A new model of central-bank intervention: some examples

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Abstract

Optimal monetary policy is studied, by way of numerical examples, in a model with (i) heterogeneity in the degree to which different people are monitored (have publicly known histories); (ii) idiosyncratic shocks that give rise to heterogeneity in earning and spending realizations; and (iii) central-bank intervention in a "market" in claims or credit in which the participants are those who are heavily monitored. The results serve as counterexamples to two widely held views: optimal policy is unrelated to what makes money important; and, there are simple and well-known principles to guide monetary policy.

Epigraph

[For the Bank of England in 1805] knowing the direction of the wind was [important] ... If ... from the east, ships would soon be sailing up the Thames to unload goods in London. The Bank would need to supply lots of money..... If a westerly was blowing, the Bank would mop up any excess money..., thereby avoiding inflation. The 19th century Bank knew the importance of money ..., Mervyn King, the current governor, told the FT in an interview .... But money matters were much simpler in 1805 than today ("Winds of change" by Chris Giles, Financial Times, (FT), May 14, 2007).

1 Introduction

Monetary economics is split into two main strands. One consists of dynamic stochastic general equilibrium (DSGE) models, calibrated or estimated, that claim to provide guidance to monetary policy. It rests largely on various formulations of sticky prices (menu costs) and its implications do not depend on the role of money. The other strand, a good deal of which is inspired by the work on matching models of Kiyotaki and Wright (1989), uses models in which money has a distinct role based on two main assumptions: people cannot commit to future actions and at least some people are imperfectly monitored in the sense that their previous actions are not common knowledge. Despite advances within work in the second strand, several challenges remain for it. One is to depict central-bank policy in a way that resembles to at least some degree actual central-bank intervention in credit markets. We address that challenge here. We amend the Cavalcanti and Wallace (CW) (1999) model of inside and outside money in two ways: a central-bank monopoly on money is imposed and a credit market among the would-be issuers of private money is allowed to function. Central-bank policy is modeled as intervention in that market. The optima (in a class of implementable allocations) for such a model, without and with central-bank intervention, are described by way of a few numerical examples.

Because optima are studied from numerical examples in a very special model, it is reasonable to ask what can be learned from them. First, they illustrate a new conception of the role of policy. Second—and, perhaps, more importantly—they serve as counterexamples to two current views associated with the DSGE strand of monetary economics: optimal policy is unrelated to what makes money important; and, there are simple and well-known principles to guide monetary policy. Of course, to serve those purposes, the model must be attractive.

In CW, monitored people are issuers of private money and private money plays a desirable role. It frees current spending possibilities from recent earning and spending realizations—not unlike the role that credit cards play. In many countries, private money was replaced by a central-bank monopoly on money. A common view is that the monopoly helps solve a recognizability problem that CW assume away: the greater is the number of different monies, the greater is the threat of counterfeiting. In any case, in order to have our modeling parallel that history, we study policy under a central-bank monopoly using a model that closely resembles the one that CW use to study private money.

2 The model

There are two stages of trade at each discrete date:

sequence at a date	
stage 1	stage 2
pairwise meetings at random (goods and money exchanged)	money transfers and central bank intervention

Stage 1 is borrowed from Trejos-Wright (1995) and Shi (1995). There is a unit measure of each of $K > 2$ specialization types. As in CW, an exogenous fraction α of each type are perfectly monitored (m people), the rest are not monitored at all (n people). Each person maximizes expected discounted utility, where period-utility for a type k person is $u(q_{k+1}) - q_k/\delta_t$ and q_{k+1} (the addition is modulo K) is consumption of type k good and q_k is production of type k good and

$$\delta_t = \begin{cases} \delta_t = \delta_l \text{ if } t \text{ is odd} \\ \delta_t = \delta_h \text{ if } t \text{ is even} \end{cases} . \tag{1}$$

We assume that $0 < \delta_l < \delta_h$. Finally, each person's money holdings at the start of stage 1 is restricted to be in $\{0, 1\}$. Between stages, money holdings are permitted to be in $\{0, 1, 2\}$. As regards information, histories and money holdings are common knowledge for m people, but are private for n people. Each person's monitored status and specialization type are common knowledge.

The random stage-1 meetings give rise to random earning and spending realizations. Such realizations could instead be modeled as resulting from idiosyncratic preference shocks with every meeting being a single-coincidence meeting. No matter the source of the random earning and spending realizations, the problem in the economy is to free current activities of a person from recent earning and spending realizations. In particular, the best outcome subject only to physical feasibility is production and consumption equal to $\arg\max_q[u(q) - q/\delta_t]$—the output that maximizes *surplus* in every single-coincidence meeting.

If everyone is monitored, then money does not help. Given that some people are not, money is helpful but cannot fully overcome the problem even if individual holdings of money are allowed to be a large set. Those who experience a string of consumption opportunities will tend to run out of money, while those who experience a string of earnings will not want to expend much effort to earn more. The assumption that money holdings are in $\{0, 1\}$, which is adopted to limit the number of unknowns, gives rise to an exaggerated, but not misleading, version of the problem: n people without money cannot spend, while n people with money are so rich that they cannot be induced to produce.

