# A new proof of a theorem of Ramanujam-Morrow 

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#### Abstract

Morrow [9] classified all weighted dual graphs of the boundary of the minimal normal compactifications of the affine plane $\mathbf{A}^{2}$ by using a result of Ramanujam [10] that any minimal normal compactification of $\mathbf{A}^{2}$ has a linear chain as the graph of the boundary divisor. In this article, we give a new proof of the above-mentioned results of Ramanujam-Morrow [9] from a different point of view and by the purely algebro-geometric arguments. Moreover, we show that the affine plane $\mathbf{A}^{2}$ is characterized by the weighted dual graph of the boundary divisor.


## 1. Introduction

All algebraic varieties considered in the present article are defined over the field of complex numbers $\mathbf{C}$. For any nonsingular quasi-projective variety $X$, a normal compactification of $X$ is a nonsingular projective variety $\bar{X}$ such that $\bar{X}$ contains $X$ as a Zariski open subset and the boundary $D=\bar{X}-X$ is a divisor with simple normal crossings. In the two-dimensional case, we say that $\bar{X}$ is a minimal normal compactification (or m.n.c. for short) of $X$ if $\bar{X}$ is a normal compactification of $X$ and, moreover, if any ( -1 )-curve in $D$ meets at least three other irreducible components of $D$. A normal compactification of the affine line $\mathbf{A}^{1}$ is unique, and it a smooth rational curve $\mathbf{P}^{1}$. But there exist infinitely many normal compactifications of the affine plane $\mathbf{A}^{2}$. Let $\mathbf{A}^{2} \hookrightarrow V$ be any (minimal) normal compactification of $\mathbf{A}^{2}$ and $D:=V-\mathbf{A}^{2}$ be the boundary divisor. Let $C_{1}, \ldots, C_{r}$ denote the irreducible components of $D$. Then it is easy to see that $H_{1}\left(\cup_{i=1}^{r} C_{i} ; \mathbf{Z}\right)=0$, namely,
(i) each irreducible component $C_{i}$ of $D$ is isomorphic to $\mathbf{P}^{1}$ and $D$ contains no cycles.
The $\mathbf{R}$-vector space $F:=\oplus_{i=1}^{r} \mathbf{R} \cdot C_{i}$ with basis $C_{1}, \ldots, C_{r}$ has a quadratic form on it defined by intersection multiplicity. Let $b^{+}$be the number of positive eigenvalues of this quadratic form on $F$. Note that the Picard group Pic ( $V$ )

[^0]of $V$ is a free abelian group generated by the classes of $C_{1}, \ldots, C_{r}$ because the coordinate ring $\Gamma\left(\mathcal{O}_{\mathbf{A}^{2}}\right)$ of $\mathbf{A}^{2}$ is factorial and $\Gamma\left(\mathcal{O}_{\mathbf{A}^{2}}\right)^{*}=\mathbf{C}^{*}$. Therefore, $\mathrm{NS}(V) \otimes_{\mathbf{Z}} \mathbf{R}$ is isomorphic to $F$, where $\mathrm{NS}(V)$ is the Néron-Severi group of $V$. Thus, by Hodge Index Theorem, we have:
(ii) $b^{+}=1$.

In the course of the proof of the main result in the famous paper of Ramanujam [10] which gives a topological characterization of the affine plane $\mathbf{A}^{2}$, he shows that for any m.n.c. of $\mathbf{A}^{2}$ the dual graph of the boundary divisor $D$ is a linear chain. Ramanujam, in fact, proved this result using the properties (i), (ii) and the following topological condition:
(iii) the fundamental group of the boundary of a tubular neighborhood of $D$ is trivial.
(The fact that any m.n.c. of $\mathbf{A}^{2}$ has a linear chain as the graph of the boundary divisor is obtained by a more general result of Daigle [2, Lemma 5.11] which asserts that the weighted graph equivalent to a linear chain contracts to a linear chain. But we can not find a printed proof for this result in [2].) Ramanujam showed, moreover, that there exist at least one and at most two irreducible components of $D$ having non-negative self-intersection number and that if $D$ contains exactly two components having non-negative self-intersection number, then they meet each other and at least one of them has self-intersection number zero. Using these results of Ramanujam concerning the algebro-topological properties of the boundary of $\mathbf{A}^{2}$, Morrow [9] gave a complete classification of all wighted dual graphs of the boundary divisor for m.n.c.'s of $\mathbf{A}^{2}$. Once we admit the results of Ramanujam, Morrow's argument in [9] is a detailed observation for the weighted linear chain to be equivalent to a graph consisting of only one vertex with weight 1.

In this article, we shall reprove a theorem of Morrow as to the classification of the boundary of m.n.c.'s of $\mathbf{A}^{2}$ from a different point of view. Our method is purely algebro-geometric. Our argument is, roughly speaking, stated as follows: Let $\mathbf{A}^{2} \hookrightarrow V$ be any m.n.c. of the affine plane $\mathbf{A}^{2}$. Since $\mathbf{A}^{2}$ has a structure of an $\mathbf{A}^{1}$-fibration over $\mathbf{A}^{1}$, the closures in $V$ of the fibers of this $\mathbf{A}^{1}$-fibration generate an irreducible linear pencil $\Lambda$ on $V$. If $\Lambda$ is already base point free, then the observation is considerably simple and we can easily determine the weighted dual graph of the boundary. Whereas, if Bs $\Lambda$ is not empty, then it consists of only a single point, say $p_{0}$. We then consider the shortest succession of blowing-ups $\sigma: \widetilde{V} \rightarrow V$ with center $p_{0}$ including its infinitely near points such that the proper transform $\widetilde{\Lambda}:=\sigma^{\prime}(\Lambda)$ of $\Lambda$ is base point free. This process $\sigma$ is, in fact, a composite of Euclidean transformations and equi-multiplicity transformations (see Section 2) determined by the datum at the singularity $p_{0}$ of a general member of $\Lambda$. (In order to determine the weighted dual graph of the boundary, essential is an explicit description of the shortest process $\sigma$ to resolve $\operatorname{Bs} \Lambda$. In fact, the heart of this paper is devoted to a concrete description of the process $\sigma$ as a composite of Euclidean transformations and equi-multiplicity transformations. This is done in Section 3.) The base point free linear pencil $\widetilde{\Lambda}$ defines a $\mathbf{P}^{1}$-fibration on $\widetilde{V}$. By observing the singular fibers of this $\mathbf{P}^{1}$ fibration, we obtain the result of Morrow. As a result, we prove also the result
of Ramanujam as to the linearity of the boundary graph. To state the results of Ramanujam and Morrow, we have the following result:

Theorem 1.1. Let $\mathbf{A}^{2} \hookrightarrow V$ be a minimal normal compactification of the affine plane. Then the weighted dual graph of the boundary $D:=V-$ $\mathbf{A}^{2}$ is given by one of the graphs $(a)-(f)$ in Figure 1, where $n, m_{1}, \ldots, m_{\beta-1}$ are arbitrary positive integers and every vertex stands for a smooth rational curve. These graphs actually occur as the boundary graphs of minimal normal compactifications of $\mathbf{A}^{2}$.

