A new (n+1)-dimensional generalized Kadomtsev-Petviashvili equation: Integrability characteristics and localized solutions

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A new (n+1)-dimensional generalized Kadomtsev-Petviashvili equation: Integrability characteristics and localized solutions

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Abstract In this paper, an integrable generalization of the Kadomtsev-Petviashvili (KP) equation in arbitrary spatial dimension is proposed. Firstly, the singularity manifold analysis is performed to prove that the (n+1)-dimensional KP equation with general form is Painlevé integrable. Secondly, combining the truncated Painlevé expansion and binary Bell polynomial approach, the integrable characteristics of the (n+1)-dimensional KP equation are derived systematically, including N-soliton solution, bilinear Bäcklund transformation, the associated Lax pair as well as infinite conservation laws. Moreover, various types of localized solutions can be constructed starting from the N-soliton solutions. The abundant interactions including overtaking solitons, head-on solitons, one-order lump, two-order lump, breather, breather-soliton mixed solutions are analyzed by some graphs.

Keywords (n+1)-dimensional Kadomtsev-Petviashvili equation · Painlevé property · Bäcklund transformation · Infinite conservation laws · Localized solutions

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1 Introduction

During the past decades, mathematical physicists have devoted great efforts in developing a number of effective and efficient methods for solving integrable systems, including the inverse scattering transform (IST), Bäcklund transformation, Darboux transformation and Hirota's bilinear method amongst many others [1–5]. As is well known, many researches focus on the integrable systems in (1+1)-dimensions and (2+1)-dimensions [6–16]. Considering the fact that the real situation is in (3+1)-dimensions, a number of (3+1)-dimensional equations have been proposed starting from the lower dimensional integrable models, and their various types of exact solutions have been presented [17–22]. Among these (3+1)-dimensional equations, most of them do not pass the conventional integrability test. Therefore, searching for more integrable models in 3+1 dimensional or more higher dimensions, is a challenging and significant research issue in nonlinear science.

The KP equation formulated by Kadomtsev and Petviashvili in 1970 [23], as a two spatial dimensional analog of the classic Korteweg-de Vries equation, can well simulate nonlinear phenomena in fluid physics, plasma physics, Bose-Einstein condensates, optics etc. Due to the physical and mathematical significance, KP equation attracts much attention of scholars, its integrable features and various exact solutions have been investigated by using different methods [24–31]. Considering that the real situation is in 3+1 dimensions, various higher dimensional extensions of KP equation have been investigated from different viewpoints [32–37]. However, these higher dimensional extensions do not keep their original integrable characteristics since the Painlevé property, N-soliton, Lax pair, symmetry structure as well as infinite conservation laws are no longer conserved.

The question that naturally arises is whether the well-known KP equation can be extended to three spatial dimensions or more higher dimensional space. More importantly, the extended KP equations in higher dimensions are expected to possess the same integrable properties as the classic KP equation in 2+1 dimensions. This paper aims to give an affirmative answer to this question. For this purpose, we propose a new generalized KP equation in n+1 dimensions

$$(u_t+\alpha u u_{x_1}+\beta u_{x_1x_1x_1})_{x_1}+\gamma u_{x_2x_2}+\sum_{i=1}^{n}\sigma_i\, u_{x_1x_i}=0, \quad (1)$$

where $n \geq 2$, u is a differentiable function with respect to spatial variables $x_1, x_2, \cdots, x_n$ and time variable t, the subscripts represent the partial derivatives, and α, β, γ and $\sigma_i (i = 1, \cdots, n)$ are constant parameters.

When $\alpha = 6$, $\beta = 1$, $\gamma = 3\epsilon^2$ and $\lambda = \sigma_1 = \sigma_2 = 0$, and $x_1 = x$, $x_2 = y$, equation (1) becomes [23]

$$(u_t + 6uu_x + u_{xxx})_x + 3\epsilon^2 u_{yy} = 0, \ \epsilon^2 = \pm 1. \quad (2)$$

when $\epsilon = i$ and $\epsilon = 1$, (2) is exactly the the KPI equation and KPII equation, respectively. The change in sign of ϵ^2 is related to the magnitudes of gravity and surface tension.

When $x_1 = x$, $x_2 = y$, $x_3 = z$, equation (1) may be reduced to the (3+1)-dimensional generalized KP equation

$$\begin{aligned}(u_t + \alpha uu_x + \beta u_{xxx})_x + \gamma u_{yy} + \sigma_1 u_{xx}\\ + \sigma_2 u_{xy} + \sigma_3 u_{xz} = 0.\end{aligned} \quad (3)$$

Note that another (3+1)-dimensional KP equation investigated in Ref. [32] fails the integrability test due to the existence of second order dispersion term u_{zz}.

The remaining parts of the paper are arranged as follows. In Section 2, the singularity manifold analysis is conducted to prove that equation (1) is Painlevé integrable based on the WTC method. Subsequently, we employ the binary Bell polynomial approach to investigate several integrable features of Eq. (1). The bilinear form and N-soliton solution are constructed in Section 3, the bilinear Bäcklund transformation and Lax pair are derived in Section 4. Section 5 is devoted to prove that there exists infinite conservation laws for equation (1). In Section 6, some localized solutions and interactions of multiple waves are analyzed by graphs. Finally, some brief conclusions are given in Section 7.

2 Painlevé property

The singularity manifold analysis is a very useful and effective tool in the analysis of Painlevé property of nonlinear evolution models. According to the WTC method [2], Eq. (1) is said to have the Painlevé property if its solution can be written as Laurent series

$$u = \sum_{j=0}^{\infty} u_j\, \phi^{j+k} \quad (4)$$

and it is "single-valued" in the neighborhood of singularity manifold ϕ. Note that in (4), both ϕ and $u_j (j = 0, 1, \cdots)$ are analytic functions of $\{x_1, x_2, \cdots, x_n, t\}$. To reduce the computational complexity, we take the Kruskal's ansatz about singular manifold, i.e., letting $\phi = x_1 + \psi(x_2, \cdots, x_n, t)$.

First, substituting the series (4) into (1) and balancing the most dominant terms, yields the exponent and the coefficient of leading term in (4), $k = -2$ and $u_0 = -12\beta/\alpha$.

Subsequently, we need to determine the resonance points at which the coefficients in (4) are arbitrary. To this end, inserting $u = u_0\phi^{-2} + u_j\phi^{j-2}$ into (1), and vanishing the coefficients of ϕ^{j-6}, one obtains the general recursion relation about u_j. It easily follows from the recursion relation that the four resonant points occur at $j = -1$, 4, 5 and 6.

