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A New Narrowband Phase Modulation Mathematical Identity

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ABSTRACT

A new mathematical identity is suggested to describe narrow band phase modulation and other similar physical problems instead of using the Bessel function. Bessel functions are extensively used in mathematical physics [1,2], electromagnetic wave propagation and scattering [3,4], and communication system theory [3,5,6]. Such phenomena must often be approximated by appropriate formulas since there is no closed form solution or expression, which usually leads to complex mathematical solutions [5,7]. Comparisons are made between the exact solution numerically calculated and graphed with the new mathematical identities' prediction of phase modulation behavior. The proposed mathematical identity matches the results very well, leading to simpler analysis of such physical behavior.

Keywords: Phase Modulation; Bessel Functions; Optical Communication

1. Introduction

Frequency or phase modulation is an efficient form of communication, where information is transmitted over a carrier signal by changing the instantaneous frequency [4]. Phase modulation has been heavily employed over the last century, and is the bases of all modern digital communication systems. Bessel functions are used to better describe the mathematical equations that govern phase modulation.

The expressions $\varphi(t) = \delta \sin(\omega_m t)$ and $\cos[\varphi(t)] = \cos[\delta \sin(\omega_m t)]$ where δ and ω_m are the index and frequency of modulation, appear frequently in physics research and literature. They play a key role in the mathematical treatment of narrowband frequency modulation in communication and in antennas radiation analysis, design and studies [3,4].

Usually $\cos[\delta \sin(\omega_m t)]$ and $\sin[\delta \sin(\omega_m t)]$ are replaced by Bessel function series, which adds complexity to the mathematical treatment. In many cases this can lead to using mathematical approximations that reduce the accuracy of the analysis.

Previous published work done in collaboration with Prof. Salamo from the University of Arkansas dealt with laser beam propagation through two level samples [8]. While trying to better understanding the evolution of an electric field envelope as the previously studied laser beam propagated through the two level sample, and the effects of phase modulation by using custom in house developed software for a numerical based computer si-

mulation study, it was observed that at a certain beer's absorption length inside the sample, the electric field envelope looks like an analytical function. During fitting trials of the observed electric field in several trigonometric form functions, an expression for the $\cos[\delta \sin(\omega_m t)]$ function in two terms was developed, and the idea of a alterative to the complex Bessel Functions approach was conceived [9-11]. This new expression was compared to the exact $\cos[\delta \sin(\omega_m t)]$, the agreement and fit are excellent for $\delta < 1$ *i.e.* for small index of modulation, which is the case in optical frequency modulation and in communications.

2. Mathematical Development

Our first postulate is:

$$\cos[\delta \sin(\omega_m t)] = \cos\delta + (1 - \cos\delta)\cos^2(\omega_m t) \quad (1)$$

To verify the accuracy of our assumption, the end result undergoes vigorous numerical and analytical testing.

If the proposed postulate is true then it follows:

$$\cos[\delta \sin(\omega_m t)] = \cos\delta + \cos^2(\omega_m t) - \cos\delta\cos^2(\omega_m t) \quad (2)$$

$$\cos[\delta \sin(\omega_m t)] = \cos^2(\omega_m t) + (\cos\delta)[1 - \cos^2(\omega_m t)] \quad (3)$$

$$\cos[\delta \sin(\omega_m t)] = \cos^2(\omega_m t) + (\cos\delta)\sin^2(\omega_m t) \quad (4)$$

$$\sin^2\left[\delta\sin\left(\omega_m t\right)\right] = 1 - \cos^2\left[\delta\sin\left(\omega_m t\right)\right] \qquad (5)$$

by substituting Equation (4) in Equation (5),

$$\sin^2\left[\delta\sin\left(\omega_m t\right)\right] = 1 - \left[\cos^2\left(\omega_m t\right) + \left(\cos\delta\right)\sin^2\left(\omega_m t\right)\right]^2 \qquad (6)$$

$$\sin^2\left[\delta\sin\left(\omega_m t\right)\right] = \left[1 - \cos^2\left(\omega_m t\right) - \left(\cos\delta\right)\sin^2\left(\omega_m t\right)\right]$$
$$\left[1 + \cos^2\left(\omega_m t\right) + \left(\cos\delta\right)\sin^2\left(\omega_m t\right)\right] \qquad (7)$$