Furthermore, we shall prove that the converse of Theorem 1.1 holds. In fact, we have the following result:

Theorem 1.2. Let $X$ be a smooth affine surface and let $X \hookrightarrow V$ be a normal compactification of $X$. (This compactification is not necessarily minimal.) Suppose that each irreducible component of the boundary $D=V-X$ is a rational curve and the weighted dual graph of $D$ is the same as that of the boundary divisor of a suitable normal compactification of the affine plane $\mathbf{A}^{2}$. Then $X$ is isomorphic to $\mathbf{A}^{2}$.

Theorem 1.2 means that an affine plane $\mathbf{A}^{2}$ is determined by the weighted dual graph of the boundary divisor. But, as remarked in Section 4, if we replace the affine plane $\mathbf{A}^{2}$ by another $\mathbf{Q}$-homology plane in the statement of Theorem 1.2 , then a similar result fails to hold in general. Here, a Q-homology plane is, by definition, a smooth affine surface $S$ with $H_{i}(S ; \mathbf{Q})=0$ for $i>0$ (see Miyanishi and Sugie [8] for the relevant results of Q-homology planes).

In view of the importance of the result of Ramanujam-Morrow in affine algebraic geometry, or more specifically, in the theory of open algebraic surfaces, it is useful to reprove the result of Ramanujam-Morrow by a purely algebrogeometric method and from a different point of view. In Section 2, we give some preliminary results with respect to the singular fibers of a $\mathbf{P}^{1}$-fibration. Further, we define the notions of Euclidean transformations and equi-multiplicity transformations which will replace the topological and combinatorial arguments of Ramanujam-Morrow. In Section 3, we prove Theorem 1.1 by making use of the theory of an $\mathbf{A}^{1}$-fibration and a $\mathbf{P}^{1}$-fibration and the notions of an Euclidean transformation and an equi-multiplicity transformation. We prove Theorem 1.2 in Section 4. The author would like to express his hearty thanks to Prof. M. Miyanishi for helpful discussions.

## 2. Preliminaries

We denote by $\mathbf{A}^{n}$ the affine space of dimension $n$. A smooth projective, rational curve with self-intersection number $-n$ on a smooth algebraic surface is called a $(-n)$-curve. A morphism $\varphi$ from a smooth algebraic surface $V$ to a smooth algebraic curve $B$ is called a $\mathbf{P}^{1}$-fibration if a general fiber of $\varphi$ is isomorphic to $\mathbf{P}^{1}$. Similarly, an $\mathbf{A}^{1}$-fibration is defined.
(a) $\quad 1$
(b) $\quad \stackrel{\circ}{m} \quad 0 \quad(m \neq-1)$
(c)

(d)

(e) $\quad L_{1, e} \longrightarrow m_{1} \square_{0}^{\circ} \circ L_{2, e}$
(f) $\quad L_{1, f}=m_{1} \quad \stackrel{\square}{\circ} L_{2, f}$
$\beta$ is odd and $\beta \geq 3$
$\beta$ is even and $\beta \geq 4$

Figure 1

The following elementary result about singular fibers of a $\mathbf{P}^{1}$-fibration on a smooth projective surface is useful in the various situations (cf. Miyanishi [7, Chapter I, 4.4.1]).

Lemma 2.1. Let $f: V \rightarrow B$ be a $\mathbf{P}^{1}$-fibration on a smooth projective surface, where $B$ is a smooth complete algebraic curve. Let $F$ be a singular fiber of $f$, i.e., $F$ is not isomorphic to $\mathbf{P}^{1}$. Then the following assertions are true.
(1) The reduced curve $F_{\text {red }}$ is a divisor with simple normal crossings and each irreducible component of $F_{\text {red }}$ is isomorphic to $\mathbf{P}^{1}$. Furthermore, the dual graph of $F$ is a tree.
(2) At least one of the irreducible components of $F$ is a $(-1)$-curve.
(3) If a (-1)-curve $E$ occurs with multiplicity 1 in the fiber $F$, then $F$ contains another $(-1)$-curve.
(4) Any (-1)-curve in $F$ meets at most two other irreducible components in $F$.

Remark 2.2. $\quad$ Suppose that the support of a fiber $F$ of $f$ is written as

$$
\operatorname{Supp}(F)=C+\Gamma_{1}+\Gamma_{2}+\Gamma_{3},
$$

where $C$ is an irreducible component, $\operatorname{Supp}(F)-C$ is a disjoint union of three connected parts $\Gamma_{i}$ 's and $\Gamma_{1}, \Gamma_{2}$ contain no ( -1 ) components. Then $\Gamma_{3}$ is contractible to a smooth point. For otherwise, after the contraction, say $\tau$, of all possible contractible components in $\Gamma_{3}$, the part $\tau(F)-\tau(C)$ contains no $(-1)$-curves. So, $\tau(C)$ is a unique $(-1)$-curve in the fiber $\tau(F)$ by Lemma 2.1 (2). Then $\tau(C)$ meets three distinct fiber components. This is a contradiction to Lemma 2.1 (4).

We shall recall the definitions of Euclidean transformation and EM-transformation, which will play very important roles in the subsequent arguments of Section 3 to prove Theorem 1.1. Let $V_{0}$ be a smooth projective surface, let $p_{0}$ be a point on $V_{0}$ and let $l_{0}$ be an irreducible curve on $V_{0}$ such that $p_{0}$ is a simple point of $l_{0}$. Let $d_{0}$ and $d_{1}$ be positive integers such that $d_{1}<d_{0}$. By the Euclidean algorithm with respect to $d_{1}<d_{0}$, we find positive integers $d_{2}, \ldots, d_{\alpha}$ and $q_{1}, \ldots, q_{\alpha}$ :

$$
\left\{\begin{array}{rlrl}
d_{0} & =q_{1} d_{1}+d_{2}, & & d_{2}<d_{1} \\
d_{1} & =q_{2} d_{2}+d_{3}, & & d_{3}<d_{2} \\
\cdots & \cdots & \cdots \cdots \cdots & \cdots \\
d_{\alpha-2} & =q_{\alpha-1} d_{\alpha-1}+d_{\alpha}, & & d_{\alpha}<d_{\alpha-1} \\
d_{\alpha-1} & =q_{\alpha} d_{\alpha}, & q_{\alpha}>1 .
\end{array}\right.
$$

Set $N:=\sum_{s=1}^{\alpha} q_{s}$. Define the infinitely near points $p_{i}$ 's of $p_{0}$ for $1 \leq i<N$ and the blowing-up $\sigma_{i}: V_{i} \rightarrow V_{i-1}$ with center at $p_{i-1}$ for $1 \leq i \leq N$ inductively as follows:
(i) $p_{i}$ is an infinitely near point of order one of $p_{i-1}$ for $1 \leq i<N$.