Finally, one should verify the compatibility conditions for each non-negative resonant point. For this purpose, inserting the truncated series

$$u = \sum_{j=0}^{6} u_j\, \phi^{j-2} \quad (5)$$

into (1) and equating the coefficients of ϕ with different powers, it is obtained as

$$
\begin{aligned}
&u_0=-12\beta/\alpha, \qquad u_1=0,\\
&u_2=-(\sigma_1+\psi_t+\gamma\psi_{x_2}^2+\sum_{i=2}^{n}\sigma_i\psi_{x_i})/\alpha,\\
&u_3=\gamma\psi_{x_2x_2}/\alpha.
\end{aligned} \tag{6}
$$

The compatibility conditions corresponding to the remained resonant points $4,5,6$ are listed as follows,

$$
\begin{aligned}
&u_{1,t}+\gamma u_1\psi_{x_2x_2}+2\gamma u_{1,x_2}\psi_{x_2}+\sum_{i=2}^{n}\sigma_i\,u_{1,x_i}\;=\;0,\\
&u_{1,x_2x_2}\;=\;0,\\
&u_{3,t}+\gamma u_{2,x_2x_2}+2u_4\psi_t+\gamma u_3\psi_{x_2x_2}+2\gamma u_{3,x_2}\psi_{x_2}\\
&\quad+2\alpha u_1u_5+2\alpha u_2u_4+\alpha u_3^2+2\gamma u_4\psi_{x_2}^2+2\sigma_1u_4\\
&\quad+\sum_{i=2}^{n}\sigma_i(2u_4\psi_{x_i}+u_{3,x_i})\;=\;0.
\end{aligned} \tag{7}
$$

Combing the u_0, u_1, u_2 and u_3 values given by (6), the above three compatibility conditions are proven to be satisfied identically. In other words, u_4, u_5 and u_6 in (5) are arbitrary functions. Therefore, it is concluded that the new (n+1)-dimensional KP equation (1) with general form possesses the Painlevé property without any constraints between the parameters α,β,γ and $\sigma_i(i=1,\cdots,n)$.

3 N-soliton solutions

The binary Bell polynomial method [3] provides a direct and effective framework to construct Bäcklund transformation, Lax pair and other integrable characteristics in a systematic way, which is widely applied in a great number of NLEEs [38–42]. Starting from the above singularity manifold analysis, one obtains the truncated Painlevé transformation $u=\frac{12\beta}{\alpha}(\ln\phi)_{x_1x_1}+u_1$. Letting $q=2\ln\phi$ and $u_1=0$, and integrating (1) with respect to x_1 twice yields

$$
q_{x_1t}+3\beta\,q_{x_1x_1}^2+\beta\,q_{x_1x_1x_1x_1}+\gamma\,q_{x_2x_2}+\sum_{i=1}^{n}\sigma_iq_{x_1x_i}=0, \tag{8}
$$

which can be rewritten as the $\mathscr{P}$-polynomial equation with the form,

$$
\begin{aligned}
&\mathscr{P}_{x_1t}(q)+\beta\mathscr{P}_{x_1x_1x_1x_1}(q)+\gamma\mathscr{P}_{x_2x_2}(q)\\
&\quad+\sum_{i=1}^{n}\sigma_i\mathscr{P}_{x_1x_i}(q)=0.
\end{aligned} \tag{9}
$$

With the assumption $q=2\ln\phi$, the above $\mathscr{P}$-polynomial equation is reduced to the bilinear representation of the (n+1)-dimensional KP equation (1)

$$
(D_{x_1}D_t+\beta D_{x_1}^4+\gamma D_{x_2}^2+\sum_{i=1}^{n}\sigma_i\,D_{x_1}D_{x_i})\phi\,\cdot\,\phi=0. \tag{10}
$$

Based on the bilinear form (10), and employing the perturbation technique [5], one obtains the N-soliton solutions of equation (1), which reads

$$
u=\frac{12\beta}{\alpha}\cdot\frac{\partial^2}{\partial x_1^2}\ln\left(\sum_{\rho=0,1}\exp\left(\sum_{j=1}^{N}\rho_j\eta_j+\sum_{1\le j<s}^{N}\rho_j\rho_s\,a_{js}\right)\right), \tag{11}
$$

with

$$
\begin{aligned}
&\eta_j=\sum_{i=1}^{n}k_{ji}\,(x_i+\omega_j\,t)+\eta_{j0},\\
&\omega_j=-\,(\sigma_1+\beta k_{j1}^2+\gamma k_{j2}^2+\sum_{i=2}^{n}\sigma_i\,k_{ji}),\\
&a_{js}=\frac{3\beta(k_{j1}-k_{s1})^2\,-\,\gamma(k_{j2}-k_{s2})^2}{3\beta(k_{j1}+k_{s1})^2\,-\,\gamma(k_{j2}-k_{s2})^2},
\end{aligned} \tag{12}
$$

where $\sum_{\rho=0,1}$ means a summation of possible combinations about $\rho_j=0,1(j=1,\cdots,N)$. Note that k_{ji} and η_{j0} $(j=1,\cdots,N;i=1,\cdots,n)$ are arbitrary constants.

4 Bäcklund transformation and Lax pair

To construct the Bäcklund transformation, it is assumed that $q=2\ln F$ and $\bar{q}=2\ln G$ are two different solutions of equation (8). In addition, introducing w and v such that $w=(\bar{q}+q)/2$, $v=(\bar{q}-q)/2$, we get

$$
\begin{aligned}
\frac{E(\bar{q})-E(q)}{2}&=v_{x_1t}+6\beta v_{x_1x_1}w_{x_1x_1}+\beta v_{x_1x_1x_1x_1}\\
&\quad+\gamma\,v_{x_2x_2}+\sum_{i=1}^{n}\sigma_i\,v_{x_1x_i}\\
&=\frac{\partial}{\partial x_1}(\mathscr{Y}_t(v)+\beta\mathscr{Y}_{x_1x_1x_1}(v,w))+R,
\end{aligned} \tag{13}
$$

where R is given by

$$
\begin{aligned}
R&=3\beta\,v_{x_1x_1}w_{x_1x_1}-3\beta v_{x_1}w_{x_1x_1x_1}-3\beta v_{x_1}^2v_{x_1x_1}\\
&\quad+\gamma v_{x_2x_2}+\sum_{i=1}^{n}\sigma_i\,v_{x_1x_i}.
\end{aligned} \tag{14}
$$

Then, to write (14) as the $\mathscr{Y}$-polynomial, we suppose

$$w_{x_1x_1} + v_{x_1}^2 + \delta\, v_{x_2} = \lambda, \tag{15}$$

where λ and δ are constant parameters. Then, after some calculations, Eq. (14) can be rewritten as

$$\begin{aligned} R = 3\lambda\beta v_{x_1x_1} - \tfrac{\gamma}{\delta}(w_{x_1x_1x_2} + (2 - \tfrac{3\delta^2\beta}{\gamma})v_{x_1}v_{x_1x_2} \\ + \tfrac{3\delta^2\beta}{\gamma}v_{x_1x_1}v_{x_2}) + \sum_{i=1}^{n}\sigma_i\, v_{x_1x_i}. \end{aligned} \tag{16}$$

