

A New Non-Linear GARCH Model

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AKADEMISK AVHANDLING

som för framläggande av ekonomie doktorsexamen
vid Handelshögskolan i Stockholm
framlägges till offentlig granskning
måndagen den 5 MAJ, 1997
kl. 13.15 i sal K.A.W.
Handelshögskolan, Sveavägen 65

STOCKHOLM 1997



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Stockholm 1997



STOCKHOLM SCHOOL OF ECONOMICS
EFI, THE ECONOMIC RESEARCH INSTITUTE



A Dissertation for the
Doctor's Degree in Philosophy
Stockholm School of Economics

© The author 1997
ISBN 91-7258-444-0

Keywords:

High frequency data

Volatility

GARCH

Non-linear models

Asymmetry

Printed by

Gotab, Stockholm 1997

Distributed by

The Economic Research Institute at the Stockholm School of Economics
Box 6501, S-113 83 Stockholm

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Preface

This dissertation consists of four separate and self-contained papers. All four papers are concerned with problems regarding heteroskedasticity in financial time series. A general summary of the research area is given in an introductory chapter.

A number of people have contributed to this thesis. Insightful discussions with these persons have been invaluable for me in my research work, and to them I express my sincere gratitude.

My foremost thanks go to my thesis advisors Tomas Björk, Magnus Dahlqvist, and Timo Teräsvirta. I have also benefited from suggestions from other faculty members and Ph.D. students at the Department of Finance and at the Department of Economic Statistics at the Stockholm School of Economics. In particular, I am grateful to Jan Ericsson, Sune Karlsson, Johan Molin, Jens Nystedt, Joel Reneby, Patrik Säfvenblad, and Stefan E. Åsbrink.

I would also like to thank my colleagues at AROS Securities, especially Magnus Billing, Michael Malmquist, Peder Kruse, Fredrik Thor, and Sten Wachtmeister.

Special thanks for general support go to Åke G. Blomqvist, Johan A. Lybeck, Göran Sjöholm, and John Tandberg. Maria De Liseo has generously proof read the manuscript. All remaining errors are my own.

Eva Eklund, Anki Helmer, and Kerstin Lindskog have provided me with assistance in administrative matters, for which I owe them gratitude.

I am deeply grateful to Bankforskninginstitutet for their generous financial support. I would also like to thank the Conrad Hambergs Foundation for giving me financial support at an early stage of the Ph.D. program.

I dedicate this book to my parents.

Stockholm, March 1997

Gustaf Hagerud

Abstract

A Smooth Transition ARCH Model for Asset Returns

In the classical ARCH model of Engle [1982] the conditional variance is a linear function of lagged squared residuals. In this paper I introduce non-linearity, by adding a term that consists of a constant parameter multiplied by a transition function. Two different transition functions are considered, a logistic and an exponential. Furthermore, following Bollerslev [1986], I extend the model by introducing lagged conditional variances in the conditional variance equation. This specification reduces the number of parameters in the model, which proves to be important for successful estimation. The paper also describes a number of specification tests, that can determine if the smooth transition GARCH model can be the data generating process of a times series. The techniques proposed are illustrated on data from four stock index series.

Specification Tests for Asymmetric GARCH

In this paper I present two new Lagrange multiplier test statistics designed for testing the null of GARCH(1,1), against the alternative of asymmetric GARCH. For one test the alternative is the generalized QARCH(1,1) model of Sentana [1995], and for the other the alternative is the logistic smooth transition GARCH(1,1) model of Hagerud [1996], and González-

Rivera [1996]. In the study I present small sample properties for the two statistics. The empirical size is shown to be equal to the theoretical for reasonable sample sizes. Furthermore, I show that the power of both tests is superior to that of the asymmetry tests proposed by Engle and Ng [1993]. This is true even if the true data generating process is not the GQARCH or LSTGARCH model, but any of the models, EGARCH, GJR, TGARCH, A-PARCH, and VS-ARCH. Thus, the two tests are in fact tests for general GARCH asymmetry.

Modeling Nordic Stock Returns with Asymmetric GARCH Models

This paper investigates the presence of asymmetric GARCH effects in a number of equity return series, and compares the modeling performance of seven different conditional variance models, within the parametric GARCH class of models. The data consists of daily returns for 45 Nordic stocks, during the period July 1991 to July 1996. The models investigated are: EGARCH, GJR, TGARCH, A-PARCH, GQARCH, VS-ARCH, and LSTGARCH. In all these models the conditional variance is a function of the sign of lagged residuals. Thus, the models can capture the often reported negative correlation between lagged returns and conditional variance. The paper also introduces three new procedures for asymmetry testing. The proposed LM tests, which are based on the results of Wooldridge [1991], allow for heterokurtosis under the null. Asymmetries are detected for only 12 of the 45 series. The specifications GJR, TGARCH, and GQARCH appear to be superior for modeling the dynamics of the conditional variance. Furthermore, it is shown that the use of robust test statistics is advisable.

Discrete Time Hedging of OTC Options in a GARCH Environment: A Simulation Experiment

This paper examines the effect of using Black and Scholes formula for pricing and hedging options in a discrete time heteroskedastic environment. This is done by a simulation procedure where asset returns are generated from a GARCH(1,1)-t model. In the simulation a hypothetical trader writes an option and then delta-hedges his position until the option expires. It is shown that the variance of the returns on the hedged position is considerably higher in a GARCH(1,1) environment than in a homoskedastic environment. The variance of returns depends greatly on the level of kurtosis in the returns process and on the first-order autocorrelation in centered and squared returns.

1

Overview of the GARCH Literature

One of the most established characteristics of financial time series is the presence of non-constant and time dependent volatility in returns. The four papers published in this thesis are all concerned with issues regarding the modeling of asset returns with Autoregressive Conditional Heteroskedasticity (ARCH) models. Since Engle [1982] introduced the ARCH model, specifications within this class of models have become very popular in describing the dynamics of volatility in asset returns. This chapter gives the reader a short introduction to the ARCH class of models. I formulate the standard GARCH model, and describe how the model has been extended to capture more complex patterns of returns. Furthermore, I present a number of examples of applications where ARCH/GARCH models are used. By doing so, I hope that the reader will gain a clear understanding of how the four papers in this thesis relate to previous work in the econometrics and finance literature, and how they are related to practical considerations of econometric modeling.

The purpose of this introductory chapter, however, is not to provide a survey of the ARCH literature. The interested reader is referred to the comprehensive surveys by Bera and Higgins [1992], Bollerslev, Chou, and Kroner [1992], Bollerslev, Engle, and Nelson [1994], Diebold, and Lopez [1995], and Palm [1996]. A short and less technical introduction to the use of GARCH models in finance can be found in Engle [1993]. Furthermore, technical details concerning the statistical properties of the time series models will not be discussed.

The ARCH class of models is not alone in trying to characterize the dynamics of financial returns. A well-known alternative specification is the

class of stochastic volatility models. However this thesis will not compare ARCH models and stochastic volatility models. For an overview of the literature on stochastic volatility models, a well written survey by Ghysels, Harvey, and Renault [1996] is recommended .

1.1 GARCH Models

GARCH models are designed to capture the dynamics of the conditional second moment of a time series. However, before saying anything about the second moment, it is natural to first specify the first moment, i.e. the conditional mean. By way of introduction, it is assumed that the data generating process of asset returns is

$$r_t = \varepsilon_t,$$

where ε_t is an error term following a discrete-time stochastic process. In all GARCH models it is assumed that the residual has the form

$$\varepsilon_t = z_t h_t^{1/2},$$

where $z_t \sim iid(0, 1)$, and h_t is the conditional variance of the return at time t . It is the dynamics of the h_t that the GARCH models wish to capture.

The seminal work in the GARCH literature is Engle [1982], where the author introduces the standard ARCH model. In this model, the conditional variance is a deterministic function of lagged squared residuals. In the ARCH(q) model the conditional variance is given by

$$h_t = \gamma + \sum_{j=1}^q \alpha_j \varepsilon_{t-j}^2.$$

Bollerslev [1986] further develops the ARCH model into the generalized ARCH (GARCH) model. Bollerslev extends Engle's specification by introducing lagged conditional variances in the conditional variance equation. With this representation the number of parameters in the model can be reduced considerably. The GARCH model is commonly used in its most simple form, the GARCH(1,1) model, in which the conditional variance is given by

$$h_t = \gamma + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}.$$

Both Engle [1982] and Bollerslev [1986] used their specifications for modeling the time series pattern of inflation rates. It is, however, for modeling high frequency financial data that the model class has found popular use. In my view, the reason for this is threefold. First, the model captures the often detected and well documented *volatility cluster* of financial data. Volatility

clustering refers to the fact that large changes in asset prices tend to be followed by more large changes, in either direction, and small changes in prices tend to be followed by more small changes. The fact that implied volatilities for option prices vary with the time to expiration, is evidence that participants in the financial markets agree on the presence of time-varying volatility. Second, since the conditional variance equation is a *deterministic* function of past realizations of the time series, the parameters of the model can be estimated by maximum-likelihood estimation. Third, in many of the GARCH models proposed in the literature, making multi-step forecasts of the conditional variance is straightforward.

To my knowledge, at least fifteen different univariate GARCH models are currently in existence. Several of these models will be presented below along with a description of the kinds of characteristics of financial time series that they are designed to capture.

Particularly in equity return series, empirical investigations have detected a negative correlation between lagged returns and the current conditional variance. Thus, the volatility seems to increase in bear markets. This mechanism is sometimes termed the “leverage effect”, since a *ceteris paribus* decrease in the stock price will increase the financial leverage of the company. A number of GARCH models that try to capture the leverage effect have been proposed. In the GARCH literature, models with this feature are often termed asymmetric volatility models. In my view this is a more appropriate term, since the reason for the detected negative correlation is not considered in the models. The asymmetry is introduced by letting the conditional variance be a function of not only lagged squared residuals, but also of the sign of lagged residuals.

The first asymmetric model presented in the literature is the exponential GARCH (EGARCH) model of Nelson [1991]. In the EGARCH(1,1) model the natural logarithm of the conditional variance is a function of ε_{t-1}^2 , ε_{t-1} , and the natural logarithm of the lagged conditional variance. Since the conditional variance is dependent on ε_{t-1} , positive and negative residuals of the same magnitude will have a different impact on the level of conditional variance, which allows for asymmetry.

Another often used asymmetric specification is the GJR model of Glosten, Jagannathan, and Runkle [1993]. In the GJR model, the standard GARCH(1,1) model is augmented by $S_{t-1}^- \varepsilon_{t-1}^2$, where S_{t-1}^- takes the value one if ε_{t-1} is negative and zero otherwise. The model’s very simple structure makes estimation easy, and empirical investigations have proven that the GJR model has good modeling performance. In Chapter 2 of this thesis, I introduce a new asymmetric model, termed the logistic smooth transition GARCH (LSTGARCH) model, which constitutes a generalization of the GJR model. In the LSTGARCH model, not only the sign of a residual decides the residual’s influence on the level of the conditional variance, but the relative size of the residual is also influential.

In chapter 3, I derive two new tests that can be used to detect the presence of asymmetry in time series. In connection with this, a short review of previous asymmetric GARCH models is given. In the following paper, in Chapter 4, I compare the in sample performance of seven different asymmetric GARCH specifications. The investigation is performed on return observations from a number of Nordic stocks.

In the standard ARCH and GARCH models the conditional variance is a linear function of past squared residuals and lagged conditional variances. Asymmetric GARCH models can therefore be regarded as non-linear GARCH specifications. A number of other non-linear GARCH models, without the asymmetry feature also exists. Among the first to introduce non-linearities in the ARCH framework were Engle and Bollerslev [1986]. They propose a model where the dynamics of the conditional variance change with the magnitude of squared residuals. The transition between different conditional variance states is controlled by a normal cumulative distribution function. Higgins and Bera [1992] introduce the Non-linear ARCH (NARCH) model, which encompasses various functional forms for the conditional variance. Chapter 2 present a new non-linear symmetric GARCH model, the Exponential Smooth Transition GARCH (ESTGARCH) model. The ESTGARCH model is inspired by the Smooth Transition Autoregressive (STAR) model of Luukkonen, Saikkonen, and Teräsvirta [1988]. In the STAR, model the conditional mean is a non-linear function of lagged realizations of the series.

Apart from asymmetry, empirical investigations have found other factors that are correlated with the conditional variance of asset returns. For example, one has detected a day-of-the-week effect in the volatility, in which Mondays in particular seem to more volatile than other days. This issue is addressed by Baille and Bollerslev [1989], when they model the dynamics of a number of exchange rates series. Nothing prevents us for also including explanatory variables in the conditional variance equation. Thus, if it is believed that the nominal interest level can in some way explain the level of volatility, a proxy for the interest level can easily be included in the model.

If GARCH models are to be used for forecasting volatility for option pricing, a univariate model such as those presented above is adequate. However, in many applications of finance, the area of interest is asset pricing and portfolio allocation. In these cases, not only the dynamics of the conditional variance is of interest, but also in the correlation between the returns of different assets. For such purposes, the multivariate ARCH model, introduced by Bollerslev, Engle, and Wooldridge [1988], can be applied. Assume that the returns of N assets is given by

$$\mathbf{r}_t = \boldsymbol{\varepsilon}_t,$$

where the $N \times 1$ vector of residuals $\boldsymbol{\varepsilon}_t$ is specified as

$$\boldsymbol{\varepsilon}_t = \mathbf{z}_t \mathbf{H}_t^{1/2},$$

where the innovations \mathbf{z}_t are independently identically distributed, with expected value $\mathbf{0}$ and variance \mathbf{I}_N . The $N \times N$ matrix \mathbf{H}_t will therefore be the covariance matrix of the returns. The aim of the multivariate ARCH models is to find a structure for \mathbf{H}_t that can explain the time varying pattern of variances and covariances. A major problem in these models is the number of parameters. For a useful specification, the dimensionality of the parameter space must necessarily be restricted. One representation of the covariance matrix that has received acceptance in the literature is the BEKK specification of Baba, Engle, Kraft, and Kroner [1991]. Their parameterization guarantees that the covariance matrix is positive definite.

To simplify the exposition, we assumed above that the residual, ε_t , is equal to the observable return of an asset. This assumption, however, can be relaxed without any major consequences. In most applications, ε_t , is the unobservable error term in a regression model

$$r_t = f(\mathbf{x}_{t-1}; \boldsymbol{\phi}) + \varepsilon_t,$$

where $f(\cdot)$ denotes a function of lagged endogenous and exogenous variables known at time $t - 1$, \mathbf{x}_{t-1} , and $\boldsymbol{\phi}$ is a vector of parameters in the conditional mean specification. Since, in a number of empirical investigations autocorrelation has been detected, a frequently used functional form for $f(\cdot)$ is a standard linear AR-model, in which \mathbf{x}_{t-1} consists of lagged returns. In the finance literature, the current level of the conditional variance often enters the mean equation. Models with this feature are termed ARCH-in-Mean (ARCH-M) models, and was introduced by Engle, Lilien, and Robins [1987].

I have not yet addressed to issue of the distributional assumption made for the innovations $\{z_t\}$. This subject is extensively debated in the literature. In the original article by Engle [1982] residuals are assumed to be distributed conditionally normal. Bollerslev [1987] introduces Student-t distributed innovations with his GARCH-t model. In Nelson [1991] innovations are assumed to be drawn from a Generalized Error Distribution. Teräsvirta [1996] shows that a GARCH(1,1) model with normal errors cannot generate data with high excess kurtosis and low first-order autocorrelation in squared residuals often detected in financial time series. Based on Teräsvirta's results, it is clear that for a successful modeling, z_t should be assumed to be drawn from a distribution which allows for leptokurtosis. In the empirical investigation of Nordic stock returns, presented in Chapter 4, it is shown that an assumption of normally distributed innovations is inappropriate. An alternative to making an explicit assumption about the distribution of innovations is to employ a non-parametric approach. This is a method employed by Engle and González-Rivera [1991], amongst others.

Estimation of GARCH models is usually done with maximum likelihood (ML) methods, and in some cases with the general method of moments (GMM). For maximum likelihood estimation, a distributional assumption for the innovations must be made. This is complicated to do prior

to the estimation. However, Bollerslev and Wooldridge [1992] show that to estimate the parameters of a GARCH model a quasi-maximum likelihood (QML) approach can be used. In the QML method, the innovations z_t are assumed to be distributed independently normal, and the normal log-likelihood function is maximized, using standard numerical methods. Bollerslev and Wooldridge show that when normality is violated, the QML estimators are generally consistent and have a limiting normal distribution. Bollerslev and Wooldridge also present asymptotic standard errors of the estimators valid under non-normality. In this thesis all estimations presented are performed with ML or QML methods. Readers interested in GMM estimation are referred to Rich, Raymond, and Butler [1991].

If the residuals $\{\varepsilon_t\}$ are from a regression model, the parameters of the conditional mean model should, generally, be estimated simultaneously with the conditional variance model. However, Engle [1982] and Bollerslev [1986] show that when the GARCH model is symmetric with respect to lagged returns, the two models can be estimated separately without loss of asymptotic efficiency. This result can simplify estimations considerably.

To estimate the unknown parameters of a model in the ARCH/GARCH class, iterative numerical methods are most often required. These procedures are time consuming, and furthermore, if the model in question explains the data badly, the estimation might not converge. It is therefore essential to use reliable specification tests in an estimation procedure. These tests give the econometrician indications of which models can be the data generating process of a time series. Many such tests have been presented in the literature. Since one of the objectives for using the tests is to avoid estimating the model under the alternative hypothesis, the tests are often formulated as Lagrange multiplier (*LM*) tests. The most commonly used test is Engle's [1982] test of no ARCH. In Chapter 3 of this thesis, I develop specification procedures that can be used to test the null hypothesis of GARCH(1,1) against the alternative of an asymmetric GARCH model. A review of the specification tests previously proposed in the literature is also included.

A majority of the GARCH specification tests are developed under the assumption that residuals are distributed conditionally normal. From the above, we know that such an assumption is unlikely to be fulfilled in a financial time series. In two articles by Wooldridge [1990] and [1991], the author describes a general method that can be used to make *LM* tests robust to non-normality. In Chapter 4, the methods proposed by Wooldridge are used to robustify the specification tests presented in Chapter 3.

1.2 Applications of GARCH Models in Finance

This section describes a number of different cases in which GARCH models can be used for modeling and forecasting the conditional variance and covariance of financial data. The applications are both of practical and academic nature, and the description is relatively brief. Interested readers are recommended to consult the survey papers mentioned above, or the articles referred to below, for more exhaustive descriptions.

1.2.1 Option Pricing and Hedging

Forecasting volatility for option pricing purposes is probably the most widespread application of GARCH models. The availability of relatively simple and reliable forecasting methods was limited in the 1970s and 1980s. It is therefore not surprising that the interest in GARCH models grew enormously when the models were introduced to practitioners in the early nineties.

The price of an option depends on the expected variance of the underlying asset during the life of the option. In many option valuation methods, volatility is the only explanatory variable that is uncertain. For participants in the option markets, reliable volatility predictions are therefore essential, and much effort is put into forecasting. As was noted above many, empirical investigations have shown that the variance of returns can be modeled with GARCH processes. Furthermore, volatility forecasts performed with GARCH models have shown to be superior to other forecasting methods in a number of studies. The evidence in favor of GARCH, however, is not unanimous. In some studies, the stochastic volatility models have shown to outperform the GARCH models in volatility prediction.

The procedure for making a volatility forecast for option valuation is the following. First, the parameters of the model are estimated on a time series for the underlying asset. Most often daily observations are used, and a suitable sample size is 1000 observations, corresponding to four years of data. Second, based on the estimated parameters and on the sample data, the conditional variance at the valuation time, h_t , is computed. Third, a forecast for the conditional variance at each trading day is calculated up to the expiration date of the option. Depending on which GARCH model is used, the formula for calculating the forecast differs. If the standard GARCH(1,1) model is chosen, the one-day-ahead conditional variance forecast, $\hat{h}_{t+1|t}$, is calculated as

$$\hat{h}_{t+1|t} = \hat{\gamma} + \hat{\alpha}\varepsilon_t^2 + \hat{\beta}h_t,$$

where $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ are the estimated parameters of the GARCH(1,1) model. The s -day-ahead forecast is

$$\hat{h}_{t+s|t} = \hat{\gamma} + (\hat{\alpha} + \hat{\beta})\hat{h}_{t+s-1|t},$$

where $\hat{h}_{t+s-1|t}$ is the forecasted conditional variance for the day prior to day s , made at day t . The series of forecasted daily conditional variances can now be used in the option valuation formula chosen.

Practitioners calculating prices of European options often use formulas where the variance is assumed to be constant during the life of the option, e.g. the Black and Scholes [1973] formula. This practice is followed even though the agents believe that the return process is conditionally heteroskedastic. The customary procedure is to use the average variance over a specific time interval as a dependent variable in the in Black and Scholes' formula. The average predicted variance during the period from $t + 1$ to $t + \tau$ is simply calculated as

$$\frac{1}{\tau} \sum_{i=1}^{\tau} \hat{h}_{t+i|t}.$$

The practice of using Black and Scholes' formula in a non-constant variance environment obviously constitutes a logical inconsistency. However, the procedure is used primarily because of calculational convenience. In other words, practitioners know that they are using the wrong model, but do so because they believe that the Black and Scholes formula gives them a good approximation to the option value calculated under an assumption of non-constant volatility. Chapter 5 of this thesis examines the effect of using this practice. This is done in a simulation procedure, where it is assumed that the agent knows the true data generating process.

For practitioners in the option markets, the value of options is not the only important factor. It is equally important to know how option prices react to changes in the price of the underlying security, measured by *delta* and *gamma*, and to changes in the volatility, measured as *vega*. These issues have been investigated by Engle and Rosenberg [1994] and [1995]. In Chapter 5, these issues are considered in some detail. The customary practice is that the option trader calculates the risk factors, delta, gamma and vega, using Black and Scholes formula, and the volatility plugged into the formula is the average conditional variance predicted, for example, with a GARCH model.

The pricing of options when returns of assets are generated by GARCH processes is a complicated subject. The major valuation problem is that the model is not complete in the sense that options cannot be replicated by portfolios of the underlying asset and a risk-free bond. However, this is a general problem for all discrete time models where the price of an asset, in the next period, can take more than two values. Duan [1995] develops a method for pricing options in GARCH(1,1) environment, but to do this he has to make assumptions about the risk preferences of agents. This will be discussed further in Chapter 5.

1.2.2 Deriving Optimal Hedge Ratios

The previous section discussed the hedging of options with the use of the underlying security of the option. This is, at least fundamentally, an uncomplicated problem. Investors who wish to eliminate the market risk in an equity position can do this by buying

$$\frac{\text{the number of stocks in the portfolio}}{\text{delta for the option}}$$

put options on the stock. The delta on the put option can be computed with for instance the Black and Scholes formula, and as the volatility plugged into the formula, they can use an average volatility forecasted with a GARCH model. In this case, the hedging procedure is relatively simple, since the value of the equity position and the value of the hedge will have a correlation almost equal to minus one, at least over a short time interval. In many cases, however, it is impossible for investors to find a instrument, at a reasonable cost, that will give an almost perfect hedge. This can be a problem for example in the commodity markets. In such cases, the agents must accept that the best they can do is to hedge its position in a contract that has the highest correlation with the asset they wish to hedge.

Let us, for the sake of simplicity, assume that we wish to hedge an asset with a financial future that has a high, but not perfect, correlation with the asset. The amount of futures contracts that should be sold to achieve the best possible hedge is commonly expressed by the *optimal hedge ratio*. The optimal hedge ratio is calculated as the share of contracts that minimizes the variance of the hedged portfolio. The standard textbook recipe for computing the optimal hedge ratio is

$$\rho_{AF} \frac{\sigma_A}{\sigma_F}$$

where σ_A is the estimated constant standard deviation in the price change of the assets during a time equal to the life of the hedge, σ_A is the same quantity for the future, and ρ_{AF} is the estimated constant coefficient of correlation between the two price series. However, knowing that the variances and covariances of asset returns are non-constant, and that they can be modeled with GARCH specifications, it is natural to try to compute an optimal hedge ratio with a multivariate GARCH model. By employing such a procedure, the agent can hopefully minimize the cost of hedging.

This issue has been considered by Baille and Myers [1991], who use a bivariate GARCH model of the form

$$\begin{aligned} h_t^A &= \gamma_A + \alpha_A \varepsilon_{t-1}^A \varepsilon_{t-1}^A + \beta_A h_{t-1}^A, \\ h_t^F &= \gamma_F + \alpha_F \varepsilon_{t-1}^F \varepsilon_{t-1}^F + \beta_F h_{t-1}^F, \\ h_t^{AF} &= \gamma_{AF} + \alpha_{AF} \varepsilon_{t-1}^A \varepsilon_{t-1}^F + \beta_{AF} h_{t-1}^{AF}, \end{aligned}$$

where h_t^A and h_t^F are the conditional variances for the asset and future respectively, h_t^{AF} is the conditional covariance between the asset and the future, and ε_t^A and ε_t^F represent the returns. Baile and Myers estimate the parameters of the model on spot and futures series for six different commodities. Based on the estimates, they then compute time-dependent optimal hedge ratios. They show that an assumption of a time-invariant optimal hedge ratio is inappropriate. This finding should be of major interest to all practitioners involved in hedging with financial futures and options.

1.2.3 Asset Pricing and Asset Allocation

The fundamental issue in almost all models of financial economics is the relation between risk and return of assets. This section gives several examples of how GARCH models have been employed to model and capture this relationship.

A central position in the field of financial economics is taken by the literature on asset pricing. The ARCH-M model of Engle, Lilien and Robins [1987] constitutes a framework for testing empirical implications of theoretical asset pricing models. As noted above, in an ARCH-M model, the conditional mean of a return process is a function of the conditional variance of the return

$$r_t = g(\mathbf{x}_{t-1}, h_t; \phi) + \varepsilon_t,$$

where $g(\cdot)$ is a function of \mathbf{x}_{t-1} and h_t , \mathbf{x}_{t-1} is a vector of predetermined variables, h_t is the conditional variance of ε_t , and ϕ is a parameter vector. In the article by Engle, Lilien and Robins, the conditional variance of ε_t is given by an ARCH(q) model, but the customary procedure is to assume that h_t follows a GARCH model, in which the model is denoted GARCH-M.

The intertemporal capital asset pricing model (ICAPM) of Merton [1973] is one of the most well-known theories in the field. Merton [1980] makes an approximation of the ICAPM model in which the excess return of the market portfolio is a linear function of the variance. This model is studied empirically by French, Schwert, and Stambaugh [1987]. Their investigation was performed on the daily values of Standard and Poor's composite portfolio from 1928 to 1984. This portfolio is regarded as a proxy for the aggregate wealth in the economy. French, Schwert, and Stambaugh estimate an ARCH-M model and a GARCH-M model on the excess return on the portfolio. The conditional mean equation is considered when the excess return is both a linear function of the conditional variance and a linear function of the conditional standard deviation. They find a positive relation between the expected risk premium and the predictable level of volatility. Their estimates of volatility and expected risk premiums indicate that these variables have fluctuated over the sample period. The authors also report a strong negative correlation between the unpredictable component of volatility and

excess holding period returns, indicating the existence of a leverage effect. However, they show that the magnitude of the detected negative correlation is too large to be explained solely by increased financial leverage.

A large number of articles have been written following the work of French, Schwert, and Stambaugh. One often cited article in this tradition is the paper by Glosten, Jagannathan, and Runkle [1993]. They estimate the GARCH-M on data from the CRSP value-weighted index of NYSE equities from 1951 to 1989. When the conditional variance is modeled with a symmetric GARCH model, they find a positive but insignificant relation between the conditional mean and the conditional variance of the excess returns. However, they empirically show that the standard GARCH-M model is misspecified. Based on this finding they introduce a new specification for the conditional variance equation, in which the sign of excess returns is made an explanatory variable of the conditional variance. This is the GJR model mentioned in Section 1.1. When they estimate this model on the data they detect a negative relation between the conditional mean and the conditional variance of excess returns. The authors also introduce seasonal dummies in both the conditional mean and the conditional variance equation. Furthermore, the nominal risk-free interest rate is used as an explanatory variable in the equation for the conditional variance. When this augmented model is estimated on data, the negative relation between the conditional mean and the conditional variance of excess returns is shown to increase.

The two applications discussed above consider both the relation between the excess return on an aggregate portfolio for the economy and its risk measured as the conditional variance of the market return. When an aggregate market portfolio is considered, the analysis can be performed with a univariate GARCH model. But as soon as individual securities or sub-portfolios of the aggregate portfolio are studied, the correlation between the returns on assets has to be modeled, in which a multivariate specification has to be employed. We will below briefly describe two studies in which multivariate GARCH models have been used to model the relationship between risk and returns on assets.

The pioneering work in this field is by Bollerslev, Engle, and Wooldridge [1988]. They consider the empirical implications for excess return on assets given by a conditional CAPM model. Assume that \mathbf{r}_t is the vector of excess returns in the economy, and that $\boldsymbol{\mu}_t$ and \mathbf{H}_t are the conditional mean vector and the conditional covariance matrix of \mathbf{r}_t , given the information known at time $t - 1$, I_{t-1} . Using this notation CAPM will require that

$$\boldsymbol{\mu}_t = \delta \mathbf{H}_t \boldsymbol{\omega}_{t-1},$$

where δ is a scalar which in equilibrium will be an aggregate measure of the relative risk aversion in the economy, and $\boldsymbol{\omega}_{t-1}$ is the vector of value weights of assets in the previous period. Bollerslev, Engle, and Wooldridge assume that \mathbf{H}_t follows a multivariate GARCH model. Since the conditional mean

vector is a function of the conditional covariance matrix, the econometric model will be a multivariate version of the ARCH-M model of Engle, Lilien and Robins [1987]. The econometric model estimated by Bollerslev, Engle, and Wooldridge has the general form

$$\begin{aligned} \mathbf{r}_t &= \mathbf{b} + \delta \mathbf{H}_t \boldsymbol{\omega}_{t-1} + \boldsymbol{\varepsilon}_t, \\ \boldsymbol{\varepsilon}_t | I_{t-1} &\sim N(\mathbf{0}, \mathbf{H}_t), \\ \text{vech}(\mathbf{H}_t) &= \mathbf{C} + \mathbf{A}_1 \text{vech}(\boldsymbol{\varepsilon}_{t-1} \boldsymbol{\varepsilon}'_{t-1}) + \mathbf{B}_1 \text{vech}(\mathbf{H}_{t-1}), \end{aligned}$$

where $\text{vech}()$ denotes an operator that stacks the lower portion of a $N \times N$ matrix as a $N(N+1) \times 1$ vector, \mathbf{b} is a vector of constants, and \mathbf{C} , \mathbf{A}_1 , and \mathbf{B}_1 are constant parameter matrices. The econometric model should be viewed as a parsimonious approximation of the model derived from economic theory.

The estimation of the multivariate GARCH-M model is performed on quarterly US return data from 1959 to 1984. The aggregate market portfolio is assumed to be composed of three assets, 6-month Treasury bills, 20-year Treasury bonds, and stocks. Thus, the GARCH-M model investigated will be trivariate. The NYSE value-weighted equity index is used as a proxy for the stock portfolio. To simplify the estimation, the matrices \mathbf{A}_1 and \mathbf{B}_1 are assumed to be diagonal. The estimation results show that the conditional covariance matrix is highly autoregressive, and non-constant. There is evidence that the risk premia of the different asset classes are better represented by covariances with the market than by their own variances. But, contrary to the implications of the CAPM, information in addition to past innovations of the return series is important in explaining premia and heteroskedasticity. The growth rate in aggregate consumption expenditures and lagged excess returns seems to have additional explanatory power over the non-diversifiable risk.

Since the model considered by Bollerslev, Engle, and Wooldridge [1988] only considers three classes of assets, the specification is very simple in its structure. Naturally, it would be interesting to consider more complex portfolio constellations. However, as discussed in Section 1.1, computational difficulties arise when the dimension of the multivariate GARCH model increases. For a successful estimation additional assumptions are required that can simplify the structure on the conditional covariance matrix. A possible candidate for this purpose is the factor-ARCH model presented by Engle, Ng, and Rothschild [1990]. This model incorporates the idea that the risk of assets can be decomposed in a limited number of common factors and in an asset specific part. The model can therefore be viewed as an empirical formulation of the Arbitrage Pricing Theory (APT) model of Ross [1976] and Chamberlain and Rothschild [1983].

