A NEW POLYNOMIAL INVARIANT OF KNOTS AND LINKS[1]

BY P. FREYD, D. YETTER; J. HOSTE;

W. B. R. LICKORISH, K. MILLETT; AND A. OCNEANU

The purpose of this note is to announce a new isotopy invariant of oriented links of tamely embedded circles in 3-space.

We represent links by plane projections, using the customary conventions that the image of the link is a union of transversely intersecting immersed curves, each provided with an orientation, and undercrossings are indicated by broken lines. Following Conway [6], we use the symbols L_+, L_0, L_- to denote links having plane projections which agree except in a small disk, and inside that disk are represented by the pictures of Figure 1.

Conway showed that the one-variable Alexander polynomials of L_+, L_0, L_- (when suitably normalized) satisfy the relation

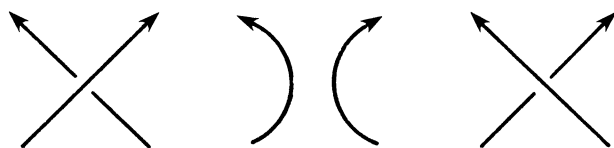$$\Delta_{L_+}(t) - \Delta_{L_-}(t) + (t^{1/2} - t^{-1/2})\Delta_{L_0}(t) = 0.$$

FIGURE 1

Recently, Vaughan Jones [10] constructed a new polynomial invariant satisfying the relation

$$tV_{L_+}(t) - t^{-1}V_{L_-}(t) + (t^{1/2} - t^{-1/2})V_{L_0}(t) = 0.$$

Our invariant, which can be regarded either as a nonhomogeneous polynomial in two variables or a homogeneous polynomial in three variables, generalizes both the Alexander-Conway [2, 6] and the Jones polynomials.

MAIN THEOREM. *There is a unique function P from the set of isotopy classes of tame oriented links to the set of homogeneous Laurent polynomials of degree 0 in x, y, z such that*
(1) $xP_{L_+}(x,y,z) + yP_{L_-}(x,y,z) + zP_{L_0}(x,y,z) = 0,$
(2) $P_L(x,y,z) = 1$ *if L consists of a single unknotted component.*

REMARKS. (1) If L consists of n unlinked and unknotted components, then

$$P_L(x,y,z) = (-(x+y)/z)^{n-1}.$$

(2) The Alexander-Conway polynomial of L is

$$\Delta_L(t) = P_L(1,-1,t^{1/2} - t^{-1/2}),$$

and the Jones polynomial is

$$V_L(t) = P_L(t, -t^{-1}, t^{1/2} - t^{-1/2}).$$

(3) Since $P_L(x,y,z)$ is homogeneous, it can be viewed (in many ways) as a polynomial in two nonhomogeneous variables. A convenient way to do this is to set $\mathcal{P}_L(l,m) = P_L(l,l^{-1},m)$. In this notation the basic relation is $l\mathcal{P}_{L_+}(l,m) + l^{-1}\mathcal{P}_{L_-}(l,m) + m\mathcal{P}_{L_0}(l,m) = 0.$
(4) Reversing the orientation of $\mathbf{R}^3$ has the effect

$$P_L(x,y,z) \leftrightarrow P_L(y,x,z), \qquad \rho_L(l,m) \leftrightarrow \rho_L(l^{-1},m).$$

However, neither polynomial is changed by simultaneous reversal of orientations on all components of L.
(5) If L is a connected sum of links L_1 and L_2, then $P_L = P_{L_1}P_{L_2}$.
(6) The following examples are easily computed by using the recurrence relation and the values of P for unlinks.

$$L = \quad \text{} \quad : P_L = yz^{-1} + x^{-1}y^2z^{-1} - x^{-1}z,$$

$$L = \quad \text{} \quad : P_L = x^{-2}z^2 - 2x^{-1}y - x^{-2}y^2,$$

$$L = \quad \text{} \quad : P_L = y^{-2}z^2 - 2xy^{-1} - x^2y^{-2},$$

$$L = \quad \text{} \quad : P_L = x^{-1}y^{-1}z^2 - xy^{-1} - x^{-1}y - 1.$$

We wish to express our appreciation to Vaughan Jones for communicating his results to us. The particular viewpoints of the various authors are outlined below.

