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A New Operation on Triangular Fuzzy Number for Solving Fuzzy Linear Programming Problem

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Abstract

The fuzzy set theory has been applied in many fields such as operation research, control theory and management sciences etc. The fuzzy numbers and fuzzy values are widely used in engineering applications because of their suitability for representing uncertain information. In standard fuzzy arithmetic operations we have some problem in subtraction and division operations. In this paper, a new operation on Triangular Fuzzy Numbers is defined, where the method of subtraction and division has been modified. These modified operators yield the exact inverse of the addition and multiplication operators.

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1. Introduction

Interval Arithmetic was first suggested by Dwyer [5] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [17, 18]. After this motivation and inspiration several authors such as Alefeld and Herzberger [1], Hasen [7,8,10], Luc.Jaulin et al [14], Lodwick and Jamison[13], Neumaier [20] etc have studied interval arithmetic. The notion of Fuzzy number's as being convex and normal fuzzy set of some referential set was introduced by Zadeh. Since then the important contributes to theory of Fuzzy Number's and its applications have been made by numerous researchers. Some of the noteworthy contributions have been due to Dubois and Prade [3, 4], Kaufmann [11], Kaufmann and Gupta [12], Mizumoto and Tanaka [16], Nahmias [19] and Nguyen [21].

In the literature various operations were defined on Fuzzy numbers. These operations do not explicitly make it clear. The proposed definitions are intrinsically natural. As an abstract mathematical system, fuzzy mathematics is very little studied subject. The same comment is true for Interval arithmetic which may be consider as degeneracy Fuzzy arithmetic.

The usual arithmetic operation on real numbers can be extended to the ones defined on fuzzy numbers by means of Zadeh's extension Principle [23, 24]. According to standard fuzzy arithmetic operation using function principle [2], we have $\tilde{A} - \tilde{A} \neq 0$ and $\frac{\tilde{A}}{\tilde{A}} \neq 1$ However, in optimization and many engineering applications, it can be desirable to have crisp values for $\tilde{A} - \tilde{A}$ and $\frac{\tilde{A}}{\tilde{A}}$. ie., the crisp values 0 and 1 respectively.

In this paper, the standard fuzzy arithmetic operation on triangular fuzzy number is modified only on subtraction and division with some conditions to overcome the above. In this paper section 2 deals with some preliminary definitions and the existing function principle operations are given. In section 3, the new fuzzy arithmetic operations on triangular fuzzy number and its properties are discussed and numerical example is given. In section 4 an application of this operation is discussed. Concluding remarks are given in section 5.

2. Preliminaries

2.1 Fuzzy set: A fuzzy set $\tilde{A}$ is defined by $\tilde{A}$= {(x, μ_A(x)): x∈A, μ_A(x)∈[0,1] }. In the pair (x, μ_A(x)), the first element x belong to the classical set A, the second element μ_A(x), belong to the interval [0, 1], called Membership function.

2.2 Support of Fuzzy Set: The support of fuzzy set $\tilde{A}$ is the set of all points x in X such that $\mu_{\tilde{A}}(x)>0$. That is Support ($\tilde{A}$) = {x | $\mu_{\tilde{A}}(x)>0$}.

2.3 α- cut: The **α-** cut of **α-** level set of fuzzy set $\tilde{A}$ is a set consisting of those elements of the universe X whose membership values exceed the threshold level **α**. That is $\tilde{A}_\alpha$ **= {x /** $\mu_{\tilde{A}}(x) \geq \alpha$}.

2.4 Convex: A fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda x_2) \geq$ min($\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)$), $x_1,x_2 \epsilon$ X and $\lambda \epsilon$ [0,1]. Alternatively, a fuzzy set is convex, if all **α**-level sets are convex.

2.5 Fuzzy Number: A fuzzy set $\tilde{A}$ on R must possess at least the following three properties to qualify as a fuzzy number,

(i) $\widetilde{A}$ must be a normal fuzzy set;

(ii) ${}^{\alpha}\widetilde{A}$ must be closed interval for every $\alpha \in [0,1]$

(iii) the support of $\widetilde{A}$, ${}^{0+}\widetilde{A}$, must be bounded.