Ng, Engle, and Rothschild [1992] use the factor-ARCH model to estimate a *dynamic market model* on stock return data. In the dynamic market

model, the $N \times 1$ vector of excess returns of assets, \mathbf{r}_t , can be written as

$$\mathbf{r}_t = \boldsymbol{\mu}_t + \boldsymbol{\beta}_m f_{mt} + \sum_{j=1}^{K_s} \boldsymbol{\beta}_j \cdot f_{jt} + \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\mu}_t$ is the $N \times 1$ vector of expected excess returns (or risk premia), f_{mt} is the unanticipated component of stock returns, $\boldsymbol{\beta}_m$ is the vector of market *betas*, f_{jt} is the *static* factor j , $\boldsymbol{\beta}_j$ is the vector of factor loadings for the factor j , K_s is the number of static factors, and $\boldsymbol{\varepsilon}_t$ is the vector of idiosyncratic noises. The term dynamic factor model stems from the fact that in the model the market excess return is assumed to be a so-called dynamic factor. A dynamic factor will have a time-varying conditional variance, and the risk premium associated with the factor will be time-varying. The dynamic factor will therefore determine the time series behavior of individual asset risk premia. A static factor has constant conditional variance and a constant risk premia, and therefore no effect on the dynamic behavior of individual asset risk premium. Using an arbitrage argument similar to that in Ross [1976], the vector of asset risk premia, $\boldsymbol{\mu}_t$, should, if the dynamic market model is correct, be given by

$$\boldsymbol{\mu}_t = \boldsymbol{\beta}_m \pi_{mt}(h_{mt}) + \boldsymbol{\mu}^s,$$

where $\pi_{mt}(h_{mt})$ is the time-varying market risk premium, which is a function of the conditional variance of the market excess return, h_{mt} . $\boldsymbol{\mu}^s$ is the static component of the vector of asset risk premia, which is related to the risk premia of the K_s static factors and the betas of the assets with respect to the static factors.

The questions asked by Bollerslev, Engle, and Wooldridge are: (i) whether the market is in fact a dynamic factor, (ii) whether the market is the sole dynamic factor, and (iii) whether there are likely to be any static factors after the dynamic nature of the model has been taken into account. The data used to evaluate the dynamic market model is a vector of monthly excess returns of ten decile portfolios of stocks traded at NYSE and AMEX. As a proxy for the market the value weighted NYSE+AMEX is used. The sample period is 1964 to 1985. Since the number of parameters in the model is large relative to the number of observations, a two-step estimation procedure is used, rather than the more efficient full maximum likelihood method.

Ng, Engle, and Rothschild find evidence that the market is a dynamic factor. However, a three-dynamic-factor model, with the market as one of the dynamic factors, seems to model excess returns better than the dynamic market model. One additional factor is related to a January dummy, and another is related to the rate of change in industrial production and a recession dummy. After accounting for the systematic risk corresponding to the three dynamic factors, Bollerslev, Engle, and Wooldridge found no clear evidence of a static component of asset risk premium.

1.3 Concluding Remarks

The survey article of the ARCH literature by Bollerslev, Engle and Nelson [1994] makes reference to approximately 200 articles, and since 1994 a large number of both theoretical and empirical papers which concern ARCH model have been published. It is therefore inevitable that this chapter can only constitute a very brief introduction to the ARCH/GARCH literature. It is hoped however, that the presentation has helped to explain how the four papers in this thesis relate to previous work within the research field.

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2

A Smooth Transition ARCH Model for Asset Returns

2.1 Introduction

In this paper I introduce a new class of ARCH/GARCH models. This new class allows for non-linearity in the equation for the conditional variance. Two forms of non-linearity are considered. First, asymmetry regarding the sign of the error term is considered. This specification allows positive and negative shocks of equal size to have different degrees of impact on the conditional variance. Second, non-linearity regarding the size of error terms is considered. The dynamics of the conditional variance will then differ depending on whether the market is relatively volatile or not. Even though models that allow for different forms of non-linearity have been presented before, these new models are believed to have novel features, that are advantageous in a modeling situation. One major advantage is that specification tests can easily be developed from this model class. These specification tests substantially simplify the procedure of finding a suitable model for representing a financial time series.

In his seminal work, Engle [1982] introduces the ARCH (Autoregressive Conditional Heteroskedasticity) model, in which the conditional variance is a linear function of lagged squared residuals, analogous to an MA model for the conditional mean. Bollerslev [1986] introduces the Generalized ARCH (GARCH) model and extends Engle's MA specification into an ARMA model by introducing lagged conditional variances in the conditional variance equation. With this representation, the number of parameters in the model can be reduced considerably. Note, however, that

a stationary GARCH model can always be rewritten as an ARCH model with an infinite number of lags.

Among the first to introduce non-linearities in the ARCH framework were Engle and Bollerslev [1986]. They propose a model where the dynamics of the conditional variance change with the magnitude of squared residuals. The transition between different conditional variance states is controlled by a normal cumulative distribution function.

Higgins and Bera [1992] introduce the Non-linear ARCH (NARCH) model, which encompasses various functional forms for the conditional variance. Their model therefore provides a framework for testing the linear ARCH model against different non-linear alternatives. In their article, the authors derive a Lagrange multiplier statistic for such a test. This test is further developed and analyzed in Bera and Higgins [1992].

In Nelson's [1991] Exponential GARCH (EGARCH) model, the natural logarithm of the conditional variance is modeled as an ARMA process. This solves some problems concerning parameter restrictions in the GARCH model. Furthermore, Nelson introduces a term that makes the conditional variance depend on the sign of lagged residuals. This is motivated by the empirical observation that in some time series there is a correlation between the current conditional variance and lagged returns. Models with this feature are often denoted "asymmetric" or "leverage" volatility models.

Another asymmetric model is the GJR model, proposed by Glosten, Jagannathan, and Runkle [1993]. In the GJR model, the standard GARCH model is extended by letting the parameter for the squared residual have one value when the residual is positive, and another when the residual is negative.

Zakořan [1994] introduces the Threshold ARCH (TARCH) model. In this model the functional form is the same as in the GJR model, but instead of modeling the conditional variance, Zakořan models the conditional standard deviation. The TARCH model is developed further in Rabemananjara and Zakořan [1993].

Ding, Granger, and Engle [1993] present the Asymmetric Power ARCH, a model characterized by a large degree of flexibility. In fact, ARCH, GARCH, NARCH, GJR and TARCH are included in the model as special cases.

A recent asymmetric model is the volatility switching (SV) model presented by Fornari and Mele [1996b]. In the SV model, the GARCH equation is augmented by a term that captures *mean reversion* in conditional variance. Mean reversion refers to the observation that when the conditional variance is high and the residual is smaller than expected, the conditional variance will tend to decrease, and when the conditional variance is low and the residual is larger than expected, the conditional variance often increases.

The model class presented in this paper is inspired to a large extent by the Smooth Transition Autoregressive (STAR) model of Luukkonen, Saikkonen, and Teräsvirta [1988]. In the STAR model, the conditional mean is

a non-linear function of lagged realizations of the series introduced via a transition function. Two commonly used transition functions are the logistic (LSTAR) and the exponential (ESTAR) (see Teräsvirta [1994]). In the non-linear ARCH model presented in this paper, the conditional variance is a non-linear function of lagged residuals. As in the STAR models, the non-linearity is, introduced via either a logistic or an exponential transition function. This gives rise two to different models: the logistic and the exponential smooth transition ARCH model.

In the logistic smooth transition ARCH model, the conditional variance will have dynamics similar to those of the GJR model. The GJR model will obtain as a limiting case of the logistic model. In fact, the GJR model will result if the logistic function is replaced by the Heaviside function. The extra flexibility in the model presented in this paper is accomplished with the introduction of one more parameter.

In the exponential smooth transition ARCH model, the dynamics of the conditional variance are independent of the sign of lagged residuals. Instead, the magnitude of lagged squared residuals control the conditional variance. This specification is similar to that proposed in Engle and Bollerslev [1986] (eq. 36). However, the transition function in Engle and Bollerslev's model is the normal cumulative distribution function. In my model it is the exponential function which means that specification tests are easier to derive.

A much debated subject in the ARCH/GARCH literature is the distributional assumptions for the innovations (see e.g. Teräsvirta [1996]). In the ARCH model of Engle [1982], residuals are assumed to be normally distributed. Bollerslev [1987] introduces Student-t distributed innovations with his GARCH-t model. In Nelson [1991], innovations are assumed to be drawn from a Generalized Error Distribution. However, this paper will not focus on these issues. For simplicity, innovations will be assumed to be normally distributed.

The empirical analysis of stock index series should not be considered as a complete investigation of the possible data generating processes; it is included only as an illustrative example. A empirical comparison of some of the asymmetric models mentioned above is given in Fornari and Mele [1996a].

In Section 2.2 the model is described, and parameter restrictions are given to guarantee stationarity of the return series and non-negativity of the conditional variance. Section 2.3 considers specification tests. In Section 2.4 parameter estimation is briefly discussed. Empirical examples of both the specification tests and estimation are given in Section 2.5, and Section 2.6 presents the conclusions.

2.2 The Model

The return of an asset is assumed to be generated by the process

$$r_t = \varepsilon_t. \quad (2.1)$$

The error term, or the residual, is assumed to have the following form

$$\varepsilon_t = z_t h_t^{1/2}, \quad (2.2)$$

with $z_t \sim \text{nid}(0,1)$, and h_t being the conditional variance at time t . In the ARCH(q) model of Engle [1982], the conditional variance is given by the process

$$h_t = \gamma + \sum_{j=1}^q \alpha_j \varepsilon_{t-j}^2, \quad (2.3)$$

where γ and α_j ($j = 1, \dots, q$) are non-negative constants, with at least one $\alpha_j > 0$. The return process will be stationary if $\sum_{j=1}^q \alpha_j < 1$, in which case the unconditional variance is $\gamma/(1 - \sum_{j=1}^q \alpha_j)$, Milhøj [1985].

In the standard ARCH model (2.3), the conditional variance is a linear function of lagged squared error terms. In this study, this linearity condition is relaxed, and the consequences of such an extension of the ARCH model are investigated. As will be clear below, I focus on a fairly narrow class of models, which is believed have attractive features.

Assume that $\{\alpha_j\}_{j=1}^q$ in (2.3) are not constants, but functions of lagged error terms, according to $\alpha_j = \alpha_j(\varepsilon_{t-j})$. This implies the following equation for the conditional variance

$$h_t = \gamma + \sum_{j=1}^q \alpha_j(\varepsilon_{t-j}) \varepsilon_{t-j}^2,$$

The functional form proposed for $\alpha_j(\varepsilon_{t-j})$ is

$$\alpha_j(\varepsilon_{t-j}) = \alpha_{1j} + \alpha_{2j} F(\varepsilon_{t-j}),$$

where α_{1j} and α_{2j} are constants, and $F(\cdot)$ is a transition function. The proposed smooth transition ARCH model will therefore have the form

$$h_t = \gamma + \sum_{j=1}^q [\alpha_{1j} + \alpha_{2j} F(\varepsilon_{t-j})] \varepsilon_{t-j}^2. \quad (2.4)$$

Below, two specific transition functions will be considered, one logistic and one exponential. Both these functions have simple expressions for the derivatives with respect to the innovations ε_{t-j} . This will prove to be advantageous when specification tests are derived. The logistic function considered has the form

$$F(\varepsilon_{t-j}) = (1 + \exp[-\theta \varepsilon_{t-j}])^{-1} - \frac{1}{2}, \quad \theta > 0, \quad (2.5)$$

and the exponential function is

$$F(\varepsilon_{t-j}) = 1 - \exp[-\theta \varepsilon_{t-j}^2], \quad \theta > 0. \quad (2.6)$$

The two functions will generate quite different dynamics for the conditional variance. The logistic function (2.5) will generate a return process where the dynamics of the conditional variance differ depending on the sign of innovations. A related non-linear model is the model of Glosten, Jagannathan, and Runkle [1993]. In the GJR model, the conditional variance follows one process when the innovations are positive and another process when the innovations are negative. In this model, however, the transition between states is smooth. For $\varepsilon_{t-j} \rightarrow -\infty$ the transition function will be equal to $-1/2$, and when $\varepsilon_{t-j} \rightarrow +\infty$ the transition function will be equal to $1/2$.

The exponential function (2.6) is symmetric with respect to the sign of the error term. This transition function will generate data for which the dynamics of the conditional variance depend on the magnitude of the innovations. For $|\varepsilon_{t-j}| \rightarrow \infty$ the transition function will be equal to unity, and when $\varepsilon_{t-j} = 0$ the transition function is equal to zero.

To derive conditions for non-negativity of the conditional variance and stationarity of the return series, results of Milhøj [1985] and Tjøstheim [1986] are used and it is noted that for (2.5), $-1/2 \leq F(\cdot) \leq 1/2$, and for (2.6), $0 \leq F(\cdot) \leq 1$. Sufficient conditions for strictly positive conditional variance in the logistic smooth transition ARCH model are

$$\begin{aligned} \gamma &> 0 \\ \alpha_{1j} &\geq 0 \quad (j = 1, \dots, q) \\ \alpha_{1j} &\geq \frac{1}{2} |\alpha_{2j}| \quad (j = 1, \dots, q). \end{aligned}$$

For stationarity of the return process it is required

$$\sum_{j=1}^q [\alpha_{1j} - \frac{1}{2} |\alpha_{2j}| + \max(\alpha_{2j}, 0)] < 1.$$

Sufficient conditions for strictly positive conditional variance in the exponential smooth transition ARCH model are

$$\begin{aligned} \gamma &> 0 \\ \alpha_{1j} &\geq 0 \quad (j = 1, \dots, q) \\ \alpha_{1j} + \alpha_{2j} &\geq 0 \quad (j = 1, \dots, q). \end{aligned}$$

For stationarity of the return process it is required

$$\sum_{j=1}^q [\alpha_{1j} + \max(\alpha_{2j}, 0)] < 1.$$

For the smooth transition ARCH process to be defined, it is required that at least one $\alpha_{i,j} > 0$.

A natural extension of the smooth transition ARCH model is to include lagged conditional variances in the equation for the conditional variance, as is done by Bollerslev [1986] for the ARCH(q) model. The standard GARCH(q, p) model of Bollerslev is in our notation equal to

$$h_t = \gamma + \sum_{j=1}^q \alpha_{1j} \varepsilon_{t-j}^2 + \sum_{i=1}^p \beta_i h_{t-i}, \quad (2.7)$$

where the inequality conditions $\gamma > 0$, $\alpha_{1j} \geq 0$ ($j = 1, \dots, q$), and $\beta_i \geq 0$ ($i = 1, \dots, p$), are imposed to ensure that the conditional variance is strictly positive. The return process will be stationary if

$$\sum_{j=1}^q \alpha_{1j} + \sum_{i=1}^p \beta_i < 1,$$

in which case the unconditional variance is $\gamma / (1 - \sum_{j=1}^q \alpha_{1j} - \sum_{i=1}^p \beta_i)$, Bollerslev [1986]. The smooth transition GARCH(q, p) model proposed is given by

$$h_t = \gamma + \sum_{j=1}^q [\alpha_{1j} + \alpha_{2j} F(\varepsilon_{t-j})] \varepsilon_{t-j}^2 + \sum_{i=1}^p \beta_i h_{t-i}, \quad (2.8)$$

where $F(\cdot)$ is either of the form (2.5) or (2.6). The two resulting models will be termed the logistic and the exponential smooth transition GARCH model. A model similar to the logistic smooth transition GARCH model has been independently proposed by González-Rivera [1996].

The GARCH model can parsimoniously represent a higher order ARCH model. Therefore, specification (2.8) has the advantage that it will generally require fewer parameters than the smooth transition ARCH model. Using the parameter restrictions of the GARCH model (2.7), in conjunction with the properties that for (2.5), $-1/2 \leq F(\cdot) \leq 1/2$, and for (2.6), $0 \leq F(\cdot) \leq 1$, sufficient parameter restriction in model (2.8) can be derived. For positive conditional variance in the logistic smooth transition GARCH model it is required that

$$\begin{aligned} \gamma &> 0 \\ \alpha_{1j} &\geq 0 \quad (j = 1, \dots, q) \\ \beta_i &\geq 0 \quad (i = 1, \dots, p) \\ \alpha_{1j} &\geq \frac{1}{2} |\alpha_{2j}|. \end{aligned}$$

For stationarity of the return process it is required

$$\sum_{j=1}^q [\alpha_{1j} - \frac{1}{2} |\alpha_{2j}| + \max(\alpha_{2j}, 0)] + \sum_{i=1}^p \beta_i < 1.$$

For positive conditional variance in the exponential smooth transition GARCH model it is required that

$$\begin{aligned}\gamma &> 0 \\ \alpha_{1j} &\geq 0 \quad (j = 1, \dots, q) \\ \beta_i &\geq 0 \quad (j = 1, \dots, p) \\ \alpha_{1j} + \alpha_{2j} &\geq 0.\end{aligned}$$

For stationarity of the return process it is required

$$\sum_{j=1}^q [\alpha_{1j} + \max(\alpha_{2j}, 0)] + \sum_{i=1}^p \beta_i < 1.$$

For the smooth transition GARCH process to be defined it is required that at least one $\alpha_{ij} > 0$.

2.3 Specification Tests

The main purpose of this section is to present procedures for testing the null of linear conditional variance against the alternative of non-linear conditional variance. However, before testing such a hypothesis, it is natural to first test the null of constant variance against the alternative of smooth transition ARCH or smooth transition GARCH, as specified in (2.4) or (2.8). This section therefore begins with a description of a test of no smooth transition ARCH. The test procedure is similar to Engle's [1982] test of no ARCH. Following that, a test procedure that can discriminate between a linear and a non-linear ARCH model is presented. Finally, tests are derived which are to be used when the smooth transition GARCH model is considered. As will be clear below the test statistics presented will differ, depending on the functional form chosen for the transition function.

Suppose we have an observed time series $r_{-q+1}, \dots, r_0, r_1, \dots, r_T$. Let $\mathbf{w}'_t = (1, \varepsilon_{t-1}^2, \dots, \varepsilon_{t-q}^2, F(\varepsilon_{t-1})\varepsilon_{t-1}^2, \dots, F(\varepsilon_{t-q})\varepsilon_{t-q}^2)$, and $\boldsymbol{\alpha}' = (\gamma, \alpha_{11}, \dots, \alpha_{1q}, \alpha_{21}, \dots, \alpha_{2q})$. Using this notation the conditional variance equation (2.4) can be written

$$h_t = \mathbf{w}'_t \boldsymbol{\alpha}. \quad (2.9)$$

To detect ARCH effects, it is necessary to test the null hypothesis that in (2.9) $H_0 : \alpha_{1j} = \alpha_{2j} = 0$ ($j = 1, \dots, q$), against $H_1 : \text{at least one } \alpha_{ij} \neq 0$. Because (2.9) is constant under the null, it is natural to apply the Lagrange multiplier principle. The Lagrange multiplier test of no smooth transition ARCH is

$$\frac{1}{2} \left\{ \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\tilde{\sigma}^2} - 1 \right] \mathbf{w}_t \right\}' \left\{ \sum_{t=1}^T \mathbf{w}_t \mathbf{w}_t' \right\}^{-1} \left\{ \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\tilde{\sigma}^2} - 1 \right] \mathbf{w}_t \right\}, \quad (2.10)$$

where $\tilde{\sigma}^2 = T^{-1} \sum_{t=1}^T \varepsilon_t^2$. For a derivation of equation (2.10), see the appendix.

However, the vector \mathbf{w}_t in (2.10) is dependent on the transition function, which under the null has an unidentified parameter θ . Therefore, the test statistic is not operational as stated in equation (2.10). Following Luukkonen, Saikkonen, and Teräsvirta [1988], this problem is solved by making a Taylor expansion of the transition function around zero. The obtained approximation of $F(\cdot)$ is then inserted into the vector \mathbf{w}_t .

If the transition function is logistic, as specified in (2.5), it is possible to approximate $F(x)$ by

$$T_l = F'(0)x = \frac{\theta}{4}x.$$

The vector \mathbf{w}'_t can therefore be approximated by

$$\widehat{\mathbf{w}}'_t = (1, \varepsilon_{t-1}^2, \dots, \varepsilon_{t-q}^2, \frac{\theta}{4}\varepsilon_{t-1}^3, \dots, \frac{\theta}{4}\varepsilon_{t-q}^3).$$

Substituting $\widehat{\mathbf{w}}'_t$ into (2.9) yields

$$h_t = \widehat{\mathbf{w}}'_t \boldsymbol{\alpha}. \quad (2.11)$$

Furthermore, equation (2.11) can be reformulated and reparameterized as

$$h_t = \widehat{\boldsymbol{\omega}}'_t \widehat{\boldsymbol{\alpha}}, \quad (2.12)$$

where $\widehat{\boldsymbol{\omega}}_t = (1, \varepsilon_{t-1}^2, \dots, \varepsilon_{t-q}^2, \varepsilon_{t-1}^3, \dots, \varepsilon_{t-q}^3)$ and $\widehat{\boldsymbol{\alpha}} = (\gamma, \alpha_{11}, \dots, \alpha_{1q}, \alpha_{21}\theta/4, \dots, \alpha_{2q}\theta/4)$. The null hypothesis of no ARCH is reformulated as $H_0 : \alpha_{1j} = \alpha_{2j}\theta/4 = 0$ ($j = 1, \dots, q$), against H_1 : at least one α_{1j} or one $\alpha_{2j}\theta/4$ is different from zero. This hypothesis can now be tested with the Lagrange multiplier principle. The test statistic is derived using the same techniques as above, and is equal to

$$\frac{1}{2} \left\{ \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\tilde{\sigma}^2} - 1 \right] \widehat{\boldsymbol{\omega}}_t \right\}' \left\{ \sum_{t=1}^T \widehat{\boldsymbol{\omega}}_t \widehat{\boldsymbol{\omega}}_t' \right\}^{-1} \left\{ \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\tilde{\sigma}^2} - 1 \right] \widehat{\boldsymbol{\omega}}_t \right\}. \quad (2.13)$$

Following the arguments of Engle [1982], it is more convenient however to perform the following test procedure that yields an asymptotically equivalent statistic:

1. Compute the residual sum of squares $SSR_0 = \sum_{t=1}^T \varepsilon_t^2$.
2. Regress ε_t^2 on $\widehat{\boldsymbol{\omega}}_t$, and compute the residual sum of squares from the regression, SSR_3 .
3. Compute the test statistic

$$LM_1 = T \cdot \frac{SSR_0 - SSR_3}{SSR_0}. \quad (2.14)$$

When H_0 is valid, LM_1 has an asymptotic χ^2 distribution with $2 \cdot q$ degrees of freedom.¹ The statistic (2.14) can alternatively be written as $T \cdot R_u^2$, where R_u^2 denotes the coefficient of multiple correlation from an OLS estimation of the artificial model

$$\varepsilon_t^2 = \tilde{\omega}'_t \mathbf{a} + \xi_t.$$

If the transition function is exponential, as specified in (2.6), $F(x)$ can be approximated by

$$T_e = F''(0)x^2 = 2\theta x^2.$$

The vector \mathbf{w}_t can therefore be approximated by

$$\tilde{\mathbf{w}}'_t = (1, \varepsilon_{t-1}^2, \dots, \varepsilon_{t-q}^2, 2\theta\varepsilon_{t-1}^4, \dots, 2\theta\varepsilon_{t-q}^4).$$

This suggests the following test procedure:

1. Compute the residual sum of squares $SSR_0 = \sum_{t=1}^T \varepsilon_t^2$.
2. Regress ε_t^2 on $\tilde{\mathbf{w}}_t = (1, \varepsilon_{t-1}^2, \dots, \varepsilon_{t-q}^2, \varepsilon_{t-1}^4, \dots, \varepsilon_{t-q}^4)$, and compute the residual sum of squares from the regression, SSR_4 .
3. Compute the test statistic

$$LM_2 = T \cdot \frac{SSR_0 - SSR_4}{SSR_0} = T \cdot R_u^2, \quad (2.15)$$

which under the null is asymptotically χ^2 distributed with $2 \cdot q$ degrees of freedom.

Again, it can be shown that the two test statistics (2.10) calculated with \mathbf{w}_t replaced by $\tilde{\mathbf{w}}_t$, and (2.15) are asymptotically equivalent. R_u^2 in (2.15) is the coefficient of multiple correlation from an OLS estimation of the artificial model

$$\varepsilon_t^2 = \tilde{\omega}'_t \mathbf{a} + \xi_t.$$

Below is a description of a procedure to test the null hypothesis that in (2.9) $H_0 : \alpha_{2j} = 0$ ($j = 1, \dots, q$), against $H_1 : \text{at least one } \alpha_{2j} \neq 0$. A Lagrange multiplier statistic for this test would have the form

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \mathbf{w}_t \right\}' \left\{ \sum_{t=1}^T \frac{1}{h_{0t}^2} \mathbf{w}_t \mathbf{w}_t' \right\}^{-1} \\ & \times \left\{ \sum_{t=1}^T \frac{1}{h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \mathbf{w}_t \right\}, \end{aligned} \quad (2.16)$$

¹The asymptotic equivalence of (2.13) and (2.14) is shown by noting that under normality

$$plim \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\sigma^2} - 1 \right]^2 = 2T.$$

where h_{0t} is the conditional variance under the null. Under the null the conditional variance equation will be given by an ARCH(q) model. h_{0t} can therefore be obtained by estimating an ARCH(q) on the data. Following the same arguments as above, (2.16) can be made operational by replacing the transition function in w_t by a Taylor expansion of $F(\cdot)$. If the transition function is logistic, the test statistic is equal to

$$\frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \tilde{\omega}_t \right\}' \left\{ \sum_{t=1}^T \frac{1}{h_{0t}^2} \tilde{\omega}_t \tilde{\omega}_t' \right\}^{-1} \times \left\{ \sum_{t=1}^T \frac{1}{h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \tilde{\omega}_t \right\}. \quad (2.17)$$

The statistic (2.17) will under the null be asymptotically distributed as χ^2 with q degrees of freedom. In a similar way to what was shown above, it is possible to derive a regression based test statistic, which is asymptotically equivalent to (2.17). This statistic would be calculated as $T \cdot R_u^2$ from the regression of

$$\left\{ \frac{\varepsilon_t^2}{h_{0t}} - 1 \right\}$$

on

$$\left\{ \frac{1}{h_{0t}}, \frac{\varepsilon_{t-1}^2}{h_{0t}}, \dots, \frac{\varepsilon_{t-q}^2}{h_{0t}}, \frac{\varepsilon_{t-1}^3}{h_{0t}}, \dots, \frac{\varepsilon_{t-q}^3}{h_{0t}} \right\}.$$

The statistic for the test of ARCH(q) against the alternative of exponential smooth transition ARCH(q) will have a form similar to (2.17), with $\tilde{\omega}_t$ replaced by $\hat{\omega}_t$. The test statistic will be asymptotically distributed χ^2 with q degrees of freedom. An asymptotically equivalent $T \cdot R_u^2$ statistic can also be derived in this case.

Note, that the method of replacing the transition function with a suitable approximation, is not restricted to the case where the transition functions have either the form (2.5) or (2.6). Thus, this test procedure can be used for all non-linear ARCH models which have a transition function that can be approximated by a second order Taylor expansion around zero, and for which $F(0) = 0$.

Above, the cases where the transition function is either the logistic or the exponential have been deliberately separated. However, it is possible to test the null of homoskedasticity against the alternative of non-linear ARCH(q) of both forms simultaneously. This can be done in the regression model of the form

$$\begin{aligned} \varepsilon_t^2 = & a_0 + a_{11}\varepsilon_{t-1}^2 + \dots + a_{1q}\varepsilon_{t-q}^2 + a_{21}\varepsilon_{t-1}^3 + \dots + a_{2q}\varepsilon_{t-q}^3 + \\ & + a_{31}\varepsilon_{t-1}^4 + \dots + a_{3q}\varepsilon_{t-q}^4 + \xi_t. \end{aligned} \quad (2.18)$$

A *LM* type test statistic for the hypothesis is calculated as $T \cdot R_u^2$, where R_u^2 is calculated from the model (2.18). The test statistic will be asymptotically distributed χ^2 with $3 \cdot q$ degrees of freedom. Furthermore, it is straightforward to derive a *LM* statistic for testing the null of logistic smooth transition ARCH(q) against the alternative of both a logistic and an exponential smooth transition ARCH(q) model. Likewise, it is possible to test the null of exponential smooth transition non-linear ARCH against the alternative of both a logistic and an exponential transition function. However, both these tests require that the series of conditional variance under the null is estimated. Since it is far more time consuming to estimate a smooth transition ARCH(q) model than a standard ARCH(q) model, the tests will be more complicated to perform.

This section will close by describing specification procedures designed to test for smooth transition GARCH, rather than for smooth transition ARCH. Bollerslev [1986] notes that under the null of no heteroskedasticity there is no general Lagrange multiplier test for GARCH(p, q). This is due to the fact that the Hessian is singular if both $p > 0$ and $q > 0$. In the smooth transition GARCH model this will also occur. To test for smooth transition GARCH, the same test procedures as described for the ARCH model are proposed, with a fairly large number of lags. This is motivated by the fact that the smooth transition GARCH model can be rewritten as a smooth transition ARCH model with an infinite number of lags.

However, it is straightforward to derive a *LM* test for the null of GARCH- (p, q) against the alternative of smooth transition GARCH(p, q). The derivation of such a statistic is performed using techniques similar to those presented in the appendix. For simplicity the description is only of tests of the null of GARCH(1,1) against the alternative of smooth transition GARCH(1,1). Thus, the model considered under the null is

$$h_t = \gamma + [\alpha_{11} + \alpha_{21} F(\varepsilon_{t-1})] \varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the hypothesis to be tested is

$$\begin{aligned} H_0 &: \alpha_{21} = 0 \\ H_1 &: \alpha_{21} \neq 0. \end{aligned}$$

Given that transition function is the logistic (2.5), and that ε_t is distributed conditionally normal, a Lagrange multiplier test statistics for the hypothesis is

$$\begin{aligned} LM_3 &= \frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \hat{\alpha}} \right\}' \left\{ \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \hat{\alpha}} \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \hat{\alpha}} \right]' \right\}^{-1} \\ &\times \left\{ \sum_{t=1}^T \frac{1}{h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \hat{\alpha}} \right\}, \end{aligned} \quad (2.19)$$

where

$$\frac{\partial h_t}{\partial \hat{\alpha}'} = \left[\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1}, \sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} \varepsilon_{t-i}^2, \sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} \varepsilon_{t-i}^3, \sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} h_{0t-i} \right],$$

h_{0t} is the conditional variance under the null of GARCH(1,1), $\hat{\alpha}'$ is the vector of parameters $(\gamma, \alpha_{11}, \alpha_{21}\theta/4, \beta_1)$, and $\hat{\beta}_1$ is the estimated parameter β_1 in the GARCH(1,1) model. The statistic LM_3 will under the null of GARCH(1,1) be asymptotically distributed χ^2 with one degree of freedom. An asymptotically equivalent statistic would be calculated as $T \cdot R_u^2$ from the regression of

$$\left\{ \frac{\varepsilon_t^2}{h_{0t}} - 1 \right\}$$

on

$$\left\{ \frac{\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} \varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} h_{0t-i}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} \varepsilon_{t-i}^3}{h_{0t}} \right\}.$$

When the transition function is the exponential (2.6), the Lagrange multiplier test statistics will be

$$\begin{aligned} LM_4 &= \frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \hat{\alpha}'} \right\}' \left\{ \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \hat{\alpha}'} \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \hat{\alpha}'} \right]' \right\}^{-1} \\ &\quad \times \left\{ \sum_{t=1}^T \frac{1}{h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \hat{\alpha}'} \right\}, \end{aligned} \quad (2.20)$$

where

$$\frac{\partial h_t}{\partial \hat{\alpha}'} = \left[\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1}, \sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} \varepsilon_{t-i}^2, \sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} \varepsilon_{t-i}^4, \sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} h_{0t-i} \right],$$

h_{0t} is the conditional variance under the null of GARCH(1,1), $\hat{\alpha}'$ is the vector of parameters $(\gamma, \alpha_{11}, \alpha_{21}2\theta, \beta_1)$, and $\hat{\beta}_1$ is the estimated parameter β_1 in the GARCH(1,1) model. When H_0 is valid LM_4 has an asymptotic χ^2 distribution with one degree of freedom. The asymptotically equivalent statistic is $T \cdot R_u^2$ from the regression of

$$\left\{ \frac{\varepsilon_t^2}{h_{0t}} - 1 \right\}$$

on

$$\left\{ \frac{\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} \varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} h_{0t-i}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} \varepsilon_{t-i}^4}{h_{0t}} \right\}.$$

2.4 Estimation

If the conditional mean follows equation (2.1), and given that the innovations z_t are Gaussian, the parameters of the models are estimated by maximizing the likelihood function $\sum_{t=1}^T l_t$, where l_t is given by (2.22). This is done using standard numerical methods. Since the magnitude of the parameters is quite different in this model, it is recommended that the parameters be scaled, so that the diagonal elements of the Hessian are roughly equal. It is also advisable to set the starting value of the parameter θ in a region where the transition function will not just take on the extreme values of the function.

If the conditional mean follows a regression model, the parameters of the conditional variance process should be estimated simultaneously with the parameters of the conditional mean model. However, the procedure can be simplified when the exponential model is considered. Engle [1982] shows that when the conditional variance is symmetric with respect to the innovations, as it is in the exponential smooth transition ARCH model, the two models can be estimated separately. Thus, first the parameters of the conditional mean model are estimated, and then estimate the parameters of the ARCH/GARCH model are estimated on the estimated residuals from the conditional mean model.