Given the randomness and the monopoly on money, m people want to borrow and lend. That is the main role of stage 2. In this model, the borrowing and lending takes the form of insurance: those who earned tend to pay out; those who consumed tend to receive a transfer. We also permit positive transfers to n people at stage 2—positive because n people cannot be induced to surrender money at stage 2.

We adopt a two-date deterministic pattern of aggregate productivity, the δ's, to give central-bank policy a potential role in a simple way. From now on we call odd dates ($\delta_t = \delta_l$) low (productivity) dates and even dates ($\delta_t = \delta_h$) high (productivity) dates. Therefore, we are studying optimal seasonal policy.

3 Stationary and implementable allocations

We study allocations that are symmetric in two senses. There is no dependence on specialization type and there is no randomization: everyone in the same state at a date does the same thing (although it may be a lottery). (Randomization is useful for proving that ex ante welfare is increasing in the fraction who are monitored, but, here, it adds too many unknowns.) We also limit ourselves to two-date periodic allocations. Given our assumptions, the state of a person at the start of a date is in the set $\{m, n\} \times \{0, 1\}$, while it is $\{m, n\} \times \{0, 1, 2\}$ at the end of pairwise meetings. An allocation describes stage-1 trades as a function of the states of the people in a meeting and date: low or high; stage-2 transfers of money as a function of the state of a person; and the fraction of each type (m or n) who have money at the start of each date, low and high. Those fractions and the stage 1 trades and stage 2 transfers must be mutually consistent. At both stages, transfers of money are modeled as lotteries.

The only punishment is banishment of individual m-people to the set of n-people. Underlying this assumption is free exit from the set of m-people and the ruling out of global punishments (like shutting down all trade) in response to individual defections. Because n people can hide money, those with money must self-select the trades intended for them. We permit individual and cooperative defection by any pair in a stage-1 meeting, but allow only individual defection at stage 2 because there are no static gains from trade at stage 2. (See the appendix for explicit expressions for the constraints.)

The objective is ex ante expected discounted utility at the start of a *low* date prior to the assignment of types, m or n, and initial money holdings, 0 or 1. (As is well-known, this objective can be written as a weighted average of the surplus, $u(q) - q/\delta_t$, over meetings with weights that reflect the frequency of meetings and the discounting implied by the first date being a low date.) The constraints are as given above: symmetry, two-date periodicity, and implementability. One optimum, labeled no policy, has a constant quantity of money; that is, it has zero net stage-2 transfers at each date. The other, labeled optimal policy, allows the quantity of money and net stage-2 transfers to be two-date periodic.

4 A high discount-factor example

We begin by describing in detail the optima for one example. Except for its sufficiently high discount factor, this example is arbitrary.

Discount factor (β)	.95
Fraction who are m-people (α)	.25
Number of specialization types (K)	3
$u(x)$ (utility of consuming x)	$2x^{1/2}$
δ_l	.8
δ_h	1.25

Before describing the optima, we describe two benchmarks. One is an upper bound: maximum surplus, production and consumption of $(\delta_t)^2$, in every single-coincidence meeting. This implies ex ante welfare equal to $(\delta_l + \beta\delta_h)/K(1-\beta^2)$. Another benchmark, a lower bound, is the best allocation that can be attained subject to treating m people like n people. (Treating m people like n people is implementable because the only recourse of an m person is defection to being an n person.) Subject to this restriction, production occurs only in those single-coincidence meetings in which the producer has no money and the consumer has money. For this example, because its discount factor is sufficiently high, this benchmark is very simple: half have money, output at a date is $(\delta_t)^2$ in 1/4 of the single-coincidence meetings (and 0 in all other meetings), and ex ante welfare is 1/4 of the benchmark upper bound.

Table 1. Optima: Money holdings and welfare (relative to upper bound)

	no policy		optimal policy	
	low	high	low	high
fraction of m with money	1	1	1	1
fraction of n with money	0.4156	0.4156	0.4161	0.4125
welfare of m	.6627		.6622	
welfare of n	.3865		.3870	
welfare	.4556		.4558	
consumption equivalent	1.0000		1.0053	

Money holdings and welfare in the optima appear in Table 1. In these optima, there are no transfers of money to n people at either stage and each m person starts with money. Only about 41% of n people start a period with money (compared to 50% in the benchmark lower-bound). Under policy, there are net stage-2 transfers at the high date—transfers that are used to replenish the money holdings of m people. That is, under policy, m people spend more than they earn at the high date and vice versa at the low date. Central bank intervention reconciles that pattern with no stage-2 transfers to n people and with constant holdings by m people at each date.

The welfare of each type is an expected value over their money holdings. Relative to the above lower-bound, both m and n types gain. Policy helps a bit—1/2% in terms of consumption equivalents. Notice that only n people benefit from policy.