THE FREYD-YETTER APPROACH

In order to reduce the unique existence of the homfly invariant to a word problem, let R be the ring of Laurent polynomials on x, y, z; W_n the set of words on $n-1, \ldots, 2, 1, -1, \ldots, -n+1$ (no zero); W the disjoint union of the W_n's; F the free R-module generated by W. An element of W is denoted $[u]_n$, where $u \in W_n$. $R \to F$ denotes the map that sends 1 to $[\ \ ]_1$. $F \to M$ denotes the quotient module obtained by imposing the following relations, in which $u, v \in W_n$ and $0 < b < a < n$:

(A0) $[ua(-a)v]_n \equiv [uv]_n \equiv [u(-a)av]_n,$
(A1) $[uabv]_n \equiv [ubav]_n$ if $b < a - 1,$
(A2) $[ua(a-1)av]_n \equiv [u(a-1)a(a-1)v]_n,$
(M1) $[uv]_n \equiv [vu]_n,$
(M2) $[unv]_{n+1} \equiv [uv]_n \equiv [u(-n)v]_{n+1},$
(P1) $x[uav]_n + y[u(-a)v]_n + z[uv]_n \equiv 0.$

LEMMA. $R \to F \to M$ *is an isomorphism.*

This lemma implies the Main Theorem by the following argument. Each element of W_n describes a braid on n strands. By the classic theorem of Artin [1], the map $W \to F \to M$ sends two elements of W_n to the same element of M if they describe equivalent braids (A0), (A1), (A2). A braid gives rise to a link by joining its top and bottom. By the classic theorem stated by Markov [12] and proved by Birman [3], two braids are sent to the same element in M if they induce the same link (M1), (M2). The resulting function from links to M satisfies condition one of the Main Theorem (P1). Invert the isomorphism to obtain an R-valued function that satisfies both conditions.

To prove the lemma consider the following six *directed* substitution rules (still with $u, v \in W_n$, $0 < b < a < n$):

(S1) $[uabv]_n \to [ubav]_n$ for $b < a - 1,$
(S2) $[ua(a-1)\cdots(b+1)bav]_n \to [u(a-1)a(a-1)\cdots(b+1)bv]_n,$
(S3) $[u(-a)v]_n \to -xy^{-1}[uav]_n - y^{-1}z[uv]_n,$
(S4) $[uaav]_n \to -x^{-1}y[uv]_n - x^{-1}z[uav]_n,$
(S5) $[unv]_{n+1} \to [uv]_n,$
(S6) $[uv]_{n+1} \to -(x+y)z^{-1}[uv]_n.$

LEMMA.   *These rules obey the DCC: that is, there is no infinite sequence of correct applications starting with a single form.*

LEMMA.   *Suppose that f is a form and that it is possible to apply a rule to obtain the form g and to make another application (of possibly another rule) to obtain the form h. Then there is a form j, a sequence of applications from g to j, and a sequence of applications from h to j.*

An inductive proof then shows that starting with any form f and arbitrarily applying the rules, one inevitably arrives at a *unique* "terminal form" $T(f)$ on which no rule applies.

LEMMA. *A form is terminal iff it is an R-multiple of* $[\ ]_1$.

The Main Theorem is thus established with

LEMMA. $T(f) = T(g)$ *iff* $f \equiv g$.

For the most part these four lemmas have straightforward (but tedious) proofs. The DCC lemma needs a "complexity" measurement on forms guaranteed to decrease under each of the substitution rules. The next lemma reduces to 21 separate cases, one for each pair of rules. Each is proven by the simple expedient of applying in sequence the unique rule that applies. The characterization of terminal forms is achieved by finding at least one rule that applies to $[u]_n$ for $n > 1$. The last lemma is easy if (M1) is deleted from the relations. ((M1) is in fact a consequence of the other relations.) The proof that $T[uv]_k = T[vu]_k$ quickly reduces to the case $T[nuv]_{n+1} = T[uvn]_{n+1}$, where u and v are descending words of positive elements, u starting with $n-1$, v with n. If v is empty or a singleton or if u is empty, a mindless verification works. Separate arguments are needed for the two remaining cases (u of length one, u of length greater than one). If long computational proofs are acceptable, a mindless approach works for both (as it must if these lemmas are true): Apply any sequence of applicable substitution rules.