2.5 Empirical example

For the empirical analysis, observations used were daily return series from four different equity indexes: the Copenhagen Stock Exchange general index (CGI), the Financial Times all share index (FT-all), the Milan Stock Exchange general index (MGI), and the Stockholm equity index (OMX).² The period investigated is January 3, 1991 to July 15, 1996. The number of observations per series is approximately 1400. Returns are calculated as $\ln(P_t/P_{t-1})$, where P_t is the index level at the end of day t .

The series are first examined for autocorrelation using a test developed by Richardson and Smith [1994]. The test, in the form used here, is a robust version of a standard Box and Pierce [1970] procedure. The statistic is calculated as

$$RS(k) = T \sum_{i=1}^k \frac{\hat{\rho}_i^2}{1 + c_i}, \quad (2.21)$$

²These series are not chosen randomly from the population of equity index series. Rather, they have been selected because they fulfill the purpose of showing how the specification tests work and how estimation results may turn out. However, it was no difficult to find series that suited this purpose.

where $\hat{\rho}_i$ is the estimated autocorrelation between the returns at time t and $t - i$. The terms c_i is an adjustment factor for heteroskedasticity, given by

$$c_i = \frac{\text{cov}[\bar{r}_t^2, \bar{r}_{t-i}^2]}{\text{var}[r_t^2]},$$

where \bar{r}_t is the demeaned return at time t . Under the null of no autocorrelation, the statistic is distributed asymptotically χ^2 with k degrees of freedom.

Richardson and Smith's test (2.21) was calculated with k equal to five. No autocorrelation was detected for FT-all and MGI, on five percent significance level, while autocorrelation was indicated for CGI and OMX. For CGI and OMX AR(1) models were fitted. To ensure that these models capture the detected autocorrelation, the test (2.21) were applied again, calculated on estimated residuals from the AR(1) models. Following the recommendations of Box and Pierce [1970] and Ljung and Box [1978], the value of the statistic in this case is compared to a χ^2 distribution with $k - 1$ degrees of freedom. No further autocorrelation was detected for CGI and OMX, on five percent significance level. It is therefore concluded that for FT-all and MGI a suitable mean equation is

$$r_t = \varphi_0 + \varepsilon_t,$$

and for CGI and OMX the mean specification chosen is

$$r_t = \varphi_0 + \varphi_1 r_{t-1} + \varepsilon_t.$$

After having considered the conditional mean specification, tests are performed regarding the conditional variance. These tests are calculated on estimated residuals from the conditional mean models.³ Results from these specification tests are showed in Table 1. Column two reports results for Engle's [1982] test of no ARCH, calculated on ten lagged squared residuals. The reported p-values show that it is possible to reject the null of constant variance against heteroskedasticity in the form of linear ARCH for all four series, on five percent significance level. Columns three and four show similar results, with respect to the smooth transition ARCH model. The last two columns report the results from the tests of GARCH(1,1) against smooth transition GARCH(1,1). According to column five, for CGI and MGI it is possible to reject the null of GARCH(1,1) against the alternative of logistic smooth transition GARCH(1,1), on five percent significance level. However,

³Bera, Higgins and Lee [1992] thoroughly analyze the problem of testing for ARCH when the conditional mean is given by an AR model. They suggest that the Lagrange multiplier statistic should be adjusted for possible autoregression. The simplistic test procedure here is motivated by the observation that even though autocorrelation is present in financial time series, any AR model can only explain a very small fraction of observed returns.

for FT-all and OMX we can reject the null against the alternative of logistic smooth transition GARCH(1,1). Column six shows that only for CGI is it possible to reject the null of GARCH(1,1) against the alternative of exponential smooth transition GARCH(1,1). Thus, it is concluded that for CGI the exponential smooth transition GARCH(1,1) model might be the data generating process of the conditional variance. For FT-all and OMX the conditional variance could have been generated from the logistic model.

For FT-all and OMX the logistic smooth transition GARCH(1,1) model was estimated. The conditional mean specification was estimated simultaneously. The exponential smooth transition GARCH(1,1) model is estimated on CGI. In this case the conditional mean is estimated separately. Since no higher order GARCH effects are detected for MGI, no further estimations are performed on the series. The results from the three estimations are compared to the results from estimations where the standard GARCH(1,1) model is fitted to data. The innovations in all models are assumed to be normally distributed.

Parameter estimates for the models are shown in Table 2. The two last rows give the values on the log-likelihood function and the value on Akaike's information criterion (AIC). All estimated parameters fall into the regions where, given the sufficient conditions in Section 2.2, it is known that the return processes are stationary and that the conditional variance is non-negative. Since specification tests have been used to establish the possible data generating processes of the series, standard errors of the estimates have intentionally been left unreported. If the residuals are assumed to be distributed conditionally normal, the standard errors can be computed from the inverse of the Hessian. If normality is not fulfilled, the estimation procedure used is a quasi-maximum likelihood method. In such cases, standard errors can be computed with a method presented by Bollerslev and Wooldridge [1992].

The first two columns of Table 2 report estimation results for CGI. According to AIC, the exponential smooth transition GARCH(1,1) model constitutes an improvement over the standard GARCH(1,1) model in describing the dynamics of the conditional variance. The estimated parameter value for θ gives a transition function which moves smoothly between zero and one, as can be seen from Figure 2. Figure 3 shows how the conditional variance reacts to different residual values, based on the estimated parameters. From the figure it can be observed that the *news impact curve* for exponential smooth transition GARCH model nearly coincides with that of the GARCH model when residuals are in the interval -1.5 to +1.5 percent.⁴ However, for larger absolute returns, the two curves show a different pattern. In the smooth transition GARCH(1,1) case, the relative influence

⁴The news impact curve, as a way of illustrating the level of asymmetry in a heteroskedastic model, was introduced by Engle and Ng [1993].

of large absolute returns will be lower than in the GARCH(1,1) case. Only 2.5 percent of all absolute returns are larger than 1.5 percent.

For FT-all the logistic model maximizes the likelihood function and minimizes AIC. As can be seen in Figure 1, the estimated value for θ will give a transition function that moves very slowly between its extreme values $-1/2$ and $+1/2$. In the region where most returns are present, between -1.5 and $+1.5$ percent, the transition function seems to be almost linear with respect to the residual value. Figure 4 shows how the conditional variance reacts to different residual values, based on the estimated parameters. As expected, the news impact curve for the logistic model is asymmetric around $\varepsilon_t = 0$. The reaction to negative residual values is much larger than the reaction to positive residuals of the same magnitude. Thus, a leverage effect seems to be present in the return series.

The last two columns of Table 2 report estimation results for OMX. Even in this case, the logistic model maximizes the likelihood function and minimizes the AIC. From Figure 1 it can be observed that the estimated parameter value for θ gives a transition function which is much more sensitive to different residual values, than what was found for FT-all. When absolute residuals are larger than 1.5 percent, the transition function will always be at its extreme values. 16 percent of all absolute returns in the sample are larger than 1.5 percent. Figure 5 shows the news impact curve for the two models estimated on OMX. As was the case for FT-all, the news impact curve shows leverage effects. Thus, large positive returns will increase the conditional variance less than large negative returns.

2.6 Summary and Conclusion

This paper has presented a new class of ARCH model, the smooth transition ARCH model. In these models, the conditional variance is a non-linear function of lagged squared residuals. The non-linearity is introduced by a transition function. Two specific transition functions are considered, the logistic and the exponential. These two functions will each give the conditional variance quite different dynamics. The logistic function allows for asymmetric behavior of the conditional variance with respect to the signs of residuals. In the exponential smooth transition model, the dynamics of the conditional variance will differ depending on the absolute size of lagged residuals.

Following the work of Bollerslev [1986] the smooth transition ARCH model is extended to a smooth transition GARCH model. In this model, the conditional variance is a function of both lagged residuals and lagged conditional variances. This formulation is likely to limit the number of parameters needed for a successful estimation of the model.

In Section 2.3 a number of specification tests for the smooth transition ARCH/GARCH model were presented. Since the estimation of the model requires iterative procedures, these tests are most valuable in a practical situation.

Section 2.5 contained a short empirical example, where four equity index returns series were estimated. Data is daily and the sample period is from January 1991 to July 1996. Using the specification tests it can be concluded that two of the series could have had the logistic smooth transition GARCH(1,1) model as their data generating process. One conditional variance series might have been generated from an exponential smooth transition GARCH(1,1) model. The models were estimated on the three series that indicated higher order GARCH effects, and these estimates are compared to standard GARCH(1,1) estimates. According to AIC, the smooth transition GARCH(1,1) model constitutes an improvement over the GARCH model. The estimated parameters for the logistic model indicate that large negative residuals increase the conditional variance more than positive residuals. Thus, for these two series the conditional volatility seems to increase in bear markets. The estimation results for the exponential model show that large absolute residuals are given too high an influence in the standard GARCH(1,1) model.

One question that has not been answered in this paper is whether this model can outperform other non-linear ARCH models presented in the literature. This is obviously a question that calls for further research. Another question that needs to be examined more carefully is the parameter restrictions required for stationarity. In Section 2.2, sufficient conditions for stationarity are provided, but these conditions could be too restrictive. Furthermore, small sample properties of the specification tests need to be analyzed.

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Appendix

1. Derivation of LM statistic (2.10)

The Lagrange multiplier statistic has the general form

$$LM = T\bar{q}_T(\alpha_0)'I(\alpha_0)^{-1}\bar{q}_T(\alpha_0),$$

where α_0 is the vector of parameters under the null. $\bar{q}_T(\alpha)$ is the average score and $I(\alpha)$ is the information matrix. If the innovations are assumed to be Gaussian, the log likelihood of one observation is equal to

$$l_t = -\frac{1}{2} \ln 2\pi - \frac{1}{2} \ln h_t - \frac{1}{2} \frac{\varepsilon_t^2}{h_t}. \quad (2.22)$$

It is then straightforward to show that the average score is equal to

$$\bar{q}_T(\alpha) = \frac{1}{T} \sum_{t=1}^T \frac{1}{2h_t} \left[\frac{\varepsilon_t^2}{h_t} - 1 \right] \mathbf{w}_t.$$

The information matrix is the negative expectation of the Hessian averaged over all observations

$$I(\alpha) = -E \left[\frac{1}{T} \sum_{t=1}^T \frac{\partial^2 l_t}{\partial \alpha \partial \alpha'} \right].$$

The Hessian for one observation can be shown to be equal to

$$\frac{\partial^2 l_t}{\partial \alpha \partial \alpha'} = -\frac{1}{2h_t^2} \left[\frac{\varepsilon_t^2}{h_t} \right] \mathbf{w}_t \mathbf{w}_t' - \frac{1}{2} \frac{1}{h_t^2} \left[\frac{\varepsilon_t^2}{h_t} - 1 \right] \mathbf{w}_t \mathbf{w}_t',$$

which implies that the information matrix becomes

$$I(\alpha) = \sum_{t=1}^T \frac{1}{2T} E \left[\frac{1}{h_t^2} \mathbf{w}_t \mathbf{w}_t' \right].$$

The information matrix is consistently estimated by

$$\hat{I}(\alpha) = \frac{1}{2T} \sum_{t=1}^T \left[\frac{1}{h_t^2} \mathbf{w}_t \mathbf{w}_t' \right].$$

Now consider the average score under the null, when the conditional variance is constant. Let σ^2 denote the constant conditional variance under the null. The average score can then be written

$$\bar{q}_T(\alpha_0) = \frac{1}{2T\sigma^2} \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\sigma^2} - 1 \right] \mathbf{w}_t,$$

and the consistently estimated information matrix under the null is equal to

$$\hat{I}(\alpha_0) = \frac{1}{2T(\sigma^2)^2} \sum_{t=1}^T \mathbf{w}_t \mathbf{w}_t'$$

The Lagrange multiplier test of no smooth transition ARCH can therefore be written

$$\begin{aligned} & T \left\{ \frac{1}{2T\tilde{\sigma}^2} \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\tilde{\sigma}^2} - 1 \right] \mathbf{w}_t \right\}' \left\{ \frac{1}{2T(\tilde{\sigma}^2)^2} \sum_{t=1}^T \mathbf{w}_t \mathbf{w}_t' \right\}^{-1} \\ & \times \left\{ \frac{1}{2T\tilde{\sigma}^2} \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\tilde{\sigma}^2} - 1 \right] \mathbf{w}_t \right\}, \end{aligned} \quad (2.23)$$

where $\tilde{\sigma}^2 = T^{-1} \sum_{t=1}^T \varepsilon_t^2$. Formula (2.23) can be simplified to

$$\frac{1}{2} \left\{ \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\tilde{\sigma}^2} - 1 \right] \mathbf{w}_t \right\}' \left\{ \sum_{t=1}^T \mathbf{w}_t \mathbf{w}_t' \right\}^{-1} \left\{ \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{\tilde{\sigma}^2} - 1 \right] \mathbf{w}_t \right\},$$

which corresponds to formula (2.10). ■

Table 1. Results from specification tests

The table reports p-values for specification tests performed on the four series of estimated residuals. For the Financial Times all share index and for the Milan general index, the residuals are calculated as returns minus mean return. For the Copenhagen general index and for the Stockholm OMX index, residuals are from an AR(1) model. The investigated period is January 3, 1991 to July 15, 1996. The column labeled No ARCH gives the results for Engle's [1982] test of no ARCH, calculated on ten lagged squared residuals. The statistic has an approximate $\chi^2(10)$ distribution under the null. The column labeled LM_1 reports the results from the test of no ARCH, against the alternative of smooth transition ARCH with a logistic transition function, as specified in equation (2.14), calculated on ten lagged residuals. The column labeled LM_2 reports the results from the test of no ARCH, against the alternative of smooth transition ARCH with an exponential transition function, as specified in equation (2.15), calculated on ten lagged residuals. The column labeled LM_3 reports the results from the test of GARCH(1,1), against the alternative of smooth transition GARCH(1,1) with a logistic transition function, as specified in equation (2.19). The column labeled LM_4 reports the results from the test of GARCH(1,1), against the alternative of smooth transition GARCH(1,1) with an exponential transition function, as specified in equation (2.20).

Index	No ARCH	LM_1	LM_2	LM_3	LM_4
Copenhagen	$1.85 \cdot 10^{-17}$	$1.84 \cdot 10^{-20}$	$3.42 \cdot 10^{-23}$	0.7762	0.0156
FT-all share	$3.28 \cdot 10^{-21}$	$4.82 \cdot 10^{-27}$	$4.35 \cdot 10^{-33}$	0.0461	0.6547
Milan	$9.52 \cdot 10^{-61}$	$4.43 \cdot 10^{-83}$	$1.24 \cdot 10^{-84}$	0.3142	0.8617
OMX	$9.49 \cdot 10^{-18}$	$2.54 \cdot 10^{-17}$	$5.12 \cdot 10^{-19}$	0.0123	0.4348

Table 2. Results from estimations

The table shows results from estimations of the model

$$r_t = \varphi_0 + \varphi_1 r_{t-1} + \varepsilon_t$$

with $\varepsilon_t \sim N(0, h_t)$. Three models for h_t are estimated: the logistic smooth transition GARCH(1,1) model

$$h_t = \gamma + \left(\alpha_{11} + \alpha_{21} \left[(1 + \exp[-\theta \varepsilon_{t-1}])^{-1} - \frac{1}{2} \right] \right) \varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

the exponential smooth transition GARCH(1,1) model

$$h_t = \gamma + (\alpha_{11} + \alpha_{21} [1 - \exp[-\theta \varepsilon_{t-1}^2]]) \varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the standard GARCH(1,1) model

$$h_t = \gamma + \alpha_{11} \varepsilon_{t-1}^2 + \beta_1 h_{t-1}.$$

The three estimated series are daily returns for the Copenhagen Stock Exchange general index, the Financial Times all-share index, and the Stockholm equity index (OMX). The investigated period is January 3, 1991 to July 15, 1996. For FT-all no autocorrelation in returns was detected, and therefore φ_1 is excluded in the estimation. The row labeled LL gives the value of the log-likelihood function.

	Copenhagen		FT-all share index		OMX	
	EST-GARCH	GARCH	LST-GARCH	GARCH	LST-GARCH	GARCH
φ_0	$1.67 \cdot 10^{-4}$	$1.67 \cdot 10^{-4}$	$3.97 \cdot 10^{-4}$	$4.32 \cdot 10^{-4}$	$4.19 \cdot 10^{-4}$	$6.34 \cdot 10^{-4}$
φ_1	0.264	0.264	-	-	0.126	0.126
γ	$7.74 \cdot 10^{-6}$	$9.21 \cdot 10^{-6}$	$3.82 \cdot 10^{-7}$	$4.46 \cdot 10^{-7}$	$4.61 \cdot 10^{-6}$	$6.03 \cdot 10^{-6}$
α_{11}	0.215	0.158	0.037	0.036	0.086	0.086
α_{21}	-0.166	-	-0.050	-	-0.094	-
β_1	0.612	0.580	0.956	0.955	0.884	0.870
θ	2128	-	43.9	-	430	-
LL	5223.5	5219.7	4880.1	4878.1	4290.7	4281.7
AIC	-10437.0	-10433.5	-9748.3	-9748.1	-8567.5	-8553.4

Figure 1. Logistic transition function for the Financial Times all share index and the OMX index

The figure shows the value of the logistic transition function

$$F(\varepsilon_{t-j}) = (1 + \exp[-\theta\varepsilon_{t-j}])^{-1} - \frac{1}{2}, \theta > 0$$

for different values on the residual ε_{t-j} . The transition function for FT-all is created with the estimated parameter value $\theta = 43.9$. The transition function for OMX is created with the estimated parameter value $\theta = 430$.

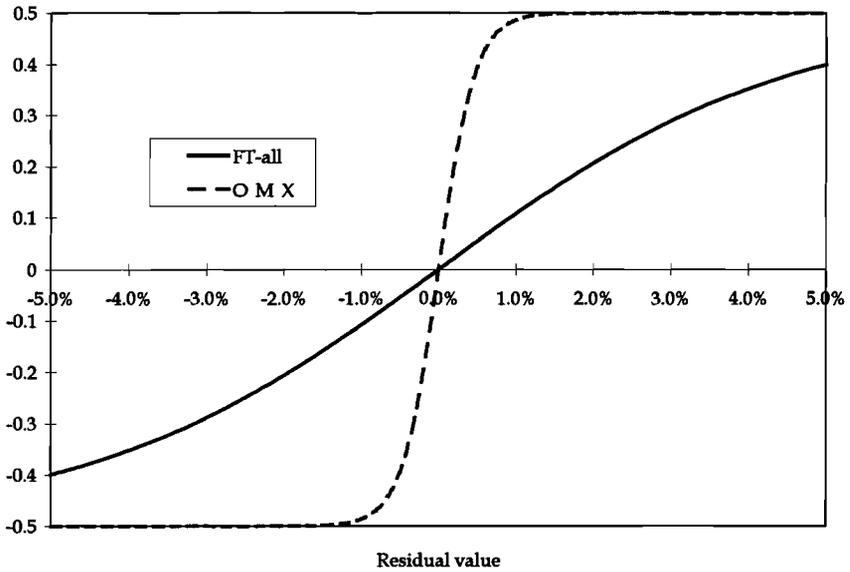


Figure 2. Exponential transition function for the Copenhagen Stock Exchange general index

The figure shows the value of the exponential transition function

$$F(\varepsilon_{t-j}) = 1 - \exp[-\theta\varepsilon_{t-j}^2], \theta > 0$$

for different values on the residual ε_{t-j} . The transition function is created with the estimated parameter value $\theta = 2128$.

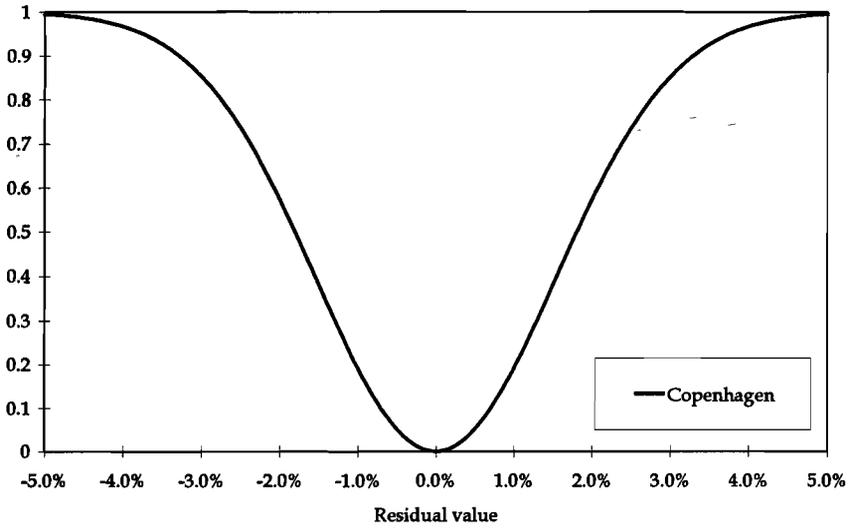


Figure 3. News impact curves for the Copenhagen Stock Exchange general index

The figure shows how the conditional variance reacts to different values on lagged residuals. The curves are created from the estimated parameter values given in Table 2. The ESTGARCH(1,1) curve is created with the formula

$$h_t = \gamma + (\alpha_{11} + \alpha_{21} [1 - \exp[-\theta \varepsilon_{t-1}^2]]) \varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the GARCH(1,1) curve with the formula

$$h_t = \gamma + \alpha_{11} \varepsilon_{t-1}^2 + \beta_1 h_{t-1}.$$

The initial conditional variance in all curves is equal to the unconditional variance for the GARCH(1,1) model, $3.52 \cdot 10^{-5}$.

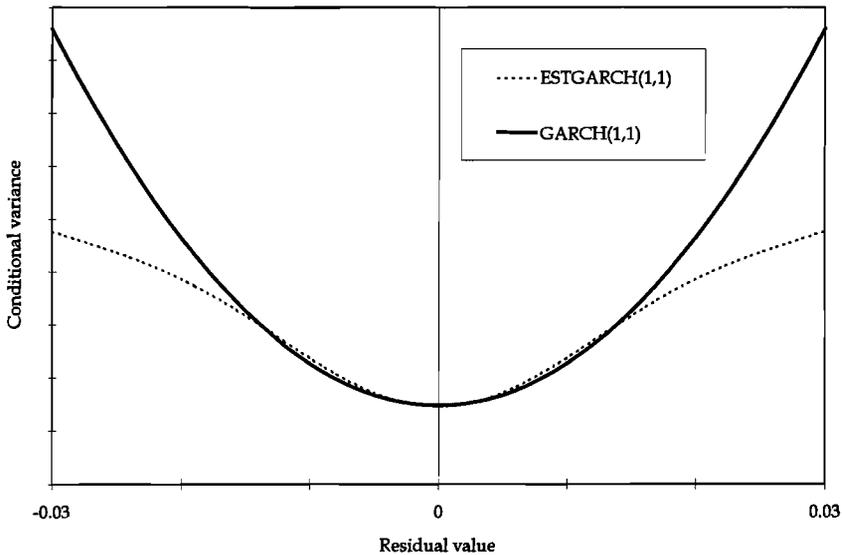


Figure 4. News impact curves for the Financial Times all share index

The figure shows how the conditional variance reacts to different values on lagged residuals. The curves are created from the estimated parameter values given in Table 2. The LSTGARCH(1,1) curve is created with the formula

$$h_t = \gamma + \left(\alpha_{11} + \alpha_{21} \left[(1 + \exp[-\theta \varepsilon_{t-1}])^{-1} - \frac{1}{2} \right] \right) \varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the GARCH(1,1) curve with the formula

$$h_t = \gamma + \alpha_{11} \varepsilon_{t-1}^2 + \beta_1 h_{t-1}.$$

The initial conditional variance in all curves is equal to the unconditional variance for the GARCH(1,1) model, $5.13 \cdot 10^{-5}$.

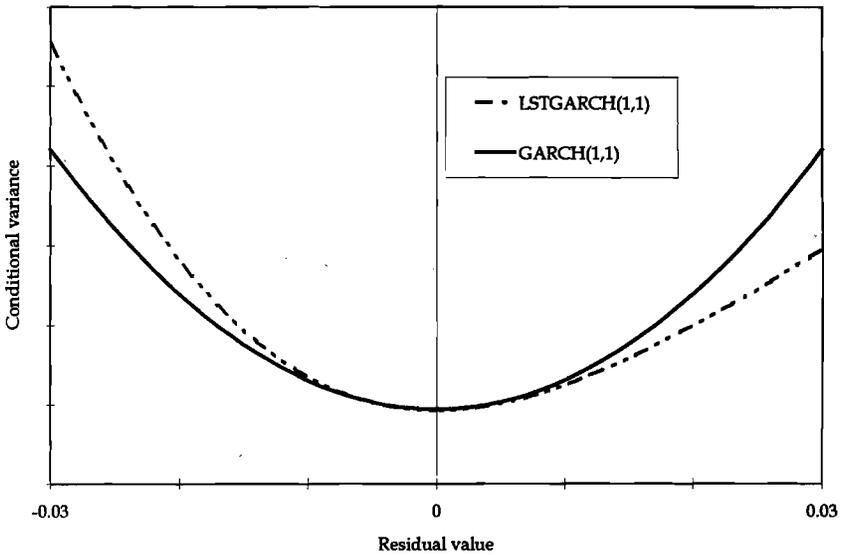


Figure 5. News impact curves for the OMX index

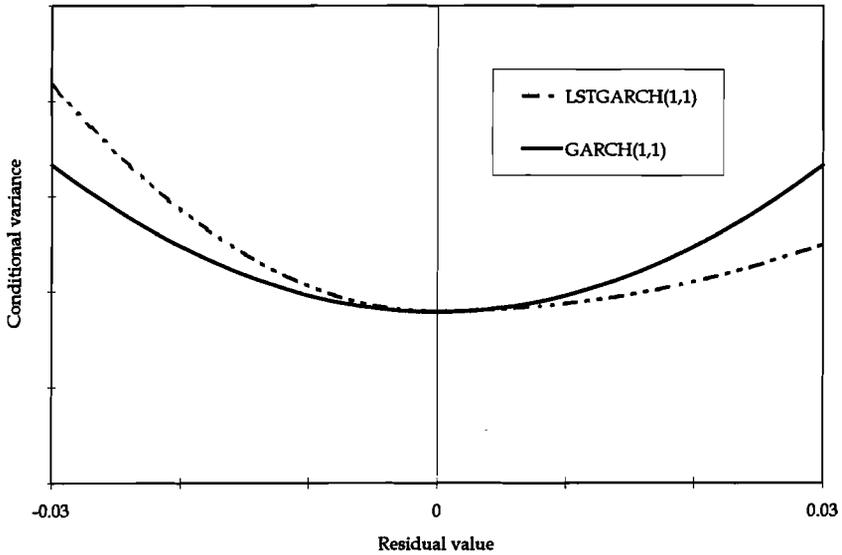
The figure shows how the conditional variance reacts to different values on lagged residuals. The curves are created from the estimated parameter values given in Table 2. The LSTGARCH(1,1) curve is created with the formula

$$h_t = \gamma + \left(\alpha_{11} + \alpha_{21} \left[(1 + \exp[-\theta \varepsilon_{t-1}])^{-1} - \frac{1}{2} \right] \right) \varepsilon_{t-1}^2 + \beta_1 h_{t-1},$$

and the GARCH(1,1) curve with the formula

$$h_t = \gamma + \alpha_{11} \varepsilon_{t-1}^2 + \beta_1 h_{t-1}.$$

The initial conditional variance in all curves is equal to the unconditional variance for the GARCH(1,1) model, $1.42 \cdot 10^{-4}$.



3

Specification Tests for Asymmetric GARCH

3.1 Introduction

To estimate the unknown parameters of a model in the ARCH/GARCH class, iterative numerical methods are very often required. These procedures are time consuming. Furthermore, if the model in question explains the data badly, the estimation might not converge. Therefore, it is essential to have reliable specification tests. These tests give the econometrician indications of which models can be the data generating process of a time series. This paper presents two Lagrange multiplier tests designed to detect higher order GARCH effects. For both tests, the null hypothesis is the GARCH model proposed by Bollerslev [1986]. In the first test, the alternative hypothesis is the generalized quadratic ARCH (GQARCH) model of Sentana [1995], and in the second test, the alternative is the logistic smooth transition GARCH (LSTGARCH) model presented by Hagerud [1996], and González-Rivera [1996]. The models are only considered in their most simple structure, when the lag lengths are equal to one, but tests for more complex models can easily be derived, using a similar method.

In both the GQARCH and LSTGARCH model, the conditional variance is asymmetric in the sign of lagged innovations. Thus, there is a correlation between current conditional variance and lagged returns. A number of other such *asymmetric* models have been proposed in the literature. The most common of these are: the EGARCH models of Nelson [1991], and the GJR model of Glosten, Jagannathan, and Runkle [1993].

In this paper, results are presented from Monte Carlo simulations performed to investigate the small sample properties of the two statistics. The empirical size is shown to coincide with the theoretical. When the power of the tests is examined, it is determined whether the tests can be used to test for the existence of asymmetry of forms other than those specified in the GQARCH and LSTGARCH models. If that is the case, this will indicate that the tests cannot distinguish between different forms of asymmetry. This is naturally a weakness of the tests. However, it might also be an advantage. If a test can detect other forms of asymmetry, one test can indicate, if the null cannot be rejected, that a large number of ARCH models can be excluded as the data generating test process of a time series. The other asymmetric models considered in this experiment are: EGARCH, GJR, TGARCH, A-PARCH, and VS-ARCH. It is shown that both tests can be used to detect asymmetries generated by these five models.

For plausible parameter values on the data generating process, the empirical power of the tests is always below 100 percent. Therefore, the relative power of the tests is compared to four other asymmetry tests proposed by Engle and Ng [1993]. The procedures of Engle and Ng are the most commonly used tests in the literature. The simulations show that the power properties of the two tests here are superior to that of Engle and Ng's tests. The major contribution of this paper, therefore, is the presentation of two tests for general asymmetry, with superior power properties.

This article is organized as follows. The next section describes the asymmetric ARCH models that will be considered in the Monte Carlo experiment. Section 3.3 surveys previous literature on specification tests in the ARCH environment. Section 3.4 contains a presentation of the specification tests. Results from the Monte Carlo experiment are given in Section 3.5. Finally, Section 3.6 concludes the paper.

3.2 Asymmetric GARCH Models

This section presents the GQARCH and the LSTGARCH models, as well as the five other asymmetric GARCH models that will be considered in the Monte Carlo experiment.¹ In all the models presented, and also for the remainder of this article, it is assumed that the return on an asset, r_t , is generated by

$$r_t = \varepsilon_t, \quad (3.1)$$

¹The EGARCH, GJR and A-PARCH models presented below, are more thoroughly surveyed by Hentschel [1996]. Based on his Asymmetric Absolute Value ARCH model, Hentschel develops a general GARCH model, which nests a large number ARCH models. Both symmetric and asymmetric models are nested, but not QARCH and VS-ARCH.

where ε_t denotes a discrete-time stochastic process with the form

$$\varepsilon_t = z_t h_t^{1/2}, \quad (3.2)$$

where $z_t \sim \text{nid}(0,1)$, and h_t will be the conditional variance at time t .

3.2.1 GQARCH

Sentana [1995] introduces the Quadratic ARCH model. The term *quadratic* is used since the QARCH model can be interpreted as a second-order Taylor approximation to the unknown conditional variance function. The Generalized QARCH(1,1) model is

$$h_t = \gamma + \zeta \varepsilon_{t-1} + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}, \quad (3.3)$$

where γ , ζ , α , and β are constant parameters. Positivity of the variance is achieved if $\alpha, \beta \geq 0$, and $\zeta < 4\alpha\gamma$. The model is covariance stationary if $\alpha + \beta < 1$. Asymmetry is introduced with the parameter ζ .

3.2.2 LSTGARCH

The logistic smooth transition GARCH model is proposed by Hagerud [1996], and González-Rivera [1996]. In the LSTGARCH(1,1) model, the conditional variance is assumed to be generated by

$$h_t = \gamma + [\alpha_1 + \alpha_2 F(\varepsilon_{t-1})] \varepsilon_{t-1}^2 + \beta h_{t-1}, \quad (3.4)$$

where $F(\cdot)$ is a transition function with the form

$$F(\varepsilon_{t-1}) = (1 + \exp[-\theta \varepsilon_{t-1}])^{-1} - \frac{1}{2}, \quad \theta > 0. \quad (3.5)$$

For positive conditional variance in the LSTGARCH model, it is required that

$$\begin{aligned} \gamma &> 0, \\ \alpha_1 &\geq 0, \\ \beta_i &\geq 0, \\ \alpha_1 &\geq \frac{1}{2} |\alpha_2|. \end{aligned}$$

For stationarity of the return process it is required

$$[\alpha_1 - \frac{1}{2} |\alpha_2| + \max(\alpha_2, 0)] + \beta < 1.$$

For the smooth transition GARCH process to be defined, it is required that at least one $\alpha_i > 0$, $i = 1, 2$. In the LSTGARCH(1,1) model, the level of asymmetry is controlled by the parameters α_2 and θ .

