rangle_{H^{1/2}(0,1)} + \langle\partial_x\phi(0,\cdot), \partial_x\tilde\phi(0,\cdot)\rangle_{H^{1/2}(0,1)}.$$

By taking $\tilde\phi \in C_0^\infty(\Omega)$ and using the fact that the subset $N = \{-\tilde\phi + \Delta\tilde\phi \colon \tilde\phi \in C_0^\infty(\Omega)\}$ is dense in $L^2(\Omega)$, we conclude that $\Delta\phi = 0$. On the other hand, consider the subset

$$M = \{\tilde\phi(\cdot,0), \tilde\phi(0,\cdot), \partial_y\tilde\phi(\cdot,0), \partial_x\tilde\phi(0,\cdot) \colon \tilde\phi \in H^2(\Omega)\cap V_a\}$$

of $H^{1/2}(0,1)\times H^{1/2}(0,1)\times H^{1/2}(0,1)\times H^{1/2}(0,1)$. It is easy to see [9, p. 42, Theorem 1.5.2.1] that M is dense in $H^{1/2}(0,1)\times H^{1/2}(0,1)\times H^{1/2}(0,1)\times H^{1/2}(0,1)$. Therefore, we have $\partial_y\phi(\cdot,0) = \partial_x\phi(0,\cdot) = 0$. Using the fact that the equation

$$\Delta\phi(x,y) = 0, \quad (x,y)\in\Omega, \quad \partial_y\phi(x,0) = \partial_x\phi(0,y) = 0, \quad x,y \in (0,1),$$

has a unique solution in V_a, we conclude that $\phi = 0$. $\square$

The above lemma implies that for any element $\phi \in U_a$ there exists a sequence of functions $\{\phi_n\} \subset H^2(\Omega) \cap V_a$ such that

$$\begin{aligned}
\lim_{n\to\infty} \phi_n &= \phi, \text{ in } V_a, \\
\lim_{n\to\infty} \Delta\phi_n &= \Delta\phi, \text{ in } L^2(\Omega), \\
\lim_{n\to\infty} \partial_y\phi_n(\cdot,0) &= \partial_y\phi(\cdot,0), \text{ in } H^{1/2}(0,1), \\
\lim_{n\to\infty} \partial_x\phi_n(0,\cdot) &= \partial_x\phi(0,\cdot), \text{ in } H^{1/2}(0,1).
\end{aligned}$$

5. Uniform exponential stability of the solutions of the model equations. Finally, in this section we shall show that there exist constants $M \geq 0$ and $\omega > 0$ such that

$$\|e^{\mathcal{A}t}(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0})\|_{\mathcal{H}} \leq Me^{-\omega t}\|(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0})\|_{\mathcal{H}}, \tag{5.1}$$

for all $(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0}) \in \mathcal{H}$. Our approach is based on the use of a multiplier functional $Q(t)$ defined by

$$\begin{aligned}
Q(t) = \frac{t}{2}&\|(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t))\|^2_{\mathcal{H}} \\
&+ \int_0^1 u_1(t,x)v_1(t,x)\,dx + \int_0^1 u_2(t,y)v_2(t,y)\,dy \\
&+ \int_\Omega [2(x-1)\partial_x\phi(t,x,y) + 2(y-1)\partial_y\phi(t,x,y) + \phi(t,x,y)]\theta(t,x,y)\,dx\,dy.
\end{aligned}$$

It is easy to see that there exists a constant T such that for all $t \geq T$,

$$\frac{t}{4}E(t) \leq Q(t) \leq tE(t),$$

where

$$E(t) = \|(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t))\|^2_{\mathcal{H}}.$$

In the remainder of this section, we will show that there exists a constant T' such that for all $t \geq T'$, $Q(t) \leq Q(T')$. Therefore, for all $t \geq \max(T, T')$, we have

$$\frac{t}{4}E(t) \leq Q(T') \leq T'E(T').$$

As a result, we have

$$E(t) \leq \frac{4}{t}T'\|e^{\mathcal{A}T'}\|^2_{L(\mathcal{H})} \cdot \|(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0})\|^2_{\mathcal{H}}.$$

Using the semigroup property of the semigroup $e^{\mathcal{A}t}$, we obtain the exponential decay given in inequality (5.1). In order to show that Q is bounded, we shall show that $Q'(t)$ is nonpositive for all $t > T'$ for some constant T'.

