Mon. Not. R. Astron. Soc. **359,** 1117–1130 (2005) doi:10.1111/j.1365-2966.2005.08967.x

A new numerical scheme for structures of rotating magnetic stars

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Accepted 2005 February 26. Received 2005 February 26; in original form 2004 November 10

ABSTRACT

We have developed a new numerical scheme for obtaining structures of rapidly rotating stars with strong magnetic fields. In our scheme, both poloidal and toroidal magnetic fields can be treated for stars with compressibility and infinite conductivity. By introducing the vector potential and its integral representation, we can treat the boundary condition for the magnetic fields across the surface properly. We show structures and distributions of magnetic fields as well as the distributions of the currents of rotating magnetic polytropic stars with polytropic index $N = 1.5$. The shapes of magnetic stars are oblate as long as the magnetic vector potential decreases as $1/r$ when $r \to \infty$. For extremely strong magnetic fields, equilibrium configurations can be of toroidal shapes.

Key words: stars: magnetic fields – stars: rotation.

1 INTRODUCTION

The surface magnetic field of magnetars is estimated to be about 10^{15} gauss (see, e.g. Kouveliotou et al. 1998, 1999; Murakami et al. 1999). The ratio of the magnetic energy H to the gravitational potential energy W is expressed as

$$\frac{H}{|W|} \simeq 10^{-4} \left(\frac{H_{\rm s}}{10^{16}\,{\rm G}}\right)^2 \left(\frac{R}{10^6\,{\rm cm}}\right)^4 \left(\frac{M}{1.4\,{\rm M}_\odot}\right)^{-2}, \tag{1}$$

where $H_{\rm s}$, R and M are the strength of the surface magnetic field, the radius and mass of the star, respectively. Thus even if the surface magnetic field $H_{\rm s}$ is of order 10^{15} gauss, this ratio is very small. However, this expression is derived by assuming a uniform magnetic field within the star and a dipole field outside the star. This estimation could not be applied to stars with extremely strong magnetic fields within the stars. The value of this ratio could be of order 0.1 for stars with strong internal magnetic fields and larger radii due to deformation of configurations. In this paper, we will show how such strong magnetic fields can affect and change the stellar configurations. We also find the theoretical range of strength of the magnetic fields for which equilibrium configurations can exist, although such values have not been found observationally.

Theoretically, the first attempt to investigate structures and stabilities of strong magnetic stars was made by Chandrasekhar & Fermi (1953). They discussed the shapes of slightly deformed equilibrium configurations and their stabilities by applying virial theorems. Ferraro (1954) found a special solution consisting only of a poloidal magnetic field, assuming that the star is nearly spherical and that the density is uniform. He imposed the continuity of the interior field and the exterior dipole field on the surface and found that the deformation was oblate. Roberts (1955) extended Ferraro's work and obtained the surface shapes by using a series expansion with respect to the small deformation due to the weak poloidal magnetic field for uniform density stars. In his models, the interior field was assumed to be connected to the exterior field which vanishes at infinity.

Prendergast (1956) investigated the influence of both poloidal and toroidal fields on incompressible stars. He assumed the surfaces of stars to be spherical. If this condition is imposed, it is required that the magnetic field must vanish on the surface. This resulted in an eigenvalue problem and discrete solutions were obtained. In other words, the magnetic field distributions and their values could not change continuously. Woltjer (1960) extended Prendergast's study and applied it to polytropic stars.

Monaghan (1965, 1966) and Roxburgh (1966) calculated equilibrium configurations of deformed polytropic stars perturbatively by assuming that the magnetic forces were small and that the interior fields were connected to the exterior dipole fields at the surfaces. Recently, Ioka (2001) extended the perturbational approach to the second order of $H/|W|$.

For stars with strong magnetic fields, Ostriker & Hartwick (1968) extended the self-consistent field (SCF) method (Ostriker & Mark 1968) to treat equilibrium configurations with magnetic fields. Although, in principle, their formulation could be applied to strong magnetic forces, they only solved equilibrium configurations of white dwarfs with small deformations. This was because they could not find a method

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to treat the fitting condition of magnetic fields across the surface. They set the same boundary conditions as those of Prendergast (1956), i.e. magnetic fields vanish on the surfaces of stars. For poloidal magnetic fields, Miketinac (1975) formulated a new numerical scheme for magnetic stars and obtained equilibrium configurations whose surface shapes are considerably different from spheres. He has shown that for stars with extremely strong magnetic fields the density distribution can be toroidal-like, i.e. the maximum density is located at an off-centre position on the equator. His solutions are the only configurations which are significantly deformed from spheres due to *poloidal* magnetic fields.

In the framework of general relativity which is needed to treat magnetized neutron stars, Bocquet et al. (1995) presented a formulation to solve equilibrium configurations of relativistic stars with interior poloidal magnetic fields (see also Konno, Obata & Kojima 1999) for the perturbative approach in general relativity).

In other contexts, magnetic fields are regarded to be important, too. One of them is related to the star formation stages. Mouschovias (1976) presented a formulation to solve equilibrium states of isothermal gases in interstellar clouds with magnetic fields. He assumed that the magnetic fields beyond a certain finite distance from the centre were uniform and that the interior fields were considered to be connected to those uniform exterior fields. Tomisaka, Ikeuchi & Nakamura (1988) extended Mouschovias' formulation and obtained a formulation which could treat rotational clouds with magnetic fields. They have solved equilibrium configurations of rotating magnetic clouds in various situations.

Since observations so far suggest that $H/|W|$ is small, the perturbational approach can give quite good solutions. However, the surface magnetic field might not represent the interior fields. Thus, from the theoretical point of view, we will consider the possibility of very strong magnetic fields inside the stars and significantly deformed configurations.

Moreover, many attempts thus far made cannot be regarded as full and satisfactory treatments of stars with strong magnetic fields. First, boundary conditions adopted in previous works are not necessarily the most general. The most natural behaviour of the magnetic fields outside the stars would be that the magnetic force decreases as $1/r^2$ or the vector potential decreases as $1/r$ for $r \to \infty$, where the spherical coordinates (r, θ, ϕ) are used. Secondly, configurations with finite deformations have not been solved for the presence of both poloidal and toroidal magnetic fields. Third, the presence of rotation in addition to the magnetic fields has not been treated except for Tomisaka et al. (1988) in which the boundary condition is not suitable for equilibrium configurations of compressible stars. Thus, in this paper, we will present a new formulation to solve equilibrium configurations of rotating compressible stars with strong magnetic fields which extend to infinity and show several solutions.

2 FORMULATION OF THE PROBLEM

2.1 Assumptions and basic equations

We will treat rotating magnetic stars under the following assumptions. (1) The stars are in stationary states. (2) The density distributions and the magnetic field distributions are axisymmetric about the same axis. (3) The sources of the magnetic fields, i.e. the current distributions, are confined only within the stars. (4) The conductivity of the gas is assumed to be infinite. (5) As for the gas, we adopt the barotropic equation of state as follows:

$$p = p(\rho), \tag{2}$$

where p and ρ are the pressure and the density, respectively.

Under the above assumptions, the basic equations can be written as follows:

$$-\frac{1}{\rho}\ \mathrm{grad}\ p - \mathrm{grad}\ \Phi_g + \mathrm{grad}\ \Phi_r + \frac{1}{\rho}\ (\boldsymbol{j} \times \boldsymbol{H}) = 0, \tag{3}$$

$$\Delta \Phi_g = 4\pi G \rho, \tag{4}$$

$$\mathrm{rot}\ \boldsymbol{H} = 4\pi \boldsymbol{j}, \tag{5}$$

$$\mathrm{div}\ \boldsymbol{H} = 0, \tag{6}$$

where Φ_g, Φ_r, $\boldsymbol{j}$, $\boldsymbol{H}$ and G are the gravitational potential, the centrifugal potential, the current density, the magnetic field, and the gravitational constant, respectively.