3.2.3 EGARCH

The seminal work in the area of asymmetric ARCH is the exponential GARCH model of Nelson [1991]. In the EGARCH(1,1) model, the natural logarithm of the conditional variance follows the process

$$\ln h_t = \gamma + \beta \ln h_{t-1} + \lambda z_{t-1} + \varphi \left[|z_{t-1}| - \sqrt{2/\pi} \right], \quad (3.6)$$

where γ , β , λ , and φ are constant parameters, and z_t is defined as in (3.2). For the process ε_t to be stationary, it is sufficient that $\beta < 1$. Nelson gives three motivations for his model compared to the standard GARCH model of Bollerslev [1986]: (i) The GARCH model cannot explain the asymmetric behavior of the conditional variance in asset price returns. (ii) For the conditional variance to be strictly positive, the parameters of the GARCH models must be non-negative, which is not required in the EGARCH model.² (iii) In the GARCH model, it is difficult to evaluate whether or no a shock to variance persists. Persistence of conditional variance in the EGARCH is controlled by the parameter β .

3.2.4 GJR

In the GJR model of Glosten, Jagannathan, and Runkle [1993] the standard GARCH model is augmented with a term that captures asymmetry. The GJR model is

$$h_t = \gamma + \alpha \varepsilon_{t-1}^2 + \omega S_{t-1}^- \varepsilon_{t-1}^2 + \beta h_{t-1}, \quad (3.7)$$

where γ , α , β , and ω are constant parameters, and S_{t-1}^- is a variable that takes the value one when $\varepsilon_{t-1} < 0$ and zero otherwise. For positive conditional variance, it is sufficient that the parameters γ , α , and β , and $(\alpha + \omega)$ are non-negative. For the process ε_t to be stationary, it is sufficient that $\alpha + \beta + \omega < 1$.

Note that the GJR model (3.7) will obtain as a limiting case of the LSTGARCH(1,1) model (3.4), when the logistic transition function (3.5), is replaced by the Heaviside function minus one half. In the GJR model, the conditional variance follows one process when the innovations are positive and another process when the innovations are negative. In the LSTGARCH model, however, the transition between states is smooth.

3.2.5 TGARCH

The Threshold GARCH model is introduced in Zakoian [1994]. In the TGARCH model, it is not the conditional variance, but the conditional

²Nelson and Cao [1992] show that the non-negativity constraint for the GARCH model given by Bollerslev [1986] is only sufficient for strictly positive conditional variance. They demonstrate that weaker conditions can be found.

standard deviation, $\sigma_t = h_t^{1/2}$, that is modeled. The TGARCH(1,1) model is

$$\sigma_t = \gamma + \alpha^+ \varepsilon_{t-1}^+ - \alpha^- \varepsilon_{t-1}^- + \beta \sigma_{t-1}, \quad (3.8)$$

where $\varepsilon_t^+ = \max(\varepsilon_t, 0)$, and $\varepsilon_t^- = \min(\varepsilon_t, 0)$. For strictly positive conditional standard deviation, it is sufficient that $\gamma > 0$, $\alpha^+ \geq 0$, $\alpha^- \geq 0$, and $\beta \geq 0$. The return series is stationary if

$$\frac{1}{2} [(\alpha^+)^2 + (\alpha^-)^2] + \beta^2 + [\alpha^+ + \alpha^-] \sqrt{\frac{2}{\pi}} < 1.$$

Note that (3.8) can be reparameterized as

$$\sigma_t = \gamma + \alpha |\varepsilon_{t-1}| + \omega S_{t-1}^- \varepsilon_{t-1} + \beta \sigma_{t-1}.$$

Thus, in the TGARCH(1,1) model, the conditional standard deviation has the same functional form as the conditional variance has in the GJR model (3.7).

3.2.6 A-PARCH

Ding, Granger, and Engle [1993] introduce the Asymmetric Power ARCH model. In the A-PARCH(1,1) model, the conditional variance is given by

$$h_t^{\delta/2} = \gamma + \alpha (|\varepsilon_{t-1}| - \eta \varepsilon_{t-1})^\delta + \beta h_{t-1}^{\delta/2}, \quad (3.9)$$

where $\gamma, \alpha, \beta, \eta$ and $\delta > 0$ are constant parameters. Asymmetry is introduced via the parameter $\eta \in (-1, 1)$. For positive conditional variance, it is required that the parameters γ, α , and β , are non-negative. Conditions for stationarity are relatively complex, and can be found in Ding, Granger, and Engle [1993].

The A-PARCH model is a generalization of previous GARCH models. The model includes seven other models as special cases. For example, $\delta = 2$ and $\eta = 0$ will give the GARCH(1,1) model. Letting $\delta = 2$ gives the GJR model. When $\delta = 1$ the dynamics of the model will be similar to that in the TGARCH model. Since these models are nested in the A-PARCH model, likelihood ratio tests can be performed to test the significance of the parameters. Thus, the null of a specific model, against the alternative of A-PARCH, can be tested with relative ease.

3.2.7 VS-ARCH

The Volatility Switching model is proposed by Fornari and Mele [1996a]. In the VS model the conditional variance follows

$$h_t = \gamma + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1} + \xi S_{t-1} v_{t-1}^2, \quad (3.10)$$

where

$$\begin{aligned} S_t &= 1 \text{ if } \varepsilon_t > 0 \\ S_t &= 0 \text{ if } \varepsilon_t = 0 \\ S_t &= -1 \text{ if } \varepsilon_t < 0, \end{aligned}$$

and v_t^2 is defined as ε_t^2/h_t . The parameters of the model are γ , α , β , and ξ . With the series $\{S_t v_t^2\}_{t=1}^T$, Fornari and Mele introduce what they call *mean reversion* in the conditional variance. v_t^2 measures how much a given squared residual deviates from its expected value, h_t , and S_t indicates the sign of the residual. The model, for example, is able to generate data where unexpectedly large negative returns increase h_t , large positive returns decrease h_t , small negative returns decrease h_t , and small positive returns increase h_t . The level of asymmetry in the model will therefore depend on parameter ξ , and on the relative size of residuals.³

3.3 Previous ARCH Specification Tests

This section is included to introduce the reader to the area of specification tests in the ARCH/GARCH literature. Readers already familiar with this literature can proceed to Section 3.4.

In the ARCH(q) model of Engle [1982], the conditional variance is given by the process

$$h_t = \gamma + \sum_{j=1}^q \alpha_j \varepsilon_{t-j}^2, \quad (3.11)$$

where γ and α_j ($j = 1..q$) are non-negative constants, with at least one $\alpha_j > 0$. To test the presence of ARCH, i.e. testing $H_0 : \alpha_j = 0$ ($j = 1..q$), against $H_1 : \text{at least one } \alpha_j \neq 0$, Engle proposes a Lagrange multiplier test. Since the conditional variance is constant under the null, a Lagrange multiplier test is particularly suitable. Engle shows that the *LM* statistic can be calculated as $T \cdot R_u^2$, where T is the number of observations, and R_u^2 is the coefficient of multiple correlation from the regression of ε_t^2 on a constant and $\varepsilon_{t-1}^2, \dots, \varepsilon_{t-q}^2$. Under the null, the statistic has an asymptotic χ^2 distribution with q degrees of freedom. As noted by Granger and Teräsvirta [1993], McLeod and Lee's [1983] test of linearity in the conditional mean against unspecified non-linearity is asymptotically equivalent to Engle's test of no ARCH.

³In Fornari and Mele[1996b], the authors proposed a slightly different VS-ARCH model, in which two extra parameters are needed.

In the GARCH(q, p) model of Bollerslev [1986] the conditional variance is given by the process

$$h_t = \gamma + \sum_{j=1}^q \alpha_j \varepsilon_{t-j}^2 + \sum_{j=1}^p \beta_j h_{t-j}, \quad (3.12)$$

where γ , α_j ($j = 1..q$) and β_j ($j = 1..p$) are non-negative constants, with at least one $\alpha_j > 0$. To test the presence of GARCH, an *LM* statistic cannot be derived in the way described by Engle. Bollerslev [1986] notes that under the null of no heteroskedasticity, there is no general test for GARCH(q, p). This is because the information matrix is singular if both $p > 0$ and $q > 0$. Based on Bollerslev's finding Lee [1991] derives a *modified LM* statistic for $H_0 : \alpha_j = \beta_i = 0$ ($j = 1..q, i = 1..p$), against H_1 : at least one $\alpha_j \neq 0$ or one $\beta_i \neq 0$. Lee shows that this test is equivalent to the test of no ARCH(q). Thus, under the null of homoskedasticity, the GARCH(q, p) effect and the ARCH(q) effects are locally equivalent alternatives. Lee notes that with his methods for deriving a modified *LM* statistic for no GARCH, it is possible to derive a test of the null of ARCH(q) against the alternative of GARCH($k_1, q + k_2$), where $k_1 > 0$ and $k_2 > 0$.

Tests of the null of linear ARCH as in (3.11), or linear GARCH as in (3.12), against different forms of non-linear ARCH/GARCH, has been proposed by, among others, Bera and Higgins [1992], Higgins and Bera [1992], Engle, and Ng [1993], Rabemananjara, and Zakoian [1993], and Sentana [1995]. Bera and Higgins [1992] and Higgins and Bera [1992] discuss testing for ARCH against NARCH (Non-linear ARCH). In the NARCH model, the conditional variance is, as in the ARCH and GARCH models, symmetric in the sign of ε_t . Since the GQARCH and the LSTGARCH models are asymmetric, their test is of less importance in this context. It is still worth noting that the test of no LSTGARCH and the test of no NARCH, have the common problem of a non-identified parameter under the null. The methods for solving this problem, however, are quite different.

Engle, and Ng [1993] present four different *LM* type tests for linear ARCH/GARCH against asymmetry. These four statistics will be discussed more thoroughly in Section 3.5, and a detailed description of how the tests may be calculated appears in the appendix. In this section, a short introduction to the four tests is given. The *Sign bias test* examines the impact of positive and negative shocks on the conditional variance not predicted by the linear model. This is done by investigating whether, in a linear regression model, the variable S_{t-1}^- has any predictive power on squared normalized residuals ε_t^2/h_{0t} , where h_{0t} is the conditional variance under the null. S_{t-1}^- is defined as in the GJR model (3.7). The test statistic is calculated as a t-ratio in the linear regression model. The other three tests are carried out using similar methods. The *Negative size bias test* investigates whether the linear model can explain the different effects that large and small negative shocks have on the conditional variance. The variable

used for this test is $S_{t-1}^- \varepsilon_{t-1}$. In the *Positive size bias test*, different effects of large and small positive shocks are investigated. The variable used for this test is $S_{t-1}^+ \varepsilon_{t-1}$, where S_{t-1}^+ is defined analogously to S_{t-1}^- . In the fourth test, the three previous hypotheses are considered simultaneously. In Monte Carlo experiments, Engle and Ng show that the empirical size and power of the tests are reasonable when the sample size is 1000.

The TGARCH model (3.8) is further developed in Rabemananjara, and Zakoïan [1993]. They allow σ_t to become negative. Thus, σ_t cannot be considered a conditional standard deviation. The TGARCH(q, p) model of Rabemananjara, and Zakoïan [1993] is

$$\sigma_t = \gamma + \sum_{j=1}^q \alpha_j |\varepsilon_{t-j}| + \sum_{j=1}^p \beta_j |\sigma_{t-j}| + \sum_{j=1}^q \alpha_j^- \varepsilon_{t-j}^- + \sum_{j=1}^p \beta_j^- \sigma_{t-j}^-, \quad (3.13)$$

where $\varepsilon_t^- = \min(\varepsilon_t, 0)$ and $\sigma_t^- = \min(\sigma_t, 0)$. The null hypothesis for the test of asymmetry in (3.13) is, $H_0 : \alpha_j^- = \beta_j^- = 0$ ($j = 1..q, i = 1..p$). To test the null against H_1 : at least one $\alpha_j^- \neq 0$ or one $\beta_j^- \neq 0$, Rabemananjara, and Zakoïan derive an *LM* test statistic. Under the null, the statistic is asymptotically distributed χ^2 with $q + p$ degrees of freedom. In a Monte Carlo experiment, the authors show that both the empirical size and power of the test are reasonable for large sample sizes (> 500).

Sentana [1995] presents a number of test procedures that can be used in conjunction with his QARCH model. To test the null of homoskedasticity against the alternative of QARCH(q), Sentana proposes an *LM* test based on the regression model

$$\varepsilon_t^2 = a_0 + \sum_{j=1}^q a_{1j} \varepsilon_{t-j}^2 + \sum_{j=1}^q \sum_{i \leq j} a_{2ij} \varepsilon_{t-j} \varepsilon_{t-i} + \xi_t. \quad (3.14)$$

The hypothesis is: $H_0 : a_{1j} = a_{2j} = 0$ ($j = 1, \dots, q, i = 1, \dots, q$), against H_1 : at least one $a_{1j} \neq 0$ or one $a_{2ij} \neq 0$. The statistic is calculated as $T \cdot R_u^2$ from the regression model (3.14). Under H_0 , the LM statistic has an asymptotic χ^2 distribution with $q(q+3)/2$ degrees of freedom.

The tests described in this section are all derived under the assumption that the residuals are distributed conditionally normal. In practice, this assumption is often not fulfilled. This is particularly the case when the investigated series contains returns of a traded asset. The problem is carefully investigated in Wooldridge [1990] and [1991]. Wooldridge points out that when normality does not hold the asymptotic size of the statistics will be wrong. In the 1990 article, Wooldridge propose a robust version of Engle's test of no ARCH(q). In the 1991 article, he presents a general procedure for robustifying Lagrange multiplier tests of the specification of conditional variance. This method is used by Sentana [1995], when he derives a robust test procedure for the null of ARCH(1), against the alternative of

QARCH(1,1). In Wooldridge [1991], the author also presents a method for performing non-nested hypothesis tests for the conditional variance.

The two tests presented in this paper are both derived under conditional normality. Furthermore, the small sample properties are only investigated for normally distributed innovations. However, using the results of Wooldridge [1991], it would be straightforward to derive robust versions of the tests.

3.4 Specification Tests for Asymmetry

In this section, the two new test statistics are presented. For both tests, the null is the standard GARCH(1,1) model, proposed by Bollerslev [1986]

$$h_t = \gamma + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}. \tag{3.15}$$

The test of GARCH(1,1), against the alternative of GQARCH(1,1) is formulated

$$\begin{aligned} H_0 & : \zeta = 0 \\ H_1 & : \zeta \neq 0. \end{aligned}$$

Given that the residual, ε_t , is distributed conditionally normal, a Lagrange multiplier test statistic for the hypothesis is

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \beta} \right\}' \left\{ \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right]' \right\}^{-1} \\ & \times \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \beta} \right\}, \end{aligned} \tag{3.16}$$

where

$$\frac{\partial h_t}{\partial \beta'} = \left[\sum_{i=1}^{t-1} \hat{\beta}^{i-1}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i}^2, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{0t-i}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i} \right],$$

h_{0t} is the conditional variance under the null of GARCH(1,1), β' is the vector of parameters $(\gamma, \alpha, \beta, \zeta)$, and $\hat{\beta}$ is the estimated parameter β in the GARCH(1,1) model. The derivation of (3.16) is given in the appendix. In the appendix, it can also be seen that based on test (3.16), it is possible to derive the asymptotically equivalent test $T \cdot R_u^2$ from the regression

$$\left\{ \frac{\varepsilon_t^2}{h_{0t}} - 1 \right\}$$

on

$$\left\{ \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{0t-i}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i}}{h_{0t}} \right\}. \quad (3.17)$$

Before a complete test procedure is presented, first a problem concerning the estimation of the model (3.15) must be considered. Given that the series of conditional variance under the null is estimated with maximum likelihood, the normalized residuals, $v_{0t} \equiv \varepsilon_t/h_{0t}^{1/2}$, should be orthogonal to

$$\left\{ \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{0t-i}}{h_{0t}} \right\}. \quad (3.18)$$

This should be true independent of whether or not the null is true. However, in practice, exact orthogonality cannot always be guaranteed. If orthogonality does not hold, the empirical size of the statistic might be distorted. To overcome this complication, the customary procedure is to replace v_{0t} with a quantity that is guaranteed to be orthogonal to (3.18), (see e.g. Eitrheim and Teräsvirta [1996]). The following procedure will accomplish that:

1. Regress

$$\left\{ \frac{\varepsilon_t^2}{h_{0t}} - 1 \right\}$$

on (3.18). Let $\{\tilde{\varepsilon}_t\}_{t=1}^T$ be the series of residuals from the regression. These residuals will by construction be orthogonal to (3.18).

2. Regress $\tilde{\varepsilon}_t$ on (3.17). The statistic is set equal $T \cdot R_u^2$ from this regression.

In the Monte Carlo experiments, it was seen that the empirical size of the statistic (3.16), and its asymptotically alternative, were slightly above the theoretical significance level. By using the method described above, it was possible to correct the size. However, it was also found that a slightly simplified method gave almost the same result. In this method, the vector (3.17) was replaced by

$$\left\{ 1, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i}}{h_{0t}} \right\}$$

Base on this conclusion, it is proposed that the null hypothesis of GARCH(1,1), against the alternative of GQARCH(1,1), should be tested using the following procedure:

1. Estimate a GARCH(1,1) model. Form the vectors

$$\mathbf{c}_t = \left\{ 1, \frac{\sum_{i=1}^{t-1} \widehat{\beta}_1^{i-1} \varepsilon_{t-i}}{h_{0t}} \right\}$$

2. Regress

$$\left\{ \frac{\varepsilon_t^2}{h_{0t}} - 1 \right\}$$

on \mathbf{c}_t , and calculate

$$LM_1 = T \cdot R_u^2 \tag{3.19}$$

from the regression.

The test statistic LM_1 is under the null asymptotically distributed χ^2 with one degree of freedom.

The test of GARCH(1,1), against the alternative of LSTGARCH(1,1) is formulated

$$\begin{aligned} H_0 &: \alpha_2 = 0 \\ H_1 &: \alpha_2 \neq 0. \end{aligned}$$

Given that the residual, ε_t , is distributed conditionally normal, a Lagrange multiplier test statistic for the hypothesis is

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \alpha} \right\}' \left\{ \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \alpha} \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \alpha} \right]' \right\}^{-1} \\ \times & \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \alpha} \right\}, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} \frac{\partial h_t}{\partial \alpha'} &= \left\{ \sum_{i=1}^{t-1} \widehat{\beta}^{i-1}, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \varepsilon_{t-i}^2, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} F(\varepsilon_t|\theta) \varepsilon_{t-i}^2, \right. \\ & \left. \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} h_{t-i}, \alpha_2 \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \frac{e^{-\theta \varepsilon_{t-i}}}{(1 + e^{-\theta \varepsilon_{t-i}})^2} \varepsilon_{t-i}^3 \right\} \end{aligned} \tag{3.21}$$

h_{0t} is the conditional variance under the null of GARCH(1,1), α' is the vector of parameters $(\gamma, \alpha_1, \alpha_2, \beta, \theta)$, $F(\varepsilon_t|\theta)$ is the value of the transition function at time t , and $\widehat{\beta}$ is the estimated parameter β in the GARCH(1,1) model. The derivation of (3.20) is done with the same methods used for the derivation of (3.16).

The statistic (3.20), however, is not operational, since the vector (3.21) is dependent on the transition function (3.5), which under the null has a non-identified parameter θ . Following Luukkonen, Saikkonen, and Teräsvirta

[1988], this problem is solved by making a second-order Taylor expansion of the transition function, around zero.⁴ The obtained approximation of $F(\cdot)$ is then inserted into formula (3.4), and this results in an approximate version of the conditional variance equation. Since $F(0) = 0$, the transition function (3.5) can be approximated by

$$T_l = F'(0)x = \frac{\theta}{4}x \tag{3.22}$$

The LSTGARCH(1,1) model can therefore be approximated by

$$\tilde{h}_t = \gamma + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \frac{\theta}{4} \varepsilon_{t-1}^3 + \beta h_{t-1}.$$

The hypothesis of GARCH(1,1), against LSTGARCH(1,1) can therefore be written

$$\begin{aligned} H_0 &: \alpha_2 \theta / 4 = 0 \\ H_1 &: \alpha_2 \theta / 4 \neq 0. \end{aligned}$$

A Lagrange multiplier for this test is equal to

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial \tilde{h}_t}{\partial \tilde{\alpha}} \right\}' \left\{ \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial \tilde{h}_t}{\partial \tilde{\alpha}} \right] \left[\frac{1}{h_{0t}} \frac{\partial \tilde{h}_t}{\partial \tilde{\alpha}} \right]' \right\}^{-1} \\ & \times \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial \tilde{h}_t}{\partial \tilde{\alpha}} \right\}, \end{aligned} \tag{3.23}$$

where

$$\frac{\partial \tilde{h}_t}{\partial \tilde{\alpha}'} = \left[\sum_{i=1}^{t-1} \hat{\beta}^{i-1}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i}^2, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i}^3, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{0t-i} \right],$$

h_{0t} is the conditional variance under the null of GARCH(1,1), $\tilde{\alpha}'$ is the vector of parameters $(\gamma, \alpha_1, \alpha_2 \theta / 4, \beta)$, and $\hat{\beta}$ is the estimated parameter β in the GARCH(1,1) model. As in the GQARCH test, to test the null of GARCH(1,1), against the alternative of LSTGARCH(1,1), the procedure proposed is:

1. Estimate a GARCH(1, 1) model. Form the vectors

$$\mathbf{d}_t = \left(1, \frac{\sum_{i=1}^{t-1} \hat{\beta}_1^{i-1} \varepsilon_{t-i}^3}{h_{0t}} \right)$$

⁴A test of GARCH(1,1) against LSTGARCH(1,1) has previously been developed by González-Rivera [1996]. However, she solves the problem with the non-identified parameter θ somewhat differently, using a method proposed by Davies [1977].

2. Regress

$$\left\{ \frac{\varepsilon_t^2}{h_{0t}} - 1 \right\}$$

on \mathbf{d}_t , and calculate

$$LM_2 = T \cdot R_u^2 \quad (3.24)$$

from the regression.

LM_2 is under the null asymptotically distributed χ^2 with one degree of freedom.

3.5 Monte Carlo Experiment

The Monte Carlo experiment for testing the empirical size of the test statistics (3.19) and (3.24) is based on a GARCH(1,1) data generating process

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= z_t h_t^{1/2} \\ h_t &= \gamma + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1} \end{aligned} \quad (3.25)$$

where $z_t \sim \text{nid}(0,1)$. Four combinations of the constant parameters γ , α , and β are studied. These values are shown in Tables 1 and 2. For each set of parameter values 2,500 samples with 250 and 1000 observations are generated. The test statistics are calculated, and compared to the critical values for one, five and ten percent confidence levels.

In Table 1, the actual rejection frequencies of the test procedure (3.19) are reported. The empirical size of the test is relatively close to the theoretical size, both when the number of observations is 1000 and 250. However, the simulated size seems to be somewhat less accurate for the smaller sample size. Table 2 reports the simulation results for test procedure (3.24). For the larger sample the empirical size is quite close to the theoretical size. When the sample size is 250 the empirical size seems to be lower than the theoretical. It is therefore concluded that for both statistics, the simulated size is fairly accurate for the larger sample size, and the simulated size is reasonable for a sample of 250 observations.

As noted in the introduction, the two tests' ability to detect asymmetry will be compared, to that of Engle and Ng's [1993] four tests, the Sign bias test (*SB*), the Negative size bias test (*NSB*), the Positive size bias test (*PSB*), and the test of the joint hypothesis of *SB*, *NSB*, and *PSB*. Such a comparison can only be made if it is known that the empirical size properties of Engle and Ng's tests are similar to those reported for tests (3.19) and (3.24). A number of different alternative formulations of Engle and Ng's tests were tried before appropriate size properties were received. The procedures that were found most promising, and which are used in the

remainder of this article, are described in the appendix. Tables 3 to 6 report the actual rejection frequencies for the four alternative tests. Based on the results reported in Tables 1 to 6, it is concluded that for the larger sample size, the empirical size properties for the six tests are quite similar. When the number of observations is 250, the empirical size properties fluctuate more across the different test procedures. The remainder of this study will therefore focus on the results based on the larger sample size.

To investigate the power of test (3.19) the data generating process considered is

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= z_t h_t^{1/2} \\ h_t &= 1.25 \cdot 10^{-6} - 1.68 \cdot 10^{-4} \varepsilon_{t-1} + 0.0355 \cdot \varepsilon_{t-1}^2 + 0.952 \cdot h_{t-1}, \end{aligned} \quad (3.26)$$

where $z_t \sim \text{nid}(0, 1)$. The parameters of the model (3.26) have been obtained from an estimation performed on daily observations for the Financial Times All-share Index. The sample period is January 1991 to July 1996. The estimated power of the test is reported on the rows labeled LM_1 in Table 7. From the results when the sample size is 250, it can be concluded that for such a small sample, the test has very low power. When the sample size is 1000 the estimated power is increased, but is still low. However, compared to the estimated power for the four tests of Engle and Ng, also reported in Table 7, the test constitutes a marked improvement. The result is not surprising, since the test (3.19) is designed to detect the kind of asymmetry caused by (3.26), whereas the tests of Engle and Ng are designed to detect general GARCH asymmetry. But surprisingly, the test of no LSTGARCH (3.24) also outperforms the tests of Engle and Ng.

To investigate the power of test (3.24) the data generating process considered is

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= z_t h_t^{1/2} \\ h_t &= 5.2 \cdot 10^{-7} + [0.295 - 0.258 \cdot F(\varepsilon_{t-1})] \varepsilon_{t-1}^2 + 0.70 \cdot h_{t-1} \\ F(\varepsilon_{t-1}) &= (1 + \exp[-200 \cdot \varepsilon_{t-1}])^{-1} - \frac{1}{2}, \end{aligned} \quad (3.27)$$

where $z_t \sim \text{nid}(0, 1)$. The parameters of the model (3.27) are in part taken from the paper by Engle and Ng [1993], when they consider the GJR model (3.7). If the parameter θ , which has been set to 200, is allowed to increase towards infinity, the data generating process (3.27) will coincide with the GJR model considered in Engle and Ng's article. By setting θ to 200, the transition function $F(\varepsilon_{t-1})$ will not just take on its extreme values $-1/2$ and $1/2$. Table 8 reports the power properties of the six statistics when the true data generating process is (3.27). The power of the test (3.24) is reasonable when the sample size is 1000. The power of the test is reduced

considerably for the smaller sample size. The test (3.19) also proves to have power to detect asymmetry generated by (3.27), but, as expected, the power is significantly lower than for the test (3.24). The tests of Engle and Ng are a clear disappointment. It is important to keep in mind that the reported power is a function of the parameter values in the data generating process. One should therefore not compare the figures in Table 7 and 8. The parameters of the data generating process (3.27) clearly give rise to more asymmetry than the process (3.26).

The two statistics will now be further evaluated by investigating the empirical power of the tests for detecting asymmetry caused by the other five GARCH models presented in Section 3.2. To investigate the power of the tests, when the true model is EGARCH, the data generating process considered is

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= z_t h_t^{1/2} \\ \ln h_t &= -0.7395 + 0.90 \cdot \ln h_{t-1} - 0.075 \cdot z_{t-1} + 0.25 \cdot \left[|z_{t-1}| - \sqrt{2/\pi} \right], \end{aligned} \quad (3.28)$$

where $z_t \sim \text{nid}(0, 1)$. Except for the value of the parameter γ , the parameters of the model (3.28) are those used by Engle and Ng [1993], when they consider the EGARCH model (3.6). The simulated power of the six tests are presented in Table 9. The highest power is reported for test (3.19), followed by test (3.24). Among the tests of Engle and Ng, the Negative size bias test perform best, but the power is still significantly lower than the power for the two tests presented in this paper. The results for the smaller sample size is again disappointing.

The power of the tests, when the true model is GJR, is investigated considering the data generating process

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= z_t h_t^{1/2} \\ h_t &= 5.20 \cdot 10^{-7} + 0.166 \cdot \varepsilon_{t-1}^2 + 0.2576 \cdot S_{t-1}^- \varepsilon_{t-1}^2 + 0.70 \cdot h_{t-1}, \end{aligned} \quad (3.29)$$

where $z_t \sim \text{nid}(0, 1)$. Except for the value of the parameter γ , the parameters of the model (3.29) are those used by Engle and Ng [1993], when they consider the GJR model (3.7). Results of the simulations are in Table 10. These parameter values appear to generate a very marked asymmetry. For the larger sample size, the simulated power for tests (3.19) and (3.24) are strikingly high, whereas the tests of Engle and Ng perform less efficiently.

To investigate the power of the tests, when the true model is TGARCH, the data generating process considered is

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= z_t \sigma_t \\ \sigma_t &= 6.54 \cdot 10^{-4} + 0.111 \cdot \varepsilon_{t-1}^+ - 0.192 \cdot \varepsilon_{t-1}^- + 0.833 \cdot \sigma_{t-1}, \end{aligned} \quad (3.30)$$

where $z_t \sim \text{nid}(0, 1)$. Except for the value of the parameter γ , the parameters of the model (3.30) have been obtained from an estimation performed on daily observations for the French CAC 240 Index, reported by Zakořan [1994]. The sample period is 1976 to 1990. Results from the simulations are shown in Table 11. Tests (3.19) and (3.24) perform well once again. The power properties of test (3.19) must, in these circumstances, be considered very good. The results for the tests *SB*, *NSB*, *PSB*, and the joint test are, compared to the results for the other two tests, unsatisfactory.

When the true model is A-PARCH, the situation is investigated considering the data generating process

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= z_t h_t^{1/2} \\ h_t^{1.43/2} &= 9.22 \cdot 10^{-6} + 0.083 \cdot (|\varepsilon_{t-1}| - 0.373 \cdot \varepsilon_{t-1})^{1.43} + 0.92 \cdot h_{t-1}^{1.43/2}, \end{aligned} \quad (3.31)$$

where $z_t \sim \text{nid}(0, 1)$. Except for the value of the parameter γ , the parameters of the model (3.31) have been obtained from an estimation performed on daily observations for the S&P 500 Index, reported by Ding, Granger, and Engle [1993]. The sample period is 1928 to 1991. The results of the simulations are shown in Table 12. It is interesting to note that the parameters of the models, which have been estimated on this very large sample, apparently give rise to a marked asymmetry. This can be seen from the high power reported for tests (3.19) and (3.24). In this case, the results for the tests of Engle and Ng are even more disappointing than when the TGARCH model was the true data generating process.

Finally, the power of tests when the true model is VS-ARCH is investigated. In this case, the data generating process studied is

$$\begin{aligned} r_t &= \varepsilon_t \\ \varepsilon_t &= z_t h_t^{1/2} \\ h_t &= 3.9 \cdot 10^{-6} + 0.043 \cdot \varepsilon_{t-1}^2 + 0.918 \cdot h_{t-1} + 2.22 \cdot 10^{-6} \cdot S_{t-1} v_{t-1}^2, \end{aligned} \quad (3.32)$$

where $z_t \sim \text{nid}(0, 1)$. Except for the value of the parameter γ , the parameters of the model (3.32) have been obtained from an estimation performed on daily observations for the S&P 500 Index, reported by Fornari and Mele [1996a]. The sample period is January 1990 to September 1994. Table 13 reports the simulation results. The power of test (3.19) and of test (3.24) almost coincide, and are at reasonable levels for the larger sample size. The tests of Engle and Ng prove to have almost no power at all.

3.6 Summary and Conclusion

In the paper two new Lagrange multiplier test procedures have been presented. The procedures are developed for testing the null hypothesis that

the conditional variance follows a GARCH(1,1) process, against the alternative that the conditional variance follows an asymmetric GARCH process. In the alternative hypotheses, well specified parametric models are considered. In test number one, the conditional variance follows a GQARCH(1,1) process under the alternative. In the second test, the alternative model is the LSTGARCH(1,1).

Small sample properties for the two tests have also been presented. These have been obtained from a number of Monte Carlo experiments. In those experiments two sample sizes are considered, 1000 and 250 observations. It is shown that the empirical size of the two tests is quite accurate for the larger sample size, and reasonable for the smaller. Since asymmetric GARCH specifications are primarily used for modeling high frequency financial data, a sample size of 1000 observations is not at all unusual.

The power of the tests is naturally a function of the parameters of the data generating process under the alternative. If the level of asymmetry in the data is low, the power falls considerably. To evaluate the power properties of the two tests, the power of the tests is compared to those of four other GARCH asymmetry tests, previously proposed in the literature. The four tests are: the Sign bias test (*SB*), the Negative size bias test (*NSB*), the Positive size bias test (*PSB*), and the test for the joint hypothesis of *SB*, *NSB*, and *PSB*. These tests are all developed by Engle and Ng [1993]. The Monte Carlo simulations show that the power of the two tests presented in this paper is much higher than the power of the four alternative tests. Furthermore, it is shown that the power properties of the two tests are also superior when the true data generating process is not the GQARCH(1,1) model or the LSTGARCH(1,1) model. The other data generating processes considered are: EGARCH(1,1), GJR, TGARCH(1,1), A-PARCH(1,1), and VS-ARCH. The test for which the alternative is the GQARCH(1,1) model generally proves to have slightly better power properties than the test for which the alternative is the LSTGARCH(1,1) model. It is therefore concluded that the two tests are in fact tests for general GARCH asymmetry, with reasonable power properties. This finding should be of importance for any econometrician working with GARCH models. The disappointing results of the tests of Engle and Ng [1993] are most likely a function of the fact that the tests have been developed without a well specified parametric alternative.