$\alpha$ : odd
$\alpha$ : even


Figure 2
(ii) Let $E_{i}:=\sigma_{i}^{-1}\left(p_{i-1}\right)$ for $1 \leq i \leq N$ and let $E(s, t):=E_{i}$ if $i=$ $q_{1}+\cdots+q_{s-1}+t$ with $1 \leq s \leq \alpha$ and $1 \leq t \leq q_{s}$, where we set $q_{0}:=0$ and $E(0,0):=l_{0}$. The point $p_{i}$ is an intersection point of the proper transform of $E\left(s-1, q_{s-1}\right)$ on $V_{i}$ and the exceptional curve $E(s, t)$ if $i=q_{1}+\cdots+q_{s-1}+t$ with $1 \leq s \leq \alpha$ and $1 \leq t \leq q_{s}\left(1 \leq t<q_{\alpha}\right.$ if $\left.s=\alpha\right)$.

Then a composite $\sigma:=\sigma_{1} \cdots \sigma_{N}$ is called an Euclidean transformation associated with the datum $\left\{p_{0}, l_{0}, d_{0}, d_{1}\right\}$ (cf. Miyanishi [5], [6, p. 92]). The weighted dual graph of $\operatorname{Supp}\left(\sigma^{-1}\left(l_{0}\right)\right)$ is given in Figure 2, where $E_{0}:=\sigma^{\prime}\left(l_{0}\right)$ which denotes the proper transform of $l_{0}$ by $\sigma$ and where we denote the proper transform of $E(s, t)$ on $V_{N}$ by the same notation. Let $C_{0}$ be an irreducible curve on $V_{0}$ such that $p_{0}$ is a one-place point of $C_{0}$, let $d_{0}$ be the local intersection number $i\left(C_{0} \cdot l_{0} ; p_{0}\right)$ of $C_{0}$ and $l_{0}$ at $p_{0}$ and let $d_{1}$ be the multiplicity mult $p_{0}\left(C_{0}\right)$ of $C_{0}$ at $p_{0}$. Assume that $d_{0}>d_{1}$. The proper transform $C_{i}:=\left(\sigma_{1} \cdots \sigma_{i}\right)^{\prime}\left(C_{0}\right)$ passes through $p_{i}$ so that $\left(C_{i} \cdot E(s, t)\right)=d_{s}$ and the intersection number of $C_{i}$ with the proper transform of $E\left(s-1, q_{s-1}\right)$ on $V_{i}$ is $d_{s-1}-t d_{s}$, where $i=q_{1}+\cdots+q_{s-1}+t$. The smaller one of $d_{s}$ and $d_{s-1}-t d_{s}$ is the multiplicity of $C_{i}$ at $p_{i}$ for $p_{i}$ is a one-place point of $C_{i}$. Note that the proper transform $\sigma^{\prime}\left(C_{0}\right)$ on $V_{N}$ meets the last exceptional curve $E\left(\alpha, q_{\alpha}\right)$ with order $d_{\alpha}$ and does not meet $E_{0}:=\sigma^{\prime}\left(l_{0}\right)$ and other exceptional curves arising in the blowing-up process $\sigma$.

We now explain EM-transformations, which is called an ( $e, i$ )-transformation in Miyanishi [5] and [6, p. 100]. Let $V_{0}, p_{0}$ and $l_{0}$ be the same as above. Let $r>0$ be a positive integer. An equi-multiplicity transformation (or EMtransformation, for short) of length $r$ with center at $p_{0}$ is a composite $\tau=$ $\tau_{1} \cdots \tau_{r}$ of blowing-ups defined as follows. For $1 \leq i \leq r, \tau_{i}: V_{i} \rightarrow V_{i-1}$ is defined inductively as the blowing-up with center at $p_{i-1}$ and $p_{i}$ is a point on $\tau_{i}^{-1}\left(p_{i-1}\right)$ other than the intersection point $\tau_{i}^{\prime}\left(\tau_{i-1}^{-1}\left(p_{i-2}\right)\right) \cap \tau_{i}^{-1}\left(p_{i-1}\right)\left(\tau_{1}^{\prime}\left(l_{0}\right) \cap\right.$ $\tau_{1}^{-1}\left(p_{0}\right)$ if $\left.i=1\right)$. Let $C_{0}$ be an irreducible curve on $V_{0}$ such that $p_{0}$ is a oneplace point of $C_{0}$. Suppose $d_{0}:=i\left(C_{0} \cdot l_{0} ; p_{0}\right)$ is equal to $d_{1}:=$ mult $p_{0}\left(C_{0}\right)$. Let $\tau_{1}: V_{1} \rightarrow V_{0}$ be the blowing-up with center $p_{0}$, and set $E_{1}:=\tau_{1}^{-1}\left(p_{0}\right)$ and $C_{1}:=\tau_{1}^{\prime}\left(C_{0}\right)$. Then the point $p_{1}:=C_{1} \cap E_{1}$ differs from $\tau_{1}^{\prime}\left(l_{0}\right) \cap E_{1}$. Set $d_{0}^{(1)}:=i\left(C_{1} \cdot E_{1} ; p_{1}\right)=d_{1}$ and $d_{1}^{(1)}:=$ mult ${ }_{p_{1}}\left(C_{1}\right)$. Suppose $d_{0}^{(1)}=d_{1}^{(1)}$. As above, let $\tau_{2}: V_{2} \rightarrow V_{1}$ be the blowing-up with center $p_{1}$, let $E_{2}:=\tau_{2}^{-1}\left(p_{1}\right)$ and let $C_{2}:=\tau_{2}^{\prime}\left(C_{1}\right)$. Then $p_{2}:=C_{2} \cap E_{2}$ differs from the point $\tau_{2}^{\prime}\left(E_{1}\right) \cap E_{2}$. Thus this process can be repeated as long as the intersection number of the proper transform of $C_{0}$ with the last exceptional curve is equal to the multiplicity of the proper transform of $C_{0}$ at the intersection point. If we perform the blowingups $r$ times, the composite of $r$ blowing-ups is an EM-transformation of length $r$.

## 3. Proof of Theorem 1.1

In this section, we shall determine all the boundary graphs for m.n.c.'s of $\mathbf{A}^{2}$. So, let $\mathbf{A}^{2} \hookrightarrow V$ be an m.n.c. and $D:=V-\mathbf{A}^{2}$ denote the boundary divisor. Let $C_{0} \cong \mathbf{A}^{1}$ be an affine line in $\mathbf{A}^{2}$ and let $g$ be an irreducible polynomial of $\Gamma\left(\mathcal{O}_{\mathbf{A}^{2}}\right)$ defining the curve $C_{0}$. This polynomial $g$ defines a polynomial map:

$$
\varphi: \mathbf{A}^{2} \rightarrow \mathbf{A}^{1}=\operatorname{Spec}(\mathbf{C}[g]), P \mapsto g(P) .
$$

By the Abhyankar-Moh Embedding Theorem (cf. [1]), all the fibers of $\varphi$ are isomorphic to $\mathbf{A}^{1}$ scheme-theoretically. Since the base curve $\mathbf{A}^{1}$ of $\varphi$ is rational, the closures of fibers of $\varphi$ in $V$ generate an irreducible linear pencil $\Lambda$ on $V$ such that $\left.\Phi_{\Lambda}\right|_{\mathbf{A}^{2}}$ coincides with $\varphi$, where $\Phi_{\Lambda}$ is the rational mapping defined by $\Lambda$. We consider two cases according as $\Lambda$ has base points or not.