THE HOSTE APPROACH

We define P as follows. Given any oriented link L first choose some projection S of L and then "resolve" S by first "changing" and "smoothing" some crossing in S, and then again changing and smoothing a crossing in each of the two projections that result from S, and so on until only unlinks remain. Next use property (1) of the Main Theorem and the values of P for the unlinks given in Remark (1) to obtain a value of P for S. We prove that the polynomial so produced is well defined and depends only on the isotopy class of L.

To avoid difficulties that arise from the fact that S can be resolved in infinitely many ways, we begin with a slightly different definition of P. Given S, first order the components of S and also distinguish a point on each component. We may then gain more control on the resolution of S by first giving a rule whereby an ordering and pointing is induced on projections obtained from S, and then demanding that the resolution ends, not just in unlinks, but in "descending" projections. (A pointed ordered projection is descending if one never crosses over one's path while traversing the components in the given order and direction starting on each at the distinguished point.) Now use this "distinguished" resolution to compute P of S, denoted P_S.

We next show P_S is well defined—i.e., independent of the choice of ordering, pointing, and distinguished resolution, by induction on the number of crossings in S. To prove the inductive step we first show the choice of resolution is immaterial. This is made possible by the pointing and ordering

scheme, which essentially implies that any two resolutions must eventually change all the same crossings, but perhaps in different orders. Employing the inductive hypothesis, it remains only to prove that (1) changing a crossing and immediately changing it back has no effect, and (2) changing the ith and then the jth crossings has the same effect as first changing the jth and then the ith crossings. Next we show that P_S is unchanged if S is repointed. This can be reduced to the case where S is descending and one distinguished point is moved forward along its component past one crossing. Finally, we prove P_S is independent of the ordering of S. This is the hardest step, although we can first reduce to the case where S has two components and is descending, but not with respect to its own ordering. Now we show that S can be transformed into a split descending projection (one where the components do not cross each other and, hence, the ordering does not matter) by a finite sequence of Reidemeister moves that never increase the number of crossings (so we can make use of the inductive hypothesis) and, furthermore, preserve P_S.

Once P_S is known to be well defined, it is not hard to show that it is preserved by Reidemeister moves. Thus it depends only on the isotopy class of L. At this point we finally know that any distinguished resolution of the unlink produces the unlink polynomial, and from this we can prove that any resolution can be used to compute P, not just ones that are distinguished with respect to some choice of pointing and ordering of S.

THE LICKORISH AND MILLETT APPROACH

Once the announced Main Theorem is believed, the only difficulty is proving that $\mathcal{P}_L$ can be well defined; uniqueness, calculations, and Remark (4) are easy. Our combinatorial proof has simplicity: Let $\mathcal{L}_n$ be all planar projections of oriented links of at most n crossings with components *ordered* and *base-pointed*. If $L \in \mathcal{L}_n$ let $\alpha L \in \mathcal{L}_n$ be L-with-changed-crossings so that αL *ascends* from the base point of the first component onwards (thus, following along the components of αL in the given order, always beginning at base points, a crossing is always first encountered as an underpass). Initially $\mathcal{P}$ is defined as a function $\mathcal{P}: \bigcup_n \mathcal{L}_n \to \mathbf{Z}[l^{\pm 1}, m^{\pm 1}]$. For $L \in \bigcup_n \mathcal{L}_n$ define $\mathcal{P}_{\alpha L} = \mu^{cL-1}$, where $\mu = -(l + l^{-1})m^{-1}$, and L has cL components. Inductively, assume that on $\mathcal{L}_{n-1}$, $\mathcal{P}$ has been defined, is independent of base points and orders, and

$$(*) \qquad\qquad l\mathcal{P}_{L_+} + l^{-1}\mathcal{P}_{L_-} + m\mathcal{P}_{L_0} = 0.$$

For L in $\mathcal{L}_n$ define $\mathcal{P}_L$ to be the polynomial obtained from $\mathcal{P}_{\alpha L}$ using $(*)$ (and the inductive definition) on a sequence of crossing changes that creates αL from L. The choice of sequence is irrelevant, $(*)$ holds in $\mathcal{L}_n$, and $\mathcal{P}_L$ is unchanged by those Reidemeister moves that remain *within* $\mathcal{L}_n$. If now βL denotes the new ascending element of $\mathcal{L}_n$ formed from L by reference to a new set of base points and component order, the induction step is completed by showing that $\mathcal{P}_{\beta L}$, as calculated from $\mathcal{P}_{\alpha L}$, is μ^{cL-1}. A straightforward argument on sliding base points along components shows (using the prescribed value of μ) that this is so if only base points are changed; a change of component order requires a very careful use of the *permitted* Reidemeister moves on