2.6 Triangular Fuzzy Number:

It is a fuzzy number represented with three points as follows: $\widetilde{A} = (a_1, a_2, a_3)$

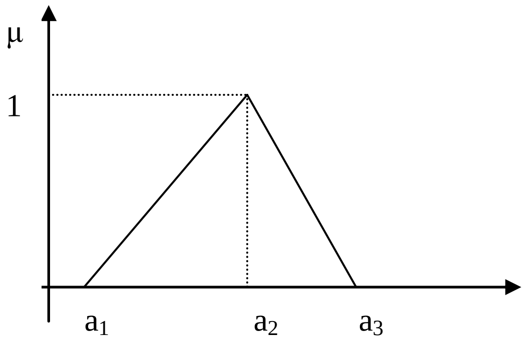
This representation is interpreted as membership functions and holds the following conditions

(i) a_1 to a_2 is increasing function

(ii) a_2 to a_3 is decreasing function

(iii) $a_1 \le a_2 \le a_3$.

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & for\ x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & for\ a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & for\ a_2 \le x \le a_3 \\ 0 & for\ x > a_3 \end{cases}$$



Triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$

2.7 α- cut of a triangular fuzzy number: We get a crisp interval by α-cut operation, interval A_α shall be obtained as follows$\forall\ \alpha \in [0, 1]$.

Thus $A_\alpha = [\alpha_1^\alpha, \alpha_3^\alpha]$ $[(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3]$.

2.8 Positive triangular fuzzy number: A positive triangular fuzzy number $\widetilde{A}$ is denoted as $\widetilde{A} = (a_1, a_2, a_3)$ where all a_i's > 0 for all i=1, 2, 3.

2.9 Negative triangular fuzzy number: A negative triangular fuzzy number $\widetilde{A}$ is denoted as $\widetilde{A} = (a_1, a_2, a_3)$ where all a_i's < 0 for all i=1, 2, 3.

Note: A negative Triangular fuzzy number can be written as the negative multiplication of a positive Triangular fuzzy number.

Eg: when $\widetilde{A} =(-3, -2, -1)$ is a negative triangular fuzzy number this can be written as $\widetilde{A} = -(1,2,3,)$.

2.10 Equal Triangular fuzzy number: Let $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. If $\widetilde{A}$ is identically equal to $\widetilde{B}$ only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

2.11 Operation of Triangular Fuzzy Number Using Function Principle:

The following are the four operations that can be performed on triangular fuzzy numbers: Let $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B}= (b_1, b_2, b_3)$ then,

(i) **Addition**: $\widetilde{A} + \widetilde{B} = (a_1+b_1, a_2+b_2, a_3+b_3)$.

(ii) **Subtraction**: $\widetilde{A} - \widetilde{B} = (a_1-b_3, a_2-b_2, a_3-b_1)$.

(iii) **Multiplication**:

$\widetilde{A} \times \widetilde{B} = (\min(a_1b_1, a_1b_3, a_3b_1, a_3b_3), a_2b_2, \max(a_1b_1, a_1b_3, a_3b_1, a_3b_3))$.

(iv) **Division**:

$\widetilde{A}/\widetilde{B} = (\min(a_1/b_1, a_1/b_3, a_3/b_1, a_3/b_3), a_2/b_2, \max(a_1/b_1, a_1/b_3, a_3/b_1, a_3/b_3))$.

2.12 Example: Let $\widetilde{A}= (2, 4, 6)$ and $\widetilde{B}=(1, 2, 3)$ be two fuzzy numbers. Then

i)$\widetilde{A} + \widetilde{B} = (3, 6, 9)$ ii) $\widetilde{A} - \widetilde{B} = (-1, 2, 5)$ iii)$\widetilde{A} \times \widetilde{B} = (2, 8, 18)$

iv) $\frac{\tilde{A}}{\tilde{B}} = (\frac{2}{3}, \frac{4}{2}, \frac{6}{1}) = (0.66, 2, 6)$ v) $\tilde{A} - \tilde{A} = (-4, 0, 4)$

vi) $\frac{\tilde{A}}{\tilde{A}} = (\frac{2}{6}, \frac{4}{4}, \frac{6}{2}) = (0.33, 1, 3)$

Remark:

As mentioned earlier that $\tilde{A} - \tilde{A} \neq 0$, $\frac{\tilde{A}}{\tilde{A}} \neq 1$, where 0 and 1 are singletons whose fuzzy representation is (0, 0, 0) and (1, 1,1). It is follows that the $\tilde{C}$ solution of the fuzzy linear equation $\tilde{A} + \tilde{B} = \tilde{C}$ is not as we would expect, $\tilde{B} = \tilde{C} - \tilde{A}$

For example, $\tilde{A} + \tilde{B} = (2, 4, 6)+(1, 2, 3) = (3, 6, 9) = \tilde{C}$

But (1,2,3) = (3,6,9) – (2,4,6) = (–3,2,6) $\neq \tilde{B}$.