First, take $(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0}) \in \mathcal{D}(\mathcal{A})$ and let

$$(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)) = e^{\mathcal{A}t}(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0}).$$

We consider the function $Q(t)$. Using the differentiability of the function $(\phi(\cdot), \theta(\cdot), u_1(\cdot), v_1(\cdot), u_2(\cdot), v_2(\cdot))$, we obtain

$$\begin{aligned}\frac{d}{dt}Q(t) = {} & \frac{1}{2}\|(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t))\|^2_{\mathcal{H}} \\ & + t\langle \mathcal{A}(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)), (\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t))\rangle_{\mathcal{H}} \\ & + \int_0^1 \partial_t u_1(t,x) v_1(t,x)\,dx + \int_0^1 u_1(t,x)\partial_t v_1(t,x)\,dx \\ & + \int_0^1 \partial_t u_2(t,y) v_2(t,y)\,dy + \int_0^1 u_2(t,y)\partial_t v_2(t,y)\,dy \\ & + \int_\Omega \partial_t\phi(t,x,y)\theta(t,x,y)\,dx\,dy + \int_\Omega \phi(t,x,y)\partial_t\theta(t,x,y)\,dx\,dy \\ & + \int_\Omega [2(x-1)\partial_x\phi(t,x,y) + 2(y-1)\partial_y\phi(t,x,y)]\partial_t\theta(t,x,y)\,dx\,dy \\ & + \int_\Omega \partial_t[2(x-1)\partial_x\phi(t,x,y) + 2(y-1)\partial_y\phi(t,x,y)]\theta(t,x,y)\,dx\,dy.\end{aligned}$$

From the definition of the operator $\mathcal{A}$, we have

$$\begin{aligned}\frac{d}{dt}\phi(t,\cdot,\cdot) &= \theta(t,\cdot,\cdot), \text{ in } V_a, \\ \frac{d}{dt}u_1(t,\cdot) &= v_1(t,\cdot), \text{ in } V_b, \\ \frac{d}{dt}u_2(t,\cdot) &= v_2(t,\cdot), \text{ in } V_b.\end{aligned}$$

Moreover, the following hold:

$$\begin{aligned}\frac{d}{dt}\theta(t,\cdot,\cdot) &= \Delta\phi(t,\cdot,\cdot), \text{ in } L^2(\Omega), \\ \frac{d}{dt}v_1(t,\cdot) &= -\partial_x^4(Eu_1(t,\cdot) + Cv_1(t,\cdot)) + \theta(t,\cdot,0), \text{ in } L^2(0,1), \\ \frac{d}{dt}v_2(t,\cdot) &= -\partial_y^4(Eu_2(t,\cdot) + Cv_2(t,\cdot)) + \theta(t,0,\cdot), \text{ in } L^2(0,1).\end{aligned}$$

The argument for establishing the above equalities is analogous to the proof of Lemma 3.2. Using the above equalities, we obtain

$$\begin{aligned}\frac{d}{dt}Q(t) = &\frac{1}{2}\|(\phi(t),\theta(t),u_1(t),v_1(t),u_2(t),v_2(t))\|^2_{\mathcal{H}}\\ &- t\gamma_a(\theta(t),\theta(t)) - t\gamma_b(v_1(t),v_1(t)) - t\gamma_b(v_2(t),v_2(t))\\ &+ \int_0^1 v_1^2(t,x)\,dx + \int_0^1 [-\partial_x^4(Eu_1(t,x)+Cv_1(t,x)) + \theta(t,x,0)]u_1(t,x)\,dx\\ &+ \int_0^1 v_2^2(t,y)\,dy + \int_0^1 [-\partial_y^4(Eu_2(t,y)+Cv_2(t,y)) + \theta(t,0,y)]u_2(t,y)\,dy\\ &+ \int_\Omega \theta^2(t,x,y)\,dx\,dy + \int_\Omega \phi(t,x,y)\Delta\phi(t,x,y)\,dx\,dy\\ &+ \int_\Omega [2(x-1)\partial_x\phi(t,x,y) + 2(y-1)\partial_y\phi(t,x,y)]\Delta\phi(t,x,y)\,dx\,dy\\ &+ \int_\Omega [2(x-1)\partial_x\theta(t,x,y) + 2(y-1)\partial_y\theta(t,x,,y)]\theta(t,x,y)\,dx\,dy.\end{aligned}$$

