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A new operational matrix of Caputo fractional derivatives of Fermat polynomials: an application for solving the Bagley-Torvik equation

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Abstract

Herein, an innovative operational matrix of fractional-order derivatives (*sensu* Caputo) of Fermat polynomials is presented. This matrix is used for solving the fractional Bagley-Torvik equation with the aid of tau spectral method. The basic approach of this algorithm depends on converting the fractional differential equation with its initial (boundary) conditions into a system of algebraic equations in the unknown expansion coefficients. The convergence and error analysis of the suggested expansion are carefully discussed in detail based on introducing some new inequalities, including the modified Bessel function of the first kind. The developed algorithm is tested via exhibiting some numerical examples with comparisons. The obtained numerical results ensure that the proposed approximate solutions are accurate and comparable to the analytical ones.

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Keywords: Fermat polynomials; operational matrix of fractional derivatives; tau method; Bagley-Torvik equation

1 Introduction

Fractional-order calculus is a vital branch of mathematical analysis. Many practical problems in various fields such as mechanics, engineering and medicine are modeled by fractional differential equations. For example, Torvik and Bagley [1] formulated a fractional differential equation that simulates the motion of a rigid plate immersed in a Newtonian fluid as follows:

$$AD^2 f(t) + BD^{\frac{3}{2}} f(t) + Cf(t) = g(t). \tag{1.1}$$

A, B and C in (1.1) are constants depending on mass and area of the plate, stiffness of spring, fluid density and viscosity. Moreover, the function $g(t)$ in (1.1) is a known function denoting the external force and $f(t)$ stands for the displacement of the plate, and it should be solved.

Spectral methods have prominent roles in treating various types of differential equations. Over the past four decades, the appeal of spectral methods for applications such as computational fluid dynamics has expanded. In fact, spectral methods are widely used in diverse applications such as wave propagation (for acoustic, elastic, seismic and electromagnetic waves), solid and structural analysis, marine engineering, biomechanics, astrophysics and even financial engineering. The main difference between these techniques is the specific choice of the trial and test functions. The main idea behind spectral methods is to assume spectral solutions of the form $\sum a_k \phi_k(x)$. The expansion coefficients a_k can be determined if a suitable spectral method is applied. The collocation method requires enforcing the differential equation to be satisfied exactly at some nodes (collocation points). The tau method is a synonym for expanding the residual function as a series of orthogonal polynomials and then applying the boundary conditions as constraints. The Galerkin method principally depends on selecting some suitable combinations of orthogonal polynomials which satisfy the underlying boundary (initial) conditions which are called 'basis functions', and after that the residual is enforced to be orthogonal with the suggested basis functions. For an intensive study on spectral methods and their applications, see Canuto et al. [2], Hesthaven et al. [3], Boyd [4], Trefethen [5], Doha et al. [6] and Bhrawy et al. [7].

Many number and polynomial sequences can be generated by difference equations of order two. Fermat polynomials are among these polynomials. This class of polynomials is considered as a special case of the general class of (p, q)-Fibonacci polynomial sequence (see [8]). This class of polynomials is of fundamental interest in mathematics since it includes some polynomials which have numerous important applications in several fields such as combinatorics and number theory. There are many papers dealing with these kinds of polynomials from a theoretical point of view (see, for example, [8–10]); however, the numerical investigations concerning these polynomials are very rare. In this respect, recently, a collocation algorithm based on employing Fibonacci polynomials has been analyzed for solving Volterra-Fredholm integral equations in [11]. Moreover, a numerical approach with error estimation to solve general integro-differential-difference equations using Dickson polynomials has been developed in [12]. This gives us a motivation for utilizing these polynomials in several numerical applications.

Recently, operational matrices have been employed for solving several kinds of differential problems, namely ordinary differential equations, fractional differential equations and integro-differential equations. The use of operational matrices of different orthogonal polynomials jointly with spectral methods produces efficient, accurate solutions for such equations (see, for example, [13–20]). The main aim of this paper can be summarized in the following two points:

- To establish a new operational matrix of fractional derivatives of Fermat polynomials.
- To analyze and present an algorithm for solving the Bagley-Torvik equation based on applying the spectral tau method.

The outline of this paper is as follows. In Section 2, we introduce some necessary definitions of the fractional calculus. Moreover, in this section, an overview on Fermat polynomials is given including some properties and also some new formulae which are useful in the sequel. Section 3 is concerned with the construction of an operational matrix of fractional derivatives (OMFD) in the Caputo sense of Fermat polynomials. In Section 4, we analyze and present a spectral tau algorithm for solving the Bagley-Torvik equation. We give a global error bound for the suggested Fermat expansion in Section 5. Some test problems with comparisons are displayed in Section 6. Finally, some concluding remarks are displayed in Section 7.

2 Preliminaries and useful formulae

This section is dedicated to presenting some fundamentals of the fractional calculus theory which will be useful throughout this article. Moreover, an overview on Fermat polynomials and some new formulae concerning these polynomials are presented.

2.1 Some fundamentals of fractional calculus

Definition 1 ([21]) The Riemann-Liouville fractional integral operator I^α of order α on the usual Lebesgue space $L_1[0,1]$ is defined as

$$\big(I^\alpha\big)f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{2.1}$$

The following properties are satisfied by this operator:

(i) $I^\alpha I^\beta = I^{\alpha+\beta}$,

(ii) $I^\alpha I^\beta = I^\beta I^\alpha$,

(iii) $I^\alpha t^\nu = \dfrac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} t^{\nu+\alpha}$,

where $\alpha, \beta \geq 0$ and $\nu > -1$.