That the two tests can detect asymmetry caused by many parametric GARCH model is, however, not only good news. The results show that, using these tests, it is very hard, to actually decide which asymmetric model might have been the data generation process of a time series. Nevertheless, the tests will indicate whether any model in the family of asymmetric GARCH models could or could not have been the data generation process. Since the tests presented above have relatively low power, I am pessimistic about the possibility of designing powerful *LM* tests for testing the different models against each other. This subject still calls for further research.

Both test statistics presented in the paper are derived under conditional normality. The small sample properties are also investigated when the innovations of the data process are drawn from a Gaussian distribution. Many empirical investigations have shown that the assumption that financial data is distributed conditionally normal is most likely incorrect. Under such circumstances, the simulated size results are of less importance. Research in the area of specifications tests under non-normality is therefore strongly called for.

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Appendix

1. Derivation of the LM statistic (3.16)

Assume that we have the observed time series $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_T$. The conditional variance is under the alternative assumed to be generated by

$$h_t = \mathbf{w}'_t \boldsymbol{\beta},$$

where $\mathbf{w}'_t = (1, \varepsilon_{t-1}^2, h_{t-1}, \varepsilon_{t-1})$, and $\boldsymbol{\beta}' = (\gamma, \alpha, \beta, \zeta)$. The test is $H_0 : \zeta = 0$, against $H_1 : \zeta \neq 0$. The Lagrange multiplier statistic has the general form

$$LM = T \bar{q}_T(\boldsymbol{\beta}_0)' I(\boldsymbol{\beta}_0)^{-1} \bar{q}_T(\boldsymbol{\beta}_0),$$

where $\boldsymbol{\beta}_0$ is the vector of parameters under the null. $\bar{q}_T(\boldsymbol{\beta})$ is the average score and $I(\boldsymbol{\beta})$ is the information matrix. If we assume that the innovations are Gaussian, the log likelihood of one observation is equal to

$$l_t = -\frac{1}{2} \ln 2\pi - \frac{1}{2} \ln h_t - \frac{1}{2} \frac{\varepsilon_t^2}{h_t}.$$

Assuming that h_1 is fixed such that $\partial h_1 / \partial \boldsymbol{\beta} = \mathbf{0}$, it can be shown that the average score is equal to

$$\bar{q}_T(\boldsymbol{\beta}_0) = \frac{1}{T} \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \boldsymbol{\beta}}, \quad (3.33)$$

where

$$\frac{\partial h_t}{\partial \boldsymbol{\beta}'} = \left[\sum_{i=1}^{t-1} \hat{\beta}^{i-1}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i}^2, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{t-i}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i} \right]. \quad (3.34)$$

In (3.33) h_{0t} is the estimated conditional variance under the null of GARCH, and $\hat{\beta}$ in (3.34) is estimated under the null. The information matrix is the negative expectation of the Hessian averaged over all observations

$$I(\boldsymbol{\beta}) = -E \left[\frac{1}{T} \sum_{t=1}^T \frac{\partial^2 l_t}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'} \right].$$

The Hessian for one observation can be shown to be equal to

$$\frac{\partial^2 l_t}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'} = \left[\frac{1}{2} - \frac{\varepsilon_t^2}{h_t} \right] \left[\frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\beta}} \right] \left[\frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\beta}} \right]',$$

which implies that the information matrix becomes

$$I(\boldsymbol{\beta}) = \frac{1}{2T} \sum_{t=1}^T E \left[\frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\beta}} \right] \left[\frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\beta}} \right]'$$

The information matrix under the null is consistently estimated by

$$\widehat{I}(\beta_0) = \frac{1}{2T} \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial h_{0t}}{\partial \beta} \right] \left[\frac{1}{h_{0t}} \frac{\partial h_{0t}}{\partial \beta} \right]'$$

The Lagrange multiplier test of GARCH against GQARCH can therefore be written

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \beta} \right\}' \left\{ \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right]' \right\}^{-1} \\ & \times \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \beta} \right\}, \end{aligned}$$

which corresponds to formula (3.16). ■

2. Derivation of asymptotically equivalent statistic $T \cdot R_u^2$

Consider the statistic (3.16). The equation can be rewritten as

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right] \right\}' \left\{ \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right]' \right\}^{-1} \\ & \times \left\{ \sum_{t=1}^T \left[\frac{\varepsilon_t^2}{h_{0t}} - 1 \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right] \right\}, \end{aligned} \tag{3.35}$$

Define $y' = (y_1, \dots, y_T)$ and $\mathbf{X}' = (\mathbf{x}_1, \dots, \mathbf{x}_T)$, where

$$y_t = \left\{ \frac{\varepsilon_t^2}{h_{0t}} - 1 \right\}$$

and

$$\begin{aligned} \mathbf{x}_t &= \frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta'} \\ &= \left\{ \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1} h_{t-i}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \varepsilon_{t-i}}{h_{0t}} \right\}. \end{aligned}$$

Then, it is straightforward to rewrite (3.35) as

$$\frac{1}{2} y' \mathbf{X} (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' y$$

Note, that given that $\varepsilon_t \sim N(0, h_{0t})$ then

$$\text{plim}_{T \rightarrow \infty} y' y = \text{plim}_{T \rightarrow \infty} \sum_{t=1}^T \left(\frac{\varepsilon_t^2}{h_{0t}} - 1 \right)^2 = 2 \cdot T.$$

This suggests that an asymptotically equivalent statistic is

$$LM_2 = T \frac{\mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}}{\mathbf{y}'\mathbf{y}} = T \cdot R_u^2$$

where R_u^2 is the squared multiple correlation between \mathbf{y} and \mathbf{X} . Thus, the statistic is equal to $T \cdot R_u^2$ from the regression y_t on \mathbf{x}_t . ■

3. Test procedure used for the Sign bias test

The sign bias test statistic is defined as the t-ratio for the coefficient b in the regression equation

$$v_t^2 = a + b \cdot S_{t-1}^- + \tau' \frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta_0} + e_t \tag{3.36}$$

where $v_t^2 = \varepsilon_t^2/h_{0t}$, h_{0t} is the conditional variance under the null, τ is a constant parameter vector, β_0 are the parameters under the null, and e_t is the residual. When the null is the GARCH(1,1) model, the test procedure used is:

1. Estimate a GARCH(1, 1) model. Form the vectors

$$v_t^2 = \left\{ \begin{matrix} \varepsilon_t^2 \\ h_{0t} \end{matrix} \right\},$$

and

$$\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta_0'} = \left[\frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{t-i}}{h_{0t}} \right]$$

2. Run the regression (3.36), and calculate the statistic SB , which is equal to t-ratio for the estimate of the parameter b .

The test statistic SB is under the null asymptotically distributed standard normal.

4. Test procedure used for the Negative size bias test

The negative size bias test statistic is defined as the t-ratio for the coefficient b in the regression equation

$$v_t^2 = a + b \cdot S_{t-1}^- \varepsilon_{t-1} + \tau' \frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta_0} + e_t \tag{3.37}$$

where $v_t^2 = \varepsilon_t^2/h_{0t}$, h_{0t} is the conditional variance under the null, τ is a constant parameter vector, β_0 are the parameters under the null, and e_t is the residual. When the null is the GARCH(1,1) model, the test procedure used is:

1. Estimate a GARCH(1, 1) model. Form the vectors

$$v_t^2 = \left\{ \begin{array}{c} \varepsilon_t^2 \\ h_{0t} \end{array} \right\},$$

and

$$\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta_0'} = \left[\frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1} h_{t-i}}{h_{0t}} \right]$$

2. Run the regression (3.37), and calculate the statistic NSB , which is equal to t-ratio for the estimate of the parameter b .

The test statistic NSB is under the null asymptotically distributed standard normal.

5. Test procedure used for the Positive size bias test

The positive size bias test statistic is defined as the t-ratio for the coefficient b in the regression equation

$$v_t^2 = a + b \cdot S_{t-1}^+ \varepsilon_{t-1} + \tau' \frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta_0'} + e_t \quad (3.38)$$

where $v_t^2 = \varepsilon_t^2/h_{0t}$, h_{0t} is the conditional variance under the null, τ is a constant parameter vector, β_0 are the parameters under the null, and e_t is the residual. When the null is the GARCH(1,1) model, we used the test procedure:

1. Estimate a GARCH(1, 1) model. Form the vectors

$$v_t^2 = \left\{ \begin{array}{c} \varepsilon_t^2 \\ h_{0t} \end{array} \right\},$$

and

$$\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta_0'} = \left[\frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \varepsilon_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1} h_{t-i}}{h_{0t}} \right]$$

2. Run the regression (3.38), and calculate the statistic PSB , which is equal to t-ratio for the estimate of the parameter b .

The test statistic PSB is under the null asymptotically distributed standard normal.

6. Test procedure used for the Joint test

The test for the joint hypothesis of SB, NSB, and PSB is formulated as

$$\begin{aligned} H_0 &: b_1 = b_2 = b_3 = 0, \\ H_1 &: b_i \neq 0, \quad i = 1, 2, 3, \end{aligned}$$

in the regression

$$v_t^2 = a + b_1 \cdot S_{t-1}^- + b_2 \cdot S_{t-1}^- \varepsilon_{t-1} + b_3 \cdot S_{t-1}^+ \varepsilon_{t-1} + \tau' \frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta_0} + e_t$$

where $v_t^2 = \varepsilon_t^2/h_{0t}$, h_{0t} is the conditional variance under the null, τ is a constant parameter vector, β_0 are the parameters under the null, and e_t is the residual. Since $\partial h_t/\partial \beta_0$ should be orthogonal to v_t^2 , the test statistic could be calculated as $T \cdot R_u^2$ from the regression. However, the simulations showed that the empirical size of such a statistic is severely distorted. To achieve an appropriate size, v_t^2 was adjusted, and $\partial h_t/\partial \beta_0$ was replaced with a slightly simplified vector. When the null is the GARCH(1,1) model, the test procedure used was:

1. Estimate a GARCH(1, 1) model. Form the vector

$$v_t^2 = \left\{ \begin{array}{c} \varepsilon_t^2 \\ h_{0t} \end{array} \right\}.$$

Run the regression v_t^2 on $\{1, \varepsilon_{t-1}^2, h_{0t-1}\}$, and calculate the series of residuals \tilde{v}_t^2 .

2. Calculate the statistic as $T \cdot R_u^2$ from the regression \tilde{v}_t^2 on

$$\left\{ 1, S_{t-1}^-, S_{t-1}^- \varepsilon_{t-1}, S_{t-1}^+ \varepsilon_{t-1}, \frac{1}{h_{0t}}, \frac{\varepsilon_{t-i}^2}{h_{0t}}, \frac{h_{t-i}}{h_{0t}} \right\}.$$

The test statistic should be compared to a χ^2 distribution with three degrees of freedom.

Table 1. Simulated Size for the Test of no GQARCH

The table shows results from a Monte Carlo experiment where the size of test statistic (3.19) is investigated. In the experiment, the data generating process is model (3.25), with the four different parameter combinations shown in column one. The column labeled Actual Rejection Frequencies report the simulated empirical size at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, thus a 95 % confidence interval is given by $\hat{\alpha} + 2 \cdot \sqrt{\hat{\alpha}(1 - \hat{\alpha})/2500}$, where $\hat{\alpha}$ is the empirical size.

Parameter Values	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.25, \beta = 0.70$	1000	1.36	4.88	9.92
	250	1.16	6.04	12.04
$\gamma = 1.0 \cdot 10^{-5}, \alpha = 0.05, \beta = 0.85$	1000	0.88	4.76	9.64
	250	0.76	5.16	10.12
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.05, \beta = 0.90$	1000	1.20	5.44	10.20
	250	0.88	5.00	11.32
$\gamma = 1.0 \cdot 10^{-6}, \alpha = 0.09, \beta = 0.90$	1000	1.12	5.64	11.96
	250	0.84	5.96	12.20

Table 2. Simulated Size for the Test of no LSTGARCH

The table shows results from a Monte Carlo experiment where the size of test statistic (3.24), is investigated. In the experiment, the data generating process is model (3.25), with the four different parameter combinations shown in column one. The columns labeled Actual Rejection Frequencies report the simulated empirical size at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, thus a 95 % confidence interval is given by $\hat{\alpha} + 2 \cdot \sqrt{\hat{\alpha}(1 - \hat{\alpha})/2500}$, where $\hat{\alpha}$ is the empirical size.

Parameter Values	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.25, \beta = 0.70$	1000	0.88	5.16	10.84
	250	0.72	3.56	8.12
$\gamma = 1.0 \cdot 10^{-5}, \alpha = 0.05, \beta = 0.85$	1000	0.88	4.40	9.16
	250	0.40	3.04	6.64
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.05, \beta = 0.90$	1000	0.92	5.16	10.20
	250	0.68	4.24	8.80
$\gamma = 1.0 \cdot 10^{-6}, \alpha = 0.09, \beta = 0.90$	1000	1.20	5.28	10.16
	250	0.76	5.20	11.24

Table 3. Simulated Size for the Sign Bias Test

The table shows results from a Monte Carlo experiment where the size of the Sign bias test, is investigated. In the experiment, the data generating process is model (3.25), with the four different parameter combinations shown in column one. The columns labeled Actual Rejection Frequencies report the simulated empirical size at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, thus a 95 % confidence interval is given by $\hat{\alpha} + 2 \cdot \sqrt{\hat{\alpha}(1 - \hat{\alpha})/2500}$, where $\hat{\alpha}$ is the empirical size.

Parameter Values	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
Actual Rejection Frequencies				
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.25, \beta = 0.70$	1000	0.96	4.80	9.44
	250	0.84	5.08	10.68
$\gamma = 1.0 \cdot 10^{-5}, \alpha = 0.05, \beta = 0.85$	1000	0.84	5.28	10.60
	250	0.72	5.24	10.84
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.05, \beta = 0.90$	1000	1.44	6.04	10.96
	250	1.00	5.64	11.12
$\gamma = 1.0 \cdot 10^{-6}, \alpha = 0.09, \beta = 0.90$	1000	1.04	5.40	9.72
	250	0.76	5.20	11.24

Table 4. Simulated Size for the Negative Size Bias Test

The table shows results from a Monte Carlo experiment where the size of the Negative size bias test, is investigated. In the experiment, the data generating process is model (3.25), with the four different parameter combinations shown in column one. The columns labeled Actual Rejection Frequencies report the simulated empirical size at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, thus a 95 % confidence interval is given by $\hat{\alpha} + 2 \cdot \sqrt{\hat{\alpha}(1 - \hat{\alpha})/2500}$, where $\hat{\alpha}$ is the empirical size.

Parameter Values	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
Actual Rejection Frequencies				
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.25, \beta = 0.70$	1000	1.08	4.24	8.96
	250	0.60	4.28	8.00
$\gamma = 1.0 \cdot 10^{-5}, \alpha = 0.05, \beta = 0.85$	1000	0.92	5.12	9.76
	250	0.60	4.28	9.20
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.05, \beta = 0.90$	1000	0.88	5.88	11.44
	250	0.72	3.68	8.80
$\gamma = 1.0 \cdot 10^{-6}, \alpha = 0.09, \beta = 0.90$	1000	0.72	4.68	9.60
	250	1.04	4.88	9.08

Table 5. Simulated Size for the Positive Size Bias Test

The table shows results from a Monte Carlo experiment where the size of the Positive size bias test, is investigated. In the experiment, the data generating process is model (3.25), with the four different parameter combinations shown in column one. The columns labeled Actual Rejection Frequencies report the simulated empirical size at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, thus a 95 % confidence interval is given by $\hat{\alpha} + 2 \cdot \sqrt{\hat{\alpha}(1 - \hat{\alpha})/2500}$, where $\hat{\alpha}$ is the empirical size.

Parameter Values	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
Actual Rejection Frequencies				
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.25, \beta = 0.70$	1000	0.80	4.20	9.28
	250	0.72	4.32	8.60
$\gamma = 1.0 \cdot 10^{-5}, \alpha = 0.05, \beta = 0.85$	1000	0.96	4.36	9.56
	250	0.84	4.64	9.32
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.05, \beta = 0.90$	1000	1.20	5.24	9.84
	250	0.56	4.20	8.64
$\gamma = 1.0 \cdot 10^{-6}, \alpha = 0.09, \beta = 0.90$	1000	1.12	5.68	10.92
	250	0.80	4.28	8.72

Table 6. Simulated Size for Engle and Ng's Joint Test

The table shows results from a Monte Carlo experiments, where the size of the Joint test of Engle and Ng[1993], is investigated. In the experiment, the data generating process is model (3.25), with the four different parameter combinations shown in column one. The columns labeled Actual Rejection Frequencies report the simulated empirical size at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, thus a 95 % confidence interval is given by $\hat{\alpha} + 2 \cdot \sqrt{\hat{\alpha}(1 - \hat{\alpha})/2500}$, where $\hat{\alpha}$ is the empirical size.

Parameter Values	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
Actual Rejection Frequencies				
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.25, \beta = 0.70$	1000	1.44	4.80	8.76
	250	0.80	3.64	7.64
$\gamma = 1.0 \cdot 10^{-5}, \alpha = 0.05, \beta = 0.85$	1000	1.12	5.08	9.72
	250	0.88	4.16	8.48
$\gamma = 5.0 \cdot 10^{-6}, \alpha = 0.05, \beta = 0.90$	1000	1.36	5.44	11.20
	250	0.76	4.00	8.92
$\gamma = 1.0 \cdot 10^{-6}, \alpha = 0.09, \beta = 0.90$	1000	1.44	5.04	9.88
	250	1.00	4.24	9.12

Table 7. Actual Rejection Frequencies when the True Model is GQARCH(1,1)

The table shows results from a Monte Carlo experiment where the empirical power of six specification tests are investigated. In the experiment, the data generating process is model (3.26). The columns labeled Actual Rejection Frequencies report the simulated empirical power at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, which give an estimated standard error of the estimated power, \hat{p} , equal to $\sqrt{\hat{p}(1 - \hat{p})}/2500$. The abbreviations are: LM_1 refers to the test of no GQARCH, LM_2 to the test of no LSTGARCH, SB to the Sign bias test, NSB to the Negative size bias test, PSB to the Positive size bias test, and Joint refers to the test of the joint hypothesis of SB, NSB, and PSB.

Test	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
LM_1	1000	6.28	20.72	32.56
	250	1.44	6.48	13.80
LM_2	1000	3.92	13.20	22.56
	250	0.84	4.60	11.36
SB	1000	1.28	5.52	10.92
	250	0.88	5.28	9.92
NSB	1000	1.60	6.00	10.92
	250	0.96	4.28	8.88
PSB	1000	1.00	4.88	10.76
	250	0.48	4.44	9.68
Joint	1000	1.24	4.92	10.48
	250	1.00	4.12	8.00

Table 8. Actual Rejection Frequencies when the True Model is LSTGARCH(1,1)

The table shows results from a Monte Carlo experiment where the empirical power of six specification tests is investigated. In the experiment, the data generating process is model (3.27). The columns labeled Actual Rejection Frequencies report the simulated empirical power at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, which give an estimated standard error of the estimated power, \hat{p} , equal to $\sqrt{\hat{p}(1-\hat{p})/2500}$. The abbreviations are: LM_1 refers to the test of no GQARCH, LM_2 to the test of no LSTGARCH, SB to the Sign bias test, NSB to the Negative size bias test, PSB to the Positive size bias test, and Joint refers to the test of the joint hypothesis of SB , NSB , and PSB .

Test	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
LM_1	1000	15.36	34.72	47.36
	250	5.12	15.36	25.12
LM_2	1000	36.12	64.40	75.60
	250	7.12	21.64	32.92
SB	1000	5.40	15.00	24.04
	250	1.64	7.76	14.48
NSB	1000	6.36	16.68	26.48
	250	1.72	7.12	12.20
PSB	1000	6.44	21.20	34.40
	250	1.00	7.40	18.08
Joint	1000	6.08	17.08	27.44
	250	2.40	6.96	12.72

Table 9. Actual Rejection Frequencies when the True Model is EGARCH(1,1)

The table shows results from a Monte Carlo experiment where the empirical power of six specification tests is investigated. In the experiment, the data generating process is model (3.28). The columns labeled Actual Rejection Frequencies report the simulated empirical power at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, which give an estimated standard error of the estimated power, \hat{p} , equal to $\sqrt{\hat{p}(1-\hat{p})/2500}$. The abbreviations are: LM_1 refers to the test of no GQARCH, LM_2 to the test of no LSTGARCH, SB to the Sign bias test, NSB to the Negative size bias test, PSB to the Positive size bias test, and *Joint* refers to the test of the joint hypothesis of SB , NSB , and PSB .

Test	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
LM_1	1000	51.08	74.00	82.80
	250	7.72	22.08	33.44
LM_2	1000	31.32	57.00	70.56
	250	4.08	15.52	24.56
SB	1000	9.16	24.44	36.80
	250	2.08	9.20	16.40
NSB	1000	12.44	29.56	41.08
	250	1.88	8.32	14.96
PSB	1000	9.52	27.84	40.68
	250	2.20	9.92	18.36
<i>Joint</i>	1000	7.32	20.84	32.12
	250	1.16	6.12	11.88

Table 10. Actual Rejection Frequencies when the True Model is GJR

The table shows results from a Monte Carlo experiment where the empirical power of six specification tests is investigated. In the experiment, the data generating process is model (3.29). The columns labeled Actual Rejection Frequencies report the simulated empirical power at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, which give an estimated standard error of the estimated power, \hat{p} , equal to $\sqrt{\hat{p}(1-\hat{p})/2500}$. The abbreviations are: LM_1 refers to the test of no GQARCH, LM_2 to the test of no LSTGARCH, SB to the Sign bias test, NSB to the Negative size bias test, PSB to the Positive size bias test, and Joint refers to the test of the joint hypothesis of SB, NSB, and PSB.

Test	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
LM_1	1000	84.16	95.00	97.44
	250	19.08	43.00	56.92
LM_2	1000	73.56	90.52	95.12
	250	14.68	37.04	51.12
SB	1000	26.68	49.76	62.20
	250	5.12	15.16	24.08
NSB	1000	18.72	39.40	51.28
	250	4.00	11.72	19.20
PSB	1000	22.16	49.08	62.96
	250	3.28	14.68	25.40
Joint	1000	20.24	44.36	58.20
	250	3.16	10.96	18.48

Table 11. Actual Rejection Frequencies when the True Model is TGARCH

The table shows results from a Monte Carlo experiment where the empirical power of six specification tests is investigated. In the experiment, the data generating process is model (3.30). The columns labeled Actual Rejection Frequencies report the simulated empirical power at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, which give an estimated standard error of the estimated power, \hat{p} , equal to $\sqrt{\hat{p}(1 - \hat{p})/2500}$. The abbreviations are: LM_1 refers to the test of no GQARCH, LM_2 to the test of no LSTGARCH, SB to the Sign bias test, NSB to the Negative size bias test, PSB to the Positive size bias test, and Joint refers to the test of the joint hypothesis of SB, NSB, and PSB.

Test	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
		Actual Rejection Frequencies (%)		
LM_1	1000	56.76	79.28	86.00
	250	10.24	26.60	38.16
LM_2	1000	32.72	58.80	71.64
	250	5.44	17.48	27.24
SB	1000	11.00	27.12	37.60
	250	2.28	8.48	15.36
NSB	1000	10.48	25.76	36.36
	250	2.76	7.80	14.52
PSB	1000	9.00	26.48	39.04
	250	2.16	8.36	16.16
Joint	1000	7.24	21.44	32.92
	250	1.32	5.68	11.44

Table 12. Actual Rejection Frequencies when the True Model is A-PARCH

The table shows results from a Monte Carlo experiment where the empirical power of six specification tests is investigated. In the experiment, the data generating process is model (3.31). The columns labeled Actual Rejection Frequencies report the simulated empirical power at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, which give an estimated standard error of the estimated power, \hat{p} , equal to $\sqrt{\hat{p}(1 - \hat{p})/2500}$. The abbreviations are: LM_1 refers to the test of no GQARCH, LM_2 to the test of no LSTGARCH, SB to the Sign bias test, NSB to the Negative size bias test, PSB to the Positive size bias test, and Joint refers to the test of the joint hypothesis of SB, NSB, and PSB.

Test	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
LM_1	1000	76.28	91.56	95.40
	250	12.76	32.40	45.84
LM_2	1000	64.16	85.36	91.32
	250	8.92	27.12	41.08
SB	1000	6.72	17.40	26.88
	250	1.48	7.36	13.16
NSB	1000	8.32	18.92	27.60
	250	2.12	8.16	14.08
PSB	1000	5.72	18.72	30.60
	250	1.20	6.88	14.00
Joint	1000	5.80	16.36	25.84
	250	1.12	5.76	11.56

Table 13. Actual Rejection Frequencies when the True Model is VS-ARCH

The table shows results from a Monte Carlo experiment where the empirical power of six specification tests is investigated. In the experiment, the data generating process is model (3.32). The columns labeled Actual Rejection Frequencies report the simulated empirical power at the three different theoretical significance levels of one, five, and ten percent. The figures are based on 2,500 samples, which give an estimated standard error of the estimated power, \hat{p} , equal to $\sqrt{\hat{p}(1-\hat{p})}/2500$. The abbreviations are: LM_1 refers to the test of no GQARCH, LM_2 to the test of no LSTGARCH, SB to the Sign bias test, NSB to the Negative size bias test, PSB to the Positive size bias test, and Joint refers to the test of the joint hypothesis of SB, NSB, and PSB.

Test	Sample Size	Nominal Significance Level		
		1 %	5 %	10%
LM_1	1000	20.96	43.32	55.92
	250	3.00	12.04	20.52
LM_2	1000	19.56	42.08	56.04
	250	1.96	10.04	17.20
SB	1000	1.60	7.52	13.68
	250	1.04	5.56	11.60
NSB	1000	2.12	9.48	17.48
	250	0.88	6.08	12.84
PSB	1000	2.52	9.64	16.96
	250	1.12	5.44	10.80
Joint	1000	1.40	7.80	15.24
	250	0.84	4.88	10.76

4

Modeling Nordic Stock Returns with Asymmetric GARCH Models

4.1 Introduction

During the last 15 years, an enormous amount of effort has been devoted to the modeling of conditional volatility in financial markets data. The seminal work in this area of research is by Engle [1982], who introduces the standard autoregressive conditional heteroskedasticity model, thereby initiating the development of the ARCH class of models. Today the ARCH literature has grown to spectacular proportions. An excellent survey of the literature is Bollerslev, Engle, and Nelson [1994].

Many of the proposed ARCH models include a term that can capture correlation between returns and conditional variance. Models with this feature are often termed “asymmetric” or “leverage” volatility models. The term *leverage* stems from the empirical observation that the conditional variance of equity returns often increases when returns are negative, i.e. when the financial leverage of the firm increases. One of the earliest asymmetric ARCH models is the EGARCH (Exponential Generalized ARCH) model of Nelson [1991]. Another popular specification is the model of Glosten, Runkle, and Jagannathan [1993], denoted GJR.

This paper investigates to what extent asymmetric GARCH models might have been the data generating process for a number of time series. Data investigated is daily observations from 45 Nordic stocks. The investigated period is July 1991 to July 1996. Furthermore, this paper is concerned with the relative in sample performance of seven different parametric asymmetric ARCH models. The models studied are: EGARCH, GJR, TGARCH of

Zakořan [1994], A-PARCH of Ding, Granger, and Engle [1993], GQARCH of Sentana [1995], VS-ARCH of Fornari and Mele [1996], and LSTGARCH of Hagerud [1996], and González-Rivera [1996].

To estimate the unknown parameters of the models, iterative numerical methods are required. These procedures are often time consuming, and if the model in question explains the data badly the estimation might not converge. Specification tests are therefore used to investigate whether a certain model might have been the data generating process of a time series. The test of no ARCH, developed by Wooldridge [1990] is used to test for general ARCH effects. This test is a robust version of Engle's [1982] test of no ARCH. To test for asymmetric effects, robust versions of two tests proposed by Hagerud [1997] are used. Hagerud's test are derived under conditional normality, and since the data shows a high level of conditional heterokurtosis, robustification is necessary.

It is found that 32 of the 45 series show signs of heteroskedasticity. Of these 32 securities, twelve could have been generated by an asymmetric GARCH model. Furthermore, for the subsample of twelve series, it is concluded that the models GJR, TGARCH, and GQARCH are superior in modeling the asymmetric dynamics of the conditional variance.

This article is organized as follows. Section 4.2 describes the conditional variance models and the estimation procedures used. In Section 4.3, the tests used for the specification of the conditional mean and variance are formulated. The data is described in Section 4.4. Results are presented in Section 4.5, and the conclusions in Section 4.6.

4.2 Models and Estimation

Let the price of a stock at time t be denoted by P_t . Returns, measured as $\ln(P_t/P_{t-1})$, are assumed to follow the AR(p)-process

$$r_t = \phi_0 + \sum_{i=1}^p \phi_i r_{t-i} + \varepsilon_t, \quad (4.1)$$

where ε_t denotes a discrete-time stochastic process with the form

$$\varepsilon_t = z_t h_t^{1/2}, \quad (4.2)$$

where $z_t \sim iid(0, 1)$, and h_t is the conditional variance of return at time t , whose dynamics GARCH specifications wish to model.

The seven asymmetric GARCH models that will be investigated, are listed below. But first is a specification of the symmetric GARCH(1,1) model that is used as the null hypothesis when the presence of asymmetry is tested for. The models will only be studied in their most simple structure, when the lag lengths are equal to one. In many empirical investigations,

these parsimonious models have proven to perform well. The description below is very brief. A more detailed presentation of the asymmetric models can be found in Hagerud [1997].

Bollerslev's [1986] GARCH(1,1) model assumes that the conditional variance is generated by

$$h_t = \gamma + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}, \quad (4.3)$$

where γ , α and β are non-negative constants. For the GARCH process to be defined, it is required that $\alpha > 0$.

The EGARCH model of Nelson [1991] is

$$\ln h_t = \gamma + \beta \ln h_{t-1} + \lambda z_{t-1} + \varphi \left[|z_{t-1}| - \sqrt{2/\pi} \right], \quad (4.4)$$

where γ , β , λ , and φ are constant parameters, and z_t is defined as in (4.2).

The GJR model of Glosten, Jagannathan, and Runkle [1993] is

$$h_t = \gamma + \alpha \varepsilon_{t-1}^2 + \omega S_{t-1}^- \varepsilon_{t-1}^2 + \beta h_{t-1}, \quad (4.5)$$

where γ , α , β , and ω are constant parameters, and S_{t-1}^- is an indicator function that takes the value one when $\varepsilon_{t-1} < 0$ and zero otherwise.

The Threshold GARCH model is introduced in Zakořan [1994]. The TGARCH(1,1) model is

$$h_t^{1/2} = \gamma + \alpha^+ \varepsilon_{t-1}^+ - \alpha^- \varepsilon_{t-1}^- + \beta h_{t-1}^{1/2}, \quad (4.6)$$

where $\varepsilon_t^+ = \max(\varepsilon_t, 0)$, and $\varepsilon_t^- = \min(\varepsilon_t, 0)$. Note that (4.6) can be reparameterized as

$$h_t^{1/2} = \gamma + \alpha |\varepsilon_{t-1}| + \omega S_{t-1}^- \varepsilon_{t-1} + \beta h_{t-1}^{1/2}.$$

Thus, in the TGARCH(1,1) model, the conditional standard deviation has the same functional form as the conditional variance has in the GJR model (4.5).

Ding, Granger, and Engle [1993] propose the Asymmetric Power ARCH model. The A-PARCH(1,1) model is

$$h_t^{\delta/2} = \gamma + \alpha (|\varepsilon_{t-1}| - \eta \varepsilon_{t-1})^\delta + \beta h_{t-1}^{\delta/2}, \quad (4.7)$$

where γ , α , β , η and $\delta \geq 0$ are constant parameters.

Sentana [1995] introduces the Quadratic ARCH model. The term *quadratic* is used since the QARCH model can be interpreted as a second-order Taylor approximation to the unknown conditional variance function. The Generalized QARCH(1,1) model is

$$h_t = \gamma + \zeta \varepsilon_{t-1} + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}, \quad (4.8)$$

where γ , ζ , α , and β are constant parameters.

The VS-ARCH (Volatility Switching) model of Fornari and Mele [1996] is

$$h_t = \gamma + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1} + \xi S_{t-1} v_{t-1}^2, \quad (4.9)$$

where

$$\begin{aligned} S_t &= 1 \text{ if } \varepsilon_t > 0 \\ S_t &= 0 \text{ if } \varepsilon_t = 0 \\ S_t &= -1 \text{ if } \varepsilon_t < 0, \end{aligned}$$

and v_t^2 is defined as ε_t^2/h_t . The parameters of the model are γ , α , β , and ξ .