By setting $\delta^2 = \gamma/(3\beta)$, equation (16) is equivalent to

$$\begin{aligned} R = \frac{\partial}{\partial x_1}(\, (3\lambda\beta + \sigma_1)\mathscr{Y}_{x_1}(v) - \frac{\gamma}{\delta}\,\mathscr{Y}_{x_1x_2}(v,w) \\ + \sum_{i=2}^{n}\sigma_i\,\mathscr{Y}_{x_i}(v)). \end{aligned} \tag{17}$$

Finally, combining (13), (15) and (17), one obtains

$$\begin{aligned} &\mathscr{Y}_{x_1x_1}(v,w) + \delta\,\mathscr{Y}_{x_2}(v) - \lambda = 0, \\ &\frac{\partial}{\partial x_1}(\mathscr{Y}_t(v) + \beta\mathscr{Y}_{x_1x_1x_1}(v,w) + (3\lambda\beta + \sigma_1)\mathscr{Y}_{x_1}(v) \\ &\quad - \frac{\gamma}{\delta}\,\mathscr{Y}_{x_1x_2}(v,w) + \sum_{i=2}^{n}\sigma_i\,\mathscr{Y}_{x_i}(v)) = 0. \end{aligned} \tag{18}$$

With proper transformation, the $\mathscr{Y}$-polynomials can be reduced to Hirota's differential operators. Based on this, the above $\mathscr{Y}$-polynomial system (18) yields the bilinear BT of Eq. (1), which reads

$$\begin{aligned} &(D_{x_1}^2 + \delta\, D_{x_2} - \lambda)F \cdot G = 0, \\ &(\, D_t + \beta\, D_{x_1}^3 + (3\lambda\beta + \sigma_1)D_{x_1} - \frac{\gamma}{\delta}D_{x_1}D_{x_2} \\ &\quad + \sum_{i=2}^{n}\sigma_i\, D_{x_i}\,)\, F \cdot G = 0. \end{aligned} \tag{19}$$

where $\delta^2 = \gamma/(3\beta)$.

To derive the Lax pair, we take

$$w = v + q, \quad v = \ln\varphi. \tag{20}$$

Starting from the bilinear Bäcklund transformation (18), through the linearizing technique, the Lax pair of (1) is as follows,

$$\begin{aligned} &\varphi_{x_1x_1} + \delta\varphi_{x_2} + q_{x_1x_1}\varphi - \lambda\,\varphi = 0, \\ &\varphi_t - (\frac{\gamma}{\delta} + \beta\delta)\varphi_{x_1x_2} - (\beta q_{x_1x_1x_1} + \frac{\gamma}{\delta}q_{x_1x_2})\varphi \\ &\quad + (2\beta q_{x_1x_1} + 4\lambda\beta + \sigma_1)\varphi_{x_1} + \sum_{i=2}^{n}\sigma_i\,\varphi_{x_i} = 0, \end{aligned} \tag{21}$$

where $\delta^2 = \gamma/(3\beta)$, and λ is an arbitrary constant parameter. The two equations in (21) are compatible provided that q satisfies Eq. (8), i.e., $\varphi_{x_1x_1t} = \varphi_{tx_1x_1}$. In other words, the system (21) is just the Lax pair of equation (1).

As a particular case of equation (1) in 3+1 dimensions, its bilinear Bäcklund transformation and the associated Lax pair may be easily derived. Here, we denote $x_1 = x$, $x_2 = y$, $x_3 = z$, (19) yields the bilinear Bäcklund transformation of equation (3) with the form,

$$\begin{aligned} &(D_x^2 + \delta\, D_y - \lambda)F \cdot G = 0, \\ &(\, D_t + \beta\, D_x^3 - \frac{\gamma}{\delta}D_xD_y + (3\lambda\beta + \sigma_1)D_x \\ &\quad + \sigma_2 D_y + \sigma_3 D_z\,)\, F \cdot G = 0. \end{aligned} \tag{22}$$

The corresponding Lax representation of (3) is given by

$$\begin{aligned} &\varphi_{xx} + \delta\varphi_y + q_{xx}\,\varphi - \lambda\,\varphi = 0, \\ &\varphi_t - (\frac{\gamma}{\delta} + \beta\delta)\varphi_{xy} + (2\beta q_{xx} + 4\lambda\beta + \sigma_1)\varphi_x \\ &\quad - (\beta q_{xxx} + \frac{\gamma}{\delta}q_{xy})\varphi + \sigma_2\,\varphi_y + \sigma_3\,\varphi_z = 0. \end{aligned} \tag{23}$$

Under the constraint $\delta^2 = \gamma/(3\beta)$, the compatibility condition $\varphi_{xxt} = \varphi_{txx}$ implies that

$$q_{xt} + \beta q_{xxxx} + 3\beta q_{xx}^2 + \gamma q_{xy} + \sigma_1 q_{xx} + \sigma_2 q_{xy} + \sigma_3 q_{xz} = 0,$$

which indicates that the system (23) is exactly the Lax pair of equation (3).

5 Infinite conservation laws

As discussed in Section 3-4, equation (1) is both Painlevé integrable and Lax integrable. The existence of infinite conservation laws is also an essential and significant integrable property. In this section, starting from the coupled $\mathscr{Y}$-polynomials system (18), we will derive the infinite conservation laws of equation (1) in arbitrary spatial dimension.

First, by introducing $\eta = (\bar{q}_{x_1} - q_{x_1})/2$, the functions v and w are related by

$$v_{x_1} = \eta, \quad w_{x_1} = q_{x_1} + \eta. \tag{24}$$

Inserting it into the coupled $\mathscr{Y}$-polynomials system (18) leads to

$$\begin{aligned}&q_{x_1x_1}+\eta_{x_1}+\eta^2+\delta\,\partial_{x_1}^{-1}\eta_{x_2}-\lambda=0,\\&\frac{\partial}{\partial t}(\eta)+\frac{\partial}{\partial x_1}[\beta\eta_{x_1x_1}+3\beta\,\eta(q_{x_1x_1}+\eta_{x_1})\\&\quad-\frac{\gamma}{\delta}(q_{x_1x_2}+\eta_{x_2}+\eta\partial_{x_1}^{-1}\eta_{x_2}))\\&\quad+(3\lambda\beta+\sigma_1)\eta+\beta\eta^3]+\sum_{i=2}^{n}\frac{\partial}{\partial x_i}(\sigma_i\eta)=0.\end{aligned}\tag{25}$$