Definition 2 ([21]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$\big(D_*^\alpha f\big)(t) = \left(\frac{d}{dt}\right)^m \big(I^{m-\alpha} f\big)(t), \quad m-1 \leq \alpha < m, m \in \mathbb{N}. \tag{2.2}$$

Definition 3 The fractional differential operator in the Caputo sense is defined as

$$\big(D^\alpha f\big)(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\,d\tau, \quad \alpha > 0, t > 0, \tag{2.3}$$

where $m-1 \leq \alpha < m$, $m \in \mathbb{N}$.

The operator D^α satisfies the following basic properties for $m-1 \leq \alpha < m$:

$$\begin{aligned} &\big(D^\alpha I^\alpha f\big)(t) = f(t), \\ &\big(I^\alpha D^\alpha f\big)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} t^k, \quad t > 0, \\ &D^\alpha t^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} t^{k-\alpha}, \quad k \in \mathbb{N}, k \geq \lceil \alpha \rceil, \end{aligned} \tag{2.4}$$

where the ceiling notation $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α.

For survey on the fractional derivatives and integrals, one can be referred to [21, 22].

2.2 An overview on Fermat polynomials

Fermat polynomials can be generated by the difference equation

$$\mathcal{F}_{i+2}(t) = 3t\mathcal{F}_{i+1}(t) - 2\mathcal{F}_i(t), \qquad \mathcal{F}_0(t) = 0, \qquad \mathcal{F}_1(t) = 1, \quad i \geq 0. \tag{2.5}$$

In fact, Fermat polynomials are particular polynomials of the so-called (p,q)-Fibonacci polynomials which were introduced in [8]. These polynomials can be generated with the aid of the recurrence relation

$$u_{i+2}(t) = p(t)u_{i+1}(t) + q(t)u_i(t), \quad i \geq 0,$$

with the initial conditions

$$u_0(t) = 0, \qquad u_1(t) = 1.$$

The Binet formula of $u_i(t)$ (see [8]) is

$$\begin{aligned} u_i(t) &= \frac{\alpha^i(t) - \beta^i(t)}{\alpha(t) - \beta(t)}, \\ \alpha(t) &= \frac{p(t) + \sqrt{p^2(t) + 4q(t)}}{2}, \qquad \beta(t) = \frac{p(t) - \sqrt{p^2(t) + 4q(t)}}{2}. \end{aligned} \tag{2.6}$$

Fermat polynomials can be obtained from these polynomials for the case corresponding to $p(t) = 3t$ and $q(t) = -2$.

In the present work we will use Fermat polynomials over the domain $t \in [0,1]$. Fermat polynomials can be written explicitly as

$$\mathcal{F}_k(t) = \begin{cases} \frac{(3t+\sqrt{9t^2-8})^k - (3t-\sqrt{9t^2-8})^k}{2^k\sqrt{9t^2-8}}, & t \neq \frac{2}{3}; \\ 2^{\frac{k}{2}} \sin(\frac{\pi}{4}k), & t = \frac{2}{3}, \end{cases} \qquad k = 1, 2, \ldots. \tag{2.7}$$

Note 1 It is worth mentioning here that $\mathcal{F}_k(t)$ is a polynomial of degree $(k-1)$ with integer coefficients.

The polynomials $\mathcal{F}_{i+1}(t)$, $i \geq 0$ have the following analytic form:

$$\mathcal{F}_{i+1}(t) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-2)^k 3^{i-2k} \binom{i-k}{k} t^{i-2k}, \tag{2.8}$$

where $\lfloor z \rfloor$ denotes the largest integer less than or equal to z. In the following, we are going to state and prove two basic theorems. The first is concerned with the inversion formula to formula (2.8); while in the second, a new expression for the first derivative of Fermat polynomials is given in terms of their original polynomials.

Theorem 1 *For every nonnegative integer m, the following inversion formula is valid*:

$$t^m = \frac{1}{3^m} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{2^i(-2i + m + 1)\binom{m}{i}}{m - i + 1} \mathcal{F}_{m-2i+1}(t). \tag{2.9}$$

Proof We proceed by induction on m. Formula (2.9) holds trivially for $m = 0$. Now, assume that formula (2.9) is valid, and we have to prove that the following formula is valid:

$$t^m = \frac{1}{3^{m+1}} \sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \frac{2^i(-2i+m+2)\binom{m+1}{i}}{m-i+2} \mathcal{F}_{m-2i+2}(t). \tag{2.10}$$

