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A NEW (PROBABILISTIC) PROOF OF THE DIAZ–METCALF AND PÓLYA–SZEGŐ INEQUALITIES AND SOME APPLICATIONS

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Abstract. The Diaz–Metcalf and Pólya–Szegő inequalities are proved in the probabilistic setting. These results generalize the classical case for both sums and integrals. Using these results we obtain some other well-known inequalities in the probabilistic setting, namely the Kantorovich, Rennie, and Schweitzer inequalities.

1. Introduction

The celebrated Cauchy–Bunyakovskiĭ–Schwarz inequality can be written in the probabilistic setting as follows:

$$|\mathsf{E}\,\xi\eta|^2 \le \mathsf{E}\,\xi^2\,\mathsf{E}\,\eta^2 \tag{1}$$

where ξ and η are random variables defined on some probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. An inequality, converse to (1), is also true but only in exceptional cases known as the Diaz–Metcalf and Pólya–Szegő inequalities. Some partial cases of the latter two results are known as Kantorovich, Rennie, and Schweitzer inequalities (see [6, §2.11], [7]).

Below we prove new inequalities that can be viewed as inverses to (1) for almost surely bounded random variables ξ and η. These inequalities involve two first moments $\mathsf{E}\,\xi$ and $\mathsf{E}\,\xi^2$ and the upper and lower bounds m and M of the random variable ξ, that is, we assume that $\mathsf{P}\{m \le \xi \le M\} = 1$. We provide necessary and sufficient conditions for all cases under consideration. These results generalize the classical Diaz–Metcalf and Pólya–Szegő inequalities as well as other inequalities mentioned above for the probabilistic setting. The proof below is given for both discrete and continuous cases. We also consider inequalities for random vectors and inequalities with weights. The methods we use are elementary and are based on the properties of the operator E.

We write $\xi \sim \psi$ if the random variable ξ has the distribution/density ψ. The symbols $\mathcal{I}_A$ and $\chi_S(t)$ stand for the indicator of a random event A and the characteristic function of a set S, respectively; $\mathbb{N}_0$ denotes the set of nonnegative integers, $\delta_{\lambda\mu}$ is the Kronecker delta. Finally $L^2_\varphi[A]$ denotes the space of functions $\{h\colon \int_A |h(t)|^2\varphi(t)\,dt < \infty\}$ such that $\operatorname{supp}(h) = \overline{\{t\colon h(t) \ne 0\}}$.

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2. Diaz–Metcalf inequality

Below we consider real, almost surely bounded random variables ξ for which there are real constants m and M, $m \le M$, such that

$$\mathsf{P}\{m \le \xi \le M\} = 1.$$

The moments $\mathsf{E}\,\xi^r$, $r > 0$, can be estimated in the case of $m \ge 0$ by $m^r \le \mathsf{E}\,\xi^r \le M^r$. The variance satisfies the inequality $\operatorname{Var}\xi \le (M-m)^2/4$. In what follows we obtain more sophisticated results.

First we provide an auxiliary result used then to derive conditions where the Diaz–Metcalf inequality becomes an equality. This result is communicated to the author by O. I. Klesov [5].

Equality Theorem. *Let $\mathfrak{X}$ be a random variable for which there are two real numbers $\mathfrak{m}$ and $\mathfrak{M}$, $\mathfrak{m} < \mathfrak{M}$, such that*

$$\mathsf{P}(\mathfrak{m} \le \mathfrak{X} \le \mathfrak{M}) = 1.$$

The following two conditions are equivalent:

$$\mathsf{E}(\mathfrak{X} - \mathfrak{m})(\mathfrak{M} - \mathfrak{X}) = 0 \tag{2}$$

and

there exist random events A and B such that

$$\mathfrak{X} = \mathfrak{m}\mathcal{I}_A + \mathfrak{M}\mathcal{I}_B \quad a.s., \tag{3}$$

$$\mathsf{P}(A \cup B) = 1, \tag{4}$$

$$\mathsf{P}(A \cap B) = 0. \tag{5}$$

Here "a.s." stands for "almost surely".

Proof. Put $\delta = (\mathfrak{X} - \mathfrak{m})(\mathfrak{M} - \mathfrak{X})$. Conditions (3)–(5) imply (2), since $\delta = 0$ almost surely in this case.