Suppose that $\eta=\bar{\eta}+\epsilon$ and $\lambda=\epsilon^2$, equation (25) is reduced to

$$\bar{\eta}_{x_1}+\bar{\eta}^2+2\epsilon\,\bar{\eta}+\delta\,\partial_{x_1}^{-1}\bar{\eta}_{x_2}+q_{x_1x_1}=0,\tag{26}$$

$$\begin{aligned}&\frac{\partial}{\partial t}(\bar{\eta})+\frac{\partial}{\partial x_1}[4\epsilon^2\beta\bar{\eta}+(2\beta q_{x_1x_1}+2\beta\bar{\eta}_{x_1}\\&-\beta\delta\partial_{x_1}^{-1}\bar{\eta}_{x_2}-\frac{\gamma}{\delta}\partial_{x_1}^{-1}\bar{\eta}_{x_2})\epsilon-(\beta\delta-\frac{\gamma}{\delta})\bar{\eta}\partial_{x_1}^{-1}\bar{\eta}_{x_2}\\&+2\beta\bar{\eta}\bar{\eta}_{x_1}-\frac{\gamma}{\delta}(q_{x_1x_2}+\bar{\eta}_{x_2})+2\beta q_{x_1x_1}\bar{\eta}\\&+\beta\bar{\eta}_{x_1x_1}+\sigma_1\bar{\eta}]+\sum_{i=2}^{n}\frac{\partial}{\partial x_i}(\sigma_i\,\bar{\eta})=0.\end{aligned}\tag{27}$$

Next, we expand the function $\bar{\eta}$ as the series with the form,

$$\bar{\eta}=\sum_{j=1}^{\infty}\mathscr{I}_j\,(q,q_{x_1},\cdots)\,\epsilon^{-j}.\tag{28}$$

Inserting the series (28) into (26) and collecting the coefficients with the same power of ϵ, we have the explicit recursion formulae for the conversed densities $\mathscr{I}_n$

$$\begin{aligned}&\mathscr{I}_1=-q_{x_1x_1}/2=-\alpha u/(12\beta),\\&\mathscr{I}_2=(\alpha u_{x_1}+\alpha\delta\partial_{x_1}^{-1}u_{x_2})/(24\beta),\\&\mathscr{I}_{j+1}=-\frac{\mathscr{I}_{j,x_1}+\delta\partial_{x_1}^{-1}\mathscr{I}_{j,x_2}+\sum_{k=1}^{j}\mathscr{I}_k\mathscr{I}_{j-k}}{2},\quad j\geq 2.\end{aligned}\tag{29}$$

Finally, combining (27) and (28), one obtains the infinite conservation laws of the (n+1)-dimensional generalized KP equation (1),

$$\mathscr{I}_{j,t}+\mathscr{F}_{j,x_1}+\sum_{i=2}^{n}\mathscr{G}_{ij,x_i}=0,\ \ j=1,2,\cdots,\tag{30}$$

where $\mathscr{I}_j$ is presented by (29). Moreover, the recursion formulae of the fluxes $\mathscr{F}_n$ is obtained as

$$\begin{aligned}&\mathscr{F}_1=-(2\alpha\beta u_{x_1x_1}+2\alpha\gamma\int\int u_{x_2x_2}dx_1dx_1\\&\quad+\alpha^2u^2+2\alpha\sigma_1 u)/(24\beta)\\&\mathscr{F}_j=4\beta\mathscr{I}_{j+2}+2\beta\mathscr{I}_{j+1,x_1}-(\beta\delta+\tfrac{\gamma}{\delta})\partial_{x_1}^{-1}\mathscr{I}_{j+1,x_2}\\&+2\beta\sum_{k=1}^{j}\mathscr{I}_k\mathscr{I}_{j-k,x_1}+\beta\mathscr{I}_{j,x_1x_1}+(2\beta q_{x_1x_1}+\sigma_1)\mathscr{I}_j\\&-\tfrac{\gamma}{\delta}\mathscr{I}_{j,x_2}-(\beta\delta+\tfrac{\gamma}{\delta})\beta\sum_{k=1}^{j}\mathscr{I}_k\partial_{x_1}^{-1}\mathscr{I}_{j-k,x_2},\,j\geq 2.\end{aligned}$$

Other fluxes $\mathscr{G}_n$ are given by

$$\begin{aligned}&\mathscr{G}_{21}=-\frac{\alpha\sigma_2}{12\beta}u,\ \ \mathscr{G}_{22}=\frac{\sigma_2(\alpha u_{x_1}+\alpha\delta\partial_{x_1}^{-1}u_{x_2})}{24\beta},\\&\mathscr{G}_{31}=-\frac{\alpha\sigma_3}{12\beta}u,\ \ \mathscr{G}_{32}=\frac{\sigma_3(\alpha u_{x_1}+\alpha\delta\partial_{x_1}^{-1}u_{x_2})}{24\beta},\\&\mathscr{G}_{ij}=\sigma_i\,\mathscr{I}_j,\quad i=2,\cdots,n;\ j=2,3,\cdots.\end{aligned}$$

In conclusion, equation (1) has N-soliton solutions, and it also possesses the Painlevé property, Lax pair, bilinear BT as well as infinite conservation laws, thus it is concluded that the proposed (n+1)-dimensional generalized KP equation (1) is completely integrable.

6 Multiple solitons and localized solutions of equation (3)

In this part, the (3+1)-dimensional KP equation (3) is chosen to illustrate the interactions between multiple solitons and localized solutions more intuitively. For convenience, we set $x_1=x$, $x_2=y$, $x_3=z$, $k_{j1}=k_j$, $k_{j2}=l_j$, $k_{j3}=m_j$, combining (11)-(12), the N-soliton solution of equation (3) is written as

$$u=\frac{12\beta}{\alpha}\cdot(\ln\phi)_{xx},\tag{31}$$

with

$$\begin{aligned}&\phi=\sum_{\rho=0,1}\exp\left(\sum_{j=1}^{N}\rho_j\eta_j+\sum_{1\leq j<s}^{N}\rho_j\rho_s\,a_{js}\right),\\&\eta_j=k_j(x+l_jy+m_jz+\omega_j\,t)+\eta_{j0},\\&\omega_j=-(\sigma_1+\beta k_j^2+\gamma l_j^2+\sigma_2 l_j+\sigma_3 m_j),\\&a_{js}=\frac{3\beta(k_j-k_s)^2-\gamma(l_j-l_s)^2}{3\beta(k_j+k_s)^2-\gamma(l_j-l_s)^2}\end{aligned}\tag{32}$$

where $\sum_{\rho=0,1}$ means a summation of possible combinations for $\rho_j=0$ and 1. The remained parameters k_j,l_j,m_j and $\eta_{j0}(j=1,\cdots,N)$ are arbitrary constants.

6.1 Two solitons and localized solutions

Taking $N = 2$ in (32), equation (31) with (32) leads to the two-soliton solution. Due to the existence of arbitrary parameters k_j, l_j and $m_j (j = 1, 2)$ in (32), there are various types of interactions of multiple waves.