If we multiply both sides of formula (2.9) by t, then, making use of the recurrence relation for Fermat polynomial (2.5), we get

$$\begin{aligned} t^{m+1} &= \frac{1}{3^{m+1}} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{2^i(-2i+m+1)\binom{m}{i}}{m-i+1} \mathcal{F}_{m-2i+2}(t) \\ &\quad + \frac{1}{3^{m+1}} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{2^{i+1}(-2i+m+1)\binom{m}{i}}{m-i+1} \mathcal{F}_{m-2i}(t), \end{aligned} \tag{2.11}$$

in the latter formula, in the second summation, letting $i \to i-1$ and collecting similar terms, we turn it into the form

$$\begin{aligned} t^{m+1} &= \frac{1}{3^{m+1}} \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \frac{2^i(-2i+m+2)\binom{m+1}{i}}{i-m-2} \mathcal{F}_{m-2i+2}(t) + \frac{1}{3^{m+1}} \mathcal{F}_{m+2}(t) \\ &\quad - \frac{(2\lfloor \frac{m}{2} \rfloor - m - 1)m! 2^{\lfloor \frac{m}{2} \rfloor + 1}}{3^{m+1} \lfloor \frac{m}{2} \rfloor ! \Gamma(m - \lfloor \frac{m}{2} \rfloor + 2)} \mathcal{F}_{m-2\lfloor \frac{m}{2} \rfloor}(t). \end{aligned} \tag{2.12}$$

If we note that

$$\frac{-(2\lfloor \frac{m}{2} \rfloor - m - 1)m! 2^{\lfloor \frac{m}{2} \rfloor + 1}}{3^{m+1} \lfloor \frac{m}{2} \rfloor ! \Gamma(m - \lfloor \frac{m}{2} \rfloor + 2)} = \begin{cases} -\frac{2^{\frac{m}{2}+1} m!}{3^{m+1} (\frac{m}{2})! (\frac{m}{2}+1)!}, & m \text{ even}, \\ -\frac{2^{\frac{m+3}{2}} m!}{3^{m+1} (\frac{m-1}{2})! (\frac{m+3}{2})!}, & m \text{ odd}, \end{cases}$$

then it is not difficult to see that (2.12) can be written in the alternative form

$$t^m = \frac{1}{3^{m+1}} \sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \frac{2^i(-2i+m+2)\binom{m+1}{i}}{m-i+2} \mathcal{F}_{m-2i+2}(t).$$

Theorem 1 is now proved. □

Theorem 2 *The first derivative of Fermat polynomials is linked with their original polynomials by the following formula*:

$$D\mathcal{F}_{i+1}(t) = 3 \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} 2^m (i-2m) \mathcal{F}_{i-2m}(t). \tag{2.13}$$

Proof The differentiation of the analytic form of the Fermat polynomials in (2.8) with respect to t yields

$$D\mathcal{F}_{i+1}(t) = \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} (-2)^k 3^{i-2k} \binom{i-k}{k} (i-2k) t^{i-2k-1}. \tag{2.14}$$

If we make use of the inversion formula (2.9), then relation (2.14) can be written as

$$\begin{aligned} D\mathcal{F}_{i+1}(t) = &\sum_{k=0}^{\lfloor\frac{i-1}{2}\rfloor}(-2)^k 3^{i-2k}(i-2k)\binom{i-k}{k} \\ &\times \sum_{r=0}^{\lfloor\frac{i-1}{2}\rfloor-k}\frac{2^r 3^{-i+2k+1}\binom{i-2k-1}{r}(2k+2r-1)}{2k+r-i}\mathcal{F}_{i-2k-2r}(t). \end{aligned}$$

In the latter relation, by letting $k+r=m$ and arranging the terms, we get

$$D\mathcal{F}_{i+1}(t) = \sum_{m=0}^{\lfloor\frac{i-1}{2}\rfloor} A_{i,m}\mathcal{F}_{i-2m}(x), \tag{2.15}$$

where $A_{i,m}$ is given by

$$A_{i,m} = 3\left(2^m\right)(i-2m)\sum_{j=0}^{m}\frac{(-1)^j(i-j)!}{j!(m-j)!(i-j-m)!}.$$

$A_{i,m}$ can be written in terms of the hypergeometric form ${}_2F_1(1)$ as

$$A_{i,m} = 3\left(2^m\right)(i-2m)\binom{i}{m}{}_2F_1\left(\left.\begin{matrix}-m,-i+m\\ -i\end{matrix}\right|1\right). \tag{2.16}$$

The Chu-Vandermonde identity implies that the hypergeometric ${}_2F_1(1)$ in (2.16) can be summed to give (see Koepf [23])

$${}_2F_1\left(\left.\begin{matrix}-m,-i+m\\ -i\end{matrix}\right|1\right) = \binom{i}{m}^{-1}, \tag{2.17}$$

and, accordingly, $A_{i,m}$ takes the following reduced form:

$$A_{i,m} = 3(2^m)(i-2m).$$

Theorem 2 is now proved. □

Now, it is useful to rewrite an analytic form of the Fermat polynomials in (2.8) and its inversion formula (2.9) in the following two equivalent formulae:

$$\mathcal{F}_{i+1}(t) = \sum_{\substack{k=0\\(k+i)\text{ even}}}^{i} 3^k(-2)^{\frac{i-k}{2}}\binom{\frac{i+k}{2}}{\frac{i-k}{2}}t^k, \quad i\geq 1, \tag{2.18}$$

and

$$t^m = \frac{1}{3^m}\sum_{\substack{r=0\\(r+m)\text{ even}}}^{m}\frac{2^{\frac{m-r}{2}+1}\binom{m}{\frac{m-r}{2}}}{m+r+2}(r+1)\mathcal{F}_{r+1}(t), \quad m\geq 1. \tag{2.19}$$

For more properties of Fermat polynomials and their related numbers, see, for example, [24, 25].