In terms of extensive margins, which are implied by the money holdings, no clear picture of the role of policy emerges. On the one hand, policy implies fewer potential trade meetings between n producers and consumers at the high date and more at the low date. On the other hand it implies more such meetings between n producers and m consumers at the high date and fewer at the low date. The intensive margins, the trades in meetings,appear in the next two tables. We start with within-group meetings.

Table 2. Optimal trades in (n, n) and (m, m) meetings: output (relative to surplus maximizing) and probability that money is transferred (in parentheses)

meeting	no policy		optimal policy	
(prod)(con)	low	high	low	high
$(n0)(n1)$	0.970 (1.000)	0.978 (1.000)	0.951 (1.000)	0.947 (1.000)
$(m1)(m1)$	1.000 (n/a)	0.832* (n/a)	1.000 (n/a)	0.836* (n/a)

In this and the subsequent tables, the state of the producer (prod) is listed first, followed by the state of the consumer (con). Here and in tables below, a (*) denotes a binding individual producer participation constraint. Because m people always start with money, acquiring money in stage 1 is useless to them. Therefore, production by them is a gift, supported by the threat of punishment. Here, that threat is binding at the high date. The binding constraint is

$$\frac{\text{high-date production by } m}{\delta_h} = \beta[\text{welfare of } m - \text{welfare of } (n,1)], \tag{2}$$

which is consistent with much higher welfare for m people than for n people. The beneficial role of policy is more obvious in the meetings between m and n people.

Table 3. Optimal trades in meetings between m and n people: output (relative to surplus maximizing) and probability that money is transferred (in parentheses)

meeting	no policy		optimal policy	
(prod)(con)	low	high	low	high
$(n0)(m1)$	0.904* (0.528)	0.723* (0.711)	0.850* (0.505)	0.777* (0.776)
$(m1)(n0)$	0.173	0.163	0.161	0.171
$(m1)(n1)$	1.106† (0.743)	0.832*† (1.000)	1.177† (0.813)	0.836*† (1.000)

Although production by an m person is a gift, output in the $(m1)(n0)$ meeting (row 2) is much lower than in the $(m1)(n1)$ meeting (row 3). That is consistent with the binding truth-telling constraints in the $(m1)(n1)$ meetings (row 3). (A (†) denotes a binding truth-telling constraint.) And, although those constraints could be loosened by collecting less money in the $(m1)(n1)$ meeting, as we now explain, that would violate other constraints.

The acquisition of money in the $(m1)(n1)$ meetings is the only source of outflow from money holdings of n people. If there is to be an inflow, in $(n0)(m1)$ meetings, then there has to an offsetting outflow. Given that m people always start with money, in the absence of policy those flows must offset each other at each date. Hence, simply collecting less money in the $(m1)(n1)$ meeting is not an option.

That inflow-outflow restriction accounts for what looks like the main beneficial effect of policy: smoother output in $(n0)(m1)$ meetings (see row 1). Consider the high date in the absence of policy. In order to raise output in the $(n0)(m1)$ meeting at the high date, there has to be higher spending by m people (because the producer participation constraint is binding). But the inflow (see row 3) is already maximal. Hence, higher spending can't happen without accompanying changes in the fractions holding money. With optimal policy, there is higher spending in the $(n0)(m1)$ meeting at the high date. The higher spending is offset by an injection of money at stage 2. And that is possible because spending by m people is lower at the low date under policy.

5 Two lower discount-factor examples

Here we report on optima for the same example except for the discount factor. For somewhat lower discount factors, the lower-bound benchmark which treats m-people like n-people has lower output at the high date; and for still lower discount factors, it has lower outputs at both dates. We present one of each type of example.

5.1 A lower discount factor: $\beta = .85$

This discount factor is chosen so that the lower bound implied by treating m like n people has surplus-maximizing output at the low date, but lower than surplus-maximizing output at the high date. As a result, ex ante welfare is now slightly less than 1/4 of the upper bound implied by surplus maximizing output at each date in each single-coincidence meeting.

Table 4. Optima ($\beta = .85$): Money holdings and Welfare (relative to upper bound)

	no policy		optimal policy	
	low	high	low	high
fraction of m with money	0.9850	1	1	1
fraction of n with money	0.4550	0.4500	0.4568	0.4490
welfare of m	0.7087		0.7089	
welfare of n	0.2947		0.2957	
welfare	0.3981		0.3990	
consumption equivalent	1.0000		1.0105	

As in the higher-discount factor example, there are no transfers to n people at either stage. At stage 2, m people who enter with either one or two units of money leave with one unit. Under policy those who enter stage 2 with nothing leave with a unit of money; however, under no policy they begin the high date with money, but begin the low date with money with probability 0.8842. As displayed in Table 4, three other main differences from the first example show up here. First, the fraction of n people with money is higher; second, welfare is a smaller fraction of the upper bound, and third, policy has a greater effect in terms of consumption equivalents—about 1%.