Case I. $\Lambda$ has no base points.
Then $\Lambda$ defines a $\mathbf{P}^{1}$-fibration $\Phi_{\Lambda}: V \rightarrow \mathbf{P}^{1}$. The boundary $D$ contains a cross-section $S$ of $\Phi_{\Lambda}$ and all other irreducible components of $D$ are contained in the fibers of it. We write $D-S$ as a disjoint union:

$$
D-S=B_{1}+\cdots+B_{r},
$$

where $B_{i}$ is a connected component of $D-S$ for $1 \leq i \leq r$. Then we have the following result:

Lemma 3.1. $\quad r=1$ and $B_{1}$ is an irreducible component with self-intersection number zero.

Proof. Since the $\mathbf{A}^{1}$-fibration $\varphi$ is parametrized by the affine line $\mathbf{A}^{1}$, one of $B_{1}, \ldots, B_{r}$, say $B_{1}$, supports a unique full fiber of $\Phi_{\Lambda}$ lying outside $\mathbf{A}^{2}$. Suppose that $r>1$. Let $Q:=\Phi_{\Lambda}\left(B_{2}\right)$ and $C$ the closure of $\varphi^{*}(Q)$ in $V$. Let $F$ be the member of $\Lambda$ containing $B_{2}$. The support of $F$ is a union of $B_{2}$ and $C$. Note that $C$ is contained in $F$ with multiplicity one because all the fibers of $\varphi$ are irreducible and reduced. Hence there exists a $(-1)$-curve $E$ in $B_{2}$ by Lemma 2.1 (3). But the minimality of $D$ and Lemma 2.1 imply that the $(-1)$-curve $E$ meets the cross-section $S$ and two distinct components in $B_{2}$. Then the contraction of $E$ leads to a contradiction. Hence $r=1$. If the fiber $B_{1}$ is reducible, it contains a $(-1)$-curve. Then we get a contradiction by the same argumemt as above. Thus $B_{1}$ is irreducible and $\left(B_{1}{ }^{2}\right)=0$.

Thus we have that if $\operatorname{Bs} \Lambda=\emptyset$ the dual graph of $D$ is of type (b) in Figure 1.

Case II. $\Lambda$ has a base point.
Since $G \cap \mathbf{A}^{2} \cong \mathbf{A}^{1}$ for a general member $G$ of $\Lambda$, Bs $\Lambda$ consists only of one point, say $p_{0}$, which is a one-place point of $G$ and is located on $D$. Let $\sigma: V \rightarrow V$ be the shortest succession of blowing-ups with centers at $p_{0}$ and its infinitely near points such that the proper transform $\widetilde{\Lambda}:=\sigma^{\prime}(\Lambda)$ of $\Lambda$ has no base points. Then $\widetilde{\Lambda}$ defines a $\mathbf{P}^{1}$-fibration $\Phi_{\widetilde{\Lambda}}: \widetilde{V} \rightarrow \mathbf{P}^{1}$ on $\widetilde{V}$ such that $\left.\Phi_{\widetilde{\Lambda}}\right|_{\mathbf{A}^{2}}$ coincides with $\varphi$, where we identify $\sigma^{-1}\left(\mathbf{A}^{2}\right)$ with $\mathbf{A}^{2}$. Among the components of $\tilde{V}-\mathbf{A}^{2}=\operatorname{Supp}\left(\sigma^{-1}(D)\right)$, the last exceptional curve of $\sigma$ is a cross-section and all the others are contained in some members of $\widetilde{\Lambda}$. Since $\varphi$ is parametrized by $\mathbf{A}^{1}$, exactly one member of $\widetilde{\Lambda}$, say $F_{\infty}$, lies outside $\mathbf{A}^{2}$. Note that the process $\sigma$ is written as a composite of Euclidean transformations and EM-transformations (cf. Section 2 for the definitions) since a general member of $\Lambda$ has the point $p_{0}$ as a one-place point. According to the location of the point $p_{0}$, we need to consider two cases:

Case II-(1). $p_{0}$ lies on only one component, say $A$, of $D$.
We then prove the following result with respect to the process $\sigma$ :
Lemma 3.2. The process $\sigma$ ends with an EM-transformation.

Proof. Otherwise $\sigma$ ends with an Euclidean transformation. Let $S$ be the last exceptional component of $\sigma$. Then $\operatorname{Supp}\left(\sigma^{-1}(D)\right)-S$ consists of two connected components, say $B_{1}$ and $B_{2}$. One of $B_{1}$ and $B_{2}$, say $B_{1}$, contains the proper transform of $D$ and the other $B_{2}$ consists of a part of the exceptional components arising from the last Euclidean transformation. Hence the selfintersection number of each component of $B_{2}$ is less than or equal to -2 . Note that one of $B_{1}$ and $B_{2}$ supports the member $F_{\infty}$ of $\widetilde{\Lambda}$ lying outside $\mathbf{A}^{2}$ and the other plus one irreducible component $C$ such that $C \cap \mathbf{A}^{2}$ is a fiber of $\varphi$ supports a member $F$ of $\widetilde{\Lambda}$ different from $F_{\infty}$. It is clear that $B_{2}$ cannot support $F_{\infty}$ because there are no $(-1)$ components contained in it. Meanwhile, it is
also impossible that $B_{2}+C$ supports $F$. Indeed, otherwise, by Lemma 2.1 (3), there exists a ( -1 )-curve in $B_{2}$ since the multiplicity of $C$ in $F$ is one (see the argument at the beginning of this section). This is a contradiction.

We write $\sigma$ as:

$$
\sigma=\tau_{1} \cdot \sigma_{1} \cdots \tau_{n-1} \cdot \sigma_{n-1} \cdot \tau_{n} \quad \text { with } \quad n \geq 1
$$

where $\sigma_{j}$ (resp. $\tau_{j}$ ) is the $j$-th Euclidean transformation (resp. EM-transformation) in $\sigma$. Although $\tau_{j}$ for $1 \leq j<n$ might be the identity morphism, $\tau_{n}$ is not so by Lemma 3.2. For $1 \leq j<n$, let $\mathcal{D}_{j}=\left\{p_{0}^{(j)}, l_{0}^{(j)}, d_{0}^{(j)}, d_{1}^{(j)}\right\}$ be the datum of $\sigma_{j}$, let $d_{2}^{(j)}, \ldots, d_{\alpha_{j}}^{(j)}$ and $q_{1}^{(j)}, \ldots, q_{\alpha_{j}}^{(j)}$ be positive integers obtained by the Euclidean algorithm with respect to $d_{0}^{(j)}>d_{1}^{(j)}$ and let $E^{(j)}(s, t)$ denote the proper transform on $\widetilde{V}$ of the exceptional component arising from the $\left(q_{1}^{(j)}+\right.$ $\left.\cdots+q_{s-1}^{(j)}+t\right)$-th blowing-up in $\sigma_{j}$ for $1 \leq s \leq \alpha_{j}$ and $1 \leq t \leq q_{s}^{(j)}$. To simplify the notations we put $C_{j}:=E^{(j)}\left(\alpha_{j}, q_{\alpha_{j}}^{(j)}\right)$ for $1 \leq j \leq n$, that is, $C_{j}$ is the proper transform on $\widetilde{V}$ of the last exceptional component arising from $\sigma_{j}$. Let $l_{j}$ be the length of $\tau_{j}$ and $E^{(j)}(k)$ the proper transform on $\widetilde{V}$ of the exceptional component from the $k$-th blowing-up in $\tau_{j}$ for $1 \leq k \leq l_{j}$ and $1 \leq j \leq n$. Note that the last exceptional component $E^{(n)}\left(l_{n}\right)$ is a cross-section of $\widetilde{\Lambda}$ and $\operatorname{Supp}\left(\sigma^{-1}(D)\right)-E^{(n)}\left(l_{n}\right)$ supports $F_{\infty}$. We can specify $\sigma_{j}$ and $\tau_{j}$ as follows:

Lemma 3.3. With the notations as above, the following hold:
(1) For $2 \leq j \leq n$, we have $l_{j}>0$, i.e., $\tau_{j}$ is not the identity. But $\tau_{1}$ might be the identity.
(2) For $1 \leq j<n, \tau_{j+1}$ determines the foregoing $\sigma_{j}$ as follows:

$$
\begin{aligned}
& \left\{\begin{array}{llll}
\alpha_{j}=1 \text { and } q_{1}^{(j)}=2 & \text { if } & l_{j+1}=1 \\
\alpha_{j}=2, q_{1}^{(j)}=1 \text { and } q_{2}^{(j)}=l_{j+1} & \text { if } & l_{j+1}>1
\end{array} \quad(j \neq n-1),\right. \\
& \left\{\begin{array}{lll}
\alpha_{n-1}=1 \text { and } q_{1}^{(n-1)}=2 & \text { if } & l_{n}=2 \\
\alpha_{n-1}=2, q_{1}^{(n-1)}=1 \text { and } q_{2}^{(n-1)}=l_{n}-1 & \text { if } & l_{n}>2 .
\end{array}\right.
\end{aligned}
$$

Proof. Near the cross-section $E^{(n)}\left(l_{n}\right)$, the fiber $F_{\infty}$ lying outside $\mathbf{A}^{2}$ has the configuration as in Figure 3, where $S_{n-1}(+)$ (resp. $S_{n-1}(-)$ ) consists of $E^{(n-1)}(s, t)$ with $s$ even (resp. odd) except for the last component $C_{n-1}$ from $\sigma_{n-1}, T_{n}$ is a linear chain supported by the (-2)-curves $E^{(n)}(1), \ldots, E^{(n)}\left(l_{n}-\right.$ 1). Note that $S_{n-1}(-)$ and $T_{n}$ contain no ( -1 -curves, so the leftside of $C_{n-1}$ in Figure 3 is contractible by Remark 2.2.

After the contraction of it, the image of $F_{\infty}$ has the configuration as described in Figure 4, where the ( -1 )-curve is the image of $C_{n-1}$. Hence $S_{n-1}(-)$ consists only of one component $E^{(n-1)}(1,1)$ with self-intersection number $-l_{n}$.


Figure 3

$$
S_{n-1}(-) \underset{(-1)}{\square} \quad T_{n} \quad \cdots \cdots \cdots \cdots \cdots \circ{ }^{E^{(n)}\left(l_{n}\right)}
$$

Figure 4

Since $\left(E^{(n-1)}(1,1)^{2}\right) \leq-2$, we have $l_{n} \geq 2$. Since $S_{n-1}(-)$ consists only of $E^{(n-1)}(1,1)$ it follows that $\alpha_{n-1} \leq 2$. If $l_{n}=2$ we have $\alpha_{n-1}=1$ and $q_{1}^{(n-1)}=2$. Indeed, if $\alpha_{n-1}=2$ the self-intersection number of $E^{(n-1)}(1,1)$ is less than -2 . Meanwhile, if $l_{n}>2$ then it is not hard to show that $\alpha_{n-1}=$ 2. If $q_{1}^{(n-1)}>1$ then $S_{n-1}(-)$ contains at least two components, which is a contradiction. Hence $q_{1}^{(n-1)}=1$. Since $\left(E^{(n-1)}(1,1)^{2}\right)=-\left(1+q_{2}^{(n-1)}\right)=-l_{n}$, we have that $q_{2}^{(n-1)}=l_{n}-1$. Note that $S_{n-1}(+)$ is void if $\alpha_{n-1}=1$ or a linear chain of $(-2)$-curves if $\alpha_{n-1}=2$. We prove the following claim:

Claim. $\tau_{n-1}$ is not the identity if $n \geq 3$.
Proof. Assume the contrary that $\tau_{n-1}=i d$. Then the fiber $F_{\infty}$ has the configuration as in Figure 5, where $S_{n-2}(+)$ (resp. $\left.S_{n-2}(-)\right)$ consists of $E^{(n-2)}(s, t)$ with $s$ even (resp. odd) except for the last component $C_{n-2}$ from $\sigma_{n-2}$. Note that $S_{n-2}(-)$ and the rightside of $C_{n-2}$ contain no ( -1 )-curves. Hence the leftside of $C_{n-2}$ is contractible (Remark 2.2). After the contraction of it, we contract $C_{n-2}$ and the components in $S_{n-1}(+)$. Then the image of $C_{n-1}$ is a $(-1)$-curve meeting three other fiber components. This is a contradiction to Lemma 2.1 (4). Hence we obtain that $\tau_{n-1}$ is not the identity.

By repeating the above arguments downward $n, n-1, \ldots$, we can prove the assertions of Lemma 3.3.

Note that the process $\sigma$ to resolve $\operatorname{Bs} \Lambda=\left\{p_{0}\right\}$ does not affect on the components of the boundary $D$ other than $A$, and that the proper transform of $D$ is contained in a member $F_{\infty}$ of $\widetilde{\Lambda}$. Every component in $F_{\infty}$ other than the proper transform $A^{\prime}:=\sigma^{\prime}(A)$ of $A$ has self-intersection number less than


Figure 5
$L \xrightarrow[\circ]{A^{\prime}}-T\left(l_{1}\right)-S\left(l_{2}\right)-T\left(l_{2}\right)-\cdots \cdots-S\left(l_{n-1}\right)-T\left(l_{n-1}\right)-S\left(l_{n}-1\right)-T_{n} \ldots \ldots{ }^{E^{(n)}\left(l_{n}\right)}$

Figure 6
or equal to -2 by the minimality of $D$ and the constructions of Euclidean and EM-transformations. Hence, unless $A^{\prime}$ itself is the fiber $F_{\infty}, A^{\prime}$ is a unique $(-1)$-curve in $F_{\infty}$. Note that the case $A^{\prime}=F_{\infty}$ occurs only if $D=A$ and $\sigma$ is a single blowing-up. In this case the dual graph of the boundary $D$ is (a) in Figure 1. We exclude the case $A^{\prime}=F_{\infty}$ in the subsequent arguments. If $D-A$ consists of two or more other connected parts, the $(-1)$-curve $A^{\prime}$ meets three or more other fiber components of $F_{\infty}$ including an exceptional component of $\sigma$. This is a contradiction to Lemma 2.1 (4). Thus $D$ is written as

$$
D=A+L,
$$

where $L$ is connected ( $L$ might be void). By Lemma 3.3, the dual graph of the fiber $F_{\infty}$ is given as in Figure 6, where the notations $S(l)$ and $T(l)$ are those defined in Figure 7, $T_{n}$ is a linear chain supported by the ( -2 )-curves $E^{(n)}(1), \ldots, E^{(n)}\left(l_{n}-1\right)$ and where the part $T\left(l_{1}\right)$ may be empty.