$$\sin^2\left[\delta\sin\left(\omega_m t\right)\right] = \left[\sin^2\left(\omega_m t\right) - \left(\cos\delta\right)\sin^2\left(\omega_m t\right)\right]$$
$$\left[2 - \sin^2\left(\omega_m t\right) + \left(\cos\delta\right)\sin^2\left(\omega_m t\right)\right] \qquad (8)$$

$$\sin^2\left[\delta\sin\left(\omega_m t\right)\right] = 2\left[\sin^2\left(\omega_m t\right) - \left(\cos\delta\right)\sin^2\left(\omega_m t\right)\right]$$
$$-\left[\sin^2\left(\omega_m t\right) - \left(\cos\delta\right)\sin^2\left(\omega_m t\right)\right]^2 \qquad (9)$$

$$\sin^2\left[\delta\sin\left(\omega_m t\right)\right] = 2\sin^2\left(\omega_m t\right)\left(1 - \cos\delta\right)$$
$$-\sin^4\left(\omega_m t\right)\left(1 - \cos\delta\right)^2 \qquad (10)$$

$$\sin^2\left[\delta\sin\left(\omega_m t\right)\right]$$
$$= 4\sin^2\left(\frac{\delta}{2}\right)\sin^2\left(\omega_m t\right)\left[1 - \sin^2\left(\omega_m t\right)\sin^2\left(\frac{\delta}{2}\right)\right] \qquad (11)$$

$$\sin\left[\delta\sin\left(\omega_m t\right)\right]$$
$$= 2\sin\left(\frac{\delta}{2}\right)\sin\left(\omega_m t\right)\sqrt{1 - \sin^2\left(\omega_m t\right)\sin^2\left(\frac{\delta}{2}\right)} \qquad (12)$$

The second postulate is:

$$\cos\left[\delta\cos\left(\omega_m t\right)\right] = \cos\delta + \left(1 - \cos\delta\right)\sin^2\left(\omega_m t\right) \qquad (13)$$

If this is a valid assumption, then:

$$\cos\left[\delta\cos\left(\omega_m t\right)\right]$$
$$= \cos\delta + \sin^2\left(\omega_m t\right) - \cos\delta\sin^2\left(\omega_m t\right) \qquad (14)$$

$$\cos\left[\delta\cos\left(\omega_m t\right)\right]$$
$$= \sin^2\left(\omega_m t\right) + \left(\cos\delta\right)\left[1 - \sin^2\left(\omega_m t\right)\right] \qquad (15)$$

$$\cos\left[\delta\cos\left(\omega_m t\right)\right] = \sin^2\left(\omega_m t\right) + \left(\cos\delta\right)\cos^2\left(\omega_m t\right) \qquad (16)$$

If that is right then:

$$\sin^2\left[\delta\cos\left(\omega_m t\right)\right] = 1 - \cos^2\left[\delta\cos\left(\omega_m t\right)\right] \qquad (17)$$

by substituting Equation (16) in Equation (17),

$$\sin^2\left[\delta\cos\left(\omega_m t\right)\right]$$
$$= 1 - \left[\sin^2\left(\omega_m t\right) + \left(\cos\delta\right)\cos^2\left(\omega_m t\right)\right]^2 \qquad (18)$$

$$\sin^2\left[\delta\cos\left(\omega_m t\right)\right] = \left[1 - \sin^2\left(\omega_m t\right) - \left(\cos\delta\right)\cos^2\left(\omega_m t\right)\right]$$
$$\left[1 + \sin^2\left(\omega_m t\right) + \left(\cos\delta\right)\cos^2\left(\omega_m t\right)\right] \qquad (19)$$

$$\sin^2\left[\delta\cos\left(\omega_m t\right)\right] = \left[\cos^2\left(\omega_m t\right) - \left(\cos\delta\right)\cos^2\left(\omega_m t\right)\right]$$
$$\left[2 - \cos^2\left(\omega_m t\right) + \left(\cos\delta\right)\cos^2\left(\omega_m t\right)\right] \qquad (20)$$