βL to change βL to another ascending projection with fewer crossings, thus determining $\mathcal{P}_{\beta L}$ by induction. This done, $\mathcal{P}_L$ is defined independent of base point and ordering; it is invariant under all Reidemeister moves, and the Main Theorem is proved.

The entire dependence of $\mathcal{P}_L$ on $(*)$ demands a redevelopment of Conway's skein theory [7] in the two-variable context. At once this new version of linear skein theory produces the elementary results that $\mathcal{P}_{L_1 \# L_2} = \mathcal{P}_{L_1} \mathcal{P}_{L_2} = \mu^{-1} \mathcal{P}_{L_1 \sqcup L_2}$. Applied to two-string tangles [11], it shows that $\mathcal{P}_L$ is unchanged by mutation (rotation through π of such a tangle) in L. Thus the two eleven-crossing knots (of distinct genera) for which $\Delta(t) = 1$ have the same $\mathcal{P}(l, m)$, which is, by calculation, not trivial (so $\mathcal{P}_L$ depends on more than the Alexander module of L). Also, mutation shows that for a pretzel knot $K(a_1, \ldots, a_r)$, $\mathcal{P}_K$ is unchanged by permutation of the a_ι. Further, if S and T are two-string tangles, if $S + T$ is

and S^N and S^D are the polynomials for
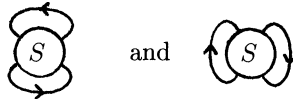
then

$$(1 - \mu^2)(S + T)^N = (S^N T^D + S^D T^N) - \mu(S^N T^N + S^D T^D).$$

This can be much applied in producing formulae for the polynomials of rational and other arborescent links. Two knots of Birman [4] with distinct signatures have the same polynomial. Thus if signature is a skein invariant, then $\mathcal{P}$ is not injective on the set of skein-equivalence classes of oriented links. Two-variable analogues exist for most of the other skein-theoretic results given in [7].

Tables will be produced (with computer aid) of the values of $\mathcal{P}$ for oriented links of low crossing number. At present the literature lacks an agreed notation and even a set of diagrams for *oriented* links. Manual calculations of the polynomials proceed via various neat recursion formulae for particular types (e.g., rational and pretzel) of links. Calculations of M. B. Thistlethwaite show that if K is the knot 11_{388} (in the notation of K. Perko) and $\overline{K}$ is its mirror image, then $\mathcal{P}_K \neq \mathcal{P}_{\overline{K}}$, but both $V_K = V_{\overline{K}}$ and $\Delta_K = \Delta_{\overline{K}}$.

THE OCNEANU APPROACH

We define for each permutation in S_n a layered braid in the n-string braid group B_n. We show combinatorially that under the equivalence relation in the Main Theorem, any braid decomposes uniquely as a linear combination of layered braids. With the coefficients of the decomposition we construct invariants to conjugation in B_n, and obtain the invariant P_L of the closed braid as an invariant to the Markov moves [3].

We then study the algebraic structure of the invariant P_L. Linear combinations of braids in B_n have an algebra structure in which layered braids form

a basis. The algebra turns out to be the Hecke algebra H_n with Dynkin diagram A_n [**5**]. In this way we explicitly construct a family of finite-dimensional irreducible representations of B_n, indexed by n-Young diagrams. The invariant appears as a weighted trace, explicitly determined via the S-functions associated to the discrete Euler β-function [**13**].

We then study the values of the variables for which the representation has additional orthogonality and positivity structure.