3 Fermat operational matrix of the Caputo fractional derivative

Let $f(t)$ be a square Lebesgue integrable function on $(0,1)$. Let $f(t)$ be a function which can be expressed in terms of the linearly independent Fermat polynomials as

$$f(t) = \sum_{j=1}^{\infty} b_j \mathcal{F}_j(t).$$

If the series is truncated, then we have

$$f(t) \approx f_M(t) = \sum_{k=1}^{M+1} b_k \mathcal{F}_k(t) = \mathbf{B}^T \mathbf{\Phi}(t), \tag{3.1}$$

where

$$\mathbf{B}^T = [b_1, b_2, \dots, b_{M+1}], \tag{3.2}$$

and

$$\mathbf{\Phi}(t) = \left[\mathcal{F}_1(t), \mathcal{F}_2(t), \dots, \mathcal{F}_{M+1}(t)\right]^T. \tag{3.3}$$

We can assume that $\frac{d\mathbf{\Phi}(t)}{dt}$ can be written as

$$\frac{d\mathbf{\Phi}(t)}{dt} = \mathbf{G}^{(1)} \mathbf{\Phi}(t), \tag{3.4}$$

where $\mathbf{G}^{(1)} = (g_{ij}^{(1)})$ is the $(M+1) \times (M+1)$ operational matrix of derivatives whose nonzero elements can be obtained directly from relation (2.13). They can be written explicitly as

$$g_{ij}^{(1)} = \begin{cases} 3(j+1)2^{\frac{i-j-1}{2}}, & i > j, (i+j) \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

For example, for $M = 6$, the operational matrix $\mathbf{G}^{(1)}$ is given by

$$\mathbf{G}^{(1)} = 3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 & 0 \\ 4 & 0 & 6 & 0 & 5 & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & 6 & 0 \end{pmatrix}_{7\times 7}.$$

3.1 Construction of Fermat OMFD

The section aims to construct the OMFD which generalizes the operational matrix of derivatives for the integer case. From relation (2.13), it is easy to observe that for every positive integer r, we have

$$\frac{d^r\mathbf{\Phi}(t)}{dt^r}=\mathbf{G}^{(r)}\mathbf{\Phi}(t)=\left(\mathbf{G}^{(1)}\right)^r\mathbf{\Phi}(t). \tag{3.5}$$

Theorem 3 *Let $\mathbf{\Phi}(t)$ be the Fermat polynomial vector defined in Eq.* (3.3). *For any $\alpha>0$ and for $t\in(0,1)$, one has*

$$D^\alpha\mathbf{\Phi}(t)=t^{-\alpha}\boldsymbol{G}^{(\alpha)}\mathbf{\Phi}(t), \tag{3.6}$$

where $\boldsymbol{G}^{(\alpha)}=(g_{i,j}^{\alpha})$ represents the $(M+1)\times(M+1)$ Fermat OMFD of order α in the Caputo sense and it is given explicitly as

$$\boldsymbol{G}^{(\alpha)}=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ \xi_\alpha(\lceil\alpha\rceil,1) & \xi_\alpha(\lceil\alpha\rceil,\lceil\alpha\rceil) & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ \xi_\alpha(i,1) & \cdots & \xi_\alpha(i,i) & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ \xi_\alpha(M+1,1) & \xi_\alpha(M+1,2) & \xi_\alpha(M+1,3) & \cdots & \xi_\alpha(M+1,M+1)\end{pmatrix}. \tag{3.7}$$

The entries $(g_{i,j}^{\alpha})$ can be written in the form

$$g_{i,j}^{\alpha}=\begin{cases}\xi_\alpha(i,j), & i\ge\lceil\alpha\rceil, i\ge j;\\ 0, & \textit{otherwise}.\end{cases}$$

where

$$\xi_\alpha(i,j)=j\sum_{\substack{k=\lceil\alpha\rceil\\ (i+k)\ odd,\ (j+k)\ odd}}^{i}\frac{k!(-1)^{\frac{1}{2}(i-2j+k+1)}2^{\frac{i-j}{2}}(\frac{i+k-1}{2})!}{(\frac{i-k-1}{2})!(\frac{k-j+1}{2})!(\frac{k+j+1}{2})!\Gamma(-\alpha+k+1)}. \tag{3.8}$$

Proof The application of the operator D^α to Eq. (2.18) along with relation (2.4) yields

$$D^\alpha\mathcal{F}_i(t)=\sum_{k=0}^{\lfloor\frac{i-1}{2}\rfloor}\frac{(-2)^k3^{i-2k-1}(i-k-1)!}{k!(i-2k-1)\Gamma(i-2k-1-\alpha)}t^{i-2k-1-\alpha}. \tag{3.9}$$

If we make use of formula (2.19) and perform some algebraic calculations, we can write

$$D^\alpha\mathcal{F}_i(t)=t^{-\alpha}\sum_{j=1}^{i}\xi_\alpha(i,j)\mathcal{F}_j(t), \tag{3.10}$$

and $\xi_\alpha(i,j)$ is given in (3.8). In a vector form, Eq. (3.10) can be alternatively written as follows:

$$D^{\alpha}\mathcal{F}_i(t) = t^{-\alpha}\big[\xi_\alpha(i,1),\xi_\alpha(i,2),\dots,\xi_\alpha(i,i),0,0,\dots,0\big]\boldsymbol{\Phi}(t), \quad \lceil\alpha\rceil \le i \le m+1. \tag{3.11}$$

Moreover, we can write

$$D^{\alpha}F_i(t) = t^{-\alpha}[0,0,\dots,0], \quad 1 \le i \le \lceil\alpha\rceil - 1. \tag{3.12}$$

Equation (3.11) along with Eq. (3.12) lead to the desired result. □

4 Numerical treatment of the Bagley-Torvik equation

In this section, we present a numerical spectral tau algorithm for solving the fractional-order linear Bagley-Torvik equation based on using the constructed Fermat operational matrix of derivatives.