It can generally be shown under the assumptions of barotropy and axisymmetry that $\boldsymbol{j}$ and $\boldsymbol{H}$ have the following relation by introducing two arbitrary functions (see e.g. Chandrasekhar 1956; Chandrasekhar & Prendergast 1956; Prendergast 1956):

$$\boldsymbol{j} = \frac{\kappa(u)}{4\pi}\boldsymbol{H} + \mu(u)\rho R \boldsymbol{e}_\phi, \tag{7}$$

where $\boldsymbol{e}_\phi$ is the unit vector in the ϕ-direction and κ and μ are two arbitrary functions of u which is a stream function and can be introduced so as to satisfy equation (6) by defining the magnetic field as follows:

$$H_R = -\frac{1}{R}\frac{\partial u}{\partial z}, \tag{8}$$

and

$$H_z = \frac{1}{R}\frac{\partial u}{\partial R}, \tag{9}$$

where H_R and H_z are the R-component and z-component of the magnetic field. Here the cylindrical coordinates (R, z, ϕ) are used.

2.2 Boundary conditions

The boundary conditions for the gravitational potential and the magnetic field are expressed as follows:

$$\Phi_g \longrightarrow \frac{1}{r}, \qquad (r \longrightarrow \infty), \tag{10}$$

$$\boldsymbol{H} \longrightarrow \frac{1}{r^2}, \qquad (r \longrightarrow \infty). \tag{11}$$

The above boundary condition for the magnetic field is very natural. This boundary condition for the magnetic field corresponds to the boundary condition for the vector magnetic potential $\boldsymbol{A}$ as follows:

$$A_\phi \longrightarrow \frac{1}{r} \qquad (r \longrightarrow \infty), \tag{12}$$

where A_ϕ is the ϕ component of the vector potential $\boldsymbol{A}$ which is introduced by the following relation:

$$\boldsymbol{H} = \mathrm{rot}\boldsymbol{A}. \tag{13}$$

2.3 Integral representation of the basic equations

From equations (5) and (13) the current density can be expressed by using the vector potential. By comparing this expression with equation (7), we obtain the following relations:

$$\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial R} - \frac{\partial A_R}{\partial z}\right) = -\kappa \frac{\partial A_\phi}{\partial z}, \tag{14}$$

$$\frac{\partial}{\partial R}\left\{R\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right)\right\} = \kappa \frac{\partial R A_\phi}{\partial R}, \tag{15}$$

$$\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial R A_\phi}{\partial R}\right) + \frac{\partial^2 A_\phi}{\partial z^2} = -\kappa\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) - 4\pi\mu\rho R. \tag{16}$$

From these relations we can see that the function κ is a function of RA_ϕ and that the stream function u is also a function of RA_ϕ. This implies that μ is a function of RA_ϕ. Furthermore, we can derive the following relation from equations (14) and (15):

$$\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} = \frac{1}{R}\int \kappa(RA_\phi)\,\mathrm{d}(RA_\phi). \tag{17}$$

By using the above equation (17), equation (16) can be rewritten as follows:

$$\frac{\partial^2 A_\phi}{\partial R^2} + \frac{1}{R}\frac{\partial A_\phi}{\partial R} - \frac{A_\phi}{R^2} + \frac{\partial^2 A_\phi}{\partial z^2} = -\frac{\kappa(RA_\phi)}{R}\int \kappa(RA_\phi)\,\mathrm{d}(RA_\phi) - 4\pi\mu\rho R. \tag{18}$$

Outside the star, the right-hand side of this equation should vanish. Thus we need to impose the following condition:

$$\frac{\kappa(RA_\phi)}{R}\int \kappa(RA_\phi)\,\mathrm{d}(RA_\phi) = 0, \qquad \text{(outside the star)}. \tag{19}$$

It would be useful to note that equation (18) can be rewritten as follows:

$$\Delta(A_\phi \sin\phi) = -\left\{\frac{\kappa(RA_\phi)}{R}\int \kappa(RA_\phi)\,\mathrm{d}(RA_\phi) + 4\pi\mu\rho R\right\}\sin\phi, \tag{20}$$

where Δ is the Laplacian in three-dimensional space.

Taking the boundary conditions for the gravitational potential and the magnetic vector potential into account, we can transform the basic equations into the integral forms as follows:

$$\int \frac{\mathrm{d}p}{\rho} = -\Phi_g + \Phi_r + \int \mu(RA_\phi)\,\mathrm{d}(RA_\phi) + C, \tag{21}$$

$$\Phi_g(\boldsymbol{r}) = -G\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}', \tag{22}$$

$$A_\phi(\boldsymbol{r})\sin\phi = -\frac{1}{4\pi}\int \frac{\left(-\frac{\kappa}{R}\int \kappa\ \mathrm{d}(RA_\phi) - 4\pi\mu\rho R'\right)}{|\boldsymbol{r} - \boldsymbol{r}'|}\sin\phi'\mathrm{d}^3\boldsymbol{r}', \tag{23}$$

where C is a constant of integration. It should be noted that all the boundary conditions are included in these expressions and we need not consider them any further.

3 NUMERICAL METHOD AND NUMERICAL RESULTS

3.1 Non-dimensional variables

In order to carry out numerical computations properly, we introduce the following non-dimensional variables by using the maximum density $\rho_{\max}$ and the equatorial radius r_e:

$$\hat{\rho} \equiv \rho/\rho_{\max}, \tag{24}$$

$$\hat{r} \equiv r/r_e = r\Big/\sqrt{\frac{1}{\beta}\frac{p_{\max}}{4\pi G{\rho_{\max}}^2}}, \tag{25}$$

$$\hat{\Omega} \equiv \Omega/\sqrt{4\pi G\rho_{\max}}, \tag{26}$$

$$\hat{C} \equiv C\Big/\left(\frac{1}{\beta}\frac{p_{\max}}{\rho_{\max}}\right) = C\Big/\left(4\pi G\rho_{\max}{r_e}^2\right), \tag{27}$$

$$\hat{\kappa} \equiv \kappa\Big/\left(\rho_{\max}\sqrt{\frac{4\pi G\beta}{p_{\max}}}\right) = \kappa\Big/\left(\frac{1}{r_e}\right), \tag{28}$$

$$\hat{\mu} \equiv \mu\Big/\left(4\pi G\rho_{\max}\sqrt{\frac{\beta}{p_{\max}}}\right) = \mu\Big/\left(\frac{\sqrt{4\pi G}}{r_e}\right), \tag{29}$$

$$\hat{A}_\phi \equiv A_\phi\Big/\left(\frac{p_{\max}}{\beta\rho_{\max}}\sqrt{\frac{1}{4\pi G}}\right) = A_\phi\Big/\left(\sqrt{4\pi G}\rho_{\max}{r_e}^2\right). \tag{30}$$

Here, $p_{\max}$, Ω and β are the maximum pressure, the angular velocity and a numerical factor which is introduced to fix the non-dimensional equatorial radius to be unity, respectively. By using these variables, the gravitational potential energy W, the rotational energy T, the internal energy U, the magnetic energy H and the mass M can be expressed as follows:

$$\hat{W} = W\Big/\left(4\pi G{r_e}^5{\rho_{\max}}^2\right), \tag{31}$$

$$\hat{T} = T\Big/\left(4\pi G{r_e}^5{\rho_{\max}}^2\right), \tag{32}$$

$$\hat{U} = U\Big/\left(\beta\ 4\pi G{r_e}^5{\rho_{\max}}^2\right), \tag{33}$$

$$\hat{H} = H\Big/\left(4\pi G{r_e}^5{\rho_{\max}}^2\right), \tag{34}$$

$$\hat{M} = M\Big/\left({r_e}^3\rho_{\max}\right), \tag{35}$$

where

$$W \equiv \frac{1}{2}\int_{\text{star}} \Phi_g\ \rho\ \mathrm{d}^3\boldsymbol{r}, \tag{36}$$

$$T \equiv \int_{\text{star}} \rho\,(R\Omega)^2\ \mathrm{d}^3\boldsymbol{r}, \tag{37}$$

$$U \equiv \int_{\text{star}} p\ \mathrm{d}^3\boldsymbol{r}, \tag{38}$$

$$H \equiv \int_{\text{all space}} \boldsymbol{r}\cdot(\boldsymbol{j}\times\boldsymbol{H})\ \mathrm{d}^3\boldsymbol{r}, \tag{39}$$

$$M \equiv \int_{\text{star}} \rho\ \mathrm{d}^3\boldsymbol{r}. \tag{40}$$

3.2 Specification of parameters and functions

In our problem, we have several parameters and functions to be specified. First, we choose the polytropic relation as one of the barotropic equations of state:

$$p = p_{\max}\lambda^{1+N}, \tag{41}$$

$$\rho = \rho_{\max}\lambda^{N}, \tag{42}$$

where λ is the Lane–Emden function and N is the polytropic index.