By integration by parts and by rearranging terms, we obtain

$$\begin{aligned}\frac{d}{dt}Q(t) = &\frac{1}{2}\int_\Omega \nabla\phi(t,x,y)\cdot\nabla\phi(t,x,y)\,dx\,dy + \int_\Omega \phi(t,x,y)\Delta\phi(t,x,y)\,dx\,dy\\ &+ \int_\Omega [2(x-1)\partial_x\phi(t,x,y) + 2(y-1)\partial_y\phi(t,x,y)]\Delta\phi(t,x,y)\,dx\,dy\\ &+ \int_\Omega [2(x-1)\partial_x\theta(t,x,y) + 2(y-1)\partial_y\theta(t,x,y)]\theta(t,x,y)\,dx\,dy\\ &+ \frac{3}{2}\int_\Omega \theta^2(t,x,y)\,dx\,dy - t\int_0^1 \alpha\theta^2(t,x,0)dx - t\int_0^1 \alpha\theta^2(t,0,y)\,dy\\ &+ \frac{3}{2}\int_0^1 v_1^2(t,x)dx + \frac{3}{2}\int_0^1 v_2^2(t,y)\,dy\\ &- \frac{1}{2}\int_0^1 E(\partial_x^2 u_1(t,x))^2dx - \frac{1}{2}\int_0^1 E(\partial_y^2 u_2(t,y))^2dy\\ &- tC\int_0^1 (\partial_x^2 v_1(t,x))^2dx - tC\int_0^1 (\partial_y^2 v_2(t,y))^2dy\\ &- C\int_0^1 \partial_x^2 v_1(t,x)\partial_x^2 u_1(t,x)dx - C\int_0^1 \partial_y^2 v_2(t,y)\partial_y^2 u_2(t,y)\,dy\\ &+ \int_0^1 \theta(t,x,0)u_1(t,x)dx + \int_0^1 \theta(t,0,y)u_2(t,y)\,dy\\ = &\sum_{k=1}^{17} I_k,\end{aligned} \tag{5.2}$$

where I_k is the kth integral in the above equality. It is easy to see that for t large enough $\sum_{k=8}^{15} I_k$ is negative for t large enough. In fact, there exists $T > 0$ such that for all $t > T$,

$$\sum_{k=8}^{15} I_k \leq -\frac{1}{4}\int_0^1 E(\partial_x^2 u_1(t,x))^2 dx - \frac{1}{4}\int_0^1 E(\partial_y^2 u_2(t,y))^2 dy \\ - \frac{t}{2}\int_0^1 (\partial_x^2 v_1(t,x))^2 dx - \frac{t}{2}\int_0^1 (\partial_y^2 v_2(t,y))^2 dy.$$

Now consider the terms I_4 and I_5 in (5.2). We have

$$\int_\Omega [2(x-1)\partial_x\theta(t,x,y) + 2(y-1)\partial_y\theta(t,x,y)]\theta(t,x,y)\,dx\,dy \\ + \frac{3}{2}\int_\Omega \theta^2(t,x,y)\,dx\,dy \\ = 2\int_\Omega \{\theta^2(t,x,y) + [(x-1)\partial_x\theta(t,x,y) + (y-1)\partial_y\theta(t,x,y)]\theta(t,x,y)\}\,dx\,dy \\ - \frac{1}{2}\int_\Omega \theta^2(t,x,y)\,dx\,dy.$$

Using the fact that

$$\operatorname{div}\left[\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}\theta^2(t,x,y)\right] = 2\theta^2(t,x,y) + 2\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}\cdot\begin{pmatrix} \partial_x\theta(t,x,y) \\ \partial_y\theta(t,x,y) \end{pmatrix}\theta(t,x,y),$$

and by the divergence theorem, we have

$$2\int_\Omega \{\theta^2(t,x,y) + [(x-1)\partial_x\theta(t,x,y) + (y-1)\partial_y\theta(t,x,y)]\theta(t,x,y)\}\,dx\,dy \\ = \int_0^1 \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} x-1 \\ -1 \end{pmatrix}\theta^2(t,x,0)\,dx + \int_0^1 \begin{pmatrix} -1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ y-1 \end{pmatrix}\theta^2(t,0,y)\,dy \\ = \int_0^1 \theta^2(t,x,0)dx + \int_0^1 \theta^2(t,0,y)dy.$$