The logistic smooth transition GARCH(1,1) model of Hagerud [1996], and González-Rivera [1996] is given by

$$h_t = \gamma + [\alpha_1 + \alpha_2 F(\varepsilon_{t-1})] \varepsilon_{t-1}^2 + \beta h_{t-1}, \quad (4.10)$$

where $F(\cdot)$ is a transition function with the form

$$F(\varepsilon_{t-1}) = (1 + \exp[-\theta \varepsilon_{t-1}])^{-1} - \frac{1}{2}, \quad \theta > 0.$$

The parameters of the model are therefore γ , α_1 , α_2 , β , and θ . Note that GJR will result as a limiting case of LSTGARCH, when $\theta \rightarrow \infty$.

To estimate the parameters of the models, a quasi-maximum likelihood approach is used. The innovations z_t are assumed to be distributed independently normal, and a normal log-likelihood function is maximized, using standard numerical methods. Bollerslev and Wooldridge [1992] show that when the normality is violated, the quasi-maximum likelihood estimators (QMLE) are generally consistent and have a limiting normal distribution. In their article, Bollerslev and Wooldridge also present asymptotic standard errors of the estimators valid under non-normality.

The parameters of the conditional mean model are estimated simultaneously with the conditional variance model. Engle [1982] and Bollerslev [1986] show that when the ARCH model is symmetric with respect to lagged returns, the two models can be estimated separately. This result is used when the standard GARCH(1,1) model is estimated, in the testing procedure. Unfortunately, this simplification cannot be used for the other models.

4.3 Specification Tests

Following the recommendations of Wooldridge [1991], a bottom-up strategy is used when performing specification tests. Thus, first the conditional

first movement is specified. Once the conditional mean is formulated and estimated satisfactorily, tests for the conditional variance specification are performed.

When attempting to specify the conditional mean, only possible autocorrelation in the returns is tested for. Thus, any possible non-linearity in the conditional mean is disregarded. Furthermore, the possibility of the conditional variance to be an explanatory variable of return is not considered. To test for autocorrelation, a test developed by Richardson and Smith [1994] is used. The test, in the form use here, is a robust version of a standard Box and Pierce [1970] procedure. Letting $\hat{\rho}_i$ be the estimated autocorrelation between the returns at time t and $t - i$, the test is formulated as

$$RS(k) = T \sum_{i=1}^k \frac{\hat{\rho}_i^2}{1 + c_i}. \quad (4.11)$$

The terms c_i is an adjustment factor for heteroskedasticity, and it is calculated as

$$c_i = \frac{\text{cov}[\bar{r}_t^2, \bar{r}_{t-i}^2]}{\text{var}[\bar{r}_t^2]}, \quad (4.12)$$

where \bar{r}_t is the demeaned return at time t . Under the null of no autocorrelation, the statistic is distributed asymptotically χ^2 with k degrees of freedom. Since the sample sizes are relatively large, any adjustments of the statistic, in the spirit of Ljung and Box [1978], are unnecessary.

If the null of no autocorrelation cannot be reject, it is concluded that returns are equal to a constant plus a residual ε_t , i.e. the conditional mean specification is model (4.1), with p equal to zero. If the null is rejected, an AR(1) model is estimated on the series. To ensure that this model captures the detected autocorrelation, test (4.11) was once more applied. In this case, the test is run on the estimated residuals from the AR(1) model. Following the recommendations of Box and Pierce [1970] and Ljung and Box [1978], the value of the statistic in this case is compared to a χ^2 distribution with $k - 1$ degrees of freedom. If the null cannot be rejected, it is concluded that returns are generated by an AR(1) model. If the null is rejected, the procedure is continued with higher order AR models, until test (4.11) is not rejected.

After the conditional mean model is deemed satisfactory, tests for possible heteroskedasticity are performed. First, the null of homoskedasticity against the alternative of heteroskedasticity is tested. The most commonly used test for this hypothesis is the LM test of no ARCH of Engle [1982]. However, Engle's test requires that the fourth conditional moment of ε_t is constant and finite, as shown by Koenker [1981]. To overcome that complication, and to ensure that the test has a satisfactory asymptotic size, a robust test of no ARCH developed by Wooldridge [1990] is used. This test is calculated using the procedure:

1. Estimate the sample variance under the null of homoskedasticity

$$\hat{\sigma}^2 = T^{-1} \sum_{t=1}^T \hat{\varepsilon}_t^2,$$

where $\hat{\varepsilon}_t$ is the consistently estimated residuals from the model (4.1).

2. Regress 1 on

$$\left(\hat{\varepsilon}_t^2 - \hat{\sigma}^2\right) \left(\hat{\varepsilon}_{t-1}^2 - \hat{\sigma}^2\right), \dots, \left(\hat{\varepsilon}_t^2 - \hat{\sigma}^2\right) \left(\hat{\varepsilon}_{t-q}^2 - \hat{\sigma}^2\right) \quad (4.13)$$

The statistic is equal to $LM_0 = T \cdot R_u^2 = T - RSS$, where R_u^2 is the uncentered coefficient of determination from the regression (4.13), and RSS is the residual sum of squares. The statistic converges in distribution to a χ^2 variable with q degrees of freedom.

If the null of no ARCH(q) cannot be rejected, the investigation continues with tests for asymmetric GARCH. This is done with two new test procedures, based on a pair of LM statistics proposed by Hagerud [1997]. In both tests, the conditional variance follows a GARCH(1,1) process under the null. In test number one, the alternative hypothesis is the GQARCH model (4.8), and in test number two, the alternative hypothesis is the LSTGARCH model (4.10). Hagerud [1997] shows that these two tests have superior power properties, compared to the standard asymmetric ARCH tests developed by Engle and Ng [1993]. This superiority remains valid even when the true data generating process is not the GQARCH or the LSTGARCH models, but any other of the asymmetric models presented in Section 4.2.

Hagerud's [1997] tests are derived under the assumption that ε_t is distributed conditionally normal. This assumption is unlikely to be fulfilled in the data set considered here. Non-normality might give the statistics the wrong asymptotic size. The statistics are therefore robustified using a method presented by Wooldridge [1991]. For these statistics to be of the correct asymptotic size, only general distributional assumptions have to be made (see Wooldridge [1990]). Furthermore, the information matrix between the conditional mean and the conditional variance parameters does not have to be block-diagonal. Derivations of statistics LM_1 and LM_2 , presented below, can be found in the appendix.

The hypothesis for the test of no GQARCH can be formulated

$$\begin{aligned} H_0 &: \zeta = 0, \\ H_1 &: \zeta \neq 0, \end{aligned}$$

and the proposed test procedure is:

1. Estimate a GARCH(1,1) model on the series of estimated residuals $\{\hat{\varepsilon}_t\}_{t=1}^T$, and form the series of conditional variance under the null $\{h_{0t}\}_{t=1}^T$. Let $\hat{\beta}$ be the estimated parameters β in the GARCH(1,1) model.

2. Regress

$$\frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}}{h_{0t}},$$

on

$$y_t = \left\{ \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{0t-i}}{h_{0t}} \right\},$$

and let \hat{a} be the vector of estimated parameters. Form the series of residuals

$$x_t = \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}}{h_{0t}} - \hat{a}y'_t.$$

3. Regress 1 on

$$\left(\frac{\hat{\varepsilon}_t^2}{h_{0t}} - 1 \right) \cdot x_t \quad (4.14)$$

The statistic is equal to $LM_1 = T \cdot R_u^2 = T - RSS$, from the regression (4.14).

The statistic LM_1 converges in distribution to a χ^2 variable with one degree of freedom, corresponding to the number of restrictions.

The test of no LSTGARCH can be formulated¹

$$H_0 : \alpha_2 = 0$$

$$H_1 : \alpha_2 \neq 0,$$

and the proposed test procedure is:

1. Estimate a GARCH(1,1) model on the series $\{\hat{\varepsilon}_t\}_{t=1}^T$, and form $\{h_{0t}\}_{t=1}^T$.

2. Regress

$$\frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^3}{h_{0t}},$$

on y_t , defined above, and let \hat{b} be the vector of estimated parameters. Form the series of residuals

$$s_t = \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^3}{h_{0t}} - \hat{b}y'_t.$$

¹To test for $\alpha_2 = 0$ in the LSTGARCH(1,1) model will cause an identification problem. Hagerud [1997] shows how this problem can be solved by making a Taylor approximation of the model.

3. Regress 1 on

$$\left(\frac{\widehat{\varepsilon}_t^2}{h_{0t}} - 1 \right) \cdot s_t \tag{4.15}$$

The statistic is equal to $LM_2 = T \cdot R_u^2 = T - RSS$.

The statistic LM_2 converges in distribution to a χ^2 variable with one degree of freedom.

While performing test LM_1 or LM_2 , and given that the series of conditional variance is estimated with quasi-maximum likelihood, the normalized residual, $v_t = \varepsilon_t/h_{0t}^{1/2}$, should be orthogonal to the vector

$$y_t = \left\{ \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \widehat{\varepsilon}_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \widehat{\beta}^{i-1} h_{0t-i}}{h_{0t}} \right\}.$$

This should be true independently of whether the null is true or not. However, in practice, exact orthogonality cannot always be guaranteed. This is noted by Engle and Ng [1993], and from a Monte Carlo study, the authors concluded that it is advisable to adjust v_t , such that orthogonality is guaranteed. However, since the small sample properties of LM_1 or LM_2 are unknown, no such adjustment will be performed.

If at least one of the tests LM_1 and LM_2 are rejected, the models (4.4) through (4.10) will be estimated on the series. To decide whether a certain model is able to capture the heteroskedasticity adequately, a number specification tests are once more performed. First, the skewness and kurtosis of the series of normalized residuals, $\widehat{\varepsilon}_t/h_t^{1/2}$, are calculated, where h_t is the estimated conditional variance. Second, a new test procedure to test for further asymmetric ARCH effects is used. This test procedure is a robust version of the *Sign bias test* of Engle and Ng's [1993]. The sign bias test is a test for general asymmetry. The robustification is, as above, done using the method of Wooldridge [1991]. The null is now the asymmetric model that was used to estimate h_t , and the alternative is the same asymmetric model augmented by a term $\tau \cdot S_{t-1}^-$. Thus, the alternative model considered is

$$g_t = h_t + \tau \cdot S_{t-1}^-,$$

where h_t is any of the asymmetric models (4.4) to (4.10). The test is formulated as follows

$$\begin{aligned} H_0 & : \tau = 0, \\ H_1 & : \tau \neq 0, \end{aligned}$$

and the robust test procedure is:

1. Estimate the asymmetric GARCH model with quasi-maximum likelihood, and form $\{h_t\}_{t=1}^T$.

2. Regress

$$\frac{1}{h_t} \frac{\partial g_t}{\partial \tau},$$

on

$$w_t = \frac{1}{h_t} \frac{\partial h_t}{\partial \beta'}$$

where β is the vector of parameters in the asymmetric model. Let \hat{c} be the vector of estimated parameters. Form the series of residuals

$$r_t = \frac{1}{h_t} \frac{\partial g_t}{\partial \tau} - \hat{c}w_t'.$$

3. Regress 1 on

$$\left(\frac{\hat{\varepsilon}_t^2}{h_t} - 1 \right) \cdot r_t. \quad (4.16)$$

The statistic is equal to $LM_3 = T \cdot R_u^2 = T - RSS$.

The statistic LM_3 converges in distribution to a χ^2 variable with one degree of freedom. Note that to perform the test, it is necessary to calculate the partial derivatives $\partial g_t / \partial \tau$, and $\partial h_t / \partial \beta$. These will differ depending on the model considered under the null. These derivatives for all the seven models are listed in the appendix.

4.4 Data

Original stock price observations are daily close prices from the exchanges in Helsinki, Stockholm, Oslo, and Copenhagen. The stocks are the most actively traded securities in each of the markets. The period investigated is July 1, 1991 to July 1, 1996. This gives approximately 1,260 price observations per security. The price observations are not adjusted for dividends, but since dividends are paid on a yearly basis, only five return observations per series are affected by dividends. The data is collected from Datasteam.

The investigated securities are listed in Table 1, and some summary statistics are given. 17 stocks are from Stockholm, 14 from Copenhagen, 8 from Oslo, and 6 from Helsinki. The prices of most of the stock have increased over the investigated period of five years. The highest return is observed for the Finnish telecommunication company Nokia, with 63 percent average yearly yield. The worst investment during the period is the Norwegian shipping company I.m. Skaugen. The average yearly yield for the 45 companies is 8.9 percent.

From the last column in Table 1, it can be seen that the volatility, measured as estimated constant standard deviation p.a., varies considerably in the sample. The lowest volatility is reported for the Danish pharmaceutical

company NovoNordisk, with 15 percent, and the highest is reported for I.m. Skaugen, with 70 percent. However, a majority of the stocks have a volatility in the interval 20 to 30 percent.

4.5 Results

Table 2 reports the results from the tests performed to specify the conditional mean. Richardson and Smith's [1994] test (4.11), calculated on eight autocorrelations, indicate that twelve companies show signs of autocorrelation, on five percent significance level. For these companies, AR(1) models were estimated. Column three in the table reports p-values for Richardson and Smith's [1994] test calculated on the estimated residuals from these AR(1) models. No autocorrelation can be detected in these series of residuals. It is therefore concluded that for 33 securities, a suitable mean equation is (4.1) with $p = 0$, and for the remaining twelve securities an AR(1) model is appropriate.

Column two in Table 3 reports the results from Wooldridge's [1990] tests of no ARCH. The test is calculated with q equal to eight.² Thirteen companies show no signs of heteroskedasticity, on five percent significance level. These companies will not be considered further. Column three contain p-values for Engle's [1982] test of no ARCH, calculated on eight lagged squared residuals. This is the non-robust alternative to Wooldridge's test. Note that if the non-robust test had been used, quite different conclusions about the presence of heteroskedasticity in the sample would have been drawn.

Column four in Table 3 shows excess kurtosis for the series of estimated residuals, and column five reports the first autocorrelation in squared residuals. Teräsvirta [1996] shows that a GARCH(1,1) model with normal errors cannot generate data with high excess kurtosis and low first-order autocorrelation in squared residuals. Many of the series in this sample show such a pattern. However, since the primary interest is finding an estimate of the series of conditional variance, $\{h_t\}_{t=1}^T$, this should not affect the results. This follows from the fact that the series of conditional variance only depends on the consistently estimated parameters α , β , and γ , and not on the distributional assumption made. This is obviously only true if the assumptions of Bollerslev and Wooldridge [1992] are fulfilled, so that the QMLEs are consistent.

Table 4 reports the results from the two tests of no asymmetric GARCH. According to column two, the null of GARCH(1,1) can be rejected against

²Wooldridge's test of no ARCH was also calculated on ten and fifteen lagged residuals. The results from those tests were only slightly different from the ones shown in Table 3. Thus, the general level of detected heteroskedasticity was not altered when the number of lags in the statistic was increased.

the alternative of GQARCH(1,1), for eleven securities on the five percent significance level. The null of GARCH(1,1) against LSTGARCH(1,1), can be rejected for three securities. In total, asymmetries have been detected for twelve securities, out of the 32 securities that showed signs of heteroskedasticity. For this subsample of twelve securities, the investigation continues by fitting the seven asymmetric models (4.4) to (4.10) to the return observations.

In Table 4, are also the p-values for the corresponding non-robust test procedures of Hagerud [1997]. The results for the test of GARCH(1,1) against GQARCH(1,1), are given in column three, and column five reports the results for the test of GARCH(1,1) against LSTGARCH(1,1). In number cases different conclusions would have been drawn if the non-robust tests had been used. Thus, it is advisable to use the proposed robust test statistics.

As pointed out in Section 4.2, a quasi-maximum likelihood approach is used to estimate the seven models on the twelve return series. For most securities, all seven models could be estimated without difficulty. Out of a total of 84 estimations, convergence was not reached in eight cases. These cases are indicated as failures in Table 7. It is primarily the A-PARCH model that is hard to estimate. Failures often occur because the series of conditional variance is given a negative value, or because stationarity conditions on the estimated parameters could not be met. The two samples that proved to be most troublesome to estimate are the price series for SCA, a Swedish forestry company, and the series for Unibank, a Danish bank.

In all models but one, the asymmetries were estimated to be in agreement with the folkloristic view that the conditional volatility increases in bear markets. Thus, the parameters of the models were estimated to such values that if a certain model had been the true data generating process, a negative correlation between lagged residuals and conditional variance would result. The exception is the VS-ARCH model estimated for I.m. Skaugen.

One of the objectives of this study was to investigate which of the asymmetric ARCH specifications (4.4) to (4.10) models the conditional variance "best." Two simplistic selection criteria for finding the best model are used: the value of the likelihood function, and the BIC information criteria of Schwarz [1978]. In addition, it is required that the best model, should not perform worse than the other models, with regard to the specification tests presented in Section 4.3.

Table 5 reports, for each security, the top three ranked models maximizing the likelihood function. Table 6 gives the analogous results for BIC. Not surprisingly, the models with most parameters, A-PARCH and LSTGARCH, often maximize the likelihood function. However, in some cases, the more parsimonious models GJR, TGARCH, and GQARCH perform better. When the number of parameters is given consideration, as in the BIC, the three models GJR, TGARCH, and GQARCH, seems to be supe-

rior. Both GJR and TGARCH minimize BIC in five cases, and GQARCH is ranked first in two cases.

P-values for the robust sign bias test, LM_3 , performed on the 76 models that were successfully estimated, are reported in Table 7. Only in two cases, GQARCH for SCA B, and EGARCH for Handelsbanken A, can the null of no additional asymmetry be rejected, on five percent significance level. In all other cases, it is hard to find any evidence which favors one model over the other. It is therefore concluded that the models seem to capture the same dynamics of heteroskedasticity, but do so more or less well. However, this conclusion should be taken with a pinch of salt, since there might be asymmetries in the data that cannot be detected with the rather general hypothesis of the sign bias test.

The investigation now focuses on the series of estimated normalized residuals, $\left\{\hat{\varepsilon}_t/h_t^{1/2}\right\}_{t=1}^T$. Table 8 reports estimated standardized third moment, *skewness*, of the series. An assumption that the distribution of z_t in (4.2) is symmetric appears to be appropriate for seven of the series, on five percent significance level. For EAC, Modo B, Orkla A, I.m. Skaugen, and Volvo B, the estimated skewness indicates that the distribution is skewed. Alternatively, this might be an indication that a different specification of the conditional variance should be used. Analogously to the results of the sign bias test, no model is better than the rest in reaching a zero skewness.

The estimated excess kurtosis of $\left\{\hat{\varepsilon}_t/h_t^{1/2}\right\}_{t=1}^T$, for the 76 models, are reported in Table 9. The hypothesis that the coefficient of excess kurtosis is equal to zero is rejected for all models. Thus, it can be concluded that an assumption that z_t is normally distributed is inappropriate. The use of a distribution that can generate large innovations more often is recommended. A possible candidate is the Student-t distribution. It is interesting to note once again that all models seem to capture a similar structure for the conditional variance series. The excess kurtosis for a particular security is almost the same across the seven models.

Based on the selection criteria, and on the results of the specification tests presented in Table 7 to 9, the GJR specification appears to have many advantageous properties. Only for EAC and I.m. Skaugen does the model not rank among the three best based on BIC. The model is also relatively easy to estimate. Convergence was reached for all twelve securities. For the Orkla A, SCA B, Skandia, and Unidanmark A, the parameter α in the GJR model was estimated to a negative value. A very large positive residual might then give a negative conditional variance. However, in the four series, this does not occur, indicating that it might only be a theoretical problem. In none of the 76 models that were successfully estimated did any negative variance occur.

The LSTGARCH model is, as noted in Section 4.2, a generalization of the GJR model. In six cases, the estimate of the parameter θ is so high that

the two models coincide. As was the case with GJR, the estimated parameters for Orkla A, SCA B, and Skandia might in extreme cases give rise to negative conditional variance. The LSTGARCH is somewhat harder to estimate, and for Unidanmark A no convergence is reached. Based on these results, it is concluded that the GJR specification is a good approximation of the LSTGARCH model.

The TGARCH model appears to be a good complement to the GJR model. For six securities, the model ranks among the best three, both according to the values on the likelihood function and BIC. The model's simple structure makes estimation easy. However, for SCA B no convergence is reached. For Orkla A, and Unidanmark A, the parameter α^+ is estimated to a negative value. This allows negative conditional variances to appear for large positive return realizations, but this does not occur for the two series.

The results for the EGARCH model are slightly disappointing. In five cases, both according to the value of the likelihood function and BIC, the model ranks among the top three models, but is always outperformed by TGARCH. The model is fairly hard to estimate. For SCA A, and Unidanmark A, no convergence was reached. The persistence parameter β is often estimated to a value just below unity. Only in two cases is β estimated to values below 0.99.

Even though the GQARCH(1,1) model is extremely simple in its structure, it performs well. For ABB A and Volvo B, the model maximizes the likelihood function. It is among the top three models, according to BIC, for four securities. The model is easily estimated for all twelve securities.

The A-PARCH model maximizes the likelihood function for five securities, and ranks among the top three models in four cases, according to BIC. The model's very general structure therefore seems to capture the dynamics of the conditional variance well. However, one of the model's disadvantages is that it is hard to estimate. For successful estimation, the choice of start values is important. For four securities convergence could not be reached.

In three cases the VS-ARCH model ranks among the top three models, both according to the value of the likelihood function and BIC. The model is easily estimated and convergence is reached for all securities. Unfortunately, the rather different asymmetry structure of the model does not appear to be successful in the sample investigated.

4.6 Summary and Conclusion

This paper has presented results from an empirical investigation of 45 equity return series, from the Nordic stock exchanges in Helsinki, Stockholm, Oslo, and Copenhagen. The study investigated whether some asymmetric GARCH model might have been the data generating process of the series.

For this, two novel test procedures that are robust to non-normality were used. Evidence of asymmetry was found for twelve securities. It was also shown that the results of the proposed tests differ from those of non-robust tests. Thus, for this sample, using robust tests seems to be advisable.

For the twelve series that indicated asymmetries, the seven GARCH models, EGARCH, GJR, TGARCH, A-PARCH, GQARCH, VS-ARCH, and LSTGARCH, were estimated. These models all allow for correlation between the conditional variance and lagged returns. In 76 cases of 84 the models were successfully estimated. Using the value on the likelihood function, and the information criteria of Schwarz [1978], BIC, an attempt was made to identify which specifications modeled the conditional variance best. The models GJR, TGARCH, and GQARCH were found to be superior.

To investigate whether the seven models were able to capture the asymmetry present in the data, a robust version of the sign bias test was performed. Only in two cases of 76 could the hypothesis of no additional asymmetry be rejected. Finally, the skewness and kurtosis of the series of normalized residuals were calculated. These coefficients were very stable across the models. For seven securities the hypothesis of zero skewness could not be rejected. The hypothesis of no excess kurtosis was rejected for all 76 models. It is therefore concluded that an assumption that the distribution of z_t is the standard normal is most likely incorrect.

This study has focused on in sample properties for a number of different parametric GARCH models. This naturally give rise to the question of how these results can be used in a practical modeling situation. This will depend on the purpose of the exercise. GARCH models are commonly used by professionals in the option markets to forecast volatility for securities. If it is believed that the in sample properties of a model reflect the forecasting ability of the model, the results presented here are of major importance. For the practitioner, it should be comforting to note that the modeling performance of the relatively simple models, GJR, TGARCH, and GQARCH are at least as good as that of the more complicated models. These models are easier to estimate, and much easier to use for forecasting. However, the reader must be warned not to equate in sample properties with forecasting ability. This subject calls for further research.

In many other situations where GARCH models are used, the mean specification is more complex than the one used here. For econometricians working with such models, it is hoped that the methodological part of this paper is of interest. The use of the proposed tests for asymmetry is not limited to the simple model structure considered here. However, in this respect, further research into the small sample properties of the statistics is needed.

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Appendix

1. Test of H_0 : GARCH(1,1), H_1 : GQARCH(1,1)

The test considered is

$$\begin{aligned} H_0 & : \zeta = 0, \\ H_1 & : \zeta \neq 0. \end{aligned}$$

Hagerud [1997] shows that given that the residuals are distributed conditionally normal, a Lagrange multiplier test statistics for the hypothesis is

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\hat{\varepsilon}_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \beta'} \right\}' \left\{ \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta'} \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta'} \right]' \right\}^{-1} \\ & \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\hat{\varepsilon}_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \beta'} \right\}, \end{aligned} \tag{4.17}$$

where

$$\frac{\partial h_t}{\partial \beta'} = \left[\sum_{i=1}^{t-1} \hat{\beta}^{i-1}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^2, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{0t-i}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i} \right],$$

h_{0t} is the conditional variance under the null of GARCH(1,1), $\hat{\varepsilon}_t$ is the consistently estimated residual, β' is the vector of conditional variance parameters $(\gamma, \alpha, \beta, \zeta)$, and $\hat{\beta}$ is the estimated parameter β in the GARCH(1,1) model. Based on test (4.17) it is straightforward to derive the asymptotically equivalent test TR_u^2 from the regression

$$\left(\frac{\hat{\varepsilon}_t^2}{h_{0t}} - 1 \right)$$

on

$$\left\{ \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{0t-i}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}}{h_{0t}} \right\}. \tag{4.18}$$

If conditional normality fails to hold, (4.18) will have the wrong asymptotic size. In *Procedure 4.1* in Wooldridge[1991], the author shows how a regression-based specification test of this type can be robustified. The resulting test statistic will be valid under the rather general regularity conditions of Wooldridge [1990]. Using the method proposed by Wooldridge [1991], the robust version of test (4.18) is:

1. Regress

$$\frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}}{h_{0t}},$$

on

$$y_t = \left\{ \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1}}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^2}{h_{0t}}, \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{0t-i}}{h_{0t}} \right\}.$$

Let \hat{a} be the vector of estimated parameters. Form the series of residuals

$$x_t = \frac{\sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}}{h_{0t}} - \hat{a}y_t'.$$

2. Regress 1 on

$$\left(\frac{\hat{\varepsilon}_t^2}{h_{0t}} - 1 \right) \cdot x_t$$

The statistic is equal to $T \cdot R_u^2 = T - RSS$, from this last regression, which is equivalent to the statistic LM_1 presented in Section 4.3. The statistic converges in distribution to a χ^2 variable with one degree of freedom.

2. Test of $H_0: \text{GARCH}(1,1)$, $H_1: \text{LSTGARCH}(1,1)$

The test considered is

$$\begin{aligned} H_0 &: \alpha_2 = 0, \\ H_1 &: \alpha_2 \neq 0. \end{aligned}$$

Hagerud [1997] shows that given that the residuals are distributed conditionally normal, a Lagrange multiplier test statistics for the hypothesis is

$$\begin{aligned} & \frac{1}{2} \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\hat{\varepsilon}_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \beta} \right\}' \left\{ \sum_{t=1}^T \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right] \left[\frac{1}{h_{0t}} \frac{\partial h_t}{\partial \beta} \right]' \right\}^{-1} \\ & \left\{ \sum_{t=1}^T \frac{1}{2h_{0t}} \left[\frac{\hat{\varepsilon}_t^2}{h_{0t}} - 1 \right] \frac{\partial h_t}{\partial \beta} \right\}, \end{aligned}$$

where

$$\frac{\partial h_t}{\partial \beta'} = \left[\sum_{i=1}^{t-1} \hat{\beta}^{i-1}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^2, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{0t-i}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^3 \right].$$

To derive the robust test LM_2 presented in Section 4.3, the same method employed in the GQARCH case above was used.

3. $\partial g_t / \partial \tau$, and $\partial h_t / \partial \beta$ for the models (4.4) to (4.10)

When the model under the null is the EGARCH model (4.4), and β' is $(\gamma, \beta, \lambda, \varphi)$, the partial derivatives are

$$\begin{aligned} \frac{\partial g_t}{\partial \tau} &= S_{t-1}^- + h_t \sum_{i=2}^{t-1} \hat{\beta}^{i-1} h_{t-i}^{-1} S_{t-i}^-, \\ \frac{\partial h_t}{\partial \beta'} &= \left[h_t \sum_{i=1}^{t-1} \hat{\beta}^{i-1}, h_t \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \ln h_{t-i}, h_t \sum_{i=1}^{t-1} \hat{\beta}^{i-1} z_{t-i}, \right. \\ &\quad \left. h_t \sum_{i=1}^{t-1} \hat{\beta}^{i-1} (|z_{t-i}| - \sqrt{2/\pi}) \right]. \end{aligned}$$

When the model under the null is the GJR model (4.5), and β' is $(\gamma, \alpha, \omega, \beta)$, the partial derivatives are

$$\begin{aligned} \frac{\partial g_t}{\partial \tau} &= \sum_{i=1}^{t-1} \hat{\beta}^{i-1} S_{t-i}^-, \\ \frac{\partial h_t}{\partial \beta'} &= \left[\sum_{i=1}^{t-1} \hat{\beta}^{i-1}, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^2, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} S_{t-i}^- \hat{\varepsilon}_{t-i}^2, \sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{t-i} \right]. \end{aligned}$$

When the model under the null is the TGARCH model (4.6), and β' is $(\gamma, \alpha^+, \alpha^-, \beta)$, the partial derivatives are

$$\begin{aligned} \frac{\partial g_t}{\partial \tau} &= S_{t-1}^- + h_t^{1/2} \sum_{i=2}^{t-1} \hat{\beta}^{i-1} h_{t+1-i}^{-1/2} S_{t-i}^-, \\ \frac{\partial h_t}{\partial \beta'} &= \left[2h_t^{1/2} \sum_{i=1}^{t-1} \hat{\beta}^{i-1}, 2h_t^{1/2} \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^+, 2h_t^{1/2} \sum_{i=1}^{t-1} \hat{\beta}^{i-1} \hat{\varepsilon}_{t-i}^-, \right. \\ &\quad \left. 2h_t^{1/2} \sum_{i=1}^{t-1} \hat{\beta}^{i-1} h_{t-i}^{1/2} \right]. \end{aligned}$$

When the model under the null is the A-PARCH model (4.7), and β' is $(\gamma, \alpha, \eta, \beta, \delta)$, the partial derivatives are

$$\frac{\partial g_t}{\partial \tau} = S_{t-1}^- + h_t^{1-\delta/2} \sum_{i=2}^{t-1} \hat{\beta}^{i-1} h_{t+1-i}^{\delta/2-1} S_{t-i}^-,$$

$$\frac{\partial h_t}{\partial \beta'} =$$

$$\left[\frac{2h_t^{1-\delta/2}}{\delta} \sum_{i=1}^{t-1} \hat{\beta}^{i-1}, \frac{2h_t^{1-\delta/2}}{\delta} \sum_{i=1}^{t-1} \hat{\beta}^{i-1} (|\hat{\varepsilon}_{t-i}| - \eta \hat{\varepsilon}_{t-i})^\delta, \right.$$

$$\begin{aligned}
& 2\alpha h_t^{1-\delta/2} \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \{(|\widehat{\varepsilon}_{t-i}| - \eta \widehat{\varepsilon}_{t-i})^\delta \widehat{\varepsilon}_{t-i}\}, \frac{2h_t^{1-\delta/2}}{\delta} \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} h_{t-i}^{\delta/2}, \\
& \frac{2h_t^{1-\delta/2}}{\delta} \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \left(-h_{t+1-i}^{\delta/2} \ln h_{t+1-i}^{\delta/2} + 2\alpha(|\widehat{\varepsilon}_{t-i}| - \eta \widehat{\varepsilon}_{t-i})^\delta \right. \\
& \left. \ln(|\widehat{\varepsilon}_{t-i}| - \eta \widehat{\varepsilon}_{t-i}) + \beta h_{t-i}^{\delta/2} \ln h_{t-i}^{\delta/2} \right)
\end{aligned}$$

When the model under the null is the GQARCH model (4.8), and β' is $(\gamma, \zeta, \alpha, \beta)$, the partial derivatives are

$$\begin{aligned}
\frac{\partial g_t}{\partial \tau} &= \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} S_{t-i}^-, \\
\frac{\partial h_t}{\partial \beta'} &= \left[\sum_{i=1}^{t-1} \widehat{\beta}^{i-1}, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \widehat{\varepsilon}_{t-i}, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \widehat{\varepsilon}_{t-i}^2, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} h_{t-i} \right].
\end{aligned}$$

When the model under the null is the VS-ARCH model (4.9), and β' is $(\gamma, \alpha, \beta, \xi)$, the partial derivatives are

$$\begin{aligned}
\frac{\partial g_t}{\partial \tau} &= \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} S_{t-i}^-, \\
\frac{\partial h_t}{\partial \beta'} &= \left[\sum_{i=1}^{t-1} \widehat{\beta}^{i-1}, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \widehat{\varepsilon}_{t-i}^2, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} h_{t-i} \widehat{\varepsilon}_{t-i}, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} S_{t-i} v_{t-i}^2 \right].
\end{aligned}$$

When the model under the null is the LSTGARCH model (4.10), and β' is $(\gamma, \alpha_1, \alpha_2, \beta, \theta)$, the partial derivatives are -

$$\begin{aligned}
\frac{\partial g_t}{\partial \tau} &= \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} S_{t-i}^-, \\
\frac{\partial h_t}{\partial \beta'} &= \left[\sum_{i=1}^{t-1} \widehat{\beta}^{i-1}, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \widehat{\varepsilon}_{t-i}^2, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \widehat{F}(\widehat{\varepsilon}_{t-i} | \widehat{\theta}) \widehat{\varepsilon}_{t-i}^2, \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} h_{t-i}, \right. \\
& \left. \alpha_2 \sum_{i=1}^{t-1} \widehat{\beta}^{i-1} \frac{e^{-\theta \widehat{\varepsilon}_{t-i}}}{(1 + e^{-\theta \widehat{\varepsilon}_{t-i}})^2} \widehat{\varepsilon}_{t-i}^3 \right],
\end{aligned}$$

where $\widehat{F}(\widehat{\varepsilon}_t | \widehat{\theta})$ is the estimated value of the transition function at time t , under the null. When θ is estimated to a high value, such that the LSTGARCH model coincides with GJR model, the columns of the matrix $\partial \mathbf{h} / \partial \beta' = \partial(h_1, \dots, h_T)' / \partial \beta'$ might not be independent. To overcome this complication the partial derivatives of the GJR model should be used instead of those of the LSTGARCH model. This is the practice followed in this paper.