Case 1. Two-soliton solution

In (32), we take the parametric choices

$$\begin{aligned} &k_1 = 0.8, k_2 = l_1 = m_1 = \sigma_1 = \beta = -\gamma = 1.0, \\ &l_2 = 0.7, m_2 = 0.6, \sigma_2 = 0.5, \sigma_3 = 1.2, \alpha = 6, \end{aligned} \tag{33}$$

the overtaking interactions between two solitons are shown in Figure 1. Fig. 1a depicts the evolution of two solitons along x direction. It is easily observed that these two solitons move along the positive x direction with different speeds, and the soliton with smaller amplitude moves faster and it overtakes the soliton with larger amplitude. In Fig. 1b and Fig. 1c, two solitons move along the positive y and z direction with different speeds, and the soliton with larger amplitude moves faster and it overtakes the soliton with smaller amplitude.

For the parametric choices

$$\begin{aligned} &k_1 = 0.8,\ k_2 = m_1 = \gamma = -1, l_1 = 0.5,\ l_2 = 0.3, \\ &m_2 = 0.6, \sigma_1 = \sigma_2 = \beta = 1, \sigma_3 = 1.2, \alpha = 6, \end{aligned} \tag{34}$$

Figure 2 shows the head-on collisions between two solitons. In Fig. 2a, the smaller amplitude soliton propagates along the positive x direction, and the larger amplitude soliton moves along the negative x direction, they still keep their original wave speeds and shapes after the head-on interactions. Similarly, Fig. 2b and Fig. 2c display the evolutions of two solitons along with y and z directions, and there also exist head-on interactions between two solitons.

Case 2. One-order lump solution

By employing the long wave limit method, the lump solutions can be constructed from solitons with even orders. Starting from the above two soliton solution, one can obtain the one-order lump solution. Taking the limit $k_j \to 0$, $e^{\eta_{j0}} = -1$ and $k_1/k_2 = O(1)$ $(j = 1, 2)$, along with (32), we have

$$\phi = \vartheta_1 \vartheta_2 + \frac{12\beta}{\gamma(l_1 - l_2)^2} \tag{35}$$

where

$$\begin{aligned} &\vartheta_j = x + l_j y + m_j z + \omega_j t, \\ &\omega_j = -(\sigma_1 + \gamma l_j^2 + \sigma_2 l_j + \sigma_3 m_j), \quad j = 1, 2. \end{aligned} \tag{36}$$

In order to rewrite (35) as quadratic functions, we set $l_1 = \rho_1 + i\nu_1$, $l_2 = \rho_1 - i\nu_1$, $m_1 = \kappa_1 + i\lambda_1$, $m_2 = \kappa_1 - i\lambda_1$. Then we have

$$\phi = (x+\rho_1 y+\kappa_1 z+\tau_1 t)^2+(\nu_1 y+\lambda_1 z+\theta_1 t)^2-\frac{3\beta}{\gamma\,\nu_1^2}. \tag{37}$$

where

$$\begin{aligned} &\tau_1 = \gamma\nu_1^2 - \gamma\rho_1^2 - \kappa_1\sigma_3 - \rho_1\sigma_2 - \sigma_1, \\ &\theta_1 = -2\gamma\rho_1\nu_1 - \lambda_1\sigma_3 - \nu_1\sigma_2. \end{aligned} \tag{38}$$

Together with the transformation (31), we get the one-order lump solution of Eq. (3) as displayed in Fig. 3.

Case 3. One-order breather solution

If taking $k_1 = k_2 = \mu_1$, $l_1 = l_2^* = \rho_1 + i\nu_1$, $m_1 = m_2^* = \kappa_1 + i\lambda_1$, $\eta_{10} = \eta_{20} = 0$, (32) is reduced to

$$\phi = 1 + 2e^{\xi_1}\ \cos(\xi_2) + \frac{\gamma\nu_1^2}{3\beta\mu_1^2 + \gamma\nu_1^2}\, e^{2\xi_1} \tag{39}$$

where

$$\begin{aligned} &\xi_1 = \mu_1(x + \rho_1\, y + \kappa_1\, z + \tau_1\, t), \\ &\xi_2 = \mu_1(\nu_1\, y + \lambda_1\, z + \theta_1\, t), \\ &\tau_1 = \gamma\nu_1^2 - \beta\mu_1^2 - \gamma\rho_1^2 - \kappa_1\sigma_3 - \rho_1\sigma_2 - \sigma_1, \\ &\theta_1 = -2\gamma\rho_1\nu_1 - \lambda_1\sigma_3 - \nu_1\sigma_2. \end{aligned} \tag{40}$$

Substituting (39) into (31), one may obtain the periodic soliton solution of equation (3). Figure 4(a)-(c) depicts the evolution of one-order breather solution in $x - y$ plane, $x - z$ plane and $y - z$ plane, respectively.

6.2 Three solitons and localized solutions

Taking $N = 3$ in (32), (31) is exactly the three-soliton solution of equation (3). Compared with the two-soliton solutions, there are more abundant interactions between three solitons. In what follows, we focus on two different cases.

Case 1. Three-soliton solution

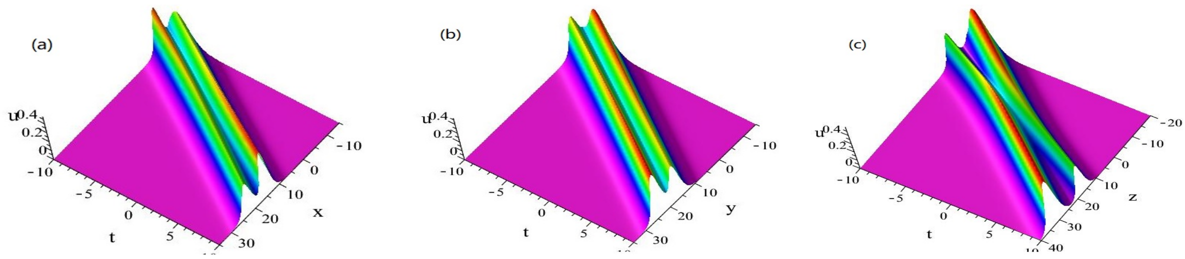

Fig. 1 (Color online) The overtaking collisions between two solitons given by (31), (32) and (33). a. $y = 0, z = 0$. b. $x = 0, z = 0$. c. $x = 0, y = 0$.

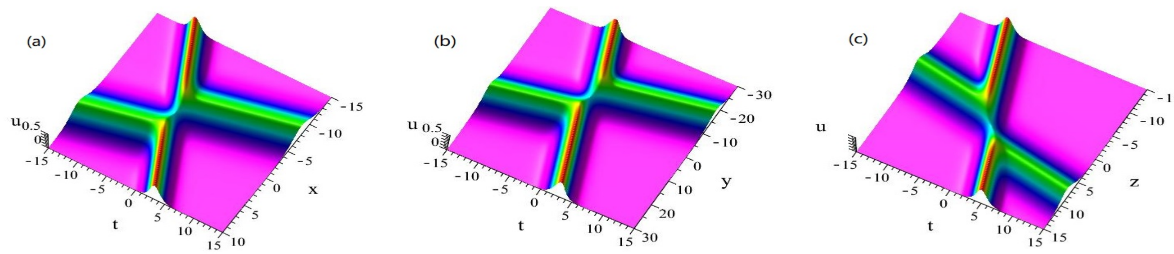

Fig. 2 (Color online) The head-on collisions between two solitons given by (31), (32) and (34). a. $y = 0, z = 0$. b. $x = 0, z = 0$. c. $x = 0, y = 0$.