Second, we need to specify the rotation law which is related to the rotational potential Φ_r. For Newtonian barotropes, the angular velocity (Ω) should be a function of R alone. In this paper, we choose the uniform rotation law, i.e. the angular velocity is constant.

As for the two functions which appear in the relation between the current density and the magnetic field, we choose the following form as one example which satisfies the requirement of equation (19):

$$\kappa(u) = \begin{cases} 0, & \text{for u} \leqslant \text{u}_{\max} \\ a(u - u_{\max}), & \text{for u}_{\max} \leqslant \text{u}, \end{cases} \tag{43}$$

where a is an arbitrary constant and $u_{\max}$ is the maximum value of the quantity RA_ϕ on the surface. Concerning the function μ, we will set it to be a constant in this paper.

Apart from these functional forms, we need to specify four parameters to obtain one equilibrium configuration. For polytropic stars, first, the polytropic index N must be specified. The second parameter is related to the deformation of the configuration. The third parameter represents the relative contribution of the rotational force and the magnetic force to the deformation. The fourth one is the representative ratio of the strength of the toroidal field to that of the poloidal field.

In this paper, we will solve $N = 1.5$ polytropes. As for the second parameter, we will choose the ratio of the smallest distance to the surface from the axis of symmetry, r_p, to the equatorial radius as follows:

$$q \equiv \frac{r_p}{r_e}. \tag{44}$$

Two other parameters can be two of the three quantities, a, μ and Ω.

3.3 Numerical scheme and numerical results

In order to solve the basic equations, we employ the Hachisu SCF scheme (Hachisu 1986). By specifying three physical parameters mentioned above, we can obtain the distributions of ρ, A_ϕ and other physical quantities such as β, C and one parameter which is not specified beforehand. By changing one parameter and keeping the other two parameters fixed, we can follow one equilibrium sequence of equilibrium configurations. For different combinations of two parameters to be specified, we obtain different kinds of equilibrium sequences.

The simplest sequence corresponds to $a = 0$, i.e. $\kappa = 0$, and $\Omega = 0$ configurations. From equation (7), this configuration contains only a toroidal current and no poloidal current. Thus the magnetic filed has only a poloidal component. In Table 1 physical quantities are shown for this sequence for which the value of q is varied. In this table, models with negative values of q mean that the configurations are of toroidal shape, i.e. the shortest distance to the surface from the rotational axis is not on the symmetry axis but on the equatorial plane. The same kind of configurations have been obtained by Miketinac (1975) who considered significantly deformed magnetic stars with poloidal fields. Although choices of model parameters are different, we can obtain corresponding values from his results. By computing the values of $\hat{\mu}_{\rm Miketinac}$ and $\hat{M}_{\rm Miketinac}$ for the $N = 1.5$ and $q = 0.6$ model of Miketinac's results, we obtain $\hat{\mu}_{\rm Miketinac} = 0.401$ and $\hat{M}_{\rm Miketinac} = 0.829$. These values should be compared with our values $\hat{\mu} = 0.402$ and $\hat{M} = 0.832$, respectively. They agree very well.

Table 1. Physical quantities for the sequence with $\hat{\kappa} = 0.0$, $\hat{\Omega}^2 = 0.0$ and $N = 1.5$ polytropic stars.

q	$H/\lvert W\rvert$	$U/\lvert W\rvert$	$\lvert\hat{W}\rvert$	$\hat{\mu}$	$\hat{C}$	$\hat{\beta}$	$\hat{M}$	VC
1.0	0.0000	0.333	3.34E−02	0.000	−0.0556	0.0300	0.699	2.75E−05
0.95	0.0125	0.329	3.34E−02	0.156	−0.0576	0.0293	0.697	3.28E−05
0.9	0.0261	0.325	3.38E−02	0.220	−0.0600	0.0286	0.698	3.66E−05
0.8	0.0574	0.314	3.56E−02	0.306	−0.0664	0.0273	0.714	4.59E−05
0.6	0.138	0.287	4.82E−02	0.402	−0.0903	0.0246	0.832	7.13E−05
0.5	0.186	0.272	4.59E−02	0.417	−0.0949	0.0219	0.818	8.90E−05
0.4	0.229	0.257	3.61E−02	0.412	−0.0892	0.0185	0.734	1.10E−04
0.2	0.274	0.242	2.13E−02	0.376	−0.0721	0.0135	0.573	1.50E−04
0.0	0.284	0.239	1.77E−02	0.363	−0.0665	0.0121	0.524	1.65E−04
−0.2	0.295	0.235	1.44E−02	0.351	−0.0608	0.0107	0.474	1.84E−04
−0.4	0.329	0.224	7.49E−03	0.325	−0.0458	0.00738	0.343	2.65E−04
−0.6	0.382	0.206	2.19E−03	0.298	−0.0267	0.00370	0.184	5.08E−04
−0.8	0.461	0.180	1.97E−04	0.275	−0.00910	0.00100	0.0535	1.63E−03

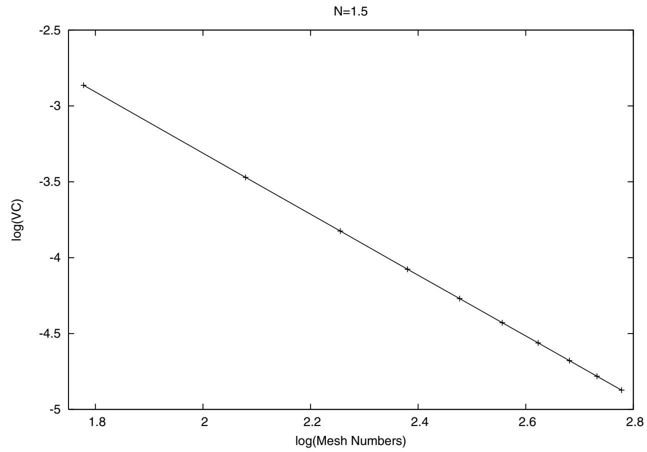

Figure 1. Log(VC) plotted against log(Mesh Numbers) for the $\hat{\kappa} = 0.0$, $\hat{\Omega}^2 = 0.0$, $q = 0.2$ and $N = 1.5$ models.

The accuracy of the numerical solutions is evaluated by the following quantity which is derived from the virial equation (e.g. Cowling 1965):

$$\mathrm{VC} = |2T + W + 3U + H|/|W|. \tag{45}$$

In our present study, we divide the interval [0, 1] in the $\hat{r}$-direction into several different numbers of meshes and the interval [0, π/2] in the θ-direction into several different numbers of meshes and carry out computations for different values of mesh numbers. In Fig. 1 we can see that VC $\sim N_r^{-2}$ where N_r is the mesh number in the r-direction. It should be noted that the values of VC represent rough accuracies of the global quantities of the solutions but do not show the accuracies of local solutions. For some methods the values of VC can be 10^{-12} (see e.g. Gourgoulhon et al. 2001).