To prove the converse put $A = \{\omega \colon \mathfrak{X}(\omega) = \mathfrak{m}\}$ and $B = \{\omega \colon \mathfrak{X}(\omega) = \mathfrak{M}\}$. Then (2) implies

$$\mathsf{P}(\mathfrak{X} \in \mathbb{R} \setminus \{\mathfrak{m}, \mathfrak{M}\}) = 0,$$

whence conditions (3) and (4) follow. Since $\mathfrak{m} < \mathfrak{M}$, condition (5) also holds. □

Now we are ready to prove the Diaz–Metcalf inequality for bounded random variables.

Theorem 1. *Let ξ and η be real random variables defined on the same probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. Assume that $\mathsf{P}\{m_1 \le \xi \le M_1\} = 1$, $\mathsf{P}\{m_2 \le \eta \le M_2\} = 1$, $m_1 \le M_1$, $m_2 \le M_2$, and $m_2 > 0$. Then*

$$\mathsf{E}\,\xi^2 + \frac{m_1 M_1}{m_2 M_2}\,\mathsf{E}\,\eta^2 \le \left(\frac{m_1}{M_2} + \frac{M_1}{m_2}\right)\mathsf{E}\,\xi\eta. \tag{6}$$

The inequality becomes an equality if and only if either (i) $m_1/M_2 = M_1/m_2$, *or* (ii) $m_1/M_2 < M_1/m_2$ *and*

$$\mathsf{P}\left(\frac{\xi}{\eta} \in \left\{\frac{m_1}{M_2}, \frac{M_1}{m_2}\right\}\right) = 1. \tag{7}$$

Proof. It is easy to see that

$$\frac{m_1}{M_2} \le \frac{\xi}{\eta} \le \frac{M_1}{m_2}. \tag{8}$$

The operator E is monotone, thus

$$\mathsf{E}\left(\frac{M_1\eta}{m_2} - \xi\right)\left(\xi - \frac{m_1\eta}{M_2}\right) \ge 0. \tag{9}$$

Now (6) follows from (9), since E is a linear operator.

It remains to treat the case of an equality in (6), that is, in (9). Since the case (i) is obvious, we consider the case (ii). Let $m_1/M_2 < M_1/m_2$ and condition (7) hold. We have in the case of (ii) that

$$\mathsf{P}(m_2\xi = M_1\eta \vee M_2\xi = m_1\eta) = 1 \quad \text{and} \quad \mathsf{P}(m_2\xi = M_1\eta \wedge M_2\xi = m_1\eta) = 0.$$

Thus inequality (9) becomes an equality if (8) holds. On the other hand, if (9) becomes an equality, then

$$\mathsf{P}\{M_2\xi = m_1\eta \vee m_2\xi = M_1\eta\} = 1. \tag{10}$$

Put $A = \{\omega \in \Omega\colon M_2\xi = m_1\eta\}$ and $B = \{\omega \in \Omega\colon m_2\xi = M_1\eta\}$. We apply the Equality Theorem to

$$\mathfrak{X} = \xi/\eta, \qquad \mathfrak{m} = m_1/M_2, \qquad \mathfrak{M} = M_1/m_2$$

and complete the proof of (7). □

Corollary 1.1 (Weighted Diaz–Metcalf type inequality in the discrete case)**.** *Assume that*

$$m_1 \le x_k \le M_1, \qquad k = 1, \dots, n.$$

Let $0 < m_2 \le y_l \le M_2$, $l = 1, \dots, m$, $p_{kl} \ge 0$, *and* $\sum_{k=1}^n \sum_{l=1}^n p_{kl} = 1$. *Then*

$$\sum_{k=1}^n x_k^2 q_k + \frac{m_1 M_1}{m_2 M_2} \sum_{l=1}^m y_l^2 r_l \le \left(\frac{m_1}{M_2} + \frac{M_1}{m_2}\right) \sum_{k=1}^n \sum_{l=1}^m x_k y_l p_{kl} \tag{11}$$

where

$$\sum_{l=1}^m p_{kl} = q_k, \quad k = 1, \dots, n, \qquad \sum_{k=1}^n p_{kl} = r_l, \quad l = 1, \dots, m.$$

Inequality (11) *becomes an equality if and only if*

(i) *either* $m_1/M_2 = M_1/m_2$,

(ii) *or* $m_1/M_2 < M_1/m_2$ *and*

$$\sum_{k,l \in \mathbf{I}_{x,y}} p_{kl} = 1, \tag{12}$$

where $\mathbf{I}_{x,y} := \{(k,l)\colon x_k/y_l \in \{m_1/M_2, M_1/m_2\}\}$.