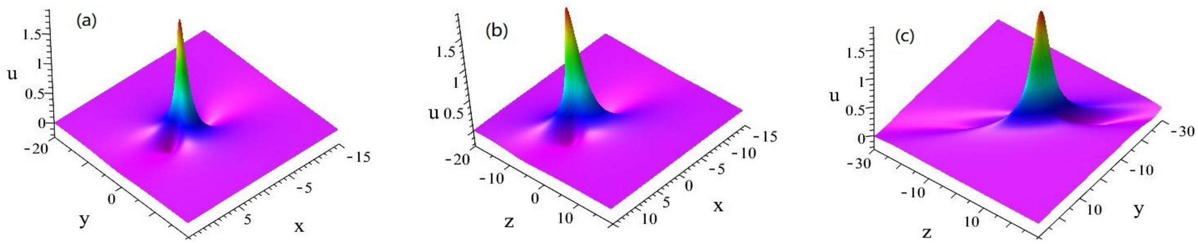

Fig. 3 (Color online) The one-order lump solution given by (31),(37) and (38) with parametric choices $\rho_1 = 0.2$, $\nu_1 = 1.2$, $\kappa_1 = 0.5$, $\lambda_1 = \sigma_1 = 1$, $\sigma_2 = 0.8$, $\sigma_3 = -1$, $\alpha = 6$, $\beta = 1$, $\gamma = -1$. a. In x-y plane. b. In x-z plane. c. In y-z plane.

Selecting proper values for parameters in (32),

$$\begin{aligned} &k_1 = l_2 = 0.8, k_3 = 1.3, l_3 = 1.2, m_2 = 0.6, \\ &k_2 = l_1 = m_1 = \beta = -\gamma = \sigma_1 = \sigma_3 = 1.0, \\ &m_3 = 0.7, \sigma_2 = 0.5, \alpha = 6, \end{aligned} \tag{41}$$

one can observe the overtaking collisions among three solitons propagating in the same direction. In Fig. 5a, three solitons propagate along the positive x direction. Among these three solitons, the one with the tallest amplitude is the lowest. Thus after interaction, the soliton with the shortest amplitude overtakes other two solitons. Fig. 5b and Fig. 5c depict the overtaking interactions of three solitons in y-direction and z-direction.

In (32), we take the parametric choices

$$\begin{aligned} &k_1 = 0.8, k_2 = m_1 = \gamma = -1.0, l_1 = \sigma_2 = 0.5, \\ &k_3 = 1.3, l_2 = 0.3, l_3 = -0.4, m_3 = 0.7, \alpha = 6, \\ &m_2 = -l_4 = 0.6, \beta = \sigma_1 = \sigma_3 = 1.0, \end{aligned} \tag{42}$$

As observed in Fig. 6a, there exists both head-on and overtaking interactions between three solitons. Two solitary waves travel along positive x-direction and one travels along negative x-direction. Note that the taller soliton overtakes the shorter soliton when both of them propagate along positive x direction. Fig. 6b and Fig. 6c depicts the collisions of three solitons propagating along y and z directions.

Case 2. Soliton-breather mixed solution

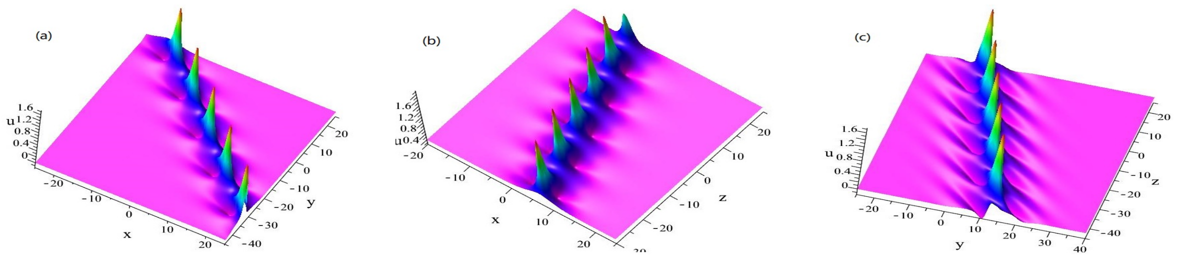

Fig. 4 (Color online) The one-order breather given by (31), (39) and (40) with the parameter choices $\mu_1 = 0.4$,$\rho_1 = 0.75$,$\nu_1 = 1.2$,$\kappa_1 = 0.25$,$\lambda_1 = 1.5$, $\sigma_1 = \sigma_2 = \sigma_3 = \beta = 1$,$\gamma = -1$,$\alpha = 6$. a. In x-y plane. b. In x-z plane. c. In y-z plane.

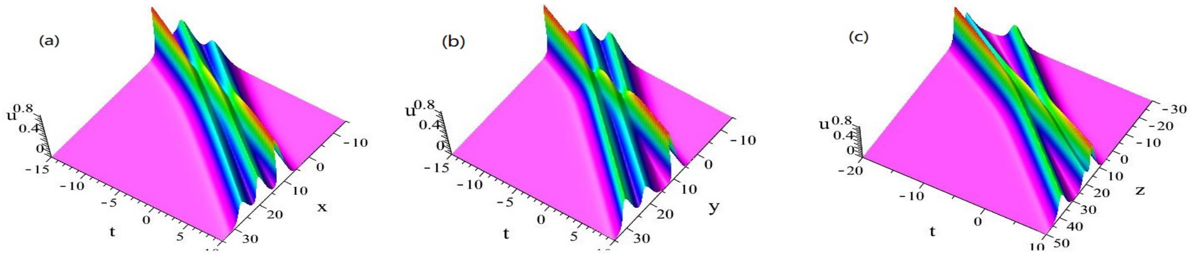

Fig. 5 (Color online) The overtaking collisions among three solitons given by (31), (32) and (41). a. $y = 0, z = 0$. b. $x = 0, z = 0$. c. $x = 0, y = 0$.