We then prove the following result:

## Lemma 3.4.

(1) The self-intersection number $\left(A^{\prime 2}\right)$ of $A^{\prime}$ is written in terms of $\left(A^{2}\right)$ as follows:

$$
\left(A^{\prime 2}\right)=\left\{\begin{array}{lll}
\left(A^{2}\right)-2 & \text { if } & T\left(l_{1}\right)=\emptyset \\
\left(A^{2}\right)-1 & \text { if } & T\left(l_{1}\right) \neq \emptyset
\end{array}\right.
$$

(2) The part $L$ is determined by $T\left(l_{1}\right)$ as follows:
(i) If $T\left(l_{1}\right)=\emptyset, L$ is empty.
(ii) If $T\left(l_{1}\right) \neq \emptyset, L$ consists only of one component with self-inter section number $-\left(1+l_{1}\right)$ if $n \geq 2$ (resp. $-l_{1}$ if $\left.n=1\right)$.

Proof. (1) If $T\left(l_{1}\right)=\emptyset$, the process $\sigma$ starts with an Euclidean transformation $\sigma_{1}$. The datum of $\sigma_{1}$ is $\left\{p_{0}, A, d_{0}^{(1)}, d_{1}^{(1)}\right\}$, where $d_{0}^{(1)}:=i(G$. $\left.A ; p_{0}\right), d_{1}^{(1)}:=\operatorname{mult}_{p_{0}} G$ for a general member $G$ of $\Lambda$. By Lemma 3.3, we have either $d_{0}^{(1)}=2 d_{1}^{(1)}$ or $d_{0}^{(1)}=d_{1}^{(1)}+d_{2}^{(1)}$ with $d_{2}^{(1)}<d_{1}^{(1)}$. Hence, we blow up in $\sigma$ two points $p_{0}$ and its infinitely near point on $A$. Thus $\left(A^{\prime 2}\right)=\left(A^{2}\right)-2$. On the other hand, if $T\left(l_{1}\right) \neq \emptyset$ then we blow up only one point $p_{0}$ on $A$ in $\sigma$. Hence $\left(A^{\prime 2}\right)=\left(A^{2}\right)-1$.
(2) The dual graph of $F_{\infty}$ (cf. Figure 6) determines the part $L$ as given in the statement in such a way that $F_{\infty}$ is brought to a smooth fiber by contractions of successive $(-1)$-curves starting with $A^{\prime}$.

In view of Lemma 3.4 and $\left(A^{\prime 2}\right)=-1$, the dual graph of the boundary $D$ is of type either (a) or (b) with $m \leq-2$ in Figure 1.

Case II-(2). $\quad p_{0}$ is an intersection point of two components of $D$.
Let $A$ and $B$ be the components of $D$ such that $A \cap B=\left\{p_{0}\right\}$. We write $D$ as

$$
D=A+B+L_{1}+L_{2}
$$

where $L_{1}$ (resp. $L_{2}$ ) is the union of all connected components of $D-(A+B)$ which are linked to $A$ (resp. $B$ ). For a general member $G$ of $\Lambda$, we put $d_{1}:=$ mult $_{p_{0}} G$. Since $p_{0}$ is a one-place point of $G$, at least one of $i\left(G \cdot A ; p_{0}\right)$ and $i\left(G \cdot B ; p_{0}\right)$ is equal to $d_{1}$. We need to consider three subcases according as how $G$ intersects $A$ and $B$ at $p_{0}$.

Case II-(2)-(i). $\quad i\left(G \cdot A ; p_{0}\right)>i\left(G \cdot B ; p_{0}\right)=d_{1}$.
We put $d_{0}:=i\left(G \cdot A ; p_{0}\right)$ and obtain positive integers $d_{2}, \ldots, d_{\alpha}$ and $q_{1}, \ldots, q_{\alpha}$ by the Euclidean algorithm with respect to $d_{0}>d_{1}$ as performed in Section 2. The shortest process $\sigma$ to resolve the base points of $\Lambda$ starts with the Euclidean transformation $\sigma_{1}$ associated with the datum $\mathcal{D}_{1}=\left\{p_{0}, A, d_{0}\right.$, $\left.d_{1}\right\}$. Let $E(s, t)$ denote the proper transform on $\widetilde{V}$ of the exceptional components arising from the $\left(q_{1}+\cdots+q_{s-1}+t\right)$-th blowing-up in $\sigma_{1}$ for $1 \leq s \leq \alpha$ and $1 \leq t \leq q_{s}$. With the indexing slightly changed, we write $\sigma$ as

$$
\sigma=\sigma_{1} \cdot \tau_{1} \cdots \sigma_{n} \cdot \tau_{n} \quad \text { with } \quad n \geq 1
$$

where $\sigma_{j}$ (resp. $\tau_{j}$ ) is the $j$-th Euclidean transformation (resp. EM-transformation) in $\sigma$ and $\tau_{j}$ might be the identity. The same argument as in the proof of Lemma 3.2 shows that the following result holds:

Lemma 3.5. If $n \geq 2$, the process $\sigma$ ends with an EM-transformation.
We suppose, for the moment, that $n \geq 2$. Let $\mathcal{D}_{j}=\left\{p_{0}^{(j)}, l_{0}^{(j)}, d_{0}^{(j)}, d_{1}^{(j)}\right\}$ be the datum of $\sigma_{j}$ and let $d_{2}^{(j)}, \ldots, d_{\alpha_{j}}^{(j)}$ and $q_{1}^{(j)}, \ldots, q_{\alpha_{j}}^{(j)}$ be positive integers obtained by the Euclidean algorithm with respect to $d_{0}^{(j)}>d_{1}^{(j)}$ for $2 \leq j \leq n$. Let $l_{j}$ be the length of $\tau_{j}$ for $1 \leq j \leq n$. Let $E^{(j)}(s, t)$ for $1 \leq s \leq \alpha_{j}, 1 \leq$ $t \leq q_{s}^{(j)}$ and $E^{(j)}(k)$ for $1 \leq k \leq l_{j}$ have the same meaning as defined after the proof of Lemma 3.2. We have the following result concerning $\sigma_{j}$ and $\tau_{j}$ for $2 \leq j \leq n$ :