As a result, there exists a constant T such that for all $t > T$

$$I_4 + I_5 \leq \frac{t\alpha}{2}\int_0^1 \theta^2(t,x,0)dx + \frac{t\alpha}{2}\int_0^1 \theta^2(t,0,y)dy.$$

On the other hand, for the terms I_{16} and I_{17}, it is easy to see that

$$\int_0^1 \theta(t,x,0)u_1(t,x)dx + \int_0^1 \theta(t,0,y)u_2(t,y)dy \leq \varepsilon\int_0^1 u_1^2(t,x)dx + \varepsilon\int_0^1 u_2^2(t,y)dy \\ + \frac{1}{\varepsilon}\int_0^1 \theta^2(t,x,0)dx + \frac{1}{\varepsilon}\int_0^1 \theta^2(t,0,y)dy.$$

Summarizing the above observations, there exists a constant T such that for all $t > T$, we have

$$\begin{aligned}\frac{d}{dt}Q(t) \le I_1 + I_2 + I_3 \\ &- \frac{t\alpha}{2}\int_0^1 \theta^2(t,x,0)dx - \frac{t\alpha}{2}\int_0^1 \theta^2(t,0,y)dy \\ &- \frac{t}{2}\int_0^1 (\partial_x^2 v_1(t,x))^2 dx - \frac{t}{2}\int_0^1 (\partial_y^2 v_2(t,y))^2 dy \\ &- \frac{1}{4}\int_0^1 E(\partial_x^2 u_1(t,x))^2 dx - \frac{1}{4}\int_0^1 E(\partial_y^2 u_2(t,y))^2 dy.\end{aligned}$$

If the first component ϕ of the solution is sufficiently smooth, the analysis of the integrals I_1, I_2, and I_3 consists only of an application of the divergence theorem. However, as we have indicated, ϕ is not in $H^2(\Omega)$. In the remainder of this section, we need to show that there exists a constant T such that for all $t > T$,

$$\begin{aligned}I_1 + I_2 + I_3 \le \frac{t\alpha}{2}\int_0^1 \theta^2(t,x,0)dx + \frac{t\alpha}{2}\int_0^1 \theta^2(t,0,y)dy \\ + \frac{t}{2}\int_0^1 (\partial_x^2 v_1(t,x))^2 dx + \frac{t}{2}\int_0^1 (\partial_y^2 v_2(t,y))^2 dy.\end{aligned} \tag{5.3}$$

Since the initial state $(\phi_0, \theta_0, u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0}) \in \mathcal{D}(\mathcal{A})$, using the properties of the C_0-semigroup of bounded linear operators, we have $(\phi(t), \theta(t), u_1(t), v_1(t), u_2(t), v_2(t)) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$. From the smoothness results in Sec. 4, we conclude that $\phi(t,\cdot,\cdot) \in U_a$ for all $t > 0$ and

$$\partial_y\phi(t,\cdot,0) = v_1(t,\cdot) + \alpha\theta(t,\cdot,0), \quad \partial_x\phi(t,0,\cdot) = v_2(t,\cdot) + \alpha\theta(t,0,\cdot), \text{ in } H^{1/2}(0,1).$$

Now for a fixed value of t, we can find a sequence of vectors $\{\phi_n(t,\cdot,\cdot)\} \subset H^2(\Omega) \cap V_a$ such that $\|\phi_n(t,\cdot,\cdot) - \phi(t,\cdot,\cdot)\|_{U_a}$ tends toward zero as n tends to infinity and

$$\begin{aligned}\int_0^1 |\partial_y\phi_n(t,x,0)|^2 dx + \int_0^1 |\partial_x\phi_n(t,0,y)|^2 dy \\ \le \int_0^1 |v_1(t,x) + \alpha\theta(t,x,0)|^2 dx + \int_0^1 |v_2(t,y) + \alpha\theta(t,0,y)|^2 dy\end{aligned} \tag{5.4}$$