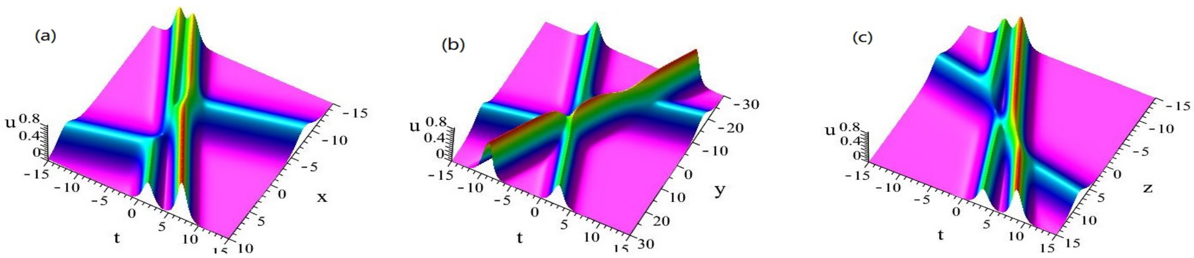

Fig. 6 (Color online) The head-on collisions among three solitons given by (31), (32) and (42). a. $y = 0, z = 0$. b. $x = 0, z = 0$. c. $x = 0, y = 0$.

For the particular choices

$$k_1 = k_2 = \mu_1, l_1 = l_2^* = \rho_1 + i\nu_1, m_1 = m_2^* = \kappa_1 + i\lambda_1,$$
$$k_3 = \mu_2, l_3 = \rho_2, m_3 = \kappa_2,\ \eta_{10} = \eta_{20} = \eta_{30} = 0,$$

(32) is reduced to

$$\begin{aligned}
\phi &= 1 + 2\,\mathrm{e}^{\xi_1}\cos(\vartheta) + B_{12}\cdot \mathrm{e}^{2\xi_1} + [B_{12}L^2\mathrm{e}^{2\xi_1}, \\
&\quad + 2L\mathrm{e}^{\xi_1}\cos(\vartheta + \Lambda) + 1]\,\mathrm{e}^{\xi_2} \\
\xi_j &= \mu_j(x + \rho_j\, y + \kappa_j\, z + \tau_j\, t),\ j = 1, 2, \\
\vartheta &= \mu_1(\nu_1\, y + \lambda_1\, z + \theta_1\, t), \\
\tau_1 &= \gamma\nu_1^2 - \beta\mu_1^2 - \gamma\rho_1^2 - \kappa_1\sigma_3 - \rho_1\sigma_2 - \sigma_1, \\
\tau_2 &= -(\sigma_1 + \beta\mu_2^2 + \gamma\rho_2^2 + \sigma_2\rho_2 + \sigma_3\kappa_2), \\
\theta_1 &= -2\gamma\rho_1\nu_1 - \lambda_1\sigma_3 - \nu_1\sigma_2,
\end{aligned} \tag{43}$$

where B_{12} and L are given by

$$\begin{aligned}
B_{12} &= \frac{\gamma\nu_1^2}{3\beta\mu_1^2 + \gamma\nu_1^2}, \\
L\mathrm{e}^{i\Lambda} &= \frac{3\beta(\mu_1 - \mu_2)^2 - \gamma(\rho_1 + i\nu_1 - \rho_2)^2}{3\beta(\mu_1 + \mu_2)^2 - \gamma(\rho_1 + i\nu_1 - \rho_2)^2}.
\end{aligned} \tag{44}$$

Substituting (43)-(44) into (31), one may obtain the soliton-breather mixed solution of equation (3). Figure 7(a)-(c) depicts the interactions between one soliton and one breather in $x - y$ plane, it can be observed that both the solitary wave and the breather wave keep their velocity and profile after interactions. Their interactions are elastic. Due to the limited space, the interactions between one soliton and one breather in $x - z$ plane and $y - z$ plane are omitted here.

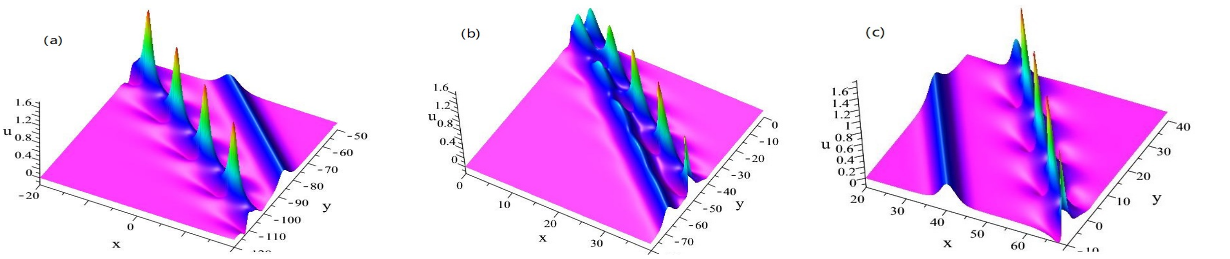

Fig. 7 (Color online) The soliton-breather mixed solution given by (31), (43) and (44). The parameters are selected as $\mu_1 = 0.4$, ρ=0.75, $\nu_1 = 1.2$, $\kappa_1 = 0.25$, $\lambda_1 = 1.5$, $\mu_2 = -1.5$, $\rho_2 = 0.3$, $\kappa_2 = 0.6$, $\sigma_1 = \sigma_2 = \sigma_3 = \beta = -\gamma = 1$, $\alpha = 6$. a. $t = -20, z = 0$. b. $t = 0, z = 0$. c. $t = 20, z = 0$.

6.3 Four solitons and localized solutions

Taking $N = 4$ in (32), equation (31) leads to the four-soliton solution of the (3+1)-dimensional generalized KP equation (3). Compared with the two-soliton solutions, there are more abundant interactions between three solitons. For the sake of simplicity, we only consider two different cases.

Case 1. Four-soliton solution

In (32), we take the parametric choices

$$\begin{aligned} &k_1 = l_2 = m_2 = 0.8, k_2 = m_1 = \sigma_1 = \sigma_3 == 1.0,\\ &k_3 = l_3 = 1.2, k_4 = l_4 = 1.5, l_1 = m_3 = \sigma_2 = 0.5, \\ &m_4 = 0.3, \alpha = 6, \beta = 1, \gamma = -1, \end{aligned} \tag{45}$$

the overtaking collisions between four solitons are shown in Figure 8. Fig. 8a depicts the evolution of four solitons propagating along x direction. It is easily observed that the shortest soliton moves fastest and overtakes three other soliton. After interactions, both of these four solitons keep their original speed and profile. Fig. 8b and Fig. 8c depicts the evolution of four solitons propagating along y and z directions.

In (32), we take the parametric choices

$$\begin{aligned} &k_1 = -k_4 = 0.8, k_2 = m_1 = \gamma = -1.0, k_3 = 1.3,\\ &l_1 = \sigma_2 = -m_4 = 0.5, m_2 = -l_4 = 0.6, m_3 = 0.7, \\ &l_2 = 0.3, l_3 = -0.4, \beta = \sigma_1 = \sigma_3 = 1.0, \alpha = 6, \end{aligned} \tag{46}$$

the interactions between four solitons are shown in Figure 9. Fig. 9a depicts the head-on collision between two left-going solitons and two right-going solitons along positive x direction. Fig. 9b and Fig. 9c depicts the evolution of four solitons propagating along y and z directions.