The same annoyance appears when solving the fuzzy equation $\tilde{A} \times \tilde{B} = \tilde{C}$ whose solution is not given by $\tilde{B} = \frac{\tilde{C}}{\tilde{A}}$.

For example, $\tilde{A} \times \tilde{B} = (2,8,18) = \tilde{C}$

But $\tilde{B} = \frac{(2,8,18)}{(2,4,6)} = \left(\frac{2}{6}, \frac{8}{4}, \frac{18}{2}\right) = (0.33, 2, 9) \neq \tilde{B}$

Therefore, the addition and subtraction (resp Multiplication and division) of fuzzy numbers are not reciprocal operations. According to this statement, it is not possible to solve inverse problems exactly using the standard fuzzy arithmetic operators.

To overcome this in function principle operation of triangular fuzzy number a new operation is proposed that allows exact inversion.

3. A New Operation for Subtraction and Division on Triangular fuzzy number

In this section our objective is to develop a new subtraction and division operators on triangular fuzzy number, which are inverse operations of the addition '+' and Multiplication '×' operators.

3.1 Subtraction:

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ then, $\tilde{A} - \tilde{B} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$.

The new subtraction operation exist only if the following condition is satisfied $DP(\tilde{A}) \geq DP(\tilde{B})$.

Where $DP(\tilde{A}) = \frac{a_3 - a_1}{2}$ and $DP(\tilde{B}) = \frac{b_3 - b_1}{2}$; where DP denotes Difference point of a Triangular fuzzy number.

3.2 Properties of Subtraction Operator:

(i) Inverse operator of +: $\tilde{B} + (\tilde{A} - \tilde{B}) = (\tilde{A} - \tilde{B}) + \tilde{B}$.

(ii) Multiplication by a Scalar: $w(\tilde{A} - \tilde{B}) = (w\tilde{A} - w\tilde{B})$

(iii) Neutral element: $(\tilde{A} - 0) = \tilde{A}$

(iv) Associativity: $\tilde{A} - (\tilde{B} - \tilde{C}) = (\tilde{A} - \tilde{B}) - \tilde{C}$

(v) Inverse element: Any Fuzzy Number is its own inverse under the modified subtraction. i.e $\tilde{A} - \tilde{A} = 0$.

(vi) Regularity: $\tilde{\mathbf{A}} - \tilde{\mathbf{B}} = \tilde{\mathbf{A}} - \tilde{\mathbf{C}} \Rightarrow \tilde{\mathbf{B}} = \tilde{\mathbf{C}}$

(v) Psedu – distributivity with regard to +:

$$(\tilde{A} + \tilde{B}) - (\tilde{C} + \tilde{D}) = (\tilde{A} - \tilde{C}) + (\tilde{B} - \tilde{D})$$

3.3 Necessary Existence Condition for Subtraction

Proposition 3.1

The new subtraction operation exist only if the following condition is satisfied $DP(\tilde{A}) \geq DP(\tilde{B})$

Proof:

From the definition 2.6 (i) and (ii) are straight forward when (iii) satisfies. Now we have derive the necessary existence condition for $\tilde{A} - \tilde{B}$ which is equal to (c_1, c_2, c_3)

Let take $c_1 \leq c_2 \leq c_3 \Rightarrow c_1 \leq c_3 \Rightarrow a_1 - b_1 \leq a_3 - b_3 \Rightarrow b_3 - b_1 \leq a_3 - a_1$

$\Rightarrow (MP(\tilde{B}) + DP(\tilde{B})) - (MP(\tilde{B}) - DP(\tilde{B})) \leq (MP(\tilde{A}) + DP(\tilde{A})) - (MP(\tilde{A}) - DP(\tilde{A}))$

$\Rightarrow MP(\tilde{B}) + DP(\tilde{B}) - MP(\tilde{B}) + DP(\tilde{B}) \leq MP(\tilde{A}) + DP(\tilde{A}) - MP(\tilde{A}) + DP(\tilde{A})$

$\Rightarrow 2DP(\tilde{B}) \leq 2DP(\tilde{A}) \Rightarrow DP(\tilde{A}) \geq DP(\tilde{B})$

This is the necessary existence condition for new subtraction operator.

3.4 Division:

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ then, $\frac{\tilde{A}}{\tilde{B}} = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3})$

The new Division operation exists only if the following conditions are satisfied $\left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right| \geq \left|\frac{DP(\tilde{B})}{MP(\tilde{B})}\right|$ *and the negative triangular fuzzy number should be changed into negative multiplication of positive number as per note in defn 2.9.*

Where $MP(\tilde{A}) = \frac{a_3 + a_1}{2}$, $DP(\tilde{A}) = \frac{a_3 - a_1}{2}$, $MP(\tilde{B}) = \frac{b_3 + b_1}{2}$ and $DP(\tilde{B}) = \frac{b_3 - b_1}{2}$ Where MP denotes Midpoint and DP denotes Difference point of a Triangular fuzzy number.