for all n. If the sum of the integrals on the right-hand side of (5.4) is different from zero, the results in Sec. 4 guarantee that we can find sequences $\{\phi_{n,k}(t,\cdot,\cdot)\} \in H^2(\Omega) \cap V_a$ that converge to $\beta_k\phi(t,\cdot,\cdot)$ in the U_a-norm as n tends to infinity where β_k is a sequence of positive real numbers in $(0,1)$ and β_k tends toward 1 as k tends to infinity. As a result, the sequence of vectors $\{\phi_{n_k,k}\}$ has the desired properties if n_k is selected such that inequality (5.4) holds. On the other hand, if the sum of the integrals on the right-hand side of (5.4) is equal to zero, we have to choose a sequence $\{\phi_n(t,\cdot,\cdot)\}$ from the subset

$$H_s^2(\Omega) = \left\{ \begin{matrix} \phi \in H^2(\Omega) \colon \phi(x,1) = \phi(1,y) = 0, \\ \partial_y\phi(x,0) = \partial_x\phi(0,y) = 0, \ x,y \in (0,1) \end{matrix} \right\}.$$

It is easy to see that $H_s^2(\Omega)$ is dense in the subset $\{\phi \in U_a \colon \partial_y\phi(x,0) = \partial_x\phi(0,y) = 0\}$ of U_a. As a result of this selection, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \nabla\phi(t,x,y)\cdot\nabla\phi(t,x,y)\,dx\,dy + \int_\Omega \phi(t,x,y)\Delta\phi(t,x,y)\,dx\,dy \\
&\qquad + \int_\Omega 2[(x-1)\partial_x\phi(t,x,y) + (y-1)\partial_y\phi(t,x,y)]\Delta\phi(t,x,y)\,dx\,dy \\
&\quad = \lim_{n\to\infty}\frac{1}{2}\int_\Omega \nabla\phi_n(t,x,y)\cdot\nabla\phi_n(t,x,y)\,dx\,dy + \int_\Omega \phi_n(t,x,y)\Delta\phi_n(t,x,y)\,dx\,dy \\
&\qquad + \int_\Omega 2[(x-1)\partial_x\phi_n(t,x,y) + (y-1)\partial_y\phi_n(t,x,y)]\Delta\phi_n(t,x,y)\,dx\,dy.
\end{aligned}$$

Now the integrals on the right-hand side only involve $\phi_n(t,\cdot,\cdot)$ which is an $H^2(\Omega)$ function. We have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \nabla\phi_n(t,x,y)\cdot\nabla\phi_n(t,x,y)\,dx\,dy + \int_\Omega \phi_n(t,x,y)\Delta\phi_n(t,x,y)\,dx\,dy \\
&\quad = -\frac{1}{2}\int_\Omega \nabla\phi_n(t,x,y)\cdot\nabla\phi_n(t,x,y)\,dx\,dy \\
&\qquad - \int_0^1 \partial_y\phi_n(t,x,0)\phi_n(t,x,0)dx - \int_0^1 \partial_x\phi_n(t,0,y)\phi_n(t,0,y)dy,
\end{aligned}$$

and we have

$$\begin{aligned}
&2((x-1)\partial_x\phi_n(t,x,y) + (y-1)\partial_y\phi_n(t,x,y))\Delta\phi_n(t,x,y) \\
&\quad = 2\,\mathrm{div}\left(\left[\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}\cdot\nabla\phi_n(t,x,y)\right]\nabla\phi_n(t,x,y) - \|\nabla\phi_n(t,x,y)\|^2\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_\Omega 2((x-1)\partial_x\phi_n(t,x,y) + (y-1)\partial_y\phi_n(t,x,y))\Delta\phi_n(t,x,y)\,dx\,dy \\
&\quad = -\int_0^1 2\left(\begin{pmatrix} x-1 \\ -1 \end{pmatrix}\cdot\nabla\phi_n(t,x,0)\right)\partial_y\phi_n(t,x,0)dx \\
&\qquad - \int_0^1 2\left(\begin{pmatrix} -1 \\ y-1 \end{pmatrix}\cdot\nabla\phi_n(t,0,y)\right)\partial_x\phi_n(t,0,y)dy \\
&\qquad - \int_0^1 \|\nabla\phi_n(t,x,0)\|^2 dx - \int_0^1 \|\nabla\phi_n(t,0,y)\|^2 dy.
\end{aligned}$$