Case 2. Two-order lump solution

By using the long wave limit method, the four-soliton solution can be transformed into

$$\begin{aligned} \phi =\ &\vartheta_1 \vartheta_2 \vartheta_3 \vartheta_4 + B_{12}\vartheta_3 \vartheta_4 + B_{13}\vartheta_2 \vartheta_4 \\ &+ B_{14}\vartheta_2 \vartheta_3 + B_{23}\vartheta_1 \vartheta_4 + B_{24}\vartheta_1 \vartheta_3 \\ &+ B_{34}\vartheta_1 \vartheta_2 + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} &\vartheta_j = x + l_j y + m_j z + \omega_j t, \\ &\omega_j = -(\sigma_1 + \gamma l_j^2 + \sigma_2 l_j + \sigma_3 m_j), \quad j = 1, \cdots, 4, \\ &B_{js} = \frac{12\beta}{\gamma(l_j - l_s)^2}, \quad 1 \le j < s \le 4. \end{aligned} \tag{48}$$

By setting $l_2 = l_1^*, m_2 = m_1^*, l_4 = l_3^*, m_4 = m_3^*$, we obtain the two-order lump solution of equation (3). Figure 10(a)-(c) depicts the evolutions of two-order lump waves in the $x - y$ plane, $x - z$ plane and $y - z$ plane, respectively.

7 Conclusions

Searching for higher dimensional integrable models is a significant and challenging issue in nonlinear mathematical physics. Our previous work presented some novel integrable models in 3+1 and 4+1 dimensions, and their integrable features as well as exact solutions are explored from different viewpoints. This paper aims to extend the classic lower dimensional integrable modes to arbitrary spatial dimension. Due to the physical and mathematical significance, we investigate the celebrated KP equation and propose its (n+1)-dimensional integrable extension. By employing the singularity manifold analysis, the (n+1)-dimensional KP equation is shown to be Painlevé integrable without any constrains of parameters. The binary Bell polynomial method is successfully used to the proposed (n+1)-dimensional KP

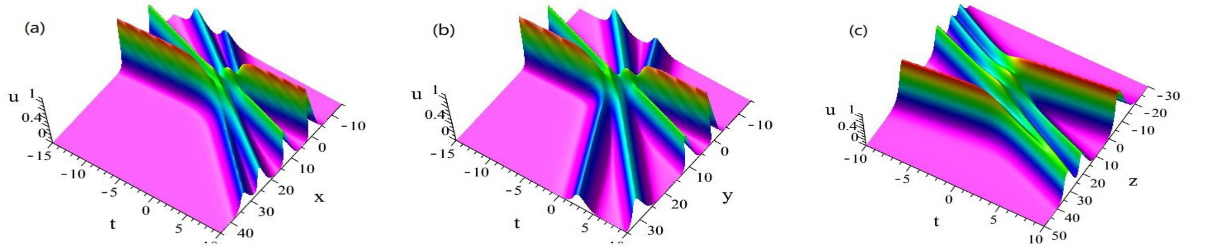

Fig. 8 (Color online) The overtaking collisions among four solitons given by (31), (32) and (45). a. $y = 0, z = 0$. b. $x = 0, z = 0$. c. $x = 0, y = 0$.

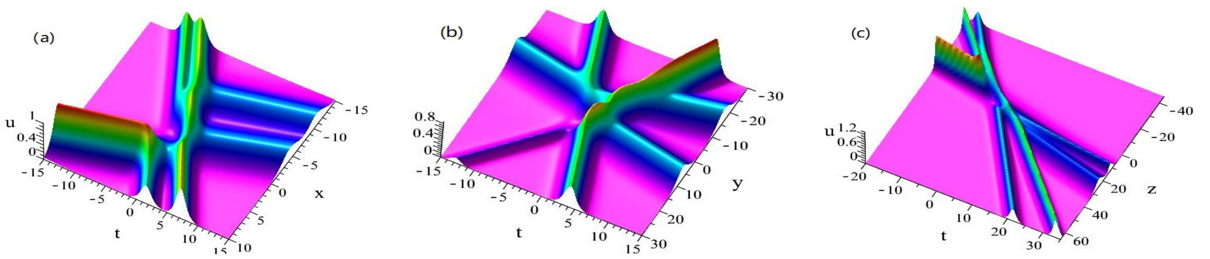

Fig. 9 (Color online) The head-on collisions among four solitons given by (31), (32) and (46). a. $y = 0, z = 0$. b. $x = 0, z = 0$. c. $x = 0, y = 0$.

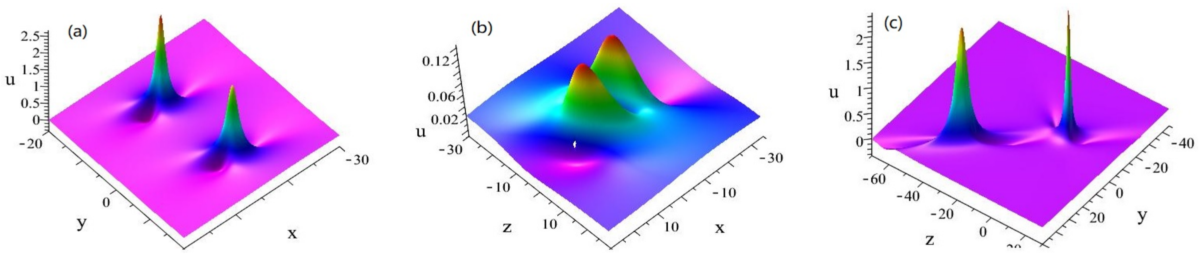

Fig. 10 (Color online) The two-order lump solution given by (31), (47) and (48). The parameters are chosen as $l_1 = l_2^* = 0.5 + 1.2i$, $l_3 = l_4^* = 0.4 - 1.5i$, $m_1 = m_2^* = 0.4 + i$, $m_3 = m_4^* = 0.8 + 1.2i$, $\sigma_1 = 1.0$, $\sigma_2 = 0.8, \sigma_3 = -1$, $\alpha = 6$, $\beta = 1$, $\gamma = -1$. a. $t = -6, z = 0$. b. $t = 0$, $y = 0$. c. $t = -3, x = 0$.

equation, and as a result, the N-soliton, Bäcklund transformation, Lax pair and infinite conservation laws are explicitly constructed systematically. Therefore, the extended KP equation in arbitrary spatial dimensions also possesses the same integrable properties as the classic KP equation in 2+1 dimensions. In addition, abundant interaction structures like overtaking and head-on solitons, one-order lump, two-order lump, breather, and breather-soliton mixed solutions are analyzed by some graphs. The research framework developed in this work may be applicable to some other lower dimensional integrable models with great physical interests. Future research is expected to construct more and more integrable models in arbitrary spatial dimension.

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Availability of data and materials Our manuscript has no associated data.

Conflict of interest The authors declare that they have no conflict of interest.

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