Proof. Let (ξ, η) be a random discrete vector whose coordinates are defined on a probability space $(\Omega, \mathcal{P}(\Omega), \mathsf{P})$. Let η be a nonnegative random variable and $\mathcal{P}(\Omega) = \{S\colon S \subseteq \Omega\}$. Let $(\xi, \eta) \sim \mathsf{P}\{\xi = x_k, \eta = y_l\} = p_{kl}$, $k = 1, \dots, n$, $l = 1, \dots, m$, and $\sum_{k=1}^n \sum_{l=1}^m p_{kl} = 1$. Theorem 1 and inequality (6) prove (11). The case of an equality in (11) follows from the corresponding part of Theorem 1. □

Remark 1. Putting $m = n$ and $p_{kl} = \delta_{kl}/n$, $k, l = 1, \dots, n$, in (11) (note that

$$q_k = r_l = 1/n$$

in this case) we obtain the original Diaz–Metcalf inequality:

$$\sum_{k=1}^n x_k^2 + \frac{m_1 M_1}{m_2 M_2} \sum_{l=1}^n y_l^2 \le \left(\frac{m_1}{M_2} + \frac{M_1}{m_2}\right) \sum_{k=1}^n x_k y_k$$

(see [2]). This inequality becomes an equality if and only if $x_k/y_l \in \{m_1/M_2, M_1/m_2\}$ for all $k = 1, \dots, n$. This follows from (12), since $p_{kl} = 0$ for all $k \ne l$ and $p_{kk} = 1/n$.

A generalization of the latter inequality is given in [2] for complex numbers (see also the list of references in [6, pp. 66–67]).

Corollary 1.2 (Weighted Diaz–Metcalf inequality for integrals)**.** *Let f and g be Borel functions such that*

$$\begin{aligned} m_1 \le f(x) \le M_1 \quad &\text{and} \quad 0 < m_2 \le g(y) \le M_2 \\ \text{for almost all } (x,y) &\in [a,b]\times[c,d]. \end{aligned} \tag{13}$$

Let $w(x,y)$ be a nonnegative function such that $\operatorname{supp}(w) = [a,b]\times[c,d]$ *and*

$$\int_a^b \int_c^d w(x,y)\,dx\,dy = 1.$$

Put

$$\int_c^d w(x,y)\,dy = w_1(x), \qquad \int_a^b w(x,y)\,dx = w_2(y).$$

Then

$$\begin{aligned} &\int_a^b f^2(x)w_1(x)\,dx + \frac{m_1M_1}{m_2M_2}\int_c^d g^2(y)w_2(y)\,dy \\ &\quad \le \left(\frac{m_1}{M_2}+\frac{M_1}{m_2}\right)\int_a^b\int_c^d f(x)g(y)w(x,y)\,dx\,dy. \end{aligned} \tag{14}$$

Inequality (14) *becomes an equality if and only if either*

(i) $m_1/M_2 = M_1/m_2$*, or*
(ii) $m_1/M_2 < M_1/m_2$ *and*

$$\int_{\mathbb{I}_{x,y}} w(x,y)\,dx\,dy = 1$$

where $\mathbb{I}_{x,y} := \{(x,y)\colon f(x)/g(y) \in \{m_1/M_2, M_1/m_2\}\}$.