Table 1. Summary data for the investigated series

This table lists the 45 Nordic stocks investigated. The column labeled " $\bar{\mu}$ in %", reports yearly yield for the security during the investigated period of five years. The column labeled " σ in %", reports the estimated constant standard deviation in return, on a yearly basis. The period investigated is July 1, 1991 to July 1, 1996.

Security	Exchange	Industry	$\bar{\mu}$ in %	σ in %
ABB A	SE	Engineering	12.9	19.8
Aker B	NO	Engineering	5.0	50.5
Astra A	SE	Pharmaceuticals	24.8	24.5
Atlas Copco A	SE	Engineering	19.2	25.6
Bergesen d.y. B	NO	Shipping	-3.8	30.5
Carlsberg B	DK	Food & bev.	1.3	20.2
Cultor 2	FI	Food & bev.	40.2	44.1
Danisco	DK	Food & bev.	8.7	19.5
Den Danske Bank	DK	Financial	2.5	21.3
D/S 1912 B	DK	Shipping	5.2	21.1
D/S Svendborg B	DK	Shipping	5.6	20.2
EAC	DK	Shipping	-7.8	35.4
Electrolux B	SE	Household durables	4.2	28.8
Enso R	FI	Forestry	21.0	33.3
FLS Industries B	DK	Engineering	-2.0	25.1
Norsk Hydro	NO	Oil and chemical	11.3	25.2
Investor B	SE	Inv. company	10.3	33.6
ISS B	DK	Service	0.3	29.3
Jyske Bank	DK	Financial	0.7	18.3
Kvaerner B	NO	Engineering	3.4	29.9

Table 1. Continued

Security	Exchange	Industry	$\bar{\mu}$ in %	σ in %
LM Ericsson B	SE	Telecom	23.9	36.4
Modo B	SE	Forestry	5.6	43.9
Nokia	FI	Telecom	63.2	40.0
Novo Nordisk B	DK	Pharmaceuticals	12.0	15.0
Norske Skog B	NO	Forestry	3.6	43.4
Orkla A	NO	Food & bev.	14.0	29.7
Outokompu A	FI	Metal and mining	31.1	36.7
Pohjola B	FI	Insurance	8.6	47.9
Saga B	NO	Oil and chemical	-5.5	33.5
Sandvik B	SE	Engineering	16.6	24.7
SAS Danmark	DK	Transportation	17.6	37.9
SCA B	SE	Forestry	3.6	30.3
Skandia	SE	Insurance	0.9	36.7
S-E Banken A	SE	Financial	1.7	35.0
Handelsbanken A	SE	Financial	8.4	47.1
Skanska B	SE	Construction	4.3	38.5
I.m. Skaugen	NO	Shipping	-17.9	69.7
SKF B	SE	Engineering	7.4	30.4
Sophus Berendsen B	DK	Service	21.7	16.5
Stora A	SE	Forestry	2.1	31.3
Topdanmark	DK	Insurance	-9.5	24.2
Trelleborg B	SE	Metal and mining	-6.7	44.2
Unidanmark A	DK	Financial	-1.1	27.1
UPM Kymmene	FI	Forestry	15.5	35.1
Volvo B	SE	Automotive	18.3	30.3

Table 2. Results from tests of autocorrelation

This table reports results from tests performed to specify the conditional mean equation. Column two gives p-values for Richardson and Smith's [1994] test for autocorrelation, (4.11), calculated on demeaned returns. Column three reports p-values for the same statistic, but calculated on estimated residuals from an AR(1) model. Both statistics are calculated on eight autocorrelations, i.e. in formula (4.11) k is equal to eight.

Security	$RS(8)$ on \bar{r}_t (p-value)	$RS(8)$ on $\hat{\varepsilon}_t$ from AR(1) (p-value)
ABB A	0.107	-
Aker B	0.305	-
Astra A	0.501	-
Atlas Copco A	0.157	-
Bergesen d.y. B	0.462	-
Carlsberg B	0.137	-
Cultor 2	0.610	-
Danisco	0.005	0.320
Den Danske Bank	0.012	0.419
D/S 1912 B	0.417	-
D/S Svendborg B	0.752	-
EAC	0.075	-
Electrolux B	0.003	0.642
Enso R	0.807	-
FLS Industries B	0.031	0.797
Norsk Hydro	0.354	-
Investor B	0.442	-
ISS B	0.039	0.803
Jyske Bank	0.018	0.739
Kvaerner B	0.250	-

Table 2. Continued

Security	$RS(8)$ on \bar{r}_t (p-value)	$RS(8)$ on $\hat{\varepsilon}_t$ from AR(1) (p-value)
LM Ericsson B	0.609	-
Modo B	0.862	-
Nokia	0.333	-
Novo Nordisk B	0.005	0.674
Norske Skog B	0.870	-
Orkla A	0.455	-
Outokompu A	0.981	-
Pohjola B	0.345	-
Saga B	0.677	-
Sandvik B	0.146	-
SAS Danmark	0.062	-
SCA B	0.519	-
Skandia	0.214	-
S-E Banken A	0.827	-
Handelsbanken A	0.361	-
Skanska B	0.434	-
I.m. Skaugen	0.041	0.365
SKF B	0.031	0.707
Sophus Berendsen B	0.047	0.826
Stora A	0.047	0.941
Topdanmark	0.431	-
Trelleborg B	0.457	-
Unidanmark A	0.001	0.607
UPM Kymmene	0.688	-
Volvo B	0.206	-

Table 3. Results from tests of ARCH(q)

This table reports results from tests performed to specify the conditional variance equation. Column two reports p-values for Wooldridge's [1990] test for no ARCH(8), (4.13). Column three gives p-values for Engle's [1982] test of no ARCH, calculated on eight squared residuals. Column four reports the coefficient of excess kurtosis calculated on estimated residuals. Column five reports first-order autocorrelation in squared estimated residuals.

Security	no ARCH(8)			
	(p-value)	non-robust	$\kappa(\hat{\varepsilon})$	$\hat{\rho}(\hat{\varepsilon}_t^2, \hat{\varepsilon}_{t-1}^2)$
ABB A	0.001	0.000	5.2	0.089
Aker B	0.057	0.000	6.9	0.136
Astra A	0.049	0.001	1.8	0.122
Atlas Copco A	0.452	0.000	4.7	0.209
Bergesen d.y. B	0.001	0.000	3.7	0.101
Carlsberg B	0.237	0.000	3.2	0.150
Cultor 2	0.502	0.059	33.7	0.001
Danisco	0.002	0.000	4.7	0.162
Den Danske Bank	0.000	0.000	3.5	0.108
D/S 1912 B	0.012	0.000	3.0	0.158
D/S Svendborg B	0.160	0.000	4.4	0.125
EAC	0.000	0.000	7.6	0.278
Electrolux B	0.000	0.000	2.8	0.202
Enso R	0.306	0.070	4.4	0.048
FLS Industries B	0.011	0.054	7.2	0.087
Norsk Hydro	0.002	0.000	6.6	0.200
Investor B	0.024	0.000	14.7	0.188
ISS B	0.202	0.000	51.8	0.028
Jyske Bank	0.488	0.090	50.1	0.102
Kvaerner B	0.031	0.000	7.2	0.116

Table 3. Continued

Security	no ARCH(8)			
	(p-value)	non-robust	$\kappa(\hat{\varepsilon})$	$\hat{\rho}(\hat{\varepsilon}_t^2, \hat{\varepsilon}_{t-1}^2)$
LM Ericsson B	0.014	0.000	3.7	0.149
Modo B	0.040	0.000	13.4	0.281
Nokia	0.025	0.000	7.5	0.100
Novo Nordisk B	0.026	0.001	3.7	0.110
Norske Skog B	0.060	0.000	10.3	0.124
Orkla A	0.039	0.000	8.5	0.115
Outokompu A	0.455	0.000	38.9	0.150
Pohjola B	0.002	0.000	7.1	0.156
Saga B	0.073	0.000	9.4	0.153
Sandvik B	0.003	0.000	3.0	0.190
SAS Danmark	0.034	0.000	8.0	0.172
SCA B	0.034	0.000	7.8	0.182
Skandia	0.031	0.000	5.7	0.353
S-E Banken A	0.004	0.000	18.9	0.347
Handelsbanken A	0.000	0.000	12.4	0.365
Skanska B	0.006	0.000	14.1	0.081
I.m. Skaugen	0.000	0.000	8.4	0.186
SKF B	0.002	0.000	2.5	0.166
Sophus Berendsen B	0.072	0.094	4.1	0.088
Stora A	0.010	0.000	6.0	0.193
Topdanmark	0.156	0.000	9.2	0.294
Trelleborg B	0.010	0.000	5.1	0.185
Unidanmark A	0.000	0.000	2.9	0.132
UPM Kymmene	0.000	0.000	2.3	0.143
Volvo B	0.000	0.000	5.4	0.202

Table 4. Results from tests of no asymmetric GARCH

This table reports results from tests performed to specify the conditional variance equation. In column two, p-values for the robust test of GARCH(1,1) against GQARCH(1,1), are reported. Column three gives the p-values for the non-robust version of the same test, as formulated in Hagerud [1997]. In column four, p-values for the robust test of GARCH(1,1) against LSTGARCH(1,1), are presented. Column five reports the p-values for the non-robust version of the same test, as formulated in Hagerud [1997].

Security	LM_1	non-robust	LM_2	non-robust
ABB A	0.001	0.009	0.199	0.129
Astra A	0.068	0.053	0.061	0.170
Bergesen d.y. B	0.144	0.419	0.558	0.397
Danisco	0.235	0.018	0.875	0.385
Den Danske Bank	0.447	0.373	0.505	0.933
D/S 1912 B	0.101	0.120	0.109	0.274
EAC	0.043	0.280	0.029	0.313
Electrolux B	0.202	0.232	0.123	0.197
FLS Industries B	0.638	0.757	0.863	0.882
Norsk Hydro	0.215	0.414	0.412	0.241
Investor B	0.063	0.740	0.188	0.753
Kvaerner B	0.572	0.785	0.780	0.631
LM Ericsson B	0.069	0.004	0.748	0.377
Modo B	0.014	0.066	0.092	0.005
Nokia	0.986	0.495	0.307	0.275
Novo Nordisk B	0.960	0.937	0.608	0.795
Orkla A	0.000	0.010	0.784	0.181
Pohjola B	0.450	0.100	0.604	0.176
Sandvik B	0.545	0.887	0.251	0.207
SAS Danmark	0.907	0.904	0.962	0.955
SCA B	0.343	0.131	0.044	0.046
Skandia	0.001	0.000	0.017	0.000
S-E Banken A	0.884	0.323	0.779	0.602
Handelsbanken A	0.032	0.002	0.140	0.033
Skanska B	0.054	0.109	0.213	0.051
I.m. Skaugen	0.046	0.495	0.779	0.552
SKF B	0.040	0.163	0.706	0.400
Stora A	0.475	0.436	0.058	0.115
Trelleborg B	0.001	0.000	0.074	0.005
Unidanmark A	0.030	0.029	0.087	0.024
UPM Kymmene	0.852	0.835	0.425	0.681
Volvo B	0.042	0.022	0.351	0.425

Table 5. Asymmetric models ranked according to value on the likelihood function

Results from quasi-maximum likelihood estimation of the EGARCH, GJR, TGARCH, A-PARCH, GQARCH, VS-ARCH, and LSTGARCH models. The table reports the three models that gave the highest value on the likelihood function, for each security. The column labelled '# 1' reports the highest value, the following column the second highest, and the last column the third highest value on the likelihood function. In those cases where the GJR model gave the same value as the LSTGARCH model, the GJR model is ranked before the LSTGARCH model.

Security	# 1	# 2	# 3
ABB A	GQARCH	LSTGARCH	A-PARCH
EAC	A-PARCH	TGARCH	EGARCH
Modo B	LSTGARCH	A-PARCH	GJR
Orkla A	TGARCH	EGARCH	GJR
SCA B	LSTGARCH	GJR	VS-ARCH
Skandia	GJR	LSTGARCH	TGARCH
Handelsbanken A	A-PARCH	GJR	LSTGARCH
I.m. Skaugen	A-PARCH	TGARCH	EGARCH
SKF B	A-PARCH	TGARCH	EGARCH
Trelleborg B	A-PARCH	TGARCH	EGARCH
Unidanmark A	GJR	VS-ARCH	TGARCH
Volvo B	GQARCH	A-PARCH	VS-ARCH

Table 6. Asymmetric models ranked according to value on BIC

Results from quasi-maximum likelihood estimation of the EGARCH, GJR, TGARCH, A-PARCH, GQARCH, VS-ARCH, and LSTGARCH models. The table reports the three models that gave the lowest value on Schwarz [1978] information criteria, BIC, for each security. The column labelled '# 1' reports the lowest value, the following column the second lowest, and the last column the third lowest value on BIC.

Security	# 1	# 2	# 3
ABB A	GQARCH	GJR	LSTGARCH
EAC	TGARCH	EGARCH	A-PARCH
Modo B	GJR	LSTGARCH	A-PARCH
Orkla A	TGARCH	EGARCH	GJR
SCA B	GJR	LSTGARCH	VS-ARCH
Skandia	GJR	TGARCH	GQARCH
Handelsbanken A	GJR	GQARCH	A-PARCH
I.m. Skaugen	TGARCH	EGARCH	A-PARCH
SKF B	TGARCH	EGARCH	GJR
Trelleborg B	TGARCH	EGARCH	GJR
Unidanmark A	GJR	VS-ARCH	TGARCH
Volvo B	GQARCH	VS-ARCH	GJR

Table 7. Test for higher order asymmetric effects

Results from the robust sign bias test LM_3 . The null hypothesis is the estimated asymmetric model, and the alternative is the asymmetric model augmented by the term $\tau \cdot S_{t-1}^-$. The table reports p-values.

Security	Model						
	EG-ARCH	GJR	TG-ARCH	A-PARCH	GQ-ARCH	VS-ARCH	LSTG-ARCH
ABB A	0.0513	0.2988	0.1291	0.2943	0.7503	0.3796	0.2942
EAC	0.8562	0.6597	0.5102	0.3837	0.5978	0.7042	0.4380
Modo B	0.7269	0.0935	0.1814	0.4840	0.3291	0.0789	0.4255
Orkla A	0.3378	0.0658	0.3928	Failure	0.3872	0.4671	0.0658
SCA B	Failure	0.9657	Failure	Failure	0.0097	0.7256	0.9570
Skandia	0.0666	0.4562	0.2707	Failure	0.9582	0.5841	0.4560
Hand. A	0.0497	0.8977	0.2342	0.9787	0.1276	0.4146	0.8778
I.m. Ska.	0.6711	0.9206	0.6615	0.6001	0.9523	0.9090	0.9510
SKF B	0.9324	0.9389	0.8305	0.8095	0.5781	0.4076	0.9897
Trell. B	0.6900	0.8063	0.5120	0.6967	0.3664	0.8149	0.7987
Unid. A	Failure	0.2440	0.6663	Failure	0.2884	0.2556	Failure
Volvo B	0.3470	0.2796	0.3299	0.2968	0.9784	0.5343	0.2793

Table 8. Skewness for the series of normalized residuals

The table reports the estimated standardized third moment (skewness) for the series of normalized residuals, $\hat{\varepsilon}_t/h_t$. The critical value on five percent significance level, for the test of zero skewness against non-zero skewness is approximately, ± 0.14 . For underlined figures, the null of zero skewness is rejected on five percent significance level.

Security	Model						
	EG-ARCH	GJR	TG-ARCH	A-PARCH	GQ-ARCH	VS-ARCH	LSTG-ARCH
ABB A	-0.1233	-0.0698	-0.1202	-0.0699	-0.0649	-0.0914	-0.0698
EAC	<u>-0.6039</u>	<u>-0.7254</u>	<u>-0.5921</u>	<u>-0.6013</u>	<u>-0.7070</u>	<u>-0.7393</u>	<u>-0.7497</u>
Modo B	<u>0.1531</u>	<u>0.2104</u>	<u>0.1482</u>	<u>0.2142</u>	<u>0.1669</u>	<u>0.1670</u>	<u>0.2206</u>
Orkla A	<u>-0.3914</u>	<u>-0.4040</u>	<u>-0.3807</u>	Failure	<u>-0.4184</u>	<u>-0.4806</u>	<u>-0.4040</u>
SCA B	Failure	0.0688	Failure	Failure	0.0420	<u>0.1825</u>	0.0685
Skandia	0.0770	0.0462	0.0776	Failure	0.0827	0.0625	0.0462
Hand. A	0.0195	0.0794	0.0109	0.0875	0.0949	0.0605	0.0794
I.m. Ska.	<u>-0.8084</u>	<u>-0.7348</u>	<u>-0.8025</u>	<u>-0.7981</u>	<u>-0.7495</u>	<u>-0.5892</u>	<u>-0.7348</u>
SKF B	0.0377	0.0285	0.0293	0.0251	0.0364	0.0370	0.0376
Trell. B	-0.0077	-0.0425	0.0014	0.0023	-0.0408	-0.0613	-0.0424
Unid. A	Failure	-0.0291	-0.0660	Failure	-0.1297	0.0576	Failure
Volvo B	<u>0.2656</u>	<u>0.2819</u>	<u>0.2656</u>	<u>0.2834</u>	<u>0.2702</u>	<u>0.2693</u>	<u>0.2596</u>

Table 9. Excess kurtosis for the series of normalized residuals

The table reports the estimated standardized fourth moment minus three (coefficient of excess kurtosis), for the series of normalized residuals, $\hat{\varepsilon}_t/h_t$. The critical value on five percent significance level, for the test of zero excess kurtosis against non-zero excess kurtosis is approximately, ± 0.27 . The null can therefore be rejected for all series, and for all models.

Security	Model						
	EG-ARCH	GJR	TG-ARCH	A-PARCH	GQ-ARCH	VS-ARCH	LSTG-ARCH
ABB A	2.5803	2.3695	2.6115	2.3692	2.3737	2.5291	2.3693
EAC	7.3740	9.4624	7.0605	7.2255	9.3352	9.7064	9.7532
Modo B	2.3141	2.0073	2.3467	1.9888	2.4548	2.4052	1.7063
Orkla A	5.2144	5.4453	5.2131	Failure	5.6993	6.2284	5.4453
SCA B	Failure	1.7293	Failure	Failure	2.0850	2.2559	1.7287
Skandia	1.5634	1.6442	1.5368	Failure	1.4757	1.7601	1.6442
Hand. A	3.0726	3.1501	3.0572	3.1512	3.0998	3.3614	3.1501
I.m. Ska.	7.8845	7.3694	7.8293	7.7823	7.4495	6.4197	7.3693
SKF B	0.7471	0.7681	0.7761	0.7909	0.7752	0.7527	0.7460
Trell. B	1.4918	1.6209	1.4497	1.4483	1.6812	1.8034	1.6202
Unid. A	Failure	1.6002	1.9712	Failure	2.4035	1.6721	Failure
Volvo B	1.6459	1.5086	1.6408	1.5365	1.5083	1.4707	1.5087

5

Discrete Time Hedging of OTC Options in a GARCH Environment

5.1 Introduction

It is well known that the price of an option depends on the expected volatility of the underlying asset during the life of the option. For participants in the option markets, it is therefore essential to have reliable volatility predictions, and a great deal of effort is put into forecasting. Many empirical investigations have shown that the variance of returns can be modeled with GARCH processes.¹ Furthermore, volatility forecasts performed with GARCH have proven to be superior to other forecasting methods in a num-

¹A introduction to the GARCH literature and a summary of empirical investigations can be found in Bollerslev, Chou and Kroner [1992].

ber of studies.² GARCH models are therefore becoming popular among practitioners in the option markets.

When practitioners calculate prices of European options this, they often use formulas where the variance is assumed to be constant during the life of the option, e.g. the Black and Scholes [1973] formula. This practice is followed even though the agents believe that the return process is heteroskedastic. This is primarily done because of calculational convenience. The practitioners know that they are using the wrong model, but do so because they believe that the Black and Scholes formula gives them a good approximation to the option value calculated under an assumption of non-constant volatility.

This paper examines the effect of using the Black and Scholes formula for valuing options when the volatility plugged into the formula is forecasted with a GARCH(1,1) model and when the variance of the asset return in fact follows a GARCH process. It is assumed that there do not exist any other options with the same underlying security, or any other assets correlated with the volatility of the underlying security. This implies that it is impossible to vega-hedge the option, and that the risk-neutral valuation technique of Harrison and Kreps [1979] cannot be used. The problem considered can therefore be compared to the situation a financial intermediary faces when it has written an OTC option on a company's stock for which no exchange traded options exist, and when it wishes to keep the option position unexposed to any stock price movements. The intermediary is forced to continuously hedge its position and the only security it that can use for this purpose is the underlying security.³

The Black and Scholes formula is derived under the assumption that securities prices move continuously in time, and that hedging can be performed at each instant. The GARCH model, however, is defined in discrete time. The investigation perform here can therefore be regarded as a study

²The effectiveness of volatility forecasts performed with GARCH models has been tested before by e.g. Engle, Hong, Kane and Noh [1993], Engle, Kane and Noh [1993], and Noh, Engle and Kane [1994]. Engle, Hong, Kane and Noh [1993] compared the profitability of using GARCH(1,1) forecasts relative to some other forecasting methods. They find the GARCH(1,1) forecasts were significantly superior for valuing one-day options on the NYSE index during the period 1962 to 1989. Engle, Kane and Noh [1993] perform a similar study, with the difference that the options priced are allowed to have longer maturities than one day. In the test, NYSE-index data from 1968 to 1991 is used. They conclude that pricing of NYSE index options of up to 90 days maturity is more accurate when a GARCH model is used. Noh, Engle and Kane [1994] report similar results when a GARCH forecast is compared to a forecast made from implied volatilities. The data used is S&P 500 index from 1986 to 1991. The evidence in favour of GARCH, however, is not unanimous. Heynen and Kat [1994] show that for a number of stock index series, a *stochastic volatility* model outperforms two GARCH specifications in volatility prediction. The period investigated is 1980 to 1992. A survey of the stochastic volatility class of models is found in Ghysels, Harvey, and Renault [1996].

³The interest rate is assumed to be constant, i.e. no interest risk is considered.

on how well a continuous time pricing formula works in a discrete time environment. This problem has previously been investigated by e.g. Bossaerts and Hillion [1995]. However, their approach is quite different to the one employed here.

This paper does not focus on the issue of the relative effectiveness of GARCH volatility predictions compared to other forecasting methods. Instead, the investigation is on how well the combination of the Black and Scholes formula and GARCH volatility forecasts performs compared to using the Black and Scholes formula when the volatility is constant. A simulation procedure is followed where the variance process, and therefore also the return process is created with a random number generator. The data is generated under the assumption that the variance follows a GARCH(1,1) process. To test the effectiveness of the combination of the Black and Scholes formula and GARCH volatility forecasts, the return on a complete option writing strategy is calculated, complete in the sense that both the initial pricing of the option and the delta-hedging strategy until maturity is included. The delta-hedging is performed on equally spaced time intervals, during the entire life of the option. Deltas are calculated with the Black and Scholes formula using a GARCH volatility forecast. Every day a new volatility forecast is made. Given this volatility forecast, a new delta is calculated and a new delta-hedging decision is made. Transaction costs are assumed to be zero. The simulation is repeated many times to give an estimate of expected return from the tested strategy.

The simulations show that the variance of profits in the heteroskedastic environment is larger than when the volatility is constant. The dispersion increases with the maturity of the option. The average profit is almost the same in both the homo- and heteroskedastic case. The effectiveness of the hedging is largely affected by, (1) the level of kurtosis in the return process, and (2) the first-order autocorrelation in centered and squared returns.

The GARCH(1,1) model is presented in Section 5.2 which also describes how volatility forecasts are performed. Section 5.3 discusses the problems concerning the pricing of options in a discrete time heteroskedastic world. Hedging techniques in discrete time under non-constant volatility are described in Section 5.4. In Section 5.5 the simulation procedure is presented. The results are given in Section 5.6 and conclusions in Section 5.7.

5.2 The GARCH(1,1)-t Model and Volatility Forecasting

The ARCH family of models was first introduced by Engle [1982] and further developed into Generalized ARCH, GARCH, by Bollerslev [1986]. As mentioned by Bollerslev, Chou and Kroner [1992], one of the most commonly used GARCH models is the GARCH(1,1) model. In the GARCH(1,1)

model the conditional variance of asset return at time t , h_t , obeys the process

$$h_t = \gamma + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}, \tag{5.1}$$

where ε_t is the error term in the return process that will be assumed to be⁴

$$r_t = \varphi + \varepsilon_t.$$

The error term is assumed to have the form

$$\varepsilon_t = z_t h_t^{1/2}, \tag{5.2}$$

where z_t is i.i.d. with expected value zero and variance, μ_2 , equal to unity. Most often in the literature z_t is assumed to be normally distributed.

For the variance process to be stationary it is required that $\alpha + \beta < 1$. Furthermore, it is required that $\gamma > 0$, $\alpha > 0$ and $\beta \geq 0$, for the variance process to be positive and the GARCH(1,1) process to be defined.⁵ Teräsvirta [1996] shows that for any $z_t \sim \text{i.i.d.}(0, \mu_2)$ for which $\mu_4 \equiv E[z_t^4] < \infty$ and $\gamma_2 \equiv E[(\beta + \alpha z_t^2)^2] < 1$, the kurtosis of ε_t is given by

$$\kappa_\varepsilon = \kappa_z \frac{1 - \gamma_1^2}{1 - \gamma_2}, \tag{5.3}$$

where κ_z is the kurtosis of the process z_t , and $\gamma_1 \equiv E[(\beta + \alpha z_t^2)]$. The autocorrelation function of squared ε_t for any $z_t \sim \text{i.i.d.}(0, \mu_2)$ for which $\mu_4 = E[z_t^4] < \infty$ and $\gamma_2 = E[(\beta + \alpha z_t^2)^2] < 1$, following Teräsvirta [1996], is equal to

$$\rho(\varepsilon_t^2, \varepsilon_{t-p}^2) = \frac{\mu_2 \gamma_1^{p-1} \alpha (1 - \beta^2 - \alpha \beta \mu_2)}{1 - \beta^2 - 2\alpha \beta \mu_2} \quad p = 1, 2, \dots \tag{5.4}$$

Below it is assumed that z_t is distributed Student-t, with ν degrees of freedom, denoted $z_t \sim t(0, 1; \nu)$.⁶ The error term, ε_t , will therefore be conditionally distributed Student-t with expected value zero and variance h_t . The reason for choosing t-distributed innovations, instead of Gaussian, is

⁴In a number of empirical investigations, the return process is estimated as being autoregressive. For example Noh, Engle, Kane [1994] specify the return process as being AR(1). Since an autoregressive return process further complicates option valuation, such return processes will not be considered here. Furthermore, no consideration is given to a conditional mean specification where the conditional variance enters as an explanatory variable.

⁵For a rigorous description of the GARCH(1,1) process see Nelson [1990a] and Teräsvirta [1996].

⁶The GARCH model with conditionally t-distributed errors, denoted the GARCH-t model, was introduced in Bollerslev [1987].

that a Student-t distribution is more likely to generate returns that resemble empirically observed high-frequency financial time series. Teräsvirta [1996] shows that the GARCH(1,1) model with normal errors is unable to generate data with the level of kurtosis and pattern of first-order autocorrelation in squared residuals empirically observed.

In this case, when $z_t \sim t(0, 1; \nu)$

$$\kappa_z = \mu_4 = \frac{3(\nu - 2)}{(\nu - 4)} \quad \nu > 4 \tag{5.5}$$

$$\gamma_1 = \beta + \alpha \tag{5.6}$$

$$\gamma_2 = \beta^2 + 2\alpha\beta + \alpha^2\kappa_z. \tag{5.7}$$

Formula (5.3) then implies that the kurtosis of ε_t is equal to

$$\kappa_\varepsilon = \kappa_z \frac{1 - (\beta + \alpha)^2}{1 - \beta^2 - 2\alpha\beta - \alpha^2\kappa_z}, \tag{5.8}$$

given that $\nu > 4$ and $\gamma_2 < 1$.

Furthermore, the autocorrelation of $\{\varepsilon_t^2\}$ when $z_t \sim t(0, 1; \nu)$ is, according to formula (5.4)

$$\rho(\varepsilon_t^2, \varepsilon_{t-p}^2) = \frac{(\alpha + \beta)^{p-1} \alpha (1 - \beta^2 - \alpha\beta)}{1 - \beta^2 - 2\alpha\beta} \quad p = 1, 2, \dots \tag{5.9}$$

From equation (5.9) it can be concluded that the autocorrelation function of squared residuals will be independent of the kurtosis of the i.i.d. process.

One important characteristic of the GARCH model is that the unconditional variance is constant. In the GARCH(1,1) the unconditional variance, σ^2 , is equal to

$$\sigma^2 = \frac{\gamma}{1 - \alpha - \beta}. \tag{5.10}$$

Given the distributional assumptions of z_t , the parameters of the model can be estimated by maximum likelihood. In the case z_t is distributed Student-t the parameters of the model will be: φ , α , β , γ , and ν . If the model is to be used for prediction of volatility for asset returns, the estimations are most often done with daily observations. Engle, Kane and Noh [1993] show that for the NYSE index, a suitable sample size consists of 1000 observations.

Given estimated parameters $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$, the one-day-ahead conditional variance forecast, $\hat{h}_{t+1|t}$, is calculated as

$$\hat{h}_{t+1|t} = \hat{\gamma} + \hat{\alpha}\varepsilon_t^2 + \hat{\beta}h_t,$$

and the s -day-ahead forecast is

$$\begin{aligned} \hat{h}_{t+s|t} &= \hat{\gamma} + \hat{\alpha}E[\varepsilon_{t+s-1}^2 | h_t] + \hat{\beta}\hat{h}_{t+s-1|t} = \\ &= \hat{\gamma} + (\hat{\alpha} + \hat{\beta})\hat{h}_{t+s-1|t}. \end{aligned} \tag{5.11}$$

In option valuation the average variance over a specific time interval is often of interest. The average predicted variance during the period from $t + 1$ until $t + \tau$ is calculated as

$$\widehat{v}_{t,\tau}^2 = \frac{1}{\tau} \sum_{i=1}^{\tau} \widehat{h}_{t+i|t}. \quad (5.12)$$

When this forecast is used in connection with option valuation, it is often rewritten in standard deviation p.a. using the formula

$$\sqrt{250 \widehat{v}_{t,\tau}^2}, \quad (5.13)$$

where 250 represents the number of trading days per year.

Recursive substitution on equation (5.11) gives

$$\begin{aligned} \widehat{h}_{t+s|t} &= \widehat{\gamma} + (\widehat{\alpha} + \widehat{\beta}) \left(\widehat{\gamma} + (\widehat{\alpha} + \widehat{\beta}) \widehat{h}_{t+s-2|t} \right) \\ &= \widehat{\gamma} \left[\sum_{i=0}^{s-2} (\widehat{\alpha} + \widehat{\beta})^i \right] + (\widehat{\alpha} + \widehat{\beta})^{s-1} \widehat{h}_{t+1|t}. \end{aligned} \quad (5.14)$$

From equations (5.14) and (5.10) it can be concluded that

$$\widehat{h}_{t+s|t} \rightarrow \frac{\widehat{\gamma}}{1 - \widehat{\alpha} - \widehat{\beta}} = \widehat{\sigma}^2 \text{ as } s \rightarrow \infty. \quad (5.15)$$

Thus the variance forecast asymptotically approaches the estimated unconditional variance, $\widehat{\sigma}^2$. The predicted variance for a period far ahead of the current period will therefore be close to the unconditional variance. This is illustrated in Figure 1. Given equations (5.12) and (5.15), it is clear that the average volatility for long periods will be close to $\widehat{\sigma}^2$. The length of the forecast needed for the prediction to be close to the unconditional variance depends on the size of $(\widehat{\alpha} + \widehat{\beta})$. If $(\widehat{\alpha} + \widehat{\beta})$ is close to unity, the variance process moves slowly towards $\widehat{\sigma}^2$. If $(\widehat{\alpha} + \widehat{\beta})$ is low the variance process will be close to $\widehat{\sigma}^2$ most of the time. When the true values of α and β are low, the behavior of the process will be similar to that of a homoskedastic process.