Table 5 shows the within-group trades. Here, we have binding producer participation constraints except in the rare meetings in which the producer is $(m, 0)$; this person will with high probability be rewarded with money at stage 2, but will not begin the next date with money if the person defects. The lower discount factor substantially reduces output at the high date. And when m people have money, the threat of defection gives rise at all dates to relatively low output. Notice that there is no smoothing of output relative to surplus-maximizing output in these within-group meetings.

Table 5. Optimal trades in (n, n) and (m, m) meetings: output and probability that money is transferred

meeting	no policy		optimal policy	
(prod)(con)	low	high	low	high
$(n0)(n1)$	0.9808* (1.000)	0.6077* (1.000)	0.9808* (1.000)	0.6074* (1.000)
$(m0)(m0)$	1.000	n/a	n/a	n/a
$(m1)(m0)$	0.4752* (n/a)	n/a	n/a	n/a
$(m0)(m1)$	1.1575 (n/a)	n/a	n/a	n/a
$(m1)(m1)$	0.4752* (n/a)	0.3013* (0.000)	0.4843* (n/a)	0.2986* (n/a)

Table 6. Optimal trades in meetings between m and n people: output and probability that money is transferred

meeting	no policy		optimal policy	
(prod)(con)	low	high	low	high
$(n0)(m1)$	0.7218* (0.736)	0.5635* (0.927)	0.6559* (0.669)	0.5980* (0.984)
$(m0)(n0)$	0.0819	n/a	n/a	n/a
$(m1)(n0)$	0.0396 (0.000)	0.0601 (0.000)	0.0422 (0.000)	0.0589 (0.000)
$(m0)(n1)$	0.6032† (1.000)	n/a	n/a	n/a
$(m1)(n1)$	0.4752*† (1.000)	0.3013*† (1.000)	0.4843*† (1.000)	0.2986*† (1.000)

Table 6 shows the between-group trades. As in the previous example, policy produces smoother output in the $(n0)(m1)$ meeting. And this is accomplished in the same way: policy is needed to allow higher spending at the high date by m people.

5.2 A still lower discount factor: $\beta = .5$

This discount factor is so low that the lower bound implied by treating m like n people has lower than surplus-maximizing output at both dates. Ex ante welfare at this lower bound is slightly less than 14% of the upper bound implied by surplus maximizing output at each date in every single-coincidence meeting.

Table 7. Optima ($\beta = .5$): Money holdings and Welfare (relative to upper bound)

	no policy		optimal policy	
	low	high	low	high
fraction of m with money	0.5320	0.5320	0.5217	0.5398
fraction of n with money	0.3472	0.3472	0.3470	0.3465
prob m with zero gets money in stage 2	0.0898	0.0898	0.1184	0.0594
welfare of m	0.2300		0.2282	
welfare of n	0.1398		0.1404	
welfare	0.16236		0.16237	
consumption equivalent	1.0000		1.0001	

Again, there are no transfers to n people at either stage and, at stage 2, m people who enter with either one or two units of money leave with one unit, while those who enter stage 2 with zero have only a small probability of receiving money. Hence, little insurance is being provided to m people. As a consequence, as shown in Table 7, the gain in welfare relative to the lower bound is quite small. Moreover, there is no gain from policy.

Table 8 displays the within-group trades. Notice that in meetings between two m people, output is close to zero except when the m producer has no money and can acquire money. That is, the threat of defection by m people with money is so severe that they behave much like n people even when they meet each other.

Table 8. Optimal trades in (n, n) and (m, m) meetings: output and probability that money is transferred

meeting	no policy		optimal policy	
(prod)(con)	low	high	low	high
$(n0)(n1)$	0.1378* (1.000)	0.0788* (1.000)	0.1376* (1.000)	0.0786* (1.000)
$(m0)(m0)$	0.0377*	0.0168*	0.0396*	0.0148*
$(m1)(m0)$	0.0124* (0.000)	0.0068* (0.000)	0.0120* (0.000)	0.0068* (0.000)
$(m0)(m1)$	0.1502* (1.000)	0.0856* (1.000)	0.1496* (1.000)	0.0854* (1.000)
$(m1)(m1)$	0.0124* (0.000)	0.0068* (0.000)	0.0120* (0.000)	0.0068* (0.000)

Table 9 displays the between-group trades. In contrast to the other examples, there is no discernible effect of policy. And, similar to the between-group trades, m producers behave much like n producers. (A (**) denotes a binding individual consumer participation constraint.)