Proof. Let (ξ,η) be a random vector with the density $w(x,y)$ whose support is $[a,b]\times[c,d]$. It is clear that $w_1(x)$ and $w_2(y)$ are the densities of the random variables ξ and η, respectively. Since f and g are Borel functions, they belong to $L^2_{w_1}[a,b]$ and $L^2_{w_2}[c,d]$, respectively. Let m_1, m_2 and M_1, M_2 be real numbers such that

$$\mathsf{P}\{m_1 \le f(\xi) \le M_1\} = \mathsf{P}\{m_2 \le g(\eta) \le M_2\} = 1.$$

It follows from Theorem 1 that

$$\mathsf{E}\, f^2(\xi) + \frac{m_1M_1}{m_2M_2}\,\mathsf{E}\, g^2(\eta) \le \left(\frac{m_1}{M_2}+\frac{M_1}{m_2}\right)\mathsf{E}\, f(\xi)g(\eta) \tag{15}$$

which is equivalent to (14).

The case of equality is obvious in the case of (i). In the case of (ii) we have

$$\begin{aligned} &\mathsf{E}\, f^2(\xi) + \frac{m_1M_1}{m_2M_2}\,\mathsf{E}\, g^2(\eta) - \left(\frac{m_1}{M_2}+\frac{M_1}{m_2}\right)\mathsf{E}\, f(\xi)g(\eta) \\ &\quad = \mathsf{E}\left(f(\xi) - \frac{M_1}{m_2}g(\eta)\right)\left(f(\xi) - \frac{m_1}{M_2}g(\eta)\right) \end{aligned}$$

and thus the statement on the equality follows from the Equality Theorem. □

Remark 2. If $(\xi,\eta) \sim w(x,y) = (b-a)^{-1}\chi_{[a,b]^2}(x,y)\delta_{xy}$, then Corollary 1.2 coincides with the classical Diaz–Metcalf inequality for integrals:

$$\int_a^b f^2(x)\,dx + \frac{m_1M_1}{m_2M_2}\int_a^b g^2(x)\,dx \le \left(\frac{m_1}{M_2}+\frac{M_1}{m_2}\right)\int_a^b f(x)g(x)\,dx,$$

since the densities of the coordinates in this case are given by

$$w_1(x) = (b-a)^{-1}\chi_{[a,b]}(x) = w_2(x).$$

The equality holds if and only if either (i) $m_1/M_2 = M_1/m_2$, or (ii) $m_1/M_2 < M_1/m_2$ and

$$\int_{\mathbb{I}_x} w_1(x)\,dx = 1,$$

where $\mathbb{I}_x \equiv \mathbb{I}_{x,x}$, $x \in [a,b]$ (see Corollary 1.2 and [2]). Other related results can be found in [6, p. 64].

3. Pólya–Szegő inequality

The Pólya–Szegő inequality is published in [6, pp. 57 and 213–214] for both sums and integrals. Below we provide a probabilistic inequality for moments covering this classical result.

Theorem 2. *Let ξ and η be real random variables defined on a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. If $\mathsf{P}\{m_1 \le \xi \le M_1\} = 1$ and $\mathsf{P}\{m_2 \le \eta \le M_2\} = 1$ for $0 < m_j \le M_j$, $j = 1, 2$, then*

$$\frac{\mathsf{E}\,\xi^2\,\mathsf{E}\,\eta^2}{(\mathsf{E}\,\xi\eta)^2} \le \frac{1}{4}\left(\sqrt{\frac{m_1m_2}{M_1M_2}} + \sqrt{\frac{M_1M_2}{m_1m_2}}\right)^2. \tag{16}$$

Inequality (16) *becomes an equality if and only if*

$$\xi = m_1\mathcal{I}_A(\omega) + M_1\mathcal{I}_{\Omega\setminus A}(\omega) \quad \text{and} \quad \eta = m_2\mathcal{I}_{\Omega\setminus A}(\omega) + M_2\mathcal{I}_A(\omega) \quad \text{a.s.} \tag{17}$$

for some random event $A \in \mathfrak{F}$ such that

$$\mathsf{P}(A) = \frac{M_1m_2}{m_1M_2 + M_1m_2}. \tag{18}$$

Proof. Assume that ξ and η are almost surely nonnegative bounded random variables.

Applying arithmetic-mean–geometric-mean inequality to the left-hand side of the Diaz–Metcalf inequality (6) we obtain

$$\mathsf{E}\,\xi^2 + \frac{m_1M_1}{m_2M_2}\,\mathsf{E}\,\eta^2 \ge 2\sqrt{\frac{m_1M_1}{m_2M_2}\,\mathsf{E}\,\xi^2\,\mathsf{E}\,\eta^2}. \tag{19}$$

Now it is easy to get the desired result.