Consider the linear Bagley-Torvik differential equation [1]

$$A_1D^{(2)}f(t) + A_2D^{(\frac{3}{2})}f(t) + A_3f(t) = g(t), \quad t \in (0,1), \tag{4.1}$$

subject to the initial conditions

$$f(0) = \gamma, \qquad f'(0) = \delta, \tag{4.2}$$

or the boundary conditions

$$f(0) = \bar{\gamma}, \qquad f(1) = \bar{\delta}. \tag{4.3}$$

Now, assume that $f(t)$ can be approximated as

$$f(t) \approx f_M(t) = \mathbf{B}^T\boldsymbol{\Phi}(t). \tag{4.4}$$

In virtue of Theorem 3, the following approximations can be obtained:

$$D^{(2)}f(t) \approx \mathbf{B}^T\mathbf{G}^{(2)}\boldsymbol{\Phi}(t) \tag{4.5}$$

and

$$D^{(\frac{3}{2})}f(t) \approx t^{-\frac{3}{2}}\mathbf{B}^T\mathbf{G}^{(\frac{3}{2})}\boldsymbol{\Phi}(t). \tag{4.6}$$

Making use of the approximations in (4.5) and (4.6), the residual of (4.1) is given by the formula

$$t^{\frac{3}{2}}R(t) = A_1t^{\frac{3}{2}}\mathbf{B}^T\mathbf{G}^{(2)}\boldsymbol{\Phi}(t) + A_2\mathbf{B}^T\mathbf{G}^{(\frac{3}{2})}\boldsymbol{\Phi}(t) + A_3t^{\frac{3}{2}}\mathbf{B}^T\boldsymbol{\Phi}(t) - t^{\frac{3}{2}}g(t), \tag{4.7}$$

and hence the application of tau method (see, for example, [26]) leads to

$$\int_0^1 t^{\frac{3}{2}}R(t)\mathcal{F}_j(t)\,dt = 0, \quad 1 \le j \le M-1. \tag{4.8}$$

Moreover, the initial conditions (4.2) yield

$$\mathbf{B}^T\Phi(0)=\gamma,\qquad \mathbf{B}^T\mathbf{G}^{(1)}\Phi(0)=\delta, \tag{4.9}$$

while the boundary conditions (4.3) yield

$$\mathbf{B}^T\Phi(0)=\bar{\gamma},\qquad \mathbf{B}^T\Phi(1)=\bar{\delta}. \tag{4.10}$$

Equations (4.8) with (4.9) or (4.10) constitute a system of algebraic equations in the unknown expansion coefficients c_i of dimension $(M+1)$. This system can be solved via the Gaussian elimination procedure or any other suitable procedure. Therefore, the approximate solution (4.4) can be found.

5 Convergence and error analysis

This section gives a detailed study for the convergence and error analysis of the proposed Fermat expansion. To proceed in such a study, the following lemmas are useful in the sequel.

Lemma 1 *Let $f(t)$ be an infinitely differentiable function at the origin. $f(t)$ has the following Fermat expansion*:

$$f(t)=\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{2^j k(2j+k-1)!\,a_{2j+k-1}}{3^{2j+k-1}j!(j+k)!}\mathcal{F}_k(t), \tag{5.1}$$

where $a_i=\frac{f^{(i)}(0)}{i!}$.

Proof Following procedures similar to those given in [27], the lemma can be obtained. □

Lemma 2 ([28], p.375) *Let $I_\nu(t)$ denote the modified Bessel function of order ν of the first kind. The following identity holds*:

$$\sum_{j=0}^{\infty}\frac{t^j}{j!(j+k)!}=t^{-\frac{k}{2}}I_k(\sqrt{2}t). \tag{5.2}$$

Lemma 3 ([29]) *The modified Bessel function of the first kind $I_\nu(t)$ satisfies the following inequality*:

$$\left|I_\nu(t)\right|\le\frac{t^\nu\cosh t}{2^\nu\Gamma(\nu+1)},\quad \forall t>0. \tag{5.3}$$

Lemma 4 *The following inequality is valid*:

$$\mathcal{F}_k(t)\le\mathcal{F}_k,\quad \forall t\in[0,1],\forall k\in\mathbb{N}, \tag{5.4}$$

where $\mathcal{F}_k=\mathcal{F}_k(1)=2^k-1$.

Now, we are in a position to state and prove the following two theorems.