3.5 Properties of Division Operator:

(i) Inverse Operator of ×: $\tilde{B} \times \frac{\tilde{A}}{\tilde{B}} = \frac{\tilde{A}}{\tilde{B}} \times \tilde{B} = \tilde{A}$.

(ii) Neutral element: The singleton $\tilde{1} = (1,1,1)$ defined by a constant profile equal to $\tilde{1}$ is a right neutral element of division $\frac{\tilde{A}}{\tilde{1}} = \left(\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}\right) = (a_1, a_2, a_3) = \tilde{A}$

(iii) Inverse element: Any fuzzy number is its own inverse under the modified division operator $\frac{\tilde{A}}{\tilde{A}} = \left(\frac{a_1}{a_1}, \frac{a_2}{a_2}, \frac{a_3}{a_3}\right) = (1, 1, 1)$.

(iv) Regularity $\frac{\tilde{A}}{\tilde{B}} = \frac{\tilde{A}}{\tilde{C}} \Rightarrow \tilde{B} = \tilde{C}$.

(v) Distributivity with regard to +: $\frac{(\tilde{A} + \tilde{B})}{\tilde{C}} = \frac{\tilde{A}}{\tilde{C}} + \frac{\tilde{B}}{\tilde{C}}$.

3.6 Necessary Existence Condition for Division:

Proposition 3.2

The new Division operation exists only if the following condition is satisfied

$$\left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right| \geq \left|\frac{DP(\tilde{B})}{MP(\tilde{B})}\right|.$$

Proof:

From the definition 2.6

(i) and (ii) are straight forward when (iii) satisfies

Now we have derive the necessary existence condition for $\tilde{A}$ / $\tilde{B}$ which is equal to (c_1,c_2,c_3)

Let take $c_1 \leq c_2 \leq c_3 \Rightarrow c_1 \leq c_3 \Rightarrow a_1 / b_1 \leq a_3 / b_3 \Rightarrow a_1 b_3 \leq a_3 b_1$

$\Rightarrow (MP(\tilde{A})-DP(\tilde{A}))\ (MP(\tilde{B}) + DP(\tilde{B})) \leq (MP(\tilde{A}) + DP(\tilde{A}))\ (MP(\tilde{B})-DP(\tilde{B}))$

$\Rightarrow MP(\tilde{A})MP(\tilde{B}) + MP(\tilde{A})DP(\tilde{B}) - DP(\tilde{A})MP(\tilde{B}) - DP(\tilde{A})DP(\tilde{B})$

$\leq MP(\tilde{A})MP(\tilde{B}) - MP(\tilde{A})DP(\tilde{B}) + DP(\tilde{A})MP(\tilde{B}) - DP(\tilde{A})DP(\tilde{B})$

$\Rightarrow MP(\tilde{A})DP(\tilde{B}) - DP(\tilde{A})MP(\tilde{B}) \leq - MP(\tilde{A})DP(\tilde{B}) + DP(\tilde{A})MP(\tilde{B})$

$\Rightarrow 2MP(\tilde{A})DP(\tilde{B}) \leq 2DP(\tilde{A})MP(\tilde{B}) \Rightarrow \frac{DP(\tilde{B})}{MP(\tilde{B})} \leq \frac{DP(\tilde{A})}{MP(\tilde{A})}$

In this $\tilde{B}$ may positive or negative. So we take the condition as $\left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right| \geq \left|\frac{DP(\tilde{B})}{MP(\tilde{B})}\right|$.

This is the necessary existence condition for new division operator.

3.7 Numerical Examples:

1. $\tilde{A}$**= (5,6,7)**

   $MP(\tilde{A}) = \frac{a_3+a_1}{2} = \frac{7+5}{2} = \frac{12}{2} = 6$; $DP(\tilde{A}) = \frac{a_3-a_1}{2} = \frac{7-5}{2} = \frac{2}{2} = 1$

   (i) Subtraction:

   Now $DP(\tilde{A}) = 1$ therefore $DP(\tilde{A}) = DP(\tilde{A})$. Hence $\tilde{A} - \tilde{A}$ is satisfying the condition. $\tilde{A} - \tilde{A} = (a_1-a_1, a_2-a_2, a_3-a_3) = (5-5, 6-6, 7-7) = (0, 0, 0)$