Lemma 3.6. Let the notations be the same as above. If $n \geq 2$, we have:
(1) For $2 \leq j \leq n, l_{j}>0$, i.e., $\tau_{j}$ is not the identity. But $\tau_{1}$ might be the identity.
(2) For $2 \leq j \leq n, \tau_{j}$ determines the foregoing $\sigma_{j}$ as follows:

$$
\left\{\begin{array}{lll}
\alpha_{j}=1 \text { and } q_{1}^{(j)}=2 & \text { if } & l_{j}=1 \\
\alpha_{j}=2, q_{1}^{(j)}=1 \text { and } q_{2}^{(j)}=l_{j} & \text { if } \quad l_{j}>1
\end{array} \quad \text { for } 2 \leq j<n\right.
$$



Figure 7


Figure 8

$$
\left\{\begin{array}{lll}
\alpha_{n}=1 \text { and } q_{1}^{(n)}=2 & \text { if } & l_{n}=2 \\
\alpha_{n}=2, q_{1}^{(n)}=1 \text { and } q_{2}^{(n)}=l_{n}-1 & \text { if } & l_{n}>2
\end{array}\right.
$$

Proof. The proof of the present lemma is the same as the one for Lemma 3.3 except for a slight difference of indices.

Let $F_{\infty}$ be the member of $\widetilde{\Lambda}$ lying outside $\mathbf{A}^{2}$. With the notations preceding Lemma 3.6, the last exceptional curve $E^{(n)}\left(l_{n}\right)$ is a cross-section of $\Lambda$ and the support of $F_{\infty}$ is $\operatorname{Supp}\left(\sigma^{-1}(D)\right)-E^{(n)}\left(l_{n}\right)$. By Lemma 3.6, the dual graph of $F_{\infty}$ is given as in Figure 8, where $A^{\prime}$ (resp. $B^{\prime}$ ) is the proper transform of $A$ (resp. B), the notations $S(l), T(l)$ are those defined in Figure $7, T_{n}$ is a linear chain supported by the $(-2)$-curves $E^{(n)}(1), \ldots, E^{(n)}\left(l_{n}-1\right)$ and where $T\left(l_{1}\right)$ may be empty. The dual graph of $E_{1}+E\left(\alpha, q_{\alpha}\right)+E_{2}$ is the same as given in Figure 2 with replaced the self-intersection number of $E\left(\alpha, q_{\alpha}\right)$ by -3 if $T\left(l_{1}\right)=\emptyset$ (resp. -2 if $\left.T\left(l_{1}\right) \neq \emptyset\right)$. Note that the process $\sigma$ does not affect on the components of $L_{1}+L_{2}$.

We prove the following result:
Lemma 3.7. With the notations as above, let $M_{1}:=L_{1}+A^{\prime}+E_{1}$ and $M_{2}:=L_{2}+B^{\prime}+E_{2}$ (see Figure 8). If $n \geq 2$ we have the following:
(1) Let $C_{2}$ be the last exceptional component of $\sigma_{2}$. (The component $C_{2}$ corresponds to the rightmost vertex of $S\left(l_{2}\right)$ in Figure 8.) Then the part lying on the leftside of $C_{2}$ in Figure 8 is contractible.
(2) Suppose $T\left(l_{1}\right)=\emptyset$. Then $E\left(\alpha, q_{\alpha}\right)$ becomes a $(-1)$-curve after the parts $M_{1}$ and $M_{2}$ are contracted to smooth points. Moreover, the contractions of the parts $M_{1}$ and $M_{2}$ start with the contractions of $A^{\prime}$ and $B^{\prime}$ and end with the contractions of the components meeting $E\left(\alpha, q_{\alpha}\right)$.
(3) Suppose $T\left(l_{1}\right) \neq \emptyset$. If $\alpha$ is odd (resp. if $\alpha$ is even) then $M_{2}$ (resp. $\left.M_{1}\right)$ is contractible and the contraction of $M_{2}\left(\right.$ resp. $\left.M_{1}\right)$ starts with $B^{\prime}$ (resp. $\left.A^{\prime}\right)$ and ends with $E\left(\alpha, q_{\alpha}-1\right)$ (cf. Figure 2).

Proof. (1) $F_{\infty}-C_{2}$ consists of three connected parts. Among them $E^{(2)}(1,1)$, which corresponds to a vertex with weight $-\left(1+l_{2}\right)$ of $S\left(l_{2}\right)$ in

Figure 8, and the rightside of $C_{2}$ contains no ( -1 )-curves. Hence, the leftside of $C_{2}$ is contractible by Remark 2.2.
(2) Suppose that $E\left(\alpha, q_{\alpha}\right)$ becomes a ( -1 )-curve before $M_{1}$ and $M_{2}$ are contracted. Then we contract all components between $E\left(\alpha, q_{\alpha}\right)$ and $C_{2}$. After this contraction the image of $C_{2}$ is a $(-1)$-curve meeting three or more other irreducible fiber components. This is a contradiction to Lemma 2.1 (4). Thus $E\left(\alpha, q_{\alpha}\right)$ becomes a ( -1 )-curve after the contractions of $M_{1}$ and $M_{2}$. Since each component in $M_{1}$ and $M_{2}$ other than $A^{\prime}$ and $B^{\prime}$ has self-intersection number less than or equal to -2 and $E\left(\alpha, q_{\alpha}\right)$ is a ( -3 )-curve, the contractions of $M_{1}$ and $M_{2}$ start with $A^{\prime}$ and $B^{\prime}$ and end with the components meeting $E\left(\alpha, q_{\alpha}\right)$.
(3) Since $E\left(\alpha, q_{\alpha}\right)$ meets three other fiber components and the rightside of $E\left(\alpha, q_{\alpha}\right)$ in Figure 8 contains no (-1)-curve, at least one of $M_{1}$ and $M_{2}$ is contractible (see Remark 2.2). But since $E\left(\alpha, q_{\alpha}\right)$ is a ( -2 )-curve, exactly one of $M_{1}$ and $M_{2}$ is contractible, and this contraction ends with the contraction of the component meeting $E\left(\alpha, q_{\alpha}\right)$. Suppose that $\alpha$ is odd and $M_{1}$ is contractible. After the contraction of $M_{1}$, the image of $F_{\infty}$ contains at least four components and one of them, which is the image of $E\left(\alpha, q_{\alpha}\right)$, is a ( -1 )-curve. If $l_{1}>1$ then the image of $E\left(\alpha, q_{\alpha}\right)$ meets two ( -2 )-curves, one in $M_{2}$ and the other in $T\left(l_{1}\right)$. This is a contradiction. If $l_{1}=1$ then $T\left(l_{1}\right)$ consists only of a ( -3 )-curve $E^{(1)}(1)$. After the contraction of $M_{1}$, we contract $E\left(\alpha, q_{\alpha}\right), E\left(\alpha, q_{\alpha}-1\right), E^{(1)}(1)$ and the components in the linear chain between $E^{(1)}(1)$ and $C_{2}$ (cf. Figure 8). But it contradicts the assertion (1) of the present lemma. Thus $M_{2}$ is contractible if $\alpha$ is odd. Similarly $M_{1}$ is contractible if $\alpha$ is even.