Table 9. Optimal trades in meetings between m and n people: output and probability that money is transferred

meeting	no policy		optimal policy	
(prod)(con)	low	high	low	high
$(n0)(m1)$	0.1378* (1.000)	0.0788* (1.000)	0.1376* (1.000)	0.0786* (1.000)
$(m0)(n0)$	0.0377*	0.0000**	0.0396*	0.0000**
$(m1)(n0)$	0.0018 (0.000)	0.0000** (0.000)	0.0017 (0.000)	0.0000** (0.000)
$(m0)(n1)$	0.1502* (1.000)	0.0540 (1.000)	0.1496* (1.000)	0.0540 (1.000)
$(m1)(n1)$	0.0124*† (1.000)	0.0068* (1.000)	0.0120*† (1.000)	0.0068* (1.000)

6 Remarks about the model

There are many controversial features of the model. One is the permanent heterogeneity in monitored status which implies the permanent heterogeneity in stage-2 participation. It comes from using the model used to study private money.

Another possibly controversial assumption is that all utility-producing activity is in pairwise meetings—in stage 1. In contrast, Lagos and Wright (2005) and the many follow-ups to it use stage 2 for centralized trade in goods *with* quasi-linear preferences. For the role it plays in Lagos-Wright, quasi-linearity is extreme. It eliminates the possibility that idiosyncratic realizations at stage 1 affect subsequent pairwise trades. As is well-known, centralized trade in goods at stage 2 without quasi-linearity would not have that consequence (see Molico 2006). Also, we regard stage 2 as mimicking activities like clearing, settlement, and borrowing and lending using commercial paper, federal funds, and the discount window. Such activities do not involve trade in goods.

Also, perhaps, controversial is the absence of a policy that resembles the Friedman rule. That rule is infeasible with $\{0,1\}$ money holdings. With a larger set of individual holdings, it could be approximated.[1] However, any such scheme would be redistributive because the net taxes needed to support it are collected from m people at stage 2, while the transfers that mimic deflation go to everyone. Because there is already ample scope to tax m people through production in pairwise meetings, it is far from obvious that any such scheme would be desirable. Also, with larger money holdings, lump-sum money creation could play a risk-sharing role as in Levine (1991), Molico (2006), and Deviatov (2006).

7 Conclusion

It goes without saying that optimal policy in the above model is related to the role of money. To reduce the role of money, we would simply assume that more people are monitored. In the limit, with everyone monitored, money plays no role. And although we have studied essentially one simple example, even its qualitative features are not obvious. Hence, it calls into question the notion that there are simple principles that guide monetary policy.

The conclusion that optimal policy is complicated is hardly a pleasant message. But is it surprising? Central banks have a monopoly on money. Therefore, optimal management of that monopoly ought to be related to

[1] Suppose the set of holdings is $\{0,1,...,B\}$, with $B>1$. Then, ignoring for a moment, the periodicity, an approximation to a constant deflation rate could be accomplished by having negative net transfers at stage 2 and using the proceeds as follows. If a person starts a period prior to pairwise meetings with $0<j<B$ units of money, then augment that holding by one unit with probability γ_j, where $\gamma_j/j=\rho\leq 1/(B-1)$.) Such a scheme approximates having the value of money grow through deflation at the rate ρ.

what gives rise to a demand for money. That demand arises from frictions that inhibit credit—namely, some specification of imperfect monitoring. Why should there be a simple, general, and compelling way to model imperfect monitoring? If not, then there is no reason to expect that optimal monetary policy is simple.

8 Appendix. Allocations and constraints

Here we formally set out the optimization problems.

Notation.

Let $S = \{m, n\} \times \{0, 1\}$ and $Q = \{m, n\} \times \{0, 1, 2\}$, where $s = (s_1, s_2)$ and $s' = (s'_1, s'_2)$ denote generic elements of S or Q. Let $t \in \{l, h\}$ denote the date. Our main notation is given in Table 1.

Table 1. Notation.

$y_t^{s,s'} \in R_+$	production by s and consumption by s', $s, s' \in S$
$\lambda_{t,s}^{s,s'}, \lambda_{t,s'}^{s,s'}$	stage 1 state transition in single-coincidence meetings with $s, s' \in S$; a lottery over $\{0, 1, 2\}$ (a 1×3 vector)
$\eta_{t,s}^{s,s'}$	stage 1 state transition for no-coincidence meetings with $s, s' \in S$; a lottery over $\{0, 1, 2\}$ (a 1×3 vector)
ϕ_t^s	stage 2 state transition for $s \in Q$; a lottery over $\{0, 1\}$ (a 1×2 vector)
$\Phi_t^{(s_1,\cdot)}$	3×2 transition matrix whose rows are lotteries ϕ_t^s
θ_t^s	fraction who are in state $s \in S$ at the start of date t
$\theta_t^{(s_1,\cdot)}$	2×1 vector whose entries are θ_t^s
v_t^s	discounted utility at start of date t for person in state s
$v_{t'}^{(s_1,\cdot)}$	2×1 vector whose entries are v_t^s

The s subscript on λ and η is the state of the person who experiences the transition.