$$\sin^2\left[\delta\cos\left(\omega_m t\right)\right] = 2\left[\cos^2\left(\omega_m t\right) - \left(\cos\delta\right)\cos^2\left(\omega_m t\right)\right]$$
$$-\left[\cos^2\left(\omega_m t\right) - \left(\cos\delta\right)\cos^2\left(\omega_m t\right)\right]^2 \qquad (21)$$

$$\sin^2\left[\delta\cos\left(\omega_m t\right)\right]$$
$$= 4\sin^2\left(\frac{\delta}{2}\right)\cos^2\left(\omega_m t\right)\left[1 - \cos^2\left(\omega_m t\right)\sin^2\left(\frac{\delta}{2}\right)\right] \qquad (22)$$

$$\sin\left[\delta\cos\left(\omega_m t\right)\right]$$
$$= 2\sin\left(\frac{\delta}{2}\right)\cos\left(\omega_m t\right)\sqrt{1 - \cos^2\left(\omega_m t\right)\sin^2\left(\frac{\delta}{2}\right)}. \qquad (23)$$

From the above derivation, we get our new identities in Equations (4), (12), (16) and (23).

3. Testing the Suggested Expression

The new expression is tested in two approaches. The first approach is to analytically compare the result to the established phase modulation of light [12]. The second approach to compare the results with computer simulations.

3.1. Phase Modulation of Light

Amnon Yariv discuses phase modulation of light in chapter 9 of his book *Introduction to Optical Electronics* [12]. The electric field for the modulated light is given by:

$$E_{out} = A\cos\left[\omega t + \delta\sin\left(\omega_m t\right)\right] \qquad (24)$$

where δ is the phase modulation index and ω_m is the phase modulation frequency.

$$E_{out} = A\cos\left(\omega_m t\right)\cos\left[\delta\sin\left(\omega_m t\right)\right]$$
$$- A\sin\left(\omega_m t\right)\sin\left[\delta\sin\left(\omega_m t\right)\right] \qquad (25)$$

The above book uses the Bessel function identities

$$\cos\left[\delta\sin\left(\omega_m t\right)\right]$$
$$= J_o\left(\delta\right) + 2J_o\left(\delta\right)\cos\left(2\omega_m t\right) + J_4\left(\delta\right)\cos\left(4\omega_m t\right) + \cdots \qquad (26)$$

$$\sin\left[\delta\sin\left(\omega_m t\right)\right]$$
$$= J_1\left(\delta\right)\sin\left(\omega_m t\right) + 2J_3\left(\delta\right)\sin\left(3\omega_m t\right) + \cdots \qquad (27)$$

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$$E_{out} = A\big[J_o(\delta)\cos(\omega t)$$
$$+J_1(\delta)\cos(\omega + \omega_m)t - J_1(\delta)\cos(\omega - \omega_m)t$$
$$+J_2(\delta)\cos(\omega + 2\omega_m)t + J_2(\delta)\cos(\omega - 2\omega_m)t \qquad (28)$$
$$+J_3(\delta)\cos(\omega + 3\omega_m)t - J_3(\delta)\cos(\omega - 3\omega_m)t$$
$$+J_4(\delta)\cos(\omega + 4\omega_m)t - J_4(\delta)\cos(\omega - 4\omega_m)t + \cdots \big]$$

However, it is generally assumed that only the first three terms are significant. This agrees with our experiments in which only the first three terms in the series are detected.

For small modulation, *i.e.* $\delta < 1$, $J_0(\delta) = 1$ and $J_1(\delta) = \sin(\delta/2)$ which leads to:

$$E_{out} = A\Big[\cos(\omega t) + \sin\Big(\frac{\delta}{2}\Big)\cos(\omega + \omega_m)t$$
$$\qquad\qquad - \sin\Big(\frac{\delta}{2}\Big)\cos(\omega - \omega_m)t \Big] \qquad (29)$$

If we use our suggested approximation instead of the Bessel function identities in E_{out}, then:

$$E_{out} = A\cos(\omega t)\cos\big[\delta\sin(\omega_m t)\big]$$
$$\qquad\qquad - A\sin(\omega_m t)\sin\big[\delta\sin(\omega_{mt})\big] \qquad (30)$$