In some studies (see e.g. Engle and Bollerslev [1986]), $(\alpha + \beta)$ has been estimated to values indistinguishable from unity. This causes the variance process to be non-stationary. This observation motivated the integrated GARCH (IGARCH) model of Engle and Bollerslev [1986]. In the simulations below, $(\alpha + \beta)$ will in some cases be close to one. Still, problems concerning non-stationarity in the variance process will not be dealt with. Furthermore, only processes where the kurtosis of the return process exists will be studied.

A number of empirical investigations have found a negative correlation between the variance and the return on some assets.⁷ This is one of the motivations for the exponential GARCH (EGARCH) model proposed by Nelson [1991]. From the specification of the GARCH(1,1) process (5.1) one can conclude that the conditional variance is independent of the path of the stock price. It is only the absolute value of the error term that affects the variance. The issue of correlation between the volatility and sign of the return is therefore not addressed in this study.

5.3 The Pricing of Options in a GARCH(1,1) Environment

When GARCH(1,1) volatility forecasts are used for options valuation, one common practice is that the predicted average volatility, calculated with equation (5.13), is plugged into the Black and Scholes [1973] formula, see e.g. Engle, Hong, Kane and Noh [1993].⁸ The purpose of this study is to evaluate how this practice affects the pricing and hedging of options. From the description in the previous section, it is evident that two of the crucial assumptions used in the Black and Scholes derivation are violated in the GARCH(1,1) world. First, in the GARCH(1,1) model, world time is discrete. Second, volatility is not constant. In this section, a few comments will be made regarding these two violations.

In this study it will be assumed that stock prices move at discrete moments in time. No assumptions will be made concerning the preferences of the agents, except that they prefer more to less. If the risk-neutral valuation techniques of Harrison and Kreps [1978] should be used, it is required that agents at all times can hedge themselves perfectly against all risk factors. In the GARCH(1,1) model, the conditional variance of the stock in the next period is deterministic, as can be seen from equation (5.1). Therefore the only risk factor to consider is the stock price movement, i.e. the delta of the option. However, since the stock price on the next discrete point in time can take an uncountable number of values, it is impossible to hedge the price risk perfectly. This is a general problem in all discrete time models which are not of the binomial type. Thus, even if the variance were constant, arbitrage arguments alone could not be used to price options. To price contingent claims in an environment where prices move discretely and where the stochastic process can give rise to more than the two pos-

⁷See Nelson [1991] for a short survey.

⁸This practice can be motivated in some way by a result presented by Hull and White [1987]. They show that when the price of an asset and its instantaneous variance follow two independent geometrical Brownian motions, the price of a European option on the asset will be equal to the Black and Scholes price, with the variance set equal to the average variance during the life of the option.

sible prices in the next period, behavioral assumptions about the agents in the economy need to be made. More precisely, the agents attitude toward risk needs to be known. This approach is taken by Duan [1995] when he develops an option pricing model in a discrete time GARCH(1,1) environment. To obtain an option pricing expression, Duan has to make very strong assumptions about the utility functions of agents and /or about the distribution of aggregate consumption in the economy.

Instead of assuming that stock prices move discretely, the discrete events in the GARCH(1,1) model could be viewed as representing a countable sample from the uncountable event space generated by a continuous stochastic process. For the sake of argument, assume that this stochastic process is of conceptually the same form as the GARCH process, such that there is only one source of risk in the economy and that the non-constant variance is given by a deterministic function of the past price trajectory. If that were the case, and hedging could be performed continuously, the model would be complete, and risk-neutral valuation techniques could be used to price contingent claims. Generally, however, nothing is known of the form of the continuous process that, when sampled once a day, generates data with GARCH(1,1) characteristics. Moreover, since the focus here is on the GARCH(1,1) process in discrete time, this insight cannot help to solve the problem considered in this paper. A topic related to these issues is the estimation of the instantaneous conditional variance in a diffusion process by a GARCH model estimation. This problem is thoroughly investigated by Nelson [1992].

Before ending this section, it might be appropriate to make some mention of the consequences of letting the length of the time steps in the GARCH model approach zero. This issue was first considered by Nelson [1990b]. In his article, Nelson presents tools for investigating the relationship between stochastic difference equations and Ito processes. Nelson then applies these techniques to two examples of ARCH models, the GARCH(1,1) model and the AR(1)-EGARCH model of Nelson [1991]. The continuous time limit of the GARCH(1,1) model is shown to have a process for the conditional variance with a stationary distribution that is the inverted gamma.

5.4 Hedging of Options in a GARCH(1,1) Environment

The previous section discussed pricing of options and it was noted, as done previously in the literature, that options cannot be priced using risk-neutral

techniques when heteroskedasticity is present.⁹ In this section, a few words will be said about hedging in the GARCH(1,1) environment.

Hedging an option entails reducing exposure to different risk factors. The factors typically correspond to different random variables. A *hedge portfolio* has zero exposure to some or all of these factors. If the random variables move continuously, the hedging should ideally be performed continuously. If the random variables move discretely, hedging should be done at each time step.

The amount of hedging needed to reduce the risk of each factor can be expressed using *hedge parameters*. If the variance of the underlying asset is assumed to be constant, only one hedge parameter, the option delta, needs to be considered. The delta used in this context will henceforth be called the Black and Scholes delta, Δ_{BS} . Δ_{BS} can be calculated as the first derivative of the Black and Scholes formula with respect to the underlying security price¹⁰

$$\Delta_{BS} = \frac{\partial FBS}{\partial S}.$$

If it is assumed that the volatility may change once and then stay at the new level until the expiration of the option, the Black and Scholes vega can be used as a hedge parameter

$$\Lambda_{BS} = \frac{\partial FBS}{\partial v}.$$

Further, the Black and Scholes gamma can be written

$$\Gamma_{BS} = \frac{\partial^2 FBS}{\partial S^2}.$$

When the volatility is stochastic, extended hedge parameters have to be developed. These can be derived using Taylor series expansion. Following Engle and Rosenberg [1994], the discrete time delta is then given by

$$\Delta_{SV} = \Delta_{BS} + \Lambda_{BS} \frac{1}{2\hat{v}} \frac{\partial \hat{v}^2}{\partial S}, \quad (5.16)$$

where \hat{v} denotes the forecasted variance. In the GARCH(1,1) environment, when the first derivative of variance with respect to price is zero, the stochastic volatility delta will, according to equation (5.16), be equal to the Black and Scholes delta, $\Delta_{SV} = \Delta_{BS}$. This will also be true for gamma, i.e. $\Gamma_{SV} = \Gamma_{BS}$.

To create a delta-neutral hedge in the GARCH(1,1) environment, the Black and Scholes delta can be used. If an agent is short the option, the

⁹One can use risk-neutral methods if there exists a traded security that is perfectly correlated with the volatility, but this is generally not the case.

¹⁰Note that in a discrete environment, the hedge parameters will not give a perfect hedge but rather one which on average reduces risk.

hedging is done by buying an amount delta of the underlying security. Since vega and gamma for the underlying security is equal to zero, the underlying security cannot be used for vega- and gamma-hedging. To hedge these two factors, other derivative securities with non-zero vega and gamma are needed. In this study, it is assumed that no other derivative securities exist, and consequently no vega and gamma hedging can be performed.

5.5 Simulation Experiment

As noted in the introduction, the object of this study is to try to evaluate the effectiveness of using GARCH(1,1) volatility prediction in combination with the Black and Scholes formula, when the return process is heteroskedastic. This is attempted by using a simulation procedure. It will be assumed that the variance of return follows a GARCH(1,1) process and that the agents know the true parameters of the process. The simulation procedure is performed in the following steps:

1. At time zero, a hypothetical trader writes a European call option. Different maturities and strike prices are considered. The premium is deposited at the risk free rate, assumed to be constant and equal to zero. The option is priced using the Black and Scholes formula. The volatility is set equal to a forecasted average volatility. The forecast is performed using a GARCH(1,1) model. The parameters of the model (5.1) will be

$$\begin{aligned}\alpha &= 0.0204 \\ \beta &= 0.9700 \\ \gamma &= 4.31 \cdot 10^{-7} \\ \nu &= 5\end{aligned}$$

The level of conditional volatility at date zero is set randomly. This is done by letting a random number generator create a return series for 250 days, assuming that the return follows the specified heteroskedastic process. At the beginning of this return series, the conditional variance is set to the unconditional variance. The last conditional variance in the series is the level assumed to prevail when the option is written.

2. After having written the option, the trader hedges the position by buying Δ_{BS} of the underlying security, where the delta is calculated with the Black and Scholes formula using the forecasted volatility.

3. Given the conditional variance for the next period, $h_{t+1|t}$, the return for the next period is generated given its distribution, $r_{t+1} \sim t(\hat{\varphi}, h_{t+1|t}; 5)$.¹¹ φ is chosen to be zero for convenience.
4. From the new stock price calculated in step 3, a new delta is calculated.¹² If this delta is different from the previous one, the hedge will be adjusted by either selling or buying the underlying security.
5. Step 3 and 4 are repeated four times every day, i.e. the stock moves four times per day and hedging is performed each time the stock price moves. Note that the conditional variance and therefore also the volatility forecast is constant during each day.
6. When one day has passed the error term, ε_t , for that day is calculated. Given this error term and given the current conditional variance, the conditional variance for the next day is calculated using equation (5.1). From this new conditional variance, the stock movements for the following day can be generated. Furthermore, a new volatility forecast is calculated.
7. Steps 3 to 6 are repeated until the option expires when the position is closed.
8. All the cash flows generated in the steps above are summarized and recorded.
9. The simulation is repeated 1000 times to give an estimate of the expected value and the variance of the return from the investigated strategy.

It should be noted that even if the price of the underlying asset does not change over time, the option price will. It will therefore be necessary to hedge the position even when the price of the underlying asset is unchanged. However, this change will usually be minimal.

The parameter values used in the simulation have been obtained from an estimation on 900 daily returns for the S&P 500 index from November 6, 1991 to July 11, 1995.¹³ In the estimation, the degrees of freedom were estimated to 4.6, but for simplicity ν is set to an integer value. The parameters give the unconditional variance $4.49 \cdot 10^{-5}$, corresponding to a standard deviation p.a. equal to 10.6 percent. The kurtosis implied by the

¹¹Note that in the GARCH(1,1) model the forecasted conditional variance, $\hat{h}_{t+1|t}$, is equal to the true conditional variance, h_{t+1} .

¹²When the deltas are calculated during the day, the time to expiration is specified in days and fractions of a day. This proved to be important for obtaining correct deltas when the time to expiration was short.

¹³I am deeply indebted to Tobias Rydén and Stefan E. Åsbrink for providing us with the estimated parameters.

parameters is 10.90, which is calculated using formulas (5.5) and (5.8). Furthermore, the first autocorrelation of squared residuals will, according to formula (5.9), be 0.04. Even though $\rho(\varepsilon_t^2, \varepsilon_{t-1}^2)$ is relatively low, the level of kurtosis and the level of autocorrelation are similar to values obtained in empirical investigations, as can be seen in Teräsvirta [1996].

The results obtained in the simulation procedure are compared to results obtained in simulations where the return process is homoskedastic. These simulations also follow the eight steps specified above, but with constant volatility. Returns are generated from a model where the error term is distributed Student-t with the same degrees of freedom as in the heteroskedastic case. The variance is held constant and equal to the unconditional variance for the GARCH(1,1) process. When the options are priced and when the deltas are calculated, the true volatility is used in the Black and Scholes formula. The level of kurtosis in these returns is equal to that of z_t , which can be calculated using formula (5.5). Thus, when ν is equal to five, the kurtosis will be equal to nine.

5.6 Results

Simulations with options of six different maturities have been performed: 1M, 2M, 3M, 4M, 5M, and 6M. The results from simulations with at-the-money call options are presented in Table 1.¹⁴ In the table, rows labelled GARCH(1,1) give the results from simulations with heteroskedastic return processes, and rows labelled Homoskedastic give the results from simulations where the variance of returns is constant. The first two lines report average profits of the hedging procedure as percentage of the initial stock price. The reported values indicate that the average profit is almost the same for the homo- and heteroskedastic cases, and close to zero. This agrees with the finding of Engle and Rosenberg [1994] that $\Delta_{SV} = \Delta_{BS}$.¹⁵

The rows labelled Standard deviation in Table 1 report the dispersion of profits as percentage of initial stock price. In the constant volatility case, the standard deviation is at the same level for all maturities, whereas the standard deviation in the GARCH(1,1) cases increases with the time to maturity. For the 1M option, the standard deviation is 10 percent higher in

¹⁴Since the interest rate is equal to zero, the at-the-money options will also be at-the-money-forward.

¹⁵Engle and Rosenberg [1995] examine whether $\Delta_{SV} = \Delta_{BS}$ is also true in practice. They estimate risk factors using Monte Carlo techniques when the variance is assumed to follow a GARCH process. Their study is performed on data from four different markets: S&P 500 index, bond index futures, weighted foreign exchange rate index, and oil futures. They assume that the error term is distributed student-t, and allows the variance to be negatively correlated with the level of return. They find that GARCH deltas are similar to those calculated with the Black and Scholes formula. GARCH gammas are found to be significantly higher than the Black and Scholes gammas.

the GARCH(1,1) case than in the constant volatility case. This difference increases to 32 percent for the 2M option, and further to 95 percent for the 6M option.

Note that the figures reported in Table 1 are based on profits in percent of initial stock prices. It might appear to be more natural to report the figures in percent of the initial option prices. However, since the initial option prices will vary with each iteration in the GARCH(1,1) case, such a procedure was found to be less suitable. To give an indication of how the figures compare to the option prices, the initial option prices in the homoskedastic case are given on the last row of Table 1. These prices are also the median prices in the heteroskedastic case. The figures show that the standard deviations reported are relatively large compared to the option prices. For example, the standard deviation of 0.20 percent reported for the 3M option in the homoskedastic case, constitutes 9 percent of the option price.

Figures 2 to 4 plot the profits from the simulations performed on options with maturities of one, three and six months, respectively. The graphs clearly show that the variance of return increases with the maturity of the option.

Since the returns generated in the constant variance case are distributed Student-t, the standard deviation would be expected to be larger in these simulation than in simulations done with normally distributed returns. To test this, a simulation was performed for a three month option with normal errors. The variance of returns was also in this case set equal to the unconditional variance of the GARCH(1,1) process. The kurtosis of this return series, like the kurtosis of any normally distributed variable will be equal to three. The standard deviation in this case was equal to 0.11, which should be compared to 0.20 for the 3M option in the Student-t case. It can therefore be concluded that with Student-t distributed errors, the standard deviation in profits is almost double that in the case with normal errors.

Table 2 contain the results from four simulations done with call options that are in-the-money and out-of-the-money. The maturity of the options is three months. The in-the-money options are 10 percent in the money and the out-of-the-money options are 10 percent out of the money. The results have a similar pattern as in the at-the-money case. For the in-the-money options, the standard deviation is 100 percent higher in the GARCH(1,1) simulation than in the constant volatility simulation. The difference for the out-of-the-money option is 171 percent. The difference in standard deviation is therefore larger in these cases than in the at-the-money case, where the difference is 50 percent for the three month options. However, it should be noted that the *level* of standard deviation is lower when the option is not written at-the-money.

The test of this section, will present results from simulations done for testing how the standard deviation of profits is dependent on the parameter

values in the GARCH(1,1)-t model. The simulations are done with European at-the-money call options with an initial maturity of three months.

The first test determines how sensitive the results are to the level of the parameter γ . Table 3 contains these results. The first two lines simply restate the results given in Table 1, whereas the last two lines give the results from two simulations done where γ is twice as high as before. Since α and β are unchanged, the kurtosis and the autocorrelation in squared residuals will be unchanged. However, the increase in γ will double the unconditional variance. As expected, the increase in the unconditional variance increases the standard deviation both in the stochastic case and in the constant volatility case. However, the relative difference is almost constant, at 50 percent when γ is equal to $4.31 \cdot 10^{-07}$, and at 53 percent when γ is equal to $8.62 \cdot 10^{-07}$. This finding is of particular interest to agents in markets where volatility is much higher than for the S&P 500 index.

Second, tests how sensitive the results are to the level of kurtosis in the return process are performed. The results from these simulations are shown in Table 4. Row one restates the results from Table 1. The following two rows show the results when the level of kurtosis is lowered by increasing the degree of freedom in the Student-t distribution. In the last row, the degree of freedom is equal to infinity, which implies that the innovations are in fact normally distributed. Since α and β are unchanged, the autocorrelation in squared residuals will also be unchanged. As can be seen in the last column, the standard deviation in profits decreases as the level of kurtosis falls.

Third, it is tested how the level of first-order autocorrelation in squared residuals influences the distribution of profits. This has been done in a number of simulations where the parameter values have been changed such that the first-order autocorrelation in squared residuals has increased, but leaving the unconditional variance and kurtosis unchanged. The results from these simulations are shown in Table 5. The values in the last column indicate that the dispersion of profits increases when the level of autocorrelation in squared residuals increases, though not monotonically. This result is of major importance since, as noted above, the level of first-order autocorrelation in the original simulations, 0.04, is relatively low.

Finally, it is investigated how the level of persistence in the conditional variance, measured by $\alpha + \beta$, affects the distribution of profits. In the simulations above, $\alpha + \beta$ has always been close to 0.99. That will give a half-life for the conditional variance of approximately 70 trading days. Table 6 shows the results from the original simulation and three other simulations where the parameters values are changed so that the level of persistence has decreased. The half-life on the last three rows are approximately 15, 11, and 10 days, respectively. As can be seen in columns six and seven, the level of kurtosis and autocorrelation in squared residuals are kept almost constant. The results presented in Table 6 indicate that the level of persistence in the conditional variance have no major effect on the standard deviation in profits.

5.7 Conclusion

The objective of this simulation study has been to investigate if it is recommendable to follow the practice of calculating option prices and deltas with the Black and Scholes formula when the return process follows a GARCH(1,1) process. In Section 5.6 it was shown that the variance of return was higher when the return followed a GARCH(1,1) process rather than a constant volatility process. The average profit, however, was almost the same in both cases. The results for in-the-money options and out-of-the-money options pointed in the same direction. Furthermore, it is shown that the dispersion of profits increases with the level of kurtosis and the level of first-order autocorrelation in squared residuals.

The findings mentioned above imply that an agent who is not risk-neutral and believes that the return follows a GARCH(1,1) process must compensate by charging a higher premium when writing options. Two good indicators of the size the risk in the hedging are, (i) the kurtosis and (ii) the level of first-order autocorrelation in squared residuals.

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Table 1. Results for at-the-money call options

The table shows results from simulations done with options that initially were at-the-money. 1000 simulations are performed for each option investigated. The rows labeled Mean return report average profit as percentage of the initial stock price, and the rows labeled Standard deviation report the dispersion of profits as percentage of initial stock price. The rows labeled GARCH(1,1) give the results from simulations with heteroskedastic return processes, with parameter values $\alpha = 0.0204$, $\beta = 0.970$, $\gamma = 4.31 \cdot 10^{-7}$ and $\nu = 5$. The rows labeled Homoskedastic report the results from simulations with a constant volatility return process. The last row reports initial option prices calculated with the Black and Scholes formula in the homoskedastic case. The number of trading days for the different maturities is: 21, 42, 63, 83, 104, and 125.

	Variance	Maturity of option					
	Procees	1M	2M	3M	4M	5M	6M
Mean return	GARCH(1,1)	0.01	0.01	0.03	0.02	0.02	0.01
	Homoskedastic	0.01	0.00	0.01	0.00	0.00	0.01
Standard deviation	GARCH(1,1)	0.21	0.25	0.30	0.32	0.35	0.39
	Homoskedastic	0.19	0.19	0.20	0.22	0.21	0.20
Initial opt. price	Homoskedastic	1.22	1.73	2.12	2.43	2.73	2.99

Table 2. Results for in-the-money and out-of-the-money call options

The table shows results from simulations done with options that initially were 10 percent in-the-money, and 10 percent out-of-the-money. 1000 simulations are performed for each option investigated. The rows labeled Mean return report average profit as percentage of the initial stock price, and the rows labeled Standard deviation report the dispersion of profits as percentage of the initial stock price. The rows labeled GARCH(1,1) give the results from simulations with heteroskedastic return processes, with parameter values $\alpha = 0.0204$, $\beta = 0.970$, $\gamma = 4.31 \cdot 10^{-7}$ and $\nu = 5$. The rows labeled Homoskedastic report the results from simulations with a constant volatility return process. The maturity of the options is three months, which corresponds to 63 trading days.

	Variance Procees	In-the-money	Out-of-the-money
Mean return	GARCH(1,1)	-0.01	0.01
	Homoskedastic	0.00	0.01
Standard deviation	GARCH(1,1)	0.08	0.12
	Homoskedastic	0.04	0.07

Table 3. Response to a higher γ

The table shows results from simulations done with options that initially were at-the-money. 1000 simulations are performed for each option investigated. The rows labeled GARCH(1,1) report the results from simulations with heteroskedastic return processes, and the rows labeled Homoskedastic report the results from simulations with a constant volatility return process. σ^2 is calculated with formula (5.10). The standard deviation figures represent the dispersion of profits as percentage of the initial stock price. The maturity of the options is three months, which corresponds to 63 trading days.

Variance Process	Parameter values			ν	σ^2	St. dev.
	α	β	γ			
GARCH(1,1)	0.0204	0.970	$4.31 \cdot 10^{-7}$	5	$4.49 \cdot 10^{-5}$	0.30
Homoskedastic				5	$4.49 \cdot 10^{-5}$	0.20
GARCH(1,1)	0.0204	0.970	$8.62 \cdot 10^{-7}$	5	$8.98 \cdot 10^{-5}$	0.43
Homoskedastic				5	$8.98 \cdot 10^{-5}$	0.28

Table 4. Response to the level of kurtosis

The table shows results from simulations done with options that initially were at-the-money. 1000 simulations are performed for each option investigated. σ^2 is calculated with formula (5.10), κ_ϵ with formulas (5.5) and (5.8), and $\rho(\epsilon_t^2, \epsilon_{t-1}^2)$ with formula (5.9). The standard deviation figures represent the dispersion of profits as percentage of the initial stock price. The maturity of the options is three months, which corresponds to 63 trading days.

α	Parameter values			ν	σ^2	κ_ϵ	$\rho(\epsilon_t^2, \epsilon_{t-1}^2)$	St. dev.
	β	γ						
0.0204	0.970	$4.31 \cdot 10^{-7}$		5	$4.49 \cdot 10^{-5}$	10.9	0.04	0.30
0.0204	0.970	$4.31 \cdot 10^{-7}$		6	$4.49 \cdot 10^{-5}$	6.7	0.04	0.27
0.0204	0.970	$4.31 \cdot 10^{-7}$		∞	$4.49 \cdot 10^{-5}$	3.1	0.04	0.19

Table 5. Response to the level of first-order autocorrelation in squared residuals

The table shows results from simulations done with options that initially were at-the-money. 1000 simulations are performed for each option investigated. The rows labeled GARCH(1,1) report the results from simulations with heteroskedastic return processes, and the rows labeled Constant Volatility report the results from simulations with homoskedastic return processes. σ^2 is calculated with formula (5.10), κ_ε with formulas (5.5) and (5.8), and $\rho(\varepsilon_t^2, \varepsilon_{t-1}^2)$ with formula (5.9). The standard deviation figures represent the dispersion of profits as percentage of the initial stock price. The maturity of the options is three months, which corresponds to 63 trading days.

Parameter values							St.
α	β	γ	ν	σ	κ_ε	$\rho(\varepsilon_t^2, \varepsilon_{t-1}^2)$	dev.
0.0204	0.970	$4.31 \cdot 10^{-7}$	5	$4.49 \cdot 10^{-5}$	10.9	0.04	0.30
0.0407	0.950	$4.16 \cdot 10^{-7}$	6	$4.49 \cdot 10^{-5}$	10.9	0.12	0.35
0.0505	0.940	$4.26 \cdot 10^{-7}$	7	$4.49 \cdot 10^{-5}$	10.9	0.16	0.47
0.0594	0.930	$4.75 \cdot 10^{-7}$	8	$4.49 \cdot 10^{-5}$	10.9	0.19	0.43

Table 6. Response to the level of persistence in the conditional variance

The table shows results from simulations done with options that initially were at-the-money. 1000 simulations are performed for each option investigated. σ^2 is calculated with formula (5.10), κ_ε with formulas (5.5) and (5.8), and $\rho(\varepsilon_t^2, \varepsilon_{t-1}^2)$ with formula (5.9). The standard deviation figures represent the dispersion of profits as percentage of the initial stock price. The maturity of the options is three months, which corresponds to 63 trading days.

Parameter values							St.
α	β	γ	ν	σ^2	κ_ε	$\rho(\varepsilon_t^2, \varepsilon_{t-1}^2)$	dev.
0.0204	0.970	$4.31 \cdot 10^{-7}$	5	$4.49 \cdot 10^{-5}$	10.9	0.04	0.30
0.0442	0.910	$20.56 \cdot 10^{-7}$	5	$4.49 \cdot 10^{-5}$	10.9	0.06	0.34
0.0503	0.890	$26.80 \cdot 10^{-7}$	5	$4.49 \cdot 10^{-5}$	10.9	0.07	0.32
0.0531	0.880	$29.85 \cdot 10^{-7}$	5	$4.49 \cdot 10^{-5}$	10.9	0.07	0.30

Figure 1. Predicted variance and predicted average variance as a function of the forecasting horizon

The forecasts are made with a GARCH(1,1) model. Parameter values are $\alpha = 0.0204$, $\beta = 0.970$ and $\gamma = 4.31 \cdot 10^{-7}$. The unconditional variance is equal to $4.49 \cdot 10^{-5}$, and the initial conditional variance is equal to $5.70 \cdot 10^{-5}$.

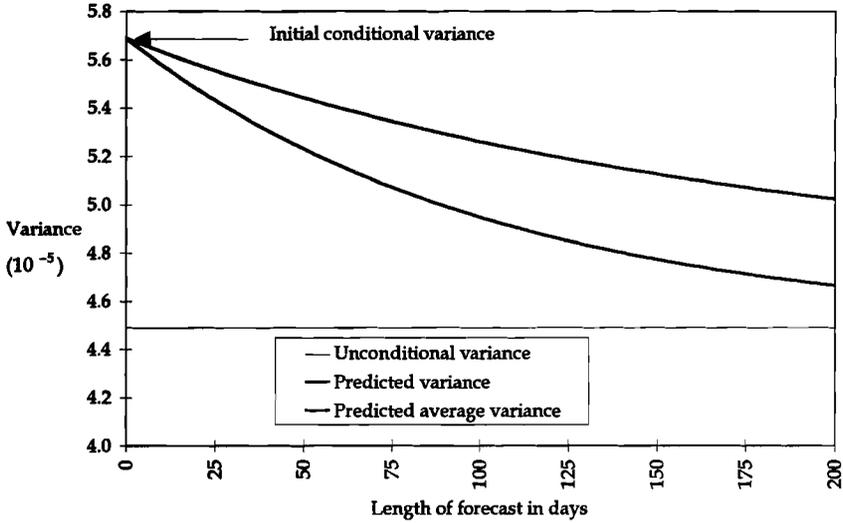


Figure 2. Distribution of profits from writing an at-the-money option with a maturity of one month and delta hedging it until expiry

The figure shows results from two simulations done with options that initially were at-the-money. 1000 simulations are performed for each option investigated. The curve labeled GARCH(1,1) gives the results from simulations with heteroskedastic return processes, with parameter values $\alpha = 0.0204$, $\beta = 0.970$, $\gamma = 4.31 \cdot 10^{-7}$ and $\nu = 5$. The curve labeled Constant Volatility gives the results from simulations with homoskedastic return processes. The maturity of the options is one month, which corresponds to 21 trading days.

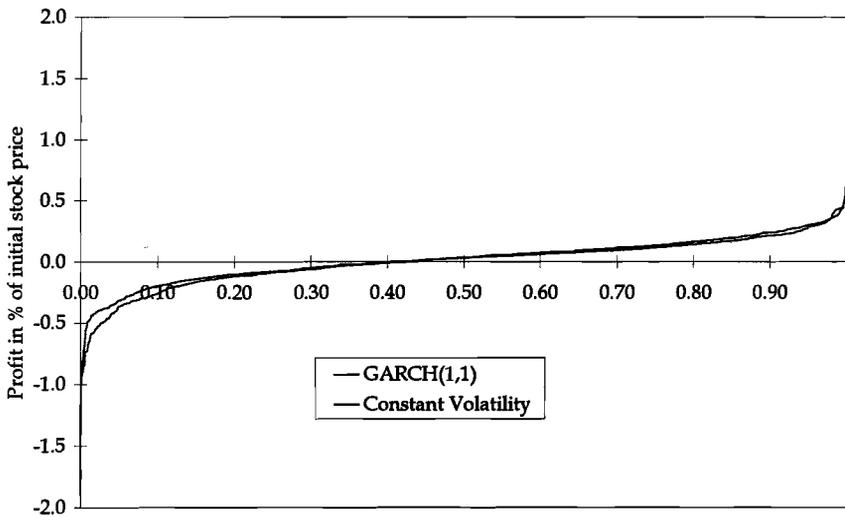


Figure 3. Distribution of profits from writing an at-the-money option with a maturity of three months and delta hedging it until expiry

The figure shows results from two simulations done with options that initially were at-the-money. 1000 simulations are performed for each option investigated. The curve labeled GARCH(1,1) gives the results from simulations with heteroskedastic return processes, with parameter values $\alpha = 0.0204$, $\beta = 0.970$, $\gamma = 4.31 \cdot 10^{-7}$ and $\nu = 5$. The curve labeled Constant Volatility gives the results from simulations with homoskedastic return processes. The maturity of the options is three months, which corresponds to 63 trading days.

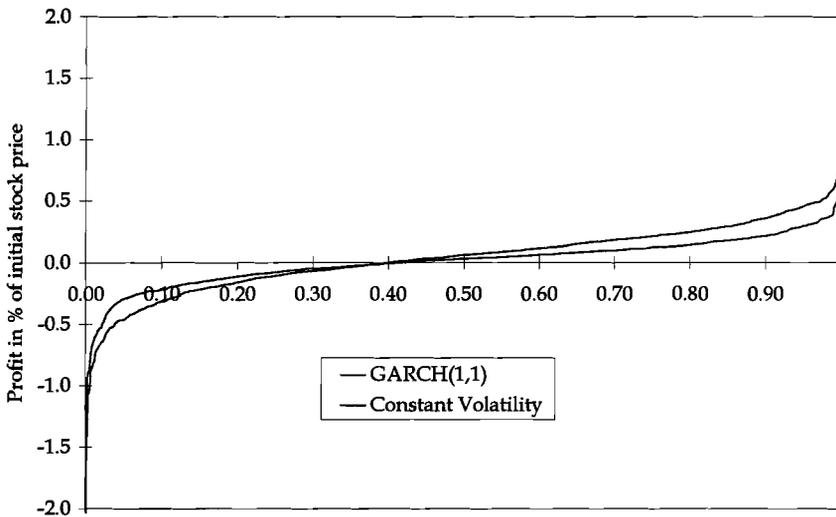
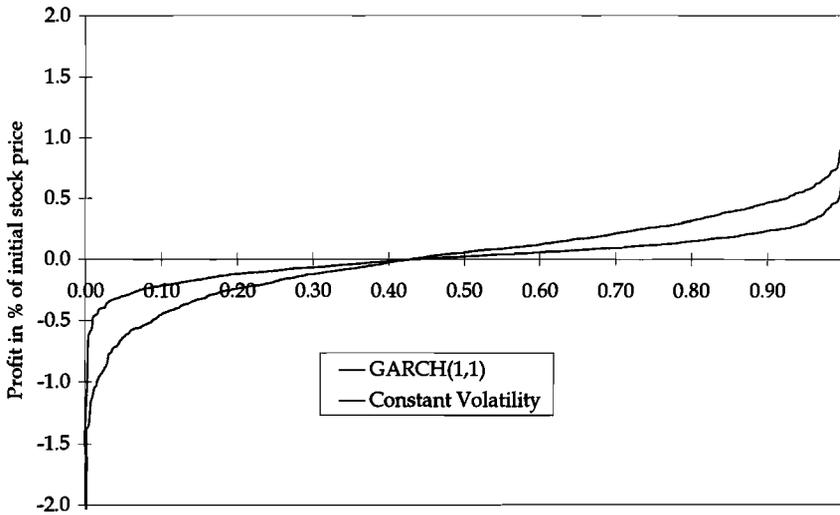


Figure 4. Distribution of profits from writing an at-the-money option with a maturity of six months and delta hedging it until expiry

The figure shows results from two simulations done with options that initially were at-the-money. 1000 simulations are performed for each option investigated. The curve labeled GARCH(1,1) gives the results from simulations with heteroskedastic return processes, with parameter values $\alpha = 0.0204$, $\beta = 0.970$, $\gamma = 4.31 \cdot 10^{-7}$ and $\nu = 5$. The curve labeled Constant Volatility gives the results from simulations with homoskedastic return processes. The maturity of the options is six months, which corresponds to 125 trading days.



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