Restrictions on Stage-1 state transitions

The transfers in stage-1 meetings must preserve total money holdings. Therefore, we have the following restrictions on λ and η. For $x = \lambda, \eta$, if $s_2 + s'_2 = 0$, then there is one possible outcome, $(0, 0)$, and, therefore, $x_{ts}^{ss'}(0) = x_{ts'}^{ss'}(0) = 1$; if $s_2 + s'_2 = 1$, then there are two possible outcomes, $(1, 0)$ and $(0, 1)$, and, therefore, $x_{ts}^{ss'}(1) = x_{ts'}^{ss'}(0)$ and $x_{ts}^{ss'}(0) = x_{ts'}^{ss'}(1) = 1 - x_{ts}^{ss'}(1)$; if $s_2 + s'_2 = 2$, then there are 3 possible outcomes, $(2, 0), (0, 2), (1, 1)$, and, therefore, $x_{ts}^{ss'}(2) = x_{ts'}^{ss'}(0)$, $x_{ts}^{ss'}(0) = x_{ts'}^{ss'}(2)$ and $x_{ts}^{ss'}(1) = x_{ts'}^{ss'}(1) = 1 - x_{ts'}^{ss'}(0) - x_{ts}^{ss'}(0)$.

Laws of motion

Pairwise trades in stage 1 and stage 2 transfers imply the transition of money among individuals. For $s_1 \in (m, n)$, let

$$T_t^{(s_1,\cdot)} = \frac{1}{K} \sum_{s' \in S} \theta_t^{s'} [\lambda_{ts}^{s,s'} \Phi_t^{(s_1,\cdot)} + \lambda_{ts}^{s',s} \Phi_t^{(s_1,\cdot)} + (K-2)\eta_{ts}^{s,s'} \Phi_t^{(s_1,\cdot)}],$$

a 2×2 transition matrix. In terms of these transition matrices (one for monitored people and one for non-monitored people), the stationarity requirements are

$$\theta_t^{(s_1,\cdot)} T_t^{(s_1,\cdot)} T_{t'}^{(s_1,\cdot)} = \theta_t^{(s_1,\cdot)} \text{ and } t \neq t'.$$

The no-intervention constraint is

$$\alpha\theta_t^{(m,1)} + (1-\alpha)\theta_t^{(n,1)} = \alpha\theta_{t'}^{(m,1)} + (1-\alpha)\theta_{t'}^{(n,1)}.$$

Discounted expected utilities and implementability restrictions

The discounted expected utility at the start of stage 1 at date t satisfies

$$v_t^s = \frac{1}{K} \sum_{s' \in S} \theta_t^{s'} [\pi^p(s, s', t) + \pi^c(s', s, t) + (K-2)\pi^0(s, s', t)], \tag{3}$$

where

$$\pi^p(s, s', t) = -\frac{y_t^{s,s'}}{\delta_t} + \beta\lambda_{ts}^{s,s'} \Phi_t^{(s_1,\cdot)} v_{t'}^{(s_1,\cdot)} \text{ and } t' \neq t, \tag{4}$$

$$\pi^c(s, s', t) = u(y_t^{s,s'}) + \beta\lambda_{ts'}^{s,s'} \Phi_t^{(s_1',\cdot)} v_{t'}^{(s_1',\cdot)} \text{ and } t' \neq t, \tag{5}$$

and

$$\pi^0(s, s', t) = \beta\eta_{ts}^{s,s'} \Phi_t^{(s_1,\cdot)} v_{t'}^{(s_1,\cdot)} \text{ and } t' \neq t. \tag{6}$$

For given productions, state transitions, and distributions, Blackwell's sufficient conditions for contraction imply that v_t^s exists and is unique. Our objective is

$$\sum_{s_1} [\theta_l^{(s_1,\cdot)} v_l^{(s_1,\cdot)} + \beta\theta_h^{(s_1,\cdot)} v_h^{(s_1,\cdot)}].$$

We next turn to the incentive constraints.

There are truth-telling constraints only for n people with money when they are consumers. They are

$$\pi^c(s,(n,1),t) \geq u(y_t^{s,(n,0)}) + \beta\phi_t^{(n,1)} v_{t'}^{(n,\cdot)} = u(y_t^{s,(n,0)}) + \beta v_{t'}^{(n,1)},$$

where the last equality follows because n people do not surrender money at stage 2.

Next, we have individual rationality constraints for stage 1 meetings. They are

$$\pi^p(s,s',t) \geq \beta\phi_t^{(n,s_2)} v_{t'}^{(n,\cdot)}, \tag{7}$$

$$\pi^c(s,s',t) \geq \beta\phi_t^{(n,s_2')} v_{t'}^{(n,\cdot)}, \tag{8}$$

and

$$\pi^0(s,s',t) \geq \beta\phi_t^{(n,s_2)} v_{t'}^{(n,\cdot)}. \tag{9}$$

We also have $\phi_t^{(n,1)} = (0,1)$ and

$$\phi_t^{(m,s_2)} v_{t'}^{(m,\cdot)} \geq v_{t'}^{(n,s_2)} \text{ for } s_2 = 0,1 \text{ and } \phi_t^{(m,2)} v_{t'}^{(m,\cdot)} \geq v_{t'}^{(n,1)}$$

where the first says that n people do not surrender money at stage 2 and the second says that m people prefer the stage 2 transfers to defecting with their start of stage-2 holdings.