By using the new identities of Equations (4) and (12) in Equation (30), we get:

$$E_{out} = A\Big[\cos(\omega t)\cos^2(\omega_m t) + \cos(\omega t)(\cos\delta)\sin^2(\omega_m t) \Big]$$
$$- A\Big[2\sin\Big(\frac{\delta}{2}\Big)\sin(\omega t)\sin\Big(\frac{\delta}{2}\Big)\sin(\omega_m t)$$
$$\sqrt{\Big(1 - \sin^2\Big(\frac{\delta}{2}\Big)\sin^2(\omega_m t)\Big)} \Big]$$
$$\qquad\qquad (31)$$

$$E_{out} = A\Big[\cos(\omega t)\big\{\cos^2(\omega_m t) + (\cos\delta)\sin^2(\omega_m t)\big\}$$
$$-2\sin\Big(\frac{\delta}{2}\Big)\sin(\omega t)\sin(\omega_m t)\sqrt{1 - \sin^2\Big(\frac{\delta}{2}\Big)\sin^2(\omega_m t)} \Big]$$
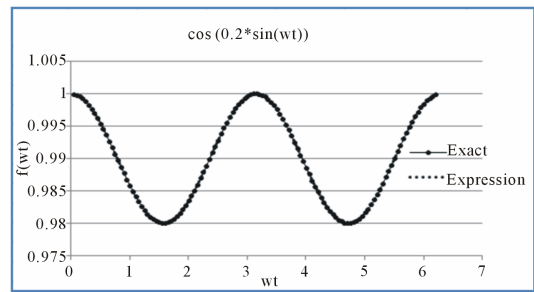$$\qquad\qquad (32)$$

To check this formula against the previous formula of E_{out} we apply it for small δ where, $\cos\delta = 1$ and $\sin^2(\delta/2) = 0$. In this case,

$$E_{out} = A\Big[\cos(\omega t) - 2\sin\Big(\frac{\delta}{2}\Big)\sin(\omega t)\sin(\omega_m t) \Big] \quad (33)$$
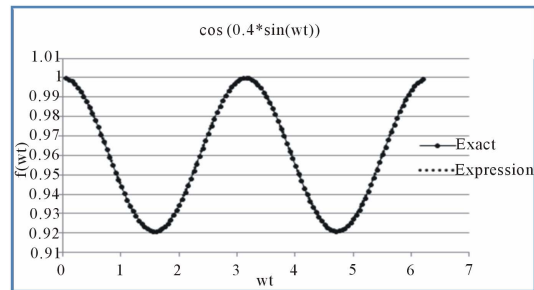
If we use the identity:

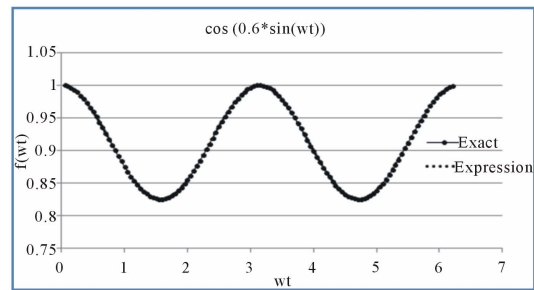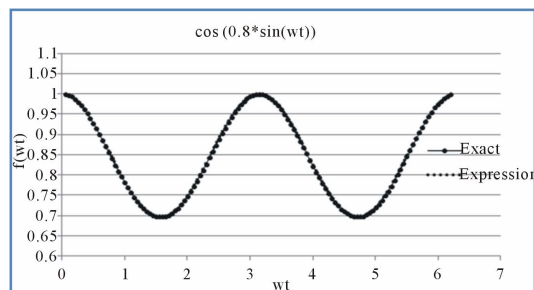$$-2\sin(\omega t)\sin(\omega_m t) = \cos(\omega + \omega_m)t - \cos(\omega - \omega_m)t \quad (34)$$
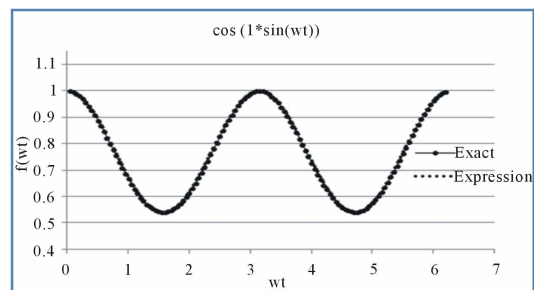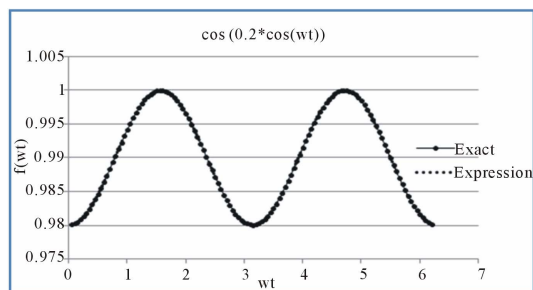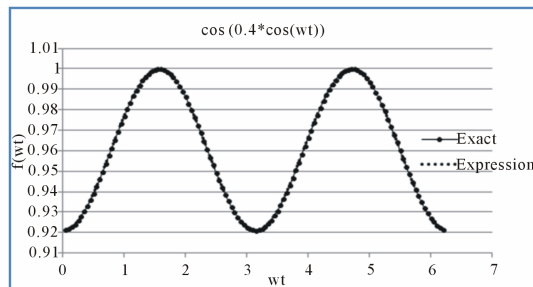
we get