To prove the necessity in the statement on equality let ξ and η be defined by (17) and (18), respectively. It is easy to see that

$$\mathsf{E}\,\xi^2 = \frac{m_1M_1(m_1m_2 + M_1M_2)}{m_1M_2 + M_1m_2},$$
$$\mathsf{E}\,\eta^2 = \frac{m_2M_2(m_1m_2 + M_1M_2)}{m_1M_2 + M_1m_2},$$
$$\mathsf{E}\,\xi\eta = \frac{2m_1m_2M_1M_2}{m_1M_2 + M_1m_2},$$

whence we obtain an equality in (16).

Inequality (16) becomes an equality if inequalities (6) and (19) become equalities. Assume that $m_1m_2 < M_1M_2$ (the theorem is obvious otherwise). Then

$$\mathsf{P}\left(\frac{\xi}{\eta} \in \left\{\frac{m_1}{M_2}, \frac{M_1}{m_2}\right\}\right) = 1, \tag{a}$$

$$\mathsf{E}\,\xi^2 = m_1M_1(m_2M_2)^{-1}\,\mathsf{E}\,\eta^2. \tag{b}$$

Relation (a) holds if and only if there exist events $A, B \in \mathfrak{F}$, $A \cup B = \Omega$, $A \cap B = \varnothing$, such that

$$\xi = m_1\mathcal{I}_A(\omega) + M_1\mathcal{I}_B(\omega), \qquad \eta = m_2\mathcal{I}_B(\omega) + M_2\mathcal{I}_A(\omega). \tag{20}$$

Substituting (20) into (b) we obtain

$$\frac{\mathsf{P}(A)}{M_1/m_1} = \frac{\mathsf{P}(B)}{M_2/m_2},$$

that is,

$$\mathsf{P}(A) = \frac{M_1 m_2}{m_1 M_2 + M_1 m_2}, \qquad \mathsf{P}(B) = \frac{M_2 m_1}{m_1 M_1 + M_1 m_2}.$$

Thus relations (17) and (18) are proved. □

Corollary 2.1 (Weighted Pólya–Szegő type inequality for sums)**.** *Assume that*

$$0 < m_1 \le x_k \le M_1, \quad k = 1, \dots, n, \qquad 0 < m_2 \le y_l \le M_2, \quad l = 1, \dots, m.$$

Also let $p_{kl} \ge 0$ *and* $\sum_{k=1}^n \sum_{l=1}^n p_{kl} = 1$. *Then*

$$\sum_{k=1}^n x_k^2 q_k \sum_{l=1}^m y_l^2 r_l \le \frac{1}{4}\left(\sqrt{\frac{m_1 m_2}{M_1 M_2}} + \sqrt{\frac{M_1 M_2}{m_1 m_2}}\right)^2 \left(\sum_{k=1}^n \sum_{l=1}^m x_k y_l p_{kl}\right)^2 \tag{21}$$

where

$$\sum_{l=1}^m p_{kl} = q_k, \quad k = 1, \dots, n, \qquad \sum_{k=1}^n p_{kl} = r_l, \quad l = 1, \dots, m.$$

Inequality (21) *becomes an equality if and only if* $m = n$ *and*

$$\kappa = \frac{M_1 m_2 n}{m_1 M_2 + M_1 m_2} \in \mathbb{N}_0, \tag{22}$$

$$\begin{gathered} x_{i_1} = \cdots = x_{i_\kappa} = m_1, \quad x_{i_{\kappa+1}} = \cdots = x_{i_n} = M_1, \\ y_{i_1} = \cdots = y_{i_\kappa} = M_2, \ y_{i_{\kappa+1}} = \cdots = y_{i_n} = m_2, \quad i_j \in \{1, \dots, n\}, \end{gathered} \tag{23}$$

$$p_{kl} = 0, \qquad (k,l) \in \{i_1, \dots, i_\kappa\} \times \{i_{\kappa+1}, \dots, i_n\} \cup \{i_{\kappa+1}, \dots, i_n\} \times \{i_1, \dots, i_\kappa\}. \tag{24}$$

Proof. Let a random vector (ξ, η) be defined on a probability space $(\Omega, \mathcal{P}(\Omega), \mathsf{P})$ and

$$0 < (\xi, \eta) \sim \mathsf{P}\{\xi = x_k, \eta = y_l\} = p_{kl}.$$

Applying (16) we easily prove (21).