Finally, we have the pairwise-core constraint for stage-1 single-coincidence meetings. We start with a definition of the pairwise core for meetings between n people who can possibly trade.

Definition. For $(s,s') = ((n,0),(n,1))$, we say that $(\pi^p\pi^c)$ is in the pairwise core if it solves the problem,

$$\max_{y_t^{s,s'},\lambda_t^{s,s'}} \gamma\pi^p(s,s',t) + (1-\gamma)\pi^c(s,s',t) \tag{10}$$

subject to (7) and (8) for some $\gamma \in [0,1]$.

We now define a function that, as we show below, gives us the above pairwise core. Let $a_t = \beta\phi_t^{(n,0)} v_{t'}^{(n,\cdot)}$ (the lowest possible payoff for the producer in the above problem) and let b_t be the solution for $\pi^p(n0,n1,t)$ to the above problem for $\gamma = 1$ (the highest possible payoff to the producer in the above problem). Let $\psi : [a_t, b_t] \to R$ be defined by

$$\psi(x) \equiv \begin{cases} \varsigma_t - x & \text{if } x \in [a_t,\ \rho_t) \\ u[\delta_\tau(\beta\phi_t^{(n,1)} v_{t'}^{(n,\cdot)} - x)] + a_t & \text{if } x \in [\rho_t,\ b_t] \end{cases}. \tag{11}$$

where

$$\rho_t = \beta\phi_t^{(n,1)} v_{t'}^{(n,\cdot)} - \frac{y_t^*}{\delta_t},$$

and

$$\varsigma_t = u(y_t^*) - \frac{y_t^*}{\delta_t} + \beta\left(\phi_t^{(n,0)} + \phi_t^{(n,1)}\right) v_{t'}^{(n,\cdot)}.$$

Notice that ρ_t is π^p for the producer who acquires money with probability 1 and who produces the surplus maximizing amount of output, while ς_t is the sum of π^p and π^c if that output is produced.

Lemma. For $(s, s') = ((n, 0), (n, 1))$, $(\pi^p \pi^c)$ is in the pairwise core iff $(\pi^p \pi^c) \in \{(x, \Psi(x)) : x \in [a_t, b_t]\}$.

Proof: First we note that the preservation-of-money constraints for this meeting imply $\lambda_{ts}^{s,s'} = (1-\lambda, \lambda, 0)$ and $\lambda_{ts'}^{s,s'} = (\lambda, 1-\lambda, 0)$ for some $\lambda \in [0, 1]$, where λ is the probability that the producer acquires money. Also, because the objective in the definition is concave in y (output) and λ and the constraint set is convex, the following first-order conditions are necessary and sufficient for that problem:

$$-\frac{1}{\delta_t}(\gamma + \sigma^s) + \left(1 - \gamma + \sigma^{s'}\right) u'(y) \begin{cases} = 0 \text{ if } y > 0 \\ \leq 0 \text{ if } y = 0 \end{cases}, \tag{12}$$

$$[(\gamma + \sigma^s) - \left(1 - \gamma + \sigma^{s'}\right)]\beta(\phi_t^{(n,1)} - \phi_t^{(n,0)}) v_{t'}^{(n,\cdot)} \begin{cases} \geq 0 \text{ if } \lambda = 1 \\ = 0 \text{ if } 0 < \lambda < 1 \\ \leq 0 \text{ if } \lambda = 0 \end{cases}, \tag{13}$$

where $\sigma^s \geq 0$ is the Lagrange multiplier associated with (7) and $\sigma^{s'} \geq 0$ is that associated with (8). Notice that if the second of these holds at equality, then $y = y_t^*$. And if $\lambda = 1$ and the second holds with strict inequality, then $y < y_t^*$. We provide a proof for the case, $a_t < \rho_t < b_t$ and $(\phi_t^{(n,1)} - \phi_t^{(n,0)}) v_{t'}^{(n,\cdot)} > 0$, the latter being valued money for n people. (The restriction on ρ_t says that $(y, \lambda) = (y_t^*, 1)$ is interior with respect to constraints (7) and (8). Cases where $\rho_t < a_t$ or $\rho_t > b_t$ are similar.

Necessity. There are three cases: $\gamma \in \left[0, \frac{1}{2}\right)$, $\gamma = \frac{1}{2}$, and $\gamma \in \left(\frac{1}{2}, 1\right]$.

If $\gamma \in \left[0, \frac{1}{2}\right)$, then condition (13) and the producer incentive compatibility constraint imply that $\sigma^s > 0$. Thus, (7) binds, which implies that $\pi^p = a_t$. Moreover, $a_t < \rho_t$ implies $y = y_t^*$ and $\lambda \in (0, 1)$. The former implies that the implied $\pi^c = \varsigma_t - a_t = \psi(a_t)$, where, as noted above, ς_t is the sum of the payoffs implied by $y = y_t^*$.