In Fig. 2 equidensity contours in the meridional cross-section are shown for different values of q. Figs 3 and 4 correspond to the magnetic fields and the current density distributions, respectively. As seen from these figures, poloidal magnetic fields can sustain toroidal configurations in their equilibrium states when the value of the ratio $H/|W|$ is large.

In Tables 2–5, we show our results for $N = 1.5$, $a = 200$, and $\hat{\mu} = 0.3, 0.35, 0.40$, and 0.41 sequences, respectively. In these tables, equilibrium configurations have both poloidal and toroidal magnetic fields in addition to uniform rotation. For all sequences, equilibrium states can exist only for a certain finite range of the value of q in contrast to the first simple sequence with only toroidal magnetic fields. As can be seen from these results, we could not obtain spherical configurations for models with $\kappa \neq 0$.

As μ becomes larger, which means stronger magnetic fields, the upper limit of the value of q below which equilibrium states can exists becomes smaller. Considering the lower limit of the value of q, we can distinguish two types of equilibrium states. We call one type rotation dominated and the other magnetic field dominated for the following reason.

When $\mu = 0.3$ and 0.35, the value of q comes to its lower limit since each model cannot support equilibrium states with further deformation because of mass shedding from the equatorial surface. This is a typical feature of equilibrium configurations with rapid rotation. Thus they are classified as the rotation dominated type. We can see from Table 1 that mass shedding never occurs for stars with magnetic forces alone.

For $\mu = 0.4$ and 0.41 sequences, the value of q comes to its lower limit since further deformation requires $\Omega^2 < 0$. Such a requirement cannot be realized in the real world. Therefore, these kinds of configurations are named magnetic field dominated configurations.

We also show the behaviours of $H/|W|$ and $T/|W|$ against the value of q in Fig. 5. In Fig. 6 equidensity contours in the meridional cross-section are displayed for models with $\hat{\mu} = 0.35$ (in the left panels) and $\hat{\mu} = 0.4$ (in the right panels). From the behaviours in these figures, it should be clear that our models can be categorized into these two types mentioned above.

In Fig. 7 we show the magnetic field distributions and the current density distributions for the model with $\hat{\mu} = 0.35$ and $q = 0.478$. The equidensity contours of this model are shown in the bottom-left panel of Fig. 6. The same quantities for the model with $\hat{\mu} = 0.4$ and $q = 0.372$, whose density contours are shown in the bottom-right panel in Fig. 6, are shown in Fig. 8. It should be noted that, in Figs 7 and 8, the magnitude of the toroidal component of the magnetic field is scaled up by a factor 10 compared with the poloidal component. Similarly, the magnitude of the poloidal current is also scaled up by a factor 10 compared with the toroidal one. However, in Fig. 8, both the magnetic field and the current are scaled down by a factor 10 compared with those of Fig. 7.

The characteristic feature of the magnetic field distribution is that the toroidal magnetic field does not cover all the interior of the star. This arises from the requirement in equation (19) and the functional form of equation (43). Although this may be strange distributions, the continuity of the magnetic field across the surface results in this kind of distribution.

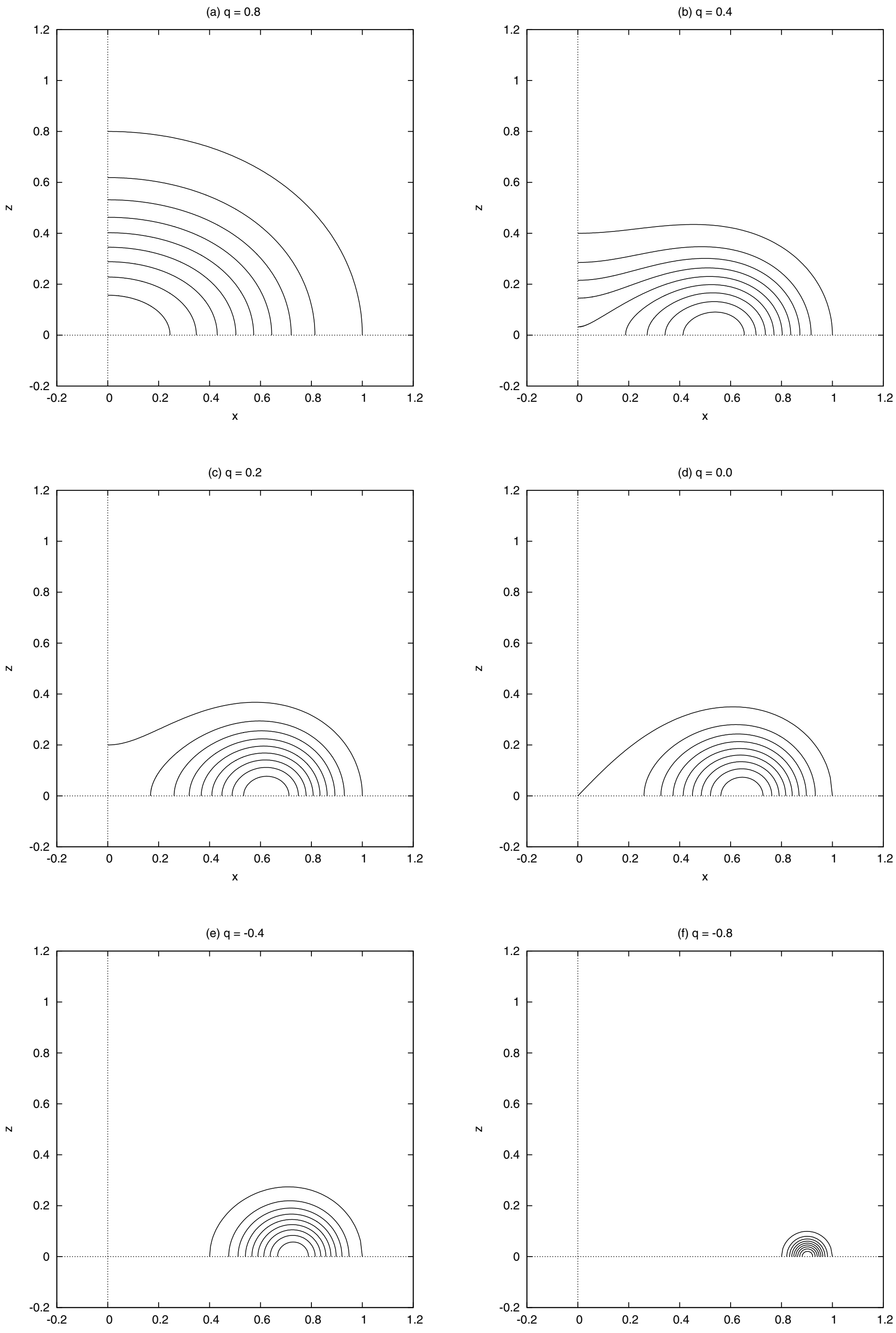

Figure 2. Equidensity contours in the meridional cross-section for the $\hat{\kappa} = 0.0$, $\hat{\Omega}^2 = 0.0$ and $N = 1.5$ polytropic stars. (a) $q = 0.8$, (b) $q = 0.4$, (c) $q = 0.2$, (d) $q = 0.0$, (e) $q = -0.4$ and (f) $q = -0.8$. The density contours are linearly spaced, i.e. the density difference between two adjacent contours is a tenth of the maximum density.

4 DISCUSSION AND SUMMARY

4.1 Discussion

In this paper, we have assumed that the arbitrary function μ is a constant and the functional form of κ is expressed by equation (43). It should be noted that our formulation proposed in this paper can be easily extended to treat more complicated functions for μ and κ.