Using the Cauchy-Schwartz inequality and the Poincaré inequality, we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega \|\nabla\phi_n(t,x,y)\|^2\,dx\,dy &+ \int_\Omega \phi_n(t,x,y)\Delta\phi(t,x,y)\,dx\,dy\\ &+\int_\Omega 2\left[\begin{pmatrix}x-1\\ y-1\end{pmatrix}\cdot\nabla\phi_n(t,x,y)\right]\Delta\phi_n(t,x,y)\,dx\,dy\\ &\le -\frac{1}{4}\int_\Omega \|\nabla\phi_n(t,x,y)\|^2\,dx\,dy + c\int_0^1(\partial_y\phi_n(t,x,0))^2dx\\ &+c\int_0^1(\partial_x\phi_n(t,0,y))^2dy,\end{aligned}$$

for some constant c independent of t and n. From the property of the sequence $\{\phi_n(t,\cdot,\cdot)\}$, we have

$$\begin{aligned}\int_\Omega \|\nabla\phi_n(t,x,y)\|^2\,dx\,dy &+ \int_\Omega \phi_n(t,x,y)\Delta\phi_n(t,x,y)\,dx\,dy\\ &+\int_\Omega 2\left[\begin{pmatrix}x-1\\ y-1\end{pmatrix}\cdot\nabla\phi_n(t,x,y)\right]\Delta\phi_n(t,x,y)\,dx\,dy\\ &\le c\int_0^1(v_1(t,x)+\alpha\theta(t,x,0))^2dx + c\int_0^1(v_2(t,y)+\alpha\theta(t,0,y))^2dy,\end{aligned}$$

for all n and t. Combining the above observations, we conclude that inequality (5.3) holds.

Finally, we can state the main result of this paper.

THEOREM 5.1. There exist constants M and $\omega>0$ such that for any vector $(\phi_0,\theta_0,u_{1,0},v_{1,0},u_{2,0},v_{2,0})\in\mathcal{H}$, the inequality (5.1) holds.

Proof. There exists a constant T such that for all $t>T$,

$$\frac{t}{4}E(t)\le Q(t),\quad Q(t)\le tE(t).$$

Moreover, in the case that the initial vector $(\phi_0,\theta_0,u_{1,0},v_{1,0},u_{2,0},v_{2,0})\in\mathcal{D}(\mathcal{A})$, $Q(t)\le Q(T)$. We obtain as a result

$$\frac{t}{4}E(t)\le TE(t)\le T\|e^{\mathcal{A}T}\|^2_{L(\mathcal{H})}\|(\phi_0,\theta_0,u_{1,0},v_{1,0},u_{2,0},v_{2,0})\|^2_{\mathcal{H}}.$$

That is, there exists a constant c such that

$$\|e^{\mathcal{A}t}(\phi_0,\theta_0,u_{1,0},v_{1,0},u_{2,0},v_{2,0})\|_{\mathcal{H}}\le\frac{c}{\sqrt{t}}\|(\phi_0,\theta_0,u_{1,0},v_{1,0},u_{2,0},v_{2,0})\|_{\mathcal{H}},$$

for all $(\phi_0,\theta_0,u_{1,0},v_{1,0},u_{2,0},v_{2,0})\in\mathcal{D}(\mathcal{A})$. By the density of the domain $\mathcal{D}(\mathcal{A})$ in $\mathcal{H}$, there exists t large enough such that

$$\|e^{\mathcal{A}t}\|_{L(\mathcal{H})}\le\tfrac{1}{2}.$$

Using the semigroup property, we conclude that (5.1) holds.

6. Conclusion. The results in our paper demonstrate that with a relatively minor change of the boundary condition from that used in [2], uniform exponential stability of the solutions of the model equations can be obtained. The proposed model is consistent with well-established theory in acoustics in the case where the flexible beam can be viewed as "porous". On the other hand, the explanation offered here is just one possible physical interpretation. It would be interesting to see whether or not an alternative physical explanation which is appropriate in more general situations can be found. Using the model proposed here, the linear quadratic control problem stated in [2] is well-posed. However, as in the case of the wave equation, in order to establish the convergence of the approximation to the optimal feedback control operator, it is critical to find numerical approximation schemes that can preserve the uniform exponential stability of the infinite-dimensional control system. As is shown in [4], many commonly used approximation schemes may not be suitable for the approximation of the optimal control problems.

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