Theorem 4 *Let $f(t)$ be defined on $[0,1]$ provided $|f^{(i)}(0)| \le L^i$, $i \ge 0$, L is a positive constant and $f(t) = \sum_{k=1}^{\infty} b_k \mathcal{F}_k(t)$. The following estimate holds for the expansion coefficients*:

$$|b_k| \le \frac{L^{k-1}}{3^k (k-1)!} \cosh\left(\frac{\sqrt{8}L}{3}\right).$$

Moreover, the series $\sum_{k=1}^{\infty} b_k \mathcal{F}_k(t)$ converges absolutely.

Proof Lemma 1 enables one to write

$$\begin{aligned} |b_k| &= k \left| \sum_{j=0}^{\infty} \frac{2^j f^{(2j+k-1)}(0)}{3^{2j+k-1} j! (j+k)!} \right| \\ &\le k \sum_{j=0}^{\infty} \frac{2^j L^{2j+k-1}}{3^{2j+k-1} j! (j+k)!}. \end{aligned}$$

In virtue of Lemma 2, the following estimate can be obtained:

$$|b_k| \le \frac{3k}{2^{\frac{k}{2}} L} I_k\left(\frac{\sqrt{8}L}{3}\right). \tag{5.5}$$

The inequality in part (i) can be obtained if Lemma 3 is applied to the inequality in (5.5). To show that the series $\sum_{k=1}^{\infty} b_k \mathcal{F}_k(t)$ converges absolutely, the comparison test is applied. Indeed, if we make use of part (i), then the application of Lemma 4 yields

$$\left|b_k \mathcal{F}_k(t)\right| \le \frac{L^{k-1}(2^k - 1)}{3^k (k-1)!} \cosh\left(\frac{\sqrt{8}L}{3}\right),$$

but

$$\sum_{k=1}^{\infty} \frac{L^{k-1}(2^k-1)}{3^k (k-1)!} = \sum_{k=1}^{\infty} \frac{L^{k-1} 2^k}{3^k (k-1)!} - \sum_{k=1}^{\infty} \frac{L^{k-1}}{3^k (k-1)!} = \frac{2e^{\frac{2L}{3}} - e^{\frac{L}{3}}}{3},$$

and this proves that the series is absolutely convergent. □

Theorem 5 *If $f(t)$ satisfies the hypothesis of Theorem 4, and $E_M(t) = \sum_{k=M+2}^{\infty} b_k \mathcal{F}_k(t)$, then we have the following error estimate*:

$$\left|E_M(t)\right| < \frac{2\mu^{M+1} \cosh(\sqrt{2}\mu)}{3(M+1)!},$$

where $\mu = \frac{2L}{3}$.

Proof By Theorem 4, we can write

$$\begin{aligned} \left|E_M(t)\right| &\le \cosh\left(\frac{\sqrt{8}L}{3}\right) \sum_{k=M+2}^{\infty} \frac{L^{k-1}(2^k-1)}{3^k (k-1)!} \\ &< \frac{2\cosh(\sqrt{2}\mu)}{3} \sum_{k=M+2}^{\infty} \frac{\mu^{k-1}}{(k-1)!}. \end{aligned} \tag{5.6}$$

The summation in (5.6) can be turned into

$$\sum_{k=M+2}^{\infty} \frac{\mu^{k-1}}{(k-1)!} = 1 - \frac{\Gamma(M+1,\mu)}{\Gamma(M+1)}, \tag{5.7}$$

where the two notations $\Gamma(\cdot)$ and $\Gamma(\cdot,\cdot)$ denote, respectively, gamma and the incomplete gamma functions. Now, we can write (5.7) in the alternative integration formula

$$\sum_{k=M+2}^{\infty} \frac{\mu^{k-1}}{(k-1)!} = \frac{1}{M!} \int_0^{\mu} t^M e^{-t}\, dt, \tag{5.8}$$

but since $e^{-t} < 1$, we get

$$\sum_{k=M+2}^{\infty} \frac{\mu^{k-1}}{(k-1)!} < \frac{\mu^{M+1}}{(M+1)!},$$

which completes the proof of the theorem. □

6 Test problems and comparisons

In this section, we use the Fermat tau operational matrix (FTM) method to solve numerically Bagley-Torvik equations. Moreover, we compare our results with some techniques existing in the literature.

Example 1 ([30]) Consider the following Bagley-Torvik equation:

$$D^2 f(t) + D^{\frac{3}{2}} f(t) + f(t) = 1 + t, \quad t \in (0,1), \tag{6.1}$$

subject to the boundary conditions

$$f(0) = 1, \qquad f(1) = 2. \tag{6.2}$$

The exact solution of Eq. (6.1) is $f(t) = 1 + t$. We apply FTM for the case $M = 1$. The residual of Eq. (6.1) is given by the formula

$$t^{\frac{3}{2}} R(t) = t^{\frac{3}{2}} \mathbf{B}^T \mathbf{G}^{(2)} \mathbf{\Phi}(t) + \mathbf{B}^T \mathbf{G}^{(\frac{3}{2})} \mathbf{\Phi}(t) + t^{\frac{3}{2}} \mathbf{B}^T \mathbf{\Phi}(t) - t^{\frac{3}{2}} g(t),$$

where the operational matrices $\mathbf{G}^{(2)}$ and $\mathbf{G}^{(\frac{3}{2})}$ are given explicitly as follows:

$$\mathbf{G}^{(2)} = \mathbf{G}^{(\frac{3}{2})} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

After some manipulations, we get

$$c_1 = 1, \qquad c_2 = \frac{1}{3},$$

and, consequently, $f_1(t) = 1 + t$, which is the exact solution.