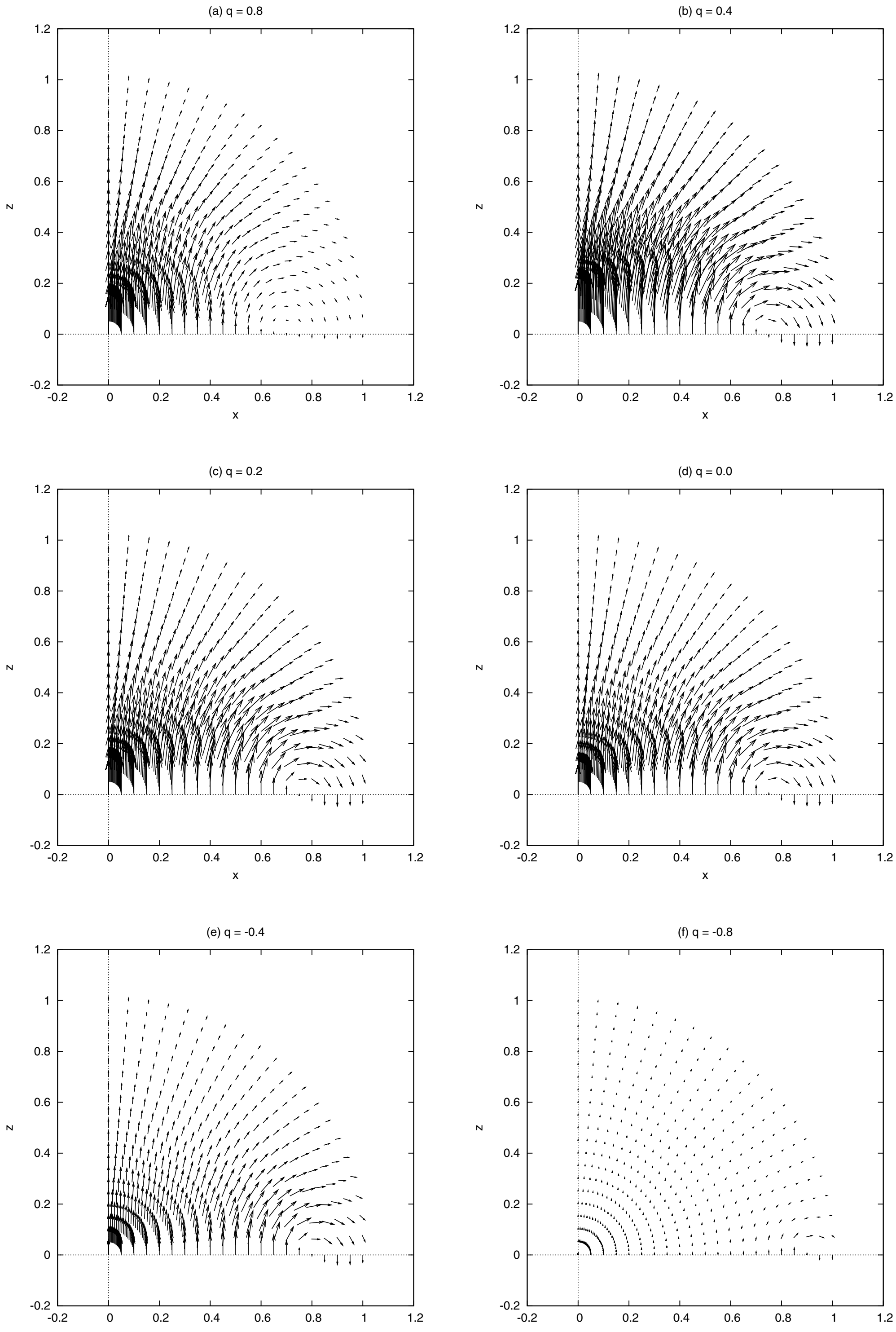

Figure 3. Magnetic fields in the meridional cross-section for the $\hat{\kappa} = 0.0$, $\hat{\Omega}^2 = 0.0$ and $N = 1.5$ polytropic stars. (a) $q = 0.8$, (b) $q = 0.4$, (c) $q = 0.2$, (d) $q = 0.0$, (e) $q = -0.4$ and (f) $q = -0.8$.

As seen from the results in the previous section, shapes of the equilibrium configurations cannot be prolate. This contrasts with the results of Ostriker & Hartwick (1968) and Ioka (2001) in which they obtained prolate equilibrium configurations. The difference comes from the different choice of the boundary condition for the magnetic fields. In our present formulation, the magnetic fields decreases as $1/r^2$ at infinity, while Ostriker & Hartwick (1968) and Ioka (2001) have assumed that the magnetic fields vanish on the surfaces of magnetic stars.

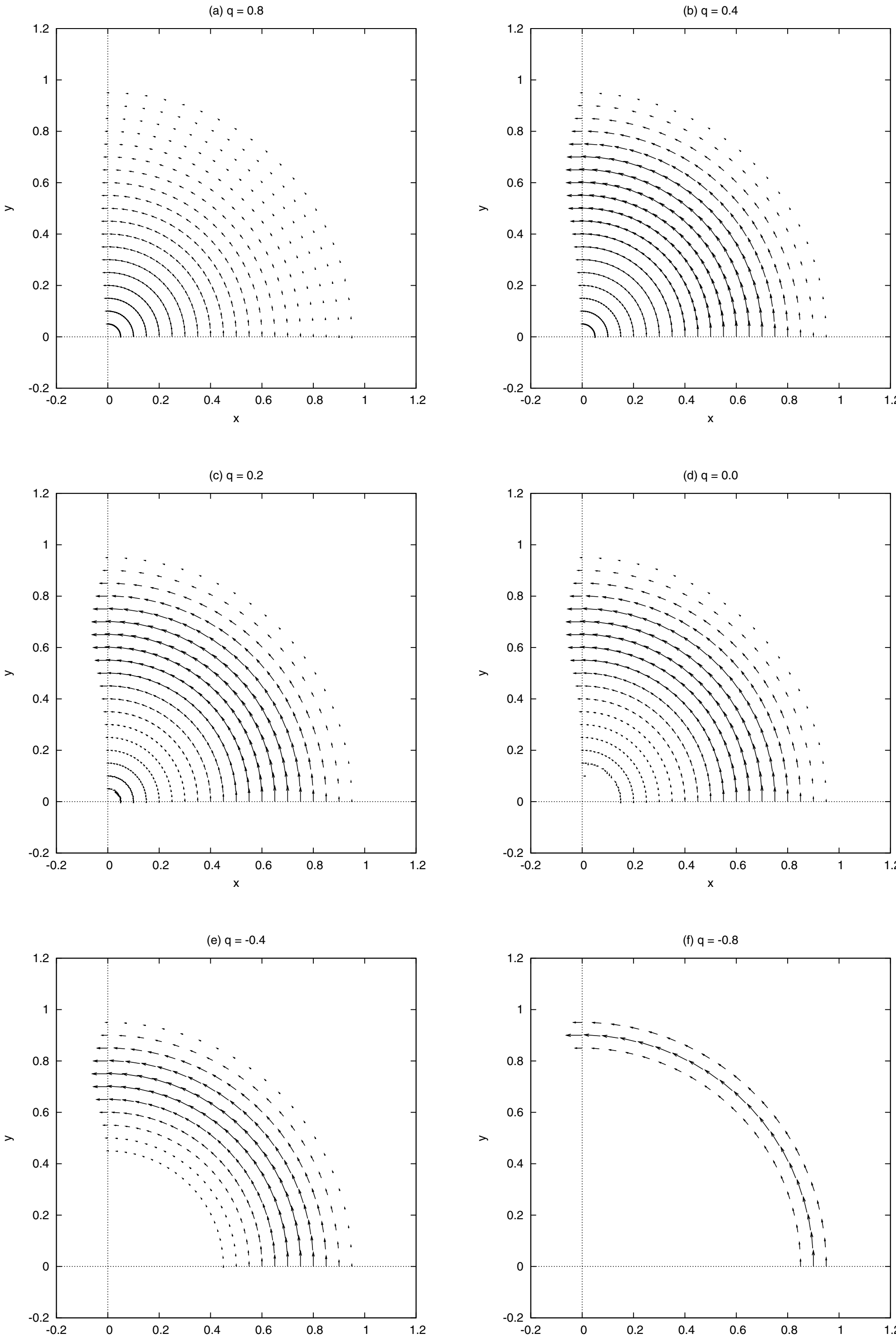

Figure 4. Current densities in the equatorial plane for the $\hat{\kappa} = 0.0$, $\hat{\Omega}^2 = 0.0$ and $N = 1.5$ polytropic stars. (a) $q = 0.8$, (b) $q = 0.4$, (c) $q = 0.2$, (d) $q = 0.0$, (e) $q = -0.4$ and (f) $q = -0.8$.