The statement on equality easily follows from conditions (22)–(24), since $m = n$. □

The Kantorovich inequality is a particular case of the weighted Pólya–Szegő inequality for sums (see [4]). Putting $m = n$, $x_j^2 = \gamma_j$, $y_k^2 = \gamma_k^{-1}$, we obtain $0 < \sqrt{\mu} \le \gamma_k \le \sqrt{M}$ where $m_1 = \mu$ and $M_1 = M$. If

$$p_{kl} = \frac{u_k^2}{\sum_{j=1}^n u_j^2} \delta_{kl}, \qquad k, l = 1, \dots, n, \tag{25}$$

where $u_k \ne 0$ for at least one k, then we get the Kantorovich inequality from (21):

$$\sum_{j=1}^n \gamma_j u_j^2 \sum_{j=1}^n \frac{u_j^2}{\gamma_j} \le \frac{1}{4}\left(\sqrt{\frac{\mu}{M}} + \sqrt{\frac{M}{\mu}}\right)^2 \left(\sum_{j=1}^n u_j^2\right)^2$$

(see also [6, p. 61]).

For $m = n$ we define the weights p_{kl} similarly to (25). Then

$$\sum_{k=1}^n x_k^2 u_k^2 \sum_{k=1}^n y_k^2 u_k^2 \le \frac{1}{4}\left(\sqrt{\frac{m_1 m_2}{M_1 M_2}} + \sqrt{\frac{M_1 M_2}{m_1 m_2}}\right)^2 \left(\sum_{k=1}^n x_k y_k u_k^2\right)^2,$$

which is exactly the Greub and Rheinboldt result (see [3] and [6, p. 61]).

Putting $m = n$ and $p_{kl} = n^{-1}\delta_{kl}$ we obtain from (21) the classical weighted Pólya–Szegő inequality for sums:

$$\sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2 \le \frac{1}{4}\left(\sqrt{\frac{m_1m_2}{M_1M_2}} + \sqrt{\frac{M_1M_2}{m_1m_2}}\right)^2 \left(\sum_{k=1}^{n} x_k y_k\right)^2 \tag{26}$$

for all positive real numbers x_k and y_k, $k = 1, \ldots, n$, bounded as in Corollary 2.1 (see for example, [6, pp. 60–61], [8]).

The condition for the equality is simpler in this case (as compared to Corollary 2.1), namely (26) becomes an equality under conditions (22)–(24).

Corollary 2.2 (Weighted Pólya–Szegő inequality for integrals)**.** *Let f and g be positive Borel functions such that*

$$\begin{gathered} 0 < m_1 \le f(x) \le M_1 \quad \text{and} \quad 0 < m_2 \le g(y) \le M_2 \\ \text{for almost all } (x,y) \in [a,b] \times [c,d]. \end{gathered} \tag{27}$$

Let $w(x,y)$ be a nonnegative weight function with $\operatorname{supp}(w) = [a,b] \times [c,d]$. *Put*

$$\int_c^d w(x,y)\,dy = w_1(x), \qquad \int_a^b w(x,y)\,dx = w_2(y).$$

Then

$$\frac{\int_a^b f^2(x)w_1(x)\,dx \int_c^d g^2(y)w_2(y)\,dy}{\left(\int_a^b \int_c^d f(x)g(y)w(x,y)\,dx\,dy\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{m_1m_2}{M_1M_2}} + \sqrt{\frac{M_1M_2}{m_1m_2}}\right)^2. \tag{28}$$

Inequality (28) *becomes an equality if and only if* $[a,b] \equiv [c,d]$ *and*

$$f(x) = m_1\chi_S(x) + M_1\chi_{[a,b]\setminus S}(x), \qquad g(y) = M_2\chi_{[a,b]\setminus S}(y) + m_2\chi_S(y), \tag{29}$$

for some Borel set $S \subseteq [a,b]$ whose Lebesgue measure is such that

$$|S| = \frac{M_1m_2(b-a)}{m_1M_2 + M_1m_2}, \tag{30}$$

and

$$w(x,y) = 0 \quad \text{for all } (x,y) \in S^2 \cup ([a,b] \setminus S)^2. \tag{31}$$

Since the proof of inequality (28) is similar to that of Corollaries 1.2 and 2.1, we omit it.