(a)

(b)

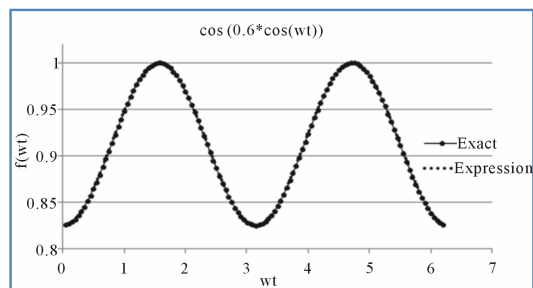(c)

(d)

(e)

Figure 1. Comparison between exact results and proposed expression for $\cos\big[\delta\sin(\omega_m t)\big]$.

(a)

(b)

(c)
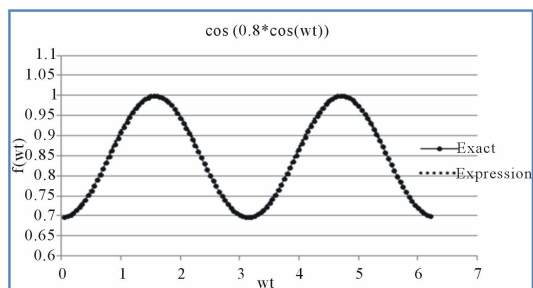
(d)
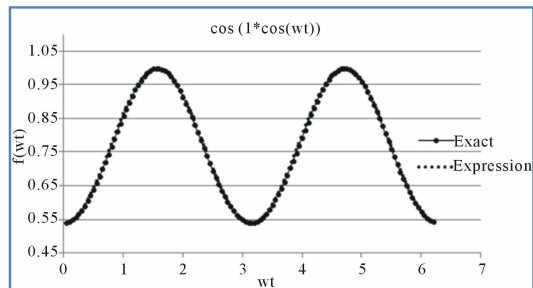
(e)

Figure 2. Comparison between exact results and proposed expression for $\cos\left[\delta\cos\left(\omega_m t\right)\right]$**.**

$$E_{out} = A\left[\cos\left(\omega t\right) + \sin\left(\frac{\delta}{2}\right)\cos\left(\omega + \omega_m\right)t\right.$$
$$\left. - \sin\left(\frac{\delta}{2}\right)\cos\left(\omega - \omega_m\right)t\right]$$

(35)

Which exactly matches the proposed solution?

3.2. Computer Generated Numerical Comparison

The graphs of **Figure 1**, compare the computer generated numerical results for the exact equation: $\cos\left[\delta\sin\left(\omega_m t\right)\right]$ at different values of δ with that of the new proposed mathematical identity. **Figure 2** shows the comparison for $\cos\left[\delta\cos\left(\omega_m t\right)\right]$.

4. Conclusion

The derivation of the four new mathematical identities in Equations (4), (12), (16) and (23) that describe narrow band phase modulation is presented. The proposed identity can also be used for similar physical phenomena's that use the Bessel function. The mathematical identity was shown to match analytical and numerical results very well. The new identities greatly reduce computation time and complexity of analytical treatment of such physical behavior.

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