   (ii) Divide:

   Now $\left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right| = \left|\frac{1}{6}\right| = 0.167$ therefore $\left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right| = \left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right|$. Hence $\frac{\tilde{A}}{\tilde{A}}$ satisfying the above condition. $\frac{\tilde{A}}{\tilde{A}} = (\frac{5}{5}, \frac{6}{6}, \frac{7}{7}) = (1, 1, 1)$

2. $\tilde{B}$ **= (-5,-3,-2)**

   $MP(\tilde{B}) = \frac{a_3+a_1}{2} = \frac{-2-5}{2} = \frac{-7}{2} = -3.5$; $DP(\tilde{B}) = \frac{a_3-a_1}{2} = \frac{-2+5}{2} = \frac{3}{2} = 1.5$

   (i) Subtraction:

   Now $DP(\tilde{B}) = 1.5$ therefore $DP(\tilde{B}) = DP(\tilde{B})$. Hence $\tilde{B} - \tilde{B}$ satisfying the condition. $\tilde{B} - \tilde{B} = (-5+5, -3+3, -2+2) = (0, 0, 0)$.

   (ii) Division

   Now $\left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right| = \left|\frac{1.5}{-3.5}\right| = 0.428$ therefore $\left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right| = \left|\frac{DP(\tilde{A})}{MP(\tilde{A})}\right|$ Hence $\frac{\tilde{A}}{\tilde{A}}$ satisfying the condition. $\frac{\tilde{A}}{\tilde{A}} = (\frac{-5}{-5}, \frac{-3}{-3}, \frac{-2}{-2}) = (1, 1, 1)$.

4. Application of the New Operations

Here we are going to solve fully fuzzy Linear programming problem using simplex algorithm:

Max $\tilde{z}$ = (5, 7, 9) $\tilde{x}_1$+ (6, 8, 10)$\tilde{x}_2$

Subject to constraint

(1,2,3) $\tilde{x}_1$+ (2,3,4) $\tilde{x}_2 \leq$ (4,6,8)

(4,5,6) $\tilde{x}_1$+(3,4,5) $\tilde{x}_2 \leq$ (8,10,12)

Rewrite as

Max $\tilde{z}$ = (5, 7, 9) $\tilde{x}_1$+ (6, 8, 10) $\tilde{x}_2$+(0,0,0) $\tilde{x}_3$ +(0,0,0)$\tilde{x}_4$

Subject to constraint

(1,2,3) $\tilde{x}_1$+ (2,3,4) $\tilde{x}_2$+(1,1,1) $\tilde{x}_3$ = (4,6,8)

(4,5,6) $\tilde{x}_1$+(3,4,5) $\tilde{x}_2$+(1,1,1) $\tilde{x}_4$= (8,10,12)

Simplex table:

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	RHS
$\tilde{x}_3$ $\tilde{x}_4$	(1,2,3) (4,5,6)	**(2,3,4)** (3,4,5)	(1,1,1) (0,0,0)	(0,0,0) (1,1,1)	(4,5,6) (8,10,12)
$\tilde{z}$	-(5,7,9)	-(6,8,10)	(0,0,0)	(0,0,0)	(0,0,0)
$\tilde{x}_2$ $\tilde{x}_4$	**(0.5,.66,.75)** (.25,2.36,4.5)	(1,1,1) (0,0,0)	(0.25,0.33,0.5) -(0.75,1.32,2.5)	(0,0,0) (1,1,1)	(1,2,4) (-12,2,9)
$\tilde{z}$	(-2,-1.72,-1.5)	(0,0,0)	(1.5,2.64,5)	(0,0,0)	(6,16,40)
$\tilde{x}_1$ $\tilde{x}_4$	(1,1,1) (0,0,0)	(1.33,1.52,2) -(0.33,3.59,9)	(0.5,0.5,0.66) -(0.88, 2.5,5.47)	(0,0,0) (1,1,1)	(1.33,3.03,8) (-48,-5.15,8.67)
$\tilde{z}$	(0,0,0)	(1.99,2.61,4)	(2.25,3.5,18.86)	(0,0,0)	(7.99,21.21,56)

Hence $\tilde{x}_1$ =(1.33,3.03,8) & $\tilde{x}_2$= (0,0,0) then $\tilde{z}$ =(7.99,21.21,56)

5. Conclusion

The main aim of this paper is to introduce a new operation for subtraction and division. The advantage of these operations is to undergo the inverse operations of the addition and multiplication. These operations may help us to solve many optimization problems.

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