Remark 3. If $[a,b] = [c,d]$ and $(\xi,\eta) \sim w(x,y) = (b-a)^{-1}\chi_{[a,b]^2}(x,y)\delta_{xy}$, then

$$\frac{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx}{\left(\int_a^b f(x)g(x)\,dx\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{m_1m_2}{M_1M_2}} + \sqrt{\frac{M_1M_2}{m_1m_2}}\right)^2 \tag{32}$$

for all $f, g \in L^2[a,b]$ such that

$$0 < m_1 \le f(x) \le M_1, \qquad 0 < m_2 \le g(x) \le M_2.$$

If functions f and g satisfy conditions (30) and (31), then inequality (32) becomes an equality, and vice versa. Inequality (32) is proved by Pólya and Szegő (see [8, pp. 71 and 253–255, Problem 93]). Note that the case of equality in (32) is not studied in [8].

4. Rennie and Schweitzer inequalities

In this section we give a generalization of the Rennie and Schweitzer inequalities for the probabilistic setting. There are two methods of proof of these results, namely

1) the proof based on the linearity and monotonicity of the operator E,
2) the proof based on an application of the Diaz–Metcalf and Pólya–Szegő inequalities.

Let ζ be a positive and bounded random variable, that is,

$$\mathsf{P}\{m \le \zeta \le M\} = 1, \qquad m > 0.$$

Then

$$\mathsf{E}\big(M\zeta^{-1} - 1\big)(\zeta - m) \ge 0.$$

Thus

$$\mathsf{E}\,\zeta + mM\,\mathsf{E}\,\zeta^{-1} \le m + M.$$

The latter is a Rennie type inequality for positive bounded random variables. According to the arithmetic-mean–geometric-mean inequality, the left-hand side of the Rennie inequality is greater than or equal to

$$2\sqrt{mM\,\mathsf{E}\,\zeta\,\mathsf{E}(1/\zeta)}.$$

Hence

$$\mathsf{E}\,\zeta\,\mathsf{E}\,\zeta^{-1} \le \frac{(m+M)^2}{4mM},$$

which is a Schweitzer type inequality.

Theorem 3. *Let ζ be an almost surely positive bounded random variable defined on a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, that is, $m > 0$ and $\mathsf{P}\{m \le \zeta \le M\} = 1$. Then*

$$\mathsf{E}\,\zeta + mM\,\mathsf{E}\,\zeta^{-1} \le m + M \tag{33}$$

where the inequality becomes an equality if and only if $\zeta = m\mathcal{I}_A(\omega) + M\mathcal{I}_{\Omega\setminus A}(\omega)$ for some random event $A \in \mathfrak{F}$. Moreover

$$\mathsf{E}\,\zeta\,\mathsf{E}\,\zeta^{-1} \le \frac{(m+M)^2}{4mM} \tag{34}$$

where the inequality becomes an equality if and only if $\zeta = m\mathcal{I}_A(\omega) + M\mathcal{I}_{\Omega\setminus A}(\omega)$ for some random event $A \in \mathfrak{F}$ such that $\mathsf{P}(A) = \frac{1}{2}$.

Proof. It is necessary to put $\xi^2 = \zeta = \eta^{-2}$ in (6) and (16). □

The following result is given in [9] without any discussion of the case of equality (see also [6, p. 63]).

Corollary 3.1 (Rennie inequality for sums)**.** *Let x_j, $1 \le j \le n$, be positive real numbers and let $\pi_j \ge 0$ for all $1 \le j \le n$ and $\sum_{j=1}^n \pi_j = 1$. Then*

$$\sum_{j=1}^n x_j\pi_j + mM\sum_{j=1}^n \frac{\pi_j}{x_j} \le m + M, \tag{35}$$

where m and M are the minimal and maximal terms of the sequence $\{x_j\}$.