If $\gamma = \frac{1}{2}$ and (7)is slack, then condition (12) implies that $y = y_t^*$. This yields,

$$\pi^p = -\frac{y_t^*}{\delta_t} + \beta(\phi_t^{(n,1)} - \phi_t^{(n,0)})v_{t'}^{(n,\cdot)}\lambda + \beta\phi_t^{(n,0)}v_{t'}^{(n,\cdot)} \tag{14}$$

and

$$\pi^c = u(y_t^*) - \beta(\phi_t^{(n,1)} - \phi_t^{(n,0)})v_{t'}^{(n,\cdot)}\lambda + \beta\phi_t^{(n,1)}v_{t'}^{(n,\cdot)} = \varsigma_t - \pi^p = \psi(\pi^p)$$

If (7) is not slack, then we have $\pi^p = a_t$ as in the first case.

If $\gamma \in \left(\frac{1}{2}, 1\right]$, then $\rho_t < b_t$ implies $\lambda = 1$. Therefore,

$$\pi^p = -\frac{y}{\delta_t} + \beta\phi_t^{(n,1)}v_{t'}^{(n,\cdot)} \tag{15}$$

and

$$\pi^c = u(y) + \beta\phi_t^{(n,0)}v_{t'}^{(n,\cdot)} = u[\delta_t(\beta\phi_t^{(n,1)}v_{t'}^{(n,\cdot)} - \pi^p)] + a_t = \psi(\pi^p),$$

where the second equality comes from substituting for y from (15).

Sufficiency. The proof proceeds by construction. For every $x \in [a_t, b_t]$ we propose a candidate solution $(\gamma, y, \lambda, \sigma^s, \sigma^{s'})$. Because the first-order conditions (12) and (13) are sufficient for the solution, it is enough to show that the candidate satisfies them. We consider four cases: $x = a_t$, $x \in (a_t, \rho_t)$, $x \in [\rho_t, b_t)$, and $x = b_t$.

Case $x = a_t$. Because $a_t < \rho_t$, our candidate solution for this case is: $\gamma \in \left[0, \frac{1}{2}\right)$, $y = y_t^*$,

$$\lambda = \frac{\frac{y_t^*}{\delta_t}}{\beta(\phi_t^{(n,1)} - \phi_t^{(n,0)})v_{t'}^{(n,\cdot)}} \in (0,1),$$

$\sigma^s = 1 - 2\gamma$, and $\sigma^{s'} = 0$.

Case $x \in (a_t, \rho_t)$. In that case the candidate is: $\gamma = \frac{1}{2}$, $y = y_t^*$,

$$\lambda = \frac{x + \frac{y_t^*}{\delta_t} - \beta\phi_t^{(n,0)}v_{t'}^{(n,\cdot)}}{\beta\left(\phi_t^{(n,1)} - \phi_t^{(n,0)}\right)v_{t'}^{(n,\cdot)}} \in (0,1),$$

and $\sigma^s = \sigma^{s'} = 0$.

Case $x \in [\rho_t, b_t)$. In that case the candidate is:

$$\gamma = \frac{\delta_t u'\left(\delta_t\left(\beta\phi_t^{(n,1)}v_{t'}^{(n,\cdot)} - x\right)\right)}{1 + \delta_t u'\left(\delta_t\left(\beta\phi_t^{(n,1)}v_{t'}^{(n,\cdot)} - x\right)\right)} \in \left[\frac{1}{2}, \overline{\gamma}_t\right). \tag{16}$$

$\lambda = 1$,

$$y = \delta_t \left(\beta \phi_t^{(n,1)} v_{t'}^{(n,\cdot)} - x \right),$$

and $\sigma^s = \sigma^{s'} = 0$. The value $\overline{\gamma}_t$, $\overline{\gamma}_t < 1$, obtains by substitution of the maximum producer payoff, $x = b_t$, in the expression (16).

Case $x = b_t$. In that case the candidate is: $\gamma \in [\overline{\gamma}_t, 1]$,

$$y = \delta_t \left(\beta \phi_t^{(n,1)} v_{t'}^{(n,\cdot)} - b_t \right),$$

$\lambda = 1$, $\sigma^s = 0$, and

$$\sigma^{s'} = \gamma \frac{1 + \delta_t u'(y)}{\delta_t u'(y)} - 1 \in \left[0, \frac{1}{\delta_t u'(y)} \right].$$

Direct substitution of the candidates in the first-order conditions (12), (13) completes the proof of the Lemma. ■

Because defection by an m person converts the person to an n person, the payoffs in a single-coincidence meeting in which one person is an m or both are m people must satisfy

$$\pi^c(s, s', t) \geq \Psi(\pi^p(s, s', t)).$$

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