Concerning the boundary conditions for the magnetic fields, there can be other choices. Since the currents should be within the star, one may impose the condition that the currents should flow along the surface. Using equations (7) and (13), currents in the r-direction, j_r, and θ-direction, j_θ, can be expressed as follows:

$$j_r = \frac{\kappa}{4\pi} \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}(\sin\theta A_\phi), \tag{46}$$

$$j_\theta = -\frac{\kappa}{4\pi} \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi). \tag{47}$$

Table 2. Physical quantities for the sequence with $a = 200$, $\hat{\mu} = 0.3$ and $N = 1.5$ polytropic stars.

q	$H/\|W\|$	$U/\|W\|$	$T/\|W\|$	$\|\hat{W}\|$	$\hat{\Omega}^2$	$\hat{C}$	$\hat{\beta}$	$\hat{M}$	VC
	–	–	–	–	–	–	–	–	$\Omega^2 < 0$
0.806	0.0548	0.315	5.70E−04	3.52E−02	2.50E−04	−0.0657	0.0273	0.709	4.61E−05
0.8	0.0548	0.314	2.09E−03	3.48E−02	9.09E−04	−0.0654	0.0271	0.703	4.60E−05
0.75	0.0542	0.305	0.0161	3.00E−02	6.73E−03	−0.0633	0.0245	0.647	4.73E−05
0.7	0.0532	0.295	0.0308	2.53E−02	1.23E−02	−0.0605	0.0226	0.586	4.96E−05
0.65	0.0513	0.286	0.0460	2.05E−02	1.74E−02	−0.0567	0.0202	0.519	5.28E−05
0.6	0.0479	0.277	0.0609	1.54E−02	2.18E−02	−0.0513	0.0176	0.440	5.68E−05
0.55	0.0411	0.271	0.0725	9.92E−03	2.47E−02	−0.0431	0.0147	0.340	6.37E−05
0.522	0.0346	0.273	0.0735	6.74E−03	2.49E−02	−0.0364	0.0128	0.270	7.07E−05
	–	–	–	–	–	–	–	–	mass shedding

Table 3. Physical quantities for the sequence with the $a = 200$, $\hat{\mu} = 0.35$ and $N = 1.5$ polytropic stars.

q	$H/\|W\|$	$U/\|W\|$	$T/\|W\|$	$\|\hat{W}\|$	$\hat{\Omega}^2$	$\hat{C}$	$\hat{\beta}$	$\hat{M}$	VC
	–	–	–	–	–	–	–	–	$\Omega^2 < 0$
0.722	0.0846	0.304	1.24E−03	3.81E−02	5.33E−04	−0.0729	0.0261	0.736	5.44E−05
0.7	0.0847	0.300	8.24E−03	3.60E−02	3.48E−03	−0.0721	0.0251	0.713	5.49E−05
0.65	0.0847	0.289	0.0248	3.14E−02	9.99E−03	−0.0702	0.0229	0.658	5.76E−05
0.6	0.0839	0.277	0.0426	2.65E−02	1.63E−02	−0.0673	0.0205	0.598	6.06E−05
0.55	0.0811	0.265	0.0618	2.11E−02	2.22E−02	−0.0627	0.0180	0.524	6.43E−05
0.5	0.0707	0.255	0.0816	1.33E−02	2.71E−02	−0.0523	0.0147	0.403	6.99E−05
0.478	0.0519	0.261	0.0823	6.96E−03	2.69E−02	−0.0388	0.0121	0.275	7.85E−05
	–	–	–	–	–	–	–	–	mass shedding

Table 4. Physical quantities for the sequence with the $a = 200$, $\hat{\mu} = 0.4$ and $N = 1.5$ polytropic stars.

q	$H/\|W\|$	$U/\|W\|$	$T/\|W\|$	$\|\hat{W}\|$	$\hat{\Omega}^2$	$\hat{C}$	$\hat{\beta}$	$\hat{M}$	VC
	–	–	–	–	–	–	–	–	$\Omega^2 < 0$
0.589	0.146	0.285	3.53E−04	4.83E−02	1.50E−04	−0.0912	0.0242	0.834	7.23E−05
0.55	0.152	0.276	0.0106	4.61E−02	4.31E−03	−0.0919	0.0227	0.812	7.61E−05
0.5	0.166	0.264	0.0205	4.34E−02	7.76E−03	−0.0927	0.0209	0.788	8.23E−05
0.45	0.190	0.256	0.0220	4.02E−02	7.44E−03	−0.0922	0.0193	0.764	9.23E−05
0.4	0.222	0.252	0.0110	3.59E−02	3.20E−03	−0.0892	0.0180	0.730	1.05E−04
0.372	0.242	0.252	7.83E−04	3.32E−02	2.07E−04	−0.0866	0.0174	0.707	1.15E−04
	–	–	–	–	–	–	–	–	$\Omega^2 < 0$

Table 5. Physical quantities for the sequence with the $a = 200$, $\hat{\mu} = 0.41$ and $N = 1.5$ polytropic stars.

q	$H/\|W\|$	$U/\|W\|$	$T/\|W\|$	$\|\hat{W}\|$	$\hat{\Omega}^2$	$\hat{C}$	$\hat{\beta}$	$\hat{M}$	VC
	–	–	–	–	–	–	–	–	$\Omega^2 < 0$
0.517	0.180	0.273	5.53E−04	4.71E−02	2.13E−04	−0.0950	0.0223	0.823	8.44E−05
0.5	0.186	0.270	2.04E−03	4.58E−02	7.60E−04	−0.0949	0.0217	0.817	8.72E−05
0.45	0.212	0.263	1.70E−04	4.15E−02	5.59E−05	−0.0931	0.0201	0.782	9.77E−05
	–	–	–	–	–	–	–	–	$\Omega^2 < 0$

If we define the surface shape as

$$r = r_s(\theta), \tag{48}$$

the condition that the currents flow along the surface can be written as follows:

$$j_r - \frac{1}{r_s(\theta)} \frac{dr_s}{d\theta} j_\theta = 0. \tag{49}$$

By introducing the surface fitted coordinates defined as

$$r^* = \frac{r}{r_s(\theta)}, \tag{50}$$

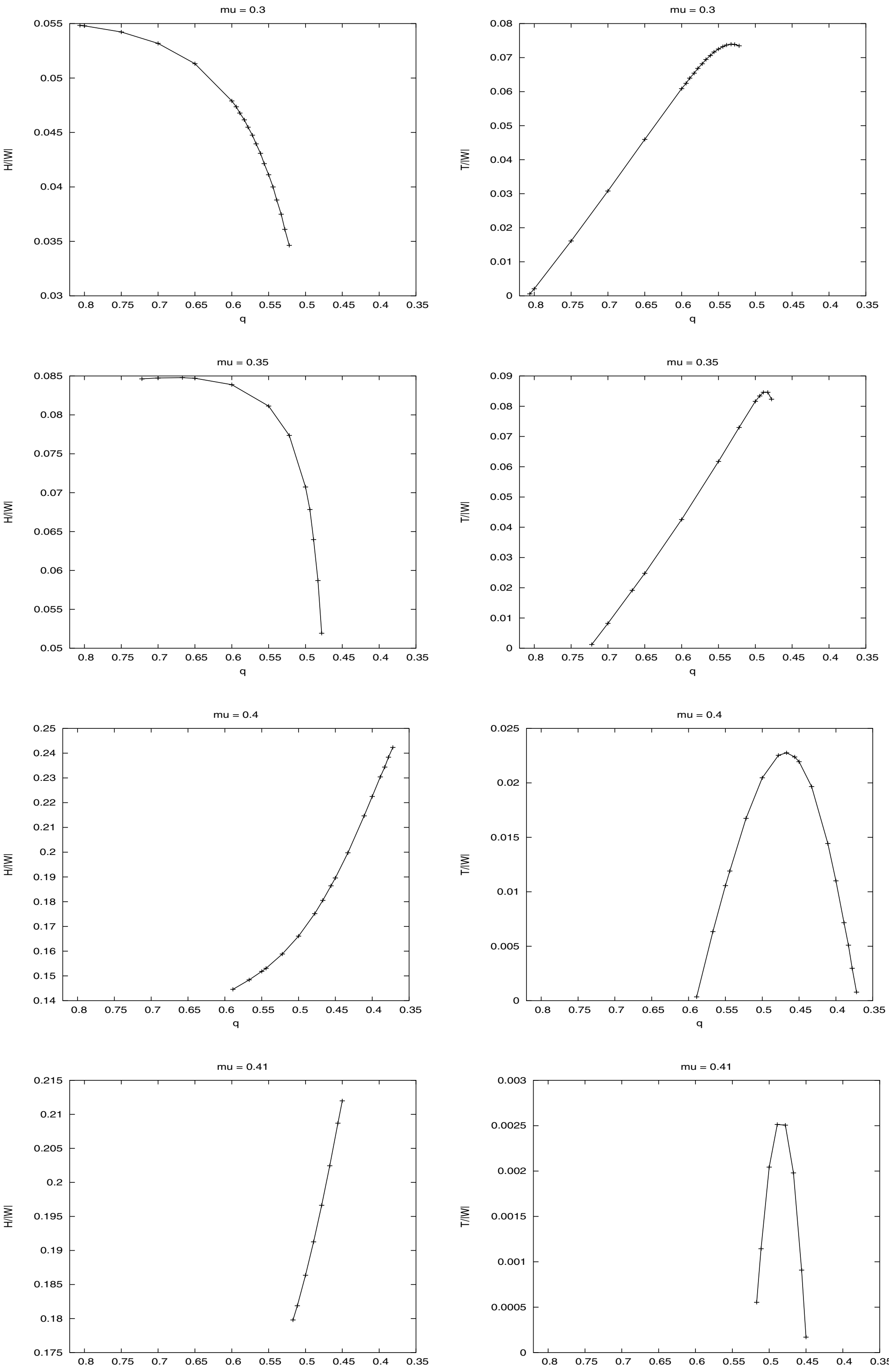

Figure 5. $H/|W|$ (left panels) and $T/|W|$ (right panels) plotted against q for $\mu = 0.3$, 0.35, 0.4 and 0.41 models whose physical quantities are shown in Tables 2–5.

$$\theta^* = \theta, \tag{51}$$

equation (49)can be expressed as

$$\frac{\partial}{\partial \theta^*}(r \sin\theta A_\phi)|_{r^*=1} = 0. \tag{52}$$

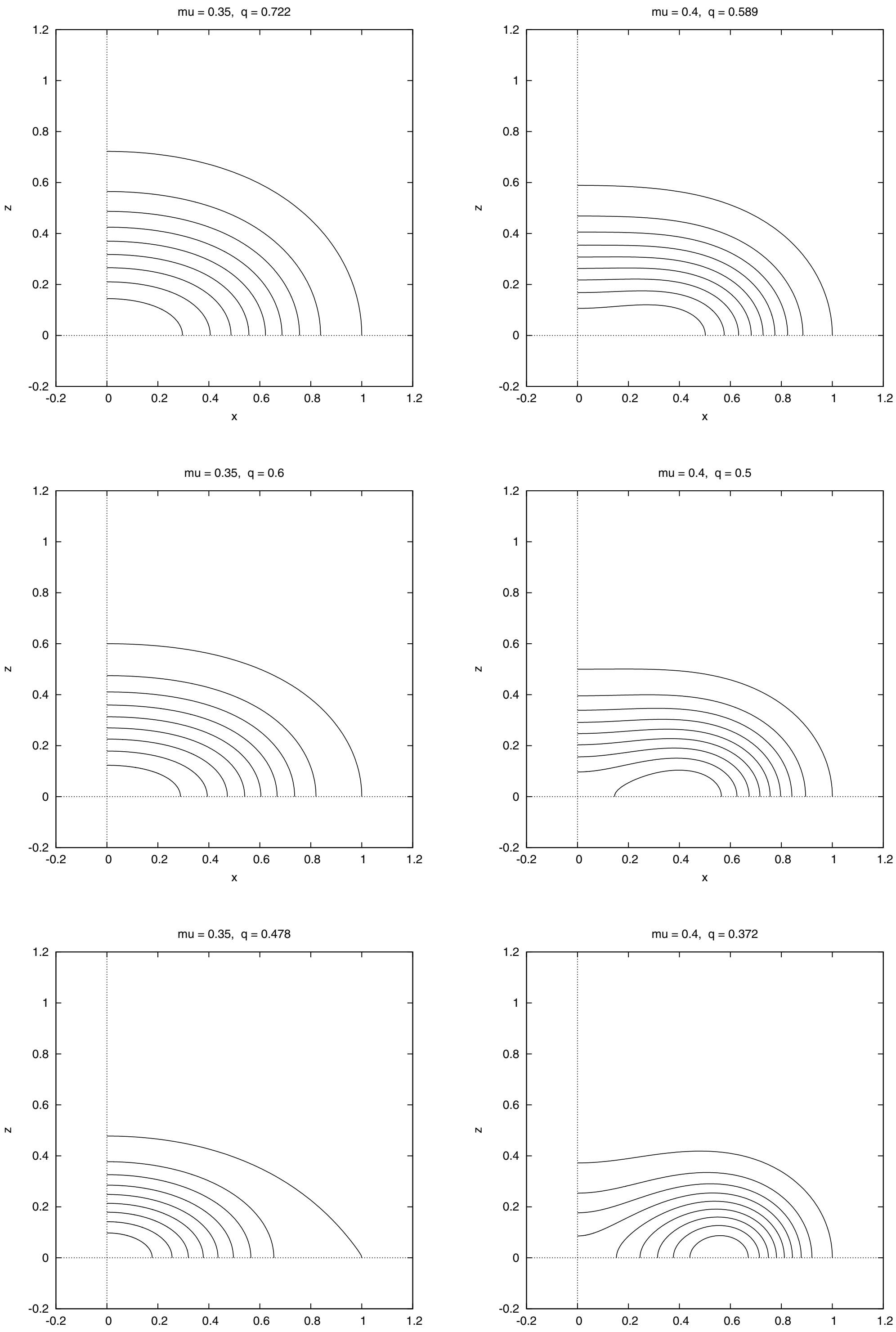

Figure 6. Equidensity contours in the meridional cross-section for the $\mu = 0.35$ (left panels) and $\mu = 0.4$ (right panels) configurations. Models in the left panels can be regarded as members of a rotation dominated sequence, while models in the right panels represent members of a magnetic field dominated sequence. The density contours are linearly spaced, i.e. the density difference between two adjacent contours is a tenth of the maximum density.