Inequality (35) becomes an equality if and only if $x_j \in \{m, M\}$, $j = 1, \dots, n$.

Proof. Consider a random variable ζ with the distribution $\mathsf{P}\{\zeta = x_j\} = \pi_j$. Applying (34) we get (35).

Let $I \subseteq \{1, \dots, n\}$. Put $x_j = m$ for $j \in I$ and $x_j = M$ for $j \notin I$. Since

$$\sum_{j=1}^{n} x_j \pi_j + mM \sum_{j=1}^{n} \frac{\pi_j}{x_j} = m \sum_{j\in I} \pi_j + M \sum_{j\notin I} \pi_j + mM \left(\frac{1}{m} \sum_{j\in I} \pi_j + \frac{1}{M} \sum_{j\notin I} \pi_j \right) = m + M,$$

Corollary 3.1 is proved. □

Corollary 3.2 (Rennie inequality for integrals). *Let $f, 1/f \in L^1_w[a,b]$ where $w(x) \geq 0$ and $\int_a^b w(x)\,dx = 1$. If $0 < m \leq f(x) \leq M$ for all $x \in [a,b]$, then*

$$\int_a^b f(x)w(x)\,dx + mM \int_a^b \frac{w(x)}{f(x)}\,dx \leq m + M. \tag{36}$$

The inequality becomes an equality if and only if $f(x) = m\chi_S(x) + M\chi_{[a,b]\setminus S}(x)$ for some Borel set $S \subseteq [a,b]$.

Corollary 3.3 (Weighted Schweitzer inequality for sums). *Let*

$$0 < m \leq x_j \leq M, \qquad \pi_j \geq 0, \qquad j = 1, \dots, n,$$

and moreover $\sum_{j=1}^n \pi_j = 1$. Then

$$\sum_{j=1}^{n} x_j \pi_j \sum_{j=1}^{n} \frac{\pi_j}{x_j} \leq \frac{(m+M)^2}{4mM}. \tag{37}$$

The inequality becomes an equality if and only if n is even ($n = 2\lambda$), $\sum_{l=1}^{\lambda} \pi_l = 1/2$, and

$$x_{j_1} = \cdots = x_{j_\lambda} = m, \qquad x_{j_{\lambda+1}} = \cdots = x_{j_n} = M, \qquad j_l \in \{1, \dots, n\}.$$

Corollary 3.4 (Weighted Schweitzer inequality for integrals). *Let*

$$f \in L^1_w[a,b] \quad \text{and} \quad 1/f \in L^1_w[a,b]$$

where $w(x) \geq 0$ and $\int_a^b w(x)\,dx = 1$. If $0 < m \leq f(x) \leq M$ for all $x \in [a,b]$, then

$$\int_a^b f(x)w(x)\,dx \int_a^b \frac{w(x)}{f(x)}\,dx \leq \frac{(m+M)^2}{4mM}. \tag{38}$$

The inequality becomes an equality if and only if $f(x) = m\chi_S(x) + M\chi_{[a,b]\setminus S}(x)$ for some Borel set $S \subseteq [a,b]$ whose Lebesgue measure $|S|$ is equal to $(b-a)/2$.

The Schweitzer inequality can be found in [10] for both sums and integrals. If a random variable

$$\xi \sim \mathsf{P}\{\xi = x_j\} = \pi_j = 1/n, \qquad j = 1, \dots, n,$$

is defined on some probability space $(\Omega, \mathcal{P}(\Omega), \mathsf{P})$, then (34) implies that

$$\sum_{j=1}^{n} x_j \sum_{j=1}^{n} \frac{1}{x_j} \leq \frac{(m+M)^2 n^2}{4mM},$$

which coincides with the Schweitzer inequality for sums.

If $\xi \sim \mathcal{U}[a,b]$ in (39), that is, $w(x) = (b-a)^{-1}\chi_{[a,b]}(x)$, then

$$\int_a^b f(x)\,dx \int_a^b \frac{dx}{f(x)} \leq \frac{(m+M)^2}{4mM}(b-a)^2.$$

This is the original Schweitzer inequality proved in [10]. Other related results can be found in [6, pp. 60–61], [8, pp. 71 and 253–255].

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