Example 2 ([31–33]) Consider the following inhomogeneous Bagley-Torvik boundary value problem:

$$D^2 f(t) + \frac{8}{17} D^{\frac{3}{2}} f(t) + \frac{13}{51} f(t) = g(t), \quad t \in (0,1), \qquad f(0) = f(1) = 0, \tag{6.3}$$

where $g(t)$ is chosen such that the exact solution of Eq. (6.3) is

$$f(t) = t^5 - \frac{29}{10} t^4 + \frac{76}{25} t^3 - \frac{339}{250} t^2 + \frac{27}{125} t.$$

We apply FTM for the case $M = 5$, we get the following four equations:

$$\begin{aligned}
&c_1 + 2.14286c_2 + 104.863c_3 + 598.271c_4 + 2052.39c_5 + 5846.82c_6 = 0.0198014,\\
&c_1 + 2.33333c_2 + 107.152c_3 + 657c_4 + 2443.43c_5 + 7377.88c_6 = 0.144502,\\
&c_1 + 2.66234c_2 + 111.08c_3 + 759.659c_4 + 3160.87c_5 + 10349.2c_6 = 0.387875,\\
&c_1 + 3.0135c_2 + 115.227c_3 + 872.033c_4 + 4024.21c_5 + 14304.9c_6 = 0.805567.
\end{aligned} \tag{6.4}$$

Moreover, the boundary conditions yield the following two equations:

$$\begin{aligned}
&c_1 - 2c_3 + 4c_5 = 0,\\
&c_1 + 3c_2 + 7c_3 + 15c_4 + 31c_5 + 63c_6 = 0.
\end{aligned} \tag{6.5}$$

Equations (6.4)-(6.5) generate a system of linear algebraic equations of dimension six. We employ the Gaussian elimination method to solve this system, the time elapsed to solve this system is 19 seconds and the number of arithmetic operations required to solve this system (see [34]) is approximately $\frac{2}{3}(M+1)^3 = 144$. The solution of this system is

$$\begin{aligned}
&c_1 = -\frac{5951}{10125}, \qquad c_2 = \frac{18367}{30375}, \qquad c_3 = -\frac{2467}{6750},\\
&c_4 = \frac{884}{6075}, \qquad c_5 = -\frac{29}{810}, \qquad c_6 = \frac{1}{243},
\end{aligned}$$

and, consequently, $f(t) = t^5 - \frac{29}{10} t^4 + \frac{76}{25} t^3 - \frac{339}{250} t^2 + \frac{27}{125} t$, which is the exact solution.

Example 3 Consider the following inhomogeneous Bagley-Torvik initial value problem:

$$D^2 f(t) + D^{\frac{3}{2}} f(t) + f(t) = g(t), \quad t \in (0,2), \qquad f(0) = f'(0) = 0, \tag{6.6}$$

where $g(t)$ is chosen such that the exact solution of Eq. (6.6) is

Case i:

$$f(t) = t^5 - t^4.$$

Case ii:

$$f(t) = t^3.$$

We apply FTM for the case $M = 5$. In this case we have a small change in the algorithm, namely integration limits will be from 0 to 2, i.e.,

$$\int_0^2 t^{\frac{3}{2}} R(t)\mathcal{F}_i(t)\,dt = 0, \quad i = 1,2,3,4. \tag{6.7}$$

These four equations with the two homogenous initial conditions in (6.6) will yield: in Case i,

$$c_1 = -\frac{8}{81}, \qquad c_2 = \frac{20}{243}, \qquad c_3 = -\frac{2}{27}, \qquad c_4 = \frac{8}{243}, \qquad c_5 = -\frac{1}{81}, \qquad c_6 = \frac{1}{243},$$

and, consequently, $f(t) = t^5 - t^4$, which is the exact solution; while in Case ii,

$$c_1 = 0, \qquad c_2 = \frac{4}{27}, \qquad c_3 = 0, \qquad c_4 = \frac{1}{27}, \qquad c_5 = 0, \qquad c_6 = 0,$$

and, consequently, $f(t) = t^3$, which is the exact solution.

Example 4 ([35]) Consider the following inhomogeneous Bagley-Torvik initial value problem:

$$D^2 f(t) + D^{\frac{3}{2}} f(t) + f(t) = g(t), \quad t \in (0,1), \qquad f(0) = 0, f'(0) = \alpha, \tag{6.8}$$

where $g(t)$ is chosen such that the exact solution of Eq. (6.8) is $f(t) = \sin(\alpha t)$. We apply FTM for various values of α and M. Table 1 displays the comparison between the results obtained by FTM with those obtained by the application of Chebyshev spectral method (CSM) which was developed in [35]. The results of this table illustrate that our algorithm gives a better error in all cases. In this table, τ denotes the time used to run the algorithm in seconds.

Example 5 Consider the following inhomogeneous Bagley-Torvik boundary value problem:

$$D^2 f(t) + D^{\frac{3}{2}} f(t) + f(t) = g(t), \quad t \in (0,2), \qquad f(0) = f(2) = 0, \tag{6.9}$$

where $g(t)$ is chosen such that the exact solution of Eq. (6.9) is $f(t) = t(2-t)e^{-t}$. We apply FTM. In Table 2, we list the maximum pointwise error E and the time elapsed τ for different values of M. The results in this table show that with few numbers of retained modes we obtained a reasonable accuracy 10^{-13}.