This equation implies that

$$RA_\phi = \text{constant}, \tag{53}$$

along the surface. If the matter extends to the axis of rotation and/or magnetic fields, i.e. $R = 0$, this condition results in the following boundary condition for the vector potential:

$$A_\phi = 0, \qquad \text{on the surface.} \tag{54}$$

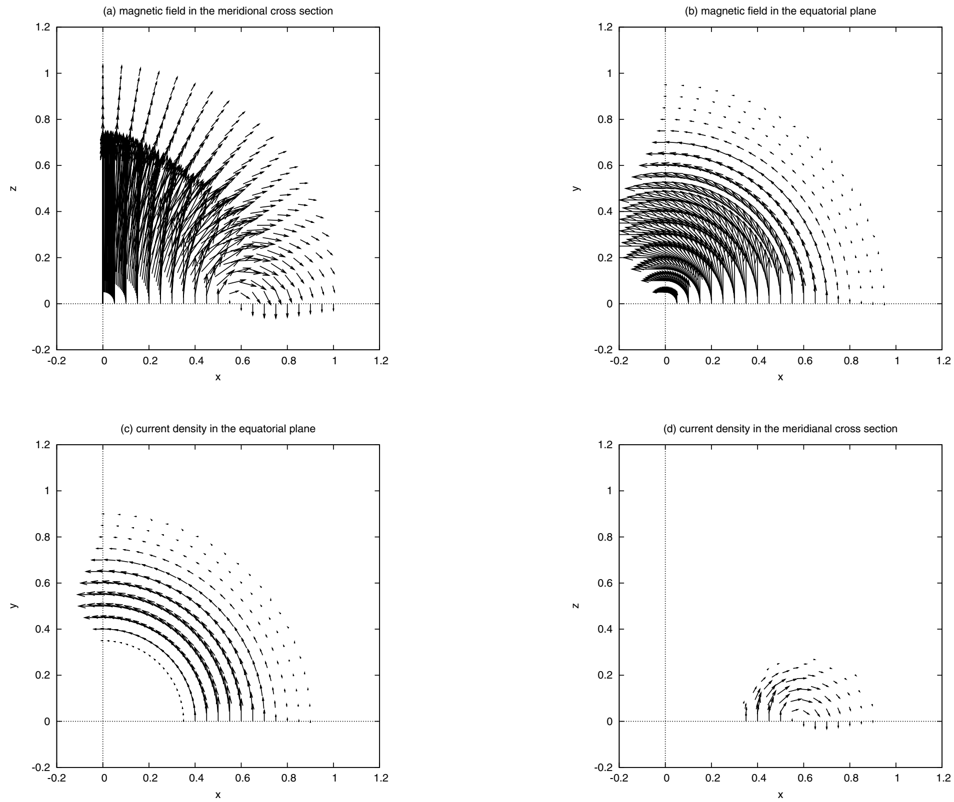

Figure 7. Physical features of a model with $\mu = 0.35$, $q = 0.478$. (a) Magnetic field in the meridional cross-section. (b) Magnetic field in the equatorial plane. (c) Current density in the equatorial plane. (d) Current density in the meridional cross-section.

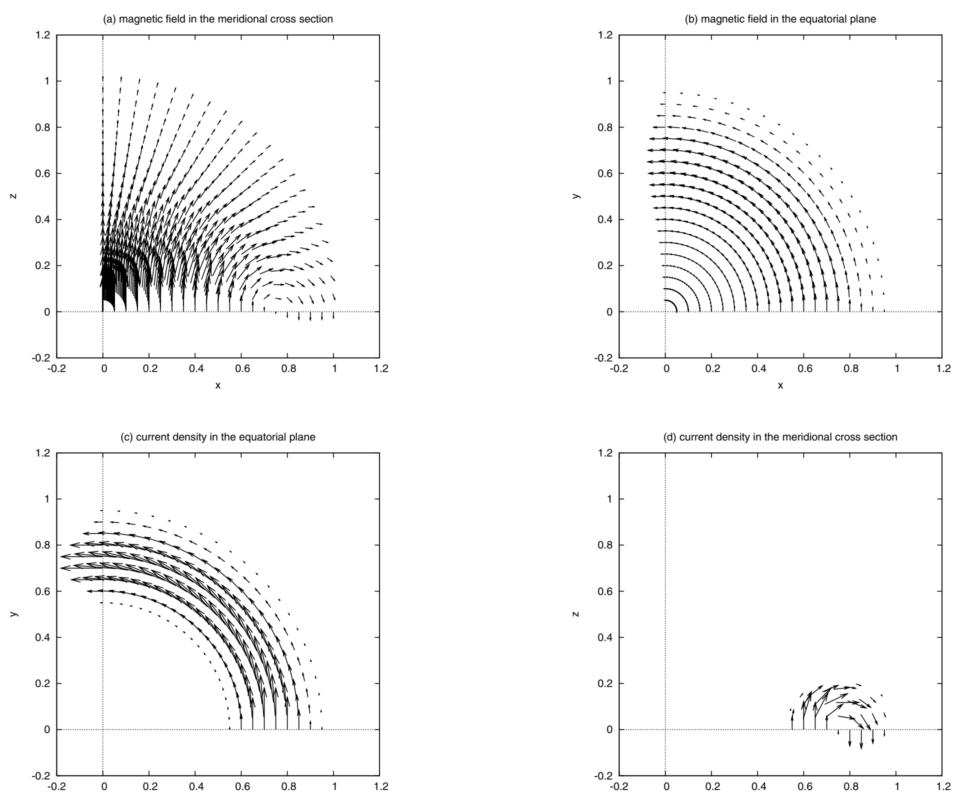

Figure 8. Physical features of a model with $\mu = 0.4$ and $q = 0.372$. (a) Magnetic field in the meridional cross-section. (b) Magnetic field in the equatorial plane. (c) Current density in the equatorial plane. (d) Current density in the meridional cross-section.

Although this boundary condition is not the same as that used by Ostriker & Hartwick (1968) and Ioka (2001), the problem is reduced to an eigenvalue problem and solutions must be discrete.

As discussed in the Introduction, the magnetic fields treated in this paper would be much stronger than those observed thus far. This can be seen if we apply some of our results to neutron stars with $\rho_{\max} = 10^{15}$ g cm^{-3} and $M = 1.4$ M$_\odot$. If we take an equilibrium configuration of the $q = 0.95$, $\hat{\kappa} = 0.0$, $\hat{\Omega}^2 = 0$ and $N = 1.5$ polytrope, the surface magnetic field and the maximum magnetic field in the interior of the star would be 2.8×10^{16} gauss at the surface and 2.5×10^{17} gauss at the maximum point in the interior of the star. If we take another equilibrium configuration of the $q = 0.5$, $\hat{\kappa} = 0.0$, $\hat{\Omega}^2 = 0$ and $N = 1.5$ polytrope, the surface magnetic field would be 2.4×10^{17} gauss and the maximum value of the magnetic field inside the star is 8.5×10^{17} gauss. These values are 'unrealistically' large. However, from the theoretical point of view, it would be necessary to develop a numerical scheme which could handle strong magnetic stars with ease.

4.2 Summary

In this paper we have presented a new numerical scheme which can solve magnetic and compressible axisymmetric stars with both *poloidal* and *toroidal* magnetic fields in addition to *rotation*. The magnetic field is assumed to decay as $1/r^2$ as $r \to \infty$ as one of the boundary conditions. This boundary condition is different from that of Ostriker & Hartwick (1968) and Ioka (2001) who have imposed vanishing magnetic fields on the stellar surfaces.

Using our new numerical scheme, we have obtained several equilibrium sequences with different model parameters. As far as our equilibrium configurations are concerned, all their shapes are oblate and some of them can be of toroidal shapes. This is a clear contrast to the prolate equilibrium configurations obtained by Ostriker & Hartwick (1968) and Ioka (2001).

We have also found that there are two types of equilibrium configurations for rotating magnetic stars, i.e. a rotation dominated type and a magnetic field dominated type.

ACKNOWLEDGMENTS

YE would like to express his gratitude to Drs Uryu and Rezzolla for their hospitality while YE stayed in SISSA where some part of this paper was written. This work was partly supported by the Grant-in-Aid for Scientific Research (C) from the Japan Society for the Promotion of Science (16540235).

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