THEOREM. *Let $\tau \in \mathbf{C}$. The free C^*-algebra A_τ over $\mathbf{C}$ generated by $\{e_k | k = 1, 2, \ldots\}$, satisfying $e_k^2 = e_k^* = e_k$ and*

$$e_k e_{k+1} e_k - \tau e_k = e_{k+1} e_k e_{k+1} - \tau e_{k+1}, \qquad k \in N,$$

$$e_k e_l = e_l e_k \quad if \ |k - l| \geq 2,$$

is nonzero if and only if $\tau \in [0, \frac{1}{4}] \cup \{(4\cos(\pi/n))^{-1}, \ n = 3, 4, 5, \ldots\}$. □

THEOREM. *Let $\alpha, \theta \in \mathbf{C}$, and let*

$$\tau = (4 \cosh^2 \theta)^{-1}, \qquad \eta = \tfrac{1}{2}(1 - \tanh \alpha \tan \theta).$$

Consider on the C^*-algebra A_τ a positive linear functional $\mu = \mu_{\tau, \eta}$ satisfying $\mu(xy) = \mu(yx)$, i.e., μ is a trace, $x, y \in A_\tau$, and $\mu(we_{n+1}) = \eta\mu(w)$, $w \in A_{\tau, n}$. Such a functional μ exists if and only if τ, η are given by one of the following values of α, θ:

    (i) $\theta \in \mathbf{R}^+$, $\alpha \in \mathbf{R}$ *(the normal range)*;
    (ii) $\theta \in \mathbf{R}^+$, $\alpha = k\theta + \pi i/2$, $k \in \mathbf{Z}\backslash\{0\}$ *(the exceptional lines)*;
    (iii) $\theta = \pi i/m$, $m = 3, 4, 5, \ldots$; $\alpha = (k/m+1/2)\pi i$, $k = 1, 2, \ldots, [(m-1)/2]$ *(the exceptional points)*,

or, if $0 \leq \eta \leq 1$ and $\tau = \eta(1 - \eta)$ (the limit line).

The trace μ is then uniquely determined and the corresponding weak closure $R_{\tau, \eta}$ of A_τ is the hyperfinite II_1 factor for $\eta \neq 0, 1$, and $\mathbf{C}$ for $\eta = 0, 1$. □

For α, θ as above consider the representation $\gamma = \gamma_{\alpha, \theta, \eta}$ of the braid group B_n into $R_{\tau, \eta}$ given by

$$\gamma(\sigma_k) = i(e^{-\alpha - \theta} e_k - e^{-\alpha + \theta}(1 - e_k)), \qquad k = 1, \ldots, n - 1.$$

Consider on $R_{\tau, \eta}$ the trace

$$\phi = \phi_{\alpha, \theta} = (i \cosh \alpha \operatorname{csch} \theta)^{n-1} \mu_{\tau, \eta}.$$

THEOREM. *For $b \in B_n$ the invariant polynomial $P_{\hat{b}}$ of the closure $\hat{b}$ is given by $P_{\hat{b}}(e^\alpha, -e^{-\alpha}, 2i \sinh \theta) = \phi(\gamma(b))$.*

P_L is given by a positive trace of a unitary representation precisely at the exceptional points. The Jones invariant corresponds to the line $k = 2$ and the Alexander invariant to $k = 0$; the Arf invariant to $m = 4$, $k = 2$. For $k = 1$, $P_L \equiv 1$, and for $k = -1$, $P_L = \pm 1$. This shows that for any knot K, $P_K - 1$ is divided by $P_{\text{trefoil}} - 1$.

We have implemented a program which computes the coefficients of P_L.

REMARK. H. Wenzl has independently obtained the structure of the simple modules of the Hecke algebras by studying subfactors of the hyperfinite II_1 factor and has computed the index and entropy of the subfactor generated by $e_2, e_3, \ldots$ in $R_{\alpha,\theta}$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19146 (Current address of P. Freyd)

DEPARTMENT OF MATHEMATICS, CLARK UNIVERSITY, WORCESTER, MASSACHUSETTS 01610 (Current address of D. Yetter)

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903 (Current address of J. Hoste)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CAMBRIDGE, CAMBRIDGE, ENGLAND (Current address of W. B. R. Lickorish)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT SANTA BARBARA, SANTA BARBARA, CALIFORNIA 93106 (Current address of K. Millett) AND MATHEMATICAL SCIENCES RESEARCH INSTITUTE, BERKELEY, CALIFORNIA 94720

MATHEMATICAL SCIENCES RESEARCH INSTITUTE, BERKELEY, CALIFORNIA 94720 (Current address of A. Ocneanu)