Table 1 Comparison between FTM and CSM in [35] for Example 4

M	τ	$\alpha = 1$		$\alpha = 4\pi$	
		FTM	CSM [35]	FTM	CSM [35]
4	23.157	$2.7 \cdot 10^{-04}$	$3.4 \cdot 10^{-04}$	$2.5 \cdot 10^{-02}$	$3.9 \cdot 10^{00}$
8	110.781	$3.5 \cdot 10^{-07}$	$4.3 \cdot 10^{-07}$	$3.5 \cdot 10^{-04}$	$4.7 \cdot 10^{-01}$
16	313.265	$4.2 \cdot 10^{-10}$	$1.8 \cdot 10^{-08}$	$4.2 \cdot 10^{-09}$	$3.5 \cdot 10^{-05}$
32	1023.854	$5.8 \cdot 10^{-12}$	$7.1 \cdot 10^{-10}$	$8.4 \cdot 10^{-12}$	$1.4 \cdot 10^{-06}$

Table 2 Maximum pointwise error of Example 5

	M		
	5	**10**	**15**
τ	15.92	131.78	321.52
E	$2.56 \cdot 10^{-3}$	$7.95 \cdot 10^{-9}$	$2.22 \cdot 10^{-13}$

Example 6 ([36, 37]) Consider the following inhomogeneous Bagley-Torvik initial value problem:

$$D^2 f(t) + \frac{1}{2} D^{\frac{3}{2}} f(t) + \frac{1}{2} f(t) = 8, \quad t \in (0,1), \qquad f(0) = 1, f'(0) = 0. \tag{6.10}$$

The exact solution of Eq. (6.10) is given by

$$f(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} t^{2k+1} E^k_{\frac{1}{2}, \frac{3}{2}k+2} \left(-\frac{\sqrt{t}}{2} \right),$$

where $E^k_{\lambda,\mu}(z)$ is the Mittag-Leffler function of the two parameters $\lambda, \mu > 0$, defined by

$$E^k_{\lambda,\mu}(z) = \sum_{j=0}^{\infty} \frac{(j+k)!}{j! \Gamma(\lambda j + \lambda k + \mu)} z^j.$$

We apply FTM for the case corresponding to $M = 15$. In Table 3, we give a comparison between the results obtained by FTM with those obtained by applying the generalized Taylor method (GTCM) [36] and the fractional Taylor method (FrTM) [37]. The results displayed in this table show that our solution is better than the solutions obtained in [36, 37].

Example 7 Consider the following initial value problem [38]:

$$D^{\frac{3}{2}} f(t) + f(t) = g(t), \quad t \in (0,2), \qquad f(0) = f'(0) = 0, \tag{6.11}$$

where $g(t)$ is chosen such that the exact solution of Eq. (6.11) is $f(t) = t^{\frac{5}{2}}$. We apply FTM for the case corresponding to $M = 9$. In Table 4, we compare the pointwise error E and the pointwise error obtained in [38]. The results in this table show that with few numbers of retained modes we obtained a maximum pointwise error of order 10^{-8}.

Table 3 Comparison between GTCM, FrTM and FTM for Example 6

t	GTCM [36]	FrTM [37]	FTM $M = 15$	Exact solution
0.0	0	0	0	0
0.1	0.036485547	0.036487480	0.036487479	0.036487479
0.2	0.140634716	0.140639621	0.140639621	0.140639621
0.3	0.307476229	0.307484627	0.307484627	0.307484627
0.4	0.533271294	0.533284110	0.533284110	0.533284110
0.5	0.814735609	0.814756949	0.814756950	0.814756950
0.6	1.148805808	1.148837422	1.148837428	1.148837428
0.7	1.532521264	1.532565426	1.532565443	1.532565443
0.8	1.962974991	1.963029255	1.963029298	1.963029298
0.9	2.437455982	2.437333971	2.437334072	2.437334072
1.0	2.954070000	2.952583880	2.952584099	2.952584099

Table 4 Maximum pointwise error of Example 7

	t					
	0	**0.2**	**0.4**	**0.6**	**0.8**	**1**
E	0	$2.28 \cdot 10^{-9}$	$1.21 \cdot 10^{-8}$	$8.60 \cdot 10^{-9}$	$1.18 \cdot 10^{-8}$	$2.45 \cdot 10^{-9}$
Error in [38]	0	$3.78 \cdot 10^{-8}$	$2.14 \cdot 10^{-7}$	$5.87 \cdot 10^{-7}$	$1.17 \cdot 10^{-6}$	$1.98 \cdot 10^{-6}$

7 Conclusions

In this paper, we have developed a new operational matrix of fractional derivatives of Fermat polynomials. This matrix is established with the aid of introducing some new identities concerning Fermat polynomials. As far as we know, the introduced operational matrix is novel, and its application in handling fractional differential equations is also new. As an application, the Bagley-Torvik equation is solved via a certain Fermat operational tau method. Some tested numerical examples including some comparisons are exhibited to demonstrate the features of the proposed method.

Competing interests
The author declares that he has no competing interests.

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