Lemma 3.8. With the notations as above, suppose that $n \geq 2$. Then we have the following:
(1) The self-intersection number of $A$ is determined as follows:

$$
\left(A^{2}\right)=\left\{\begin{array}{lll}
q_{1} & \text { if } & \alpha>1, \\
q_{1}-l_{1}-1 & \text { if } & \alpha=1
\end{array}\right.
$$

where we put $l_{1}=0$ when $\tau_{1}=i d$.
(2) $\left(B^{2}\right)=0$.
(3) If $\alpha>1$ and $\tau_{1}=i d$ (resp. if $\alpha>1$ and $\tau_{1} \neq i d$, resp. if $\alpha=1$ ) then the dual graphs of $L_{1}$ and $L_{2}$ are given as in Figure 9 (resp. as in Figure 9 with $(-2)$ components counted $\left(q_{\alpha}-l_{1}-2\right)$-times instead of $\left(q_{\alpha}-2\right)$-times, resp. $\left.L_{1}=L_{2}=\emptyset\right)$, where the $-\left(1+q_{2}\right)$-curve in $L_{1}$ meets $A$ and the $-\left(1+q_{1}\right)$-curve in $L_{2}$ meets $B$.

Proof. Suppose that $\tau_{1}=i d$, i.e., $T\left(l_{1}\right)=\emptyset$. Then by Lemma 3.7, $M_{1}$ and $M_{2}$ are contractible and these contractions start with $A^{\prime}$ and $B^{\prime}$ and end with the contractions of the components meeting $E\left(\alpha, q_{\alpha}\right)$. Hence $A^{\prime}$ and $B^{\prime}$ are $(-1)$-curves and $L_{1}$ and $L_{2}$ are uniquely determined by the parts $A^{\prime}+E_{1}$ and $B^{\prime}+E_{2}$. Noting that $\left(A^{\prime 2}\right)=\left(A^{2}\right)-\left(1+q_{1}\right)$ if $\alpha>1$ (resp. $\left(A^{\prime 2}\right)=\left(A^{2}\right)-q_{1}$ if $\alpha=1)$ and $\left(B^{\prime 2}\right)=\left(B^{2}\right)-1$, we then easily prove the assertions. Next suppose that $\tau_{1} \neq i d$, i.e., $T\left(l_{1}\right) \neq \emptyset$. We consider only the case $\alpha$ is odd because the

$\alpha$ : even


Figure 9
case $\alpha$ is even can be treated similarly. By Lemma 3.7 (3), $M_{2}$ is contractible and this contraction starts with $B^{\prime}$ and ends with $E\left(\alpha, q_{\alpha}-1\right)$. Hence, $B^{\prime}$ is a $(-1)$-curve and $L_{2}$ is determined by $B^{\prime}+E_{2}$ as described in Figure 9. We contract $E\left(\alpha, q_{\alpha}\right), E^{(1)}(1), \ldots, E^{(1)}\left(l_{1}-1\right)$ in this order after the contraction of $M_{2}$. Let $M_{1}^{\prime}$ be the image of $M_{1}$ after the above contraction and let $C$ be the image of the component meeting $E\left(\alpha, q_{\alpha}\right)$ in $M_{1}$. The self-intersection number of $C$ increases by $l_{1}$. Lemma 3.7 (1) then says that $M_{1}^{\prime}$ is contractible and that the contraction of $M_{1}^{\prime}$ starts with $A^{\prime}$ and ends with $C$. These observations imply the assertions. Note that if $\alpha=1$, the part $E_{1}$ is empty and $C$ is the image of $A^{\prime}$ satisfying $-1=\left(C^{2}\right)=\left(A^{\prime 2}\right)+l_{1}=\left(A^{2}\right)-q_{1}+l_{1}$.

By Lemma 3.8 we can determine the dual graph of the boundary $D$ as follows. At first, let us suppose $\alpha=1$. Then $L_{1}=L_{2}=\emptyset$ and the graph of $D$ is of type (b) in Figure 1. Secondly, suppose $\alpha=2$. Then $\left(A^{2}\right)=q_{1},\left(B^{2}\right)=$ $0, L_{1}=\emptyset$ and $L_{2}$ consists of $-\left(1+q_{1}\right)$-curve plus several ( -2 ) components, so the graph of $D$ is of type (c) in Figure 1. Finally, suppose $\alpha>2$. Then the graph of $D$ is of type either (e) or (f) in Figure 1 according as $\alpha$ is odd or even by Lemma 3.8.

From now on we consider the case where $\sigma$ is written as $\sigma=\sigma_{1} \cdot \tau_{1}$. Let $l_{1}$ be the length of $\tau_{1}$ and let $E(k)$ be the proper transform on $\widetilde{V}$ of the exceptional component arising from the $k$-th blowing-up in $\tau_{1}$ for $1 \leq k \leq l_{1}$ (if $\left.\tau_{1} \neq i d\right)$. If $\tau_{1}$ is not the identity, the dual graph of the fiber $F_{\infty}$ is given


Figure 10
as in Figure 10, where the component $E\left(l_{1}\right)$ is a cross-section of $\widetilde{\Lambda}$, the dual graph of $E_{1}+E\left(\alpha, q_{\alpha}\right)+E_{2}$ is the same as given in Figure 2 with the selfintersection number of $E\left(\alpha, q_{\alpha}\right)$ replaced by -2 and $A^{\prime}, B^{\prime}$ are respectively the proper transforms of $A, B$ on $\widetilde{V}$.

We prove then the following result:
Lemma 3.9. Suppose that the shortest process $\sigma$ to resolve the base points of $\Lambda$ is written as $\sigma=\sigma_{1} \cdot \tau_{1}$ with $\tau_{1} \neq i d$. Let $M_{1}:=L_{1}+A^{\prime}+E_{1}$ and let $M_{2}:=L_{2}+B^{\prime}+E_{2}$ (see Figure 10). Then we have:
(1) If $l_{1}=1$ then both of $M_{1}$ and $M_{2}$ are contractible to smooth points.
(2) If $l_{1}>1$ and $\alpha$ is odd (resp. if $l_{1}>1$ and $\alpha$ is even) then $M_{2}$ (resp. $M_{1}$ ) is contractible to a smooth point.
(3) The contractions of $M_{1}, M_{2}$ stated in (1) (resp. (2)) start with $A^{\prime}, B^{\prime}$ (resp. one of $\left.A^{\prime}, B^{\prime}\right)$ and end with the components meeting $E\left(\alpha, q_{\alpha}\right)$.

Proof. (1) If $l_{1}=1$ then $E\left(\alpha, q_{\alpha}\right)$ meets a cross-section $E(1)$ of $\widetilde{\Lambda}$. So, the multiplicity of $E\left(\alpha, q_{\alpha}\right)$ in $F_{\infty}$ is one, and we can obtain from $F_{\infty}$ a smooth fiber which is the image of $E\left(\alpha, q_{\alpha}\right)$. Therefore $M_{1}$ and $M_{2}$ are contractible to smooth points. Note that all possible $(-1)$-curves in $F_{\infty}$ are only $A^{\prime}$ and $B^{\prime}$, and that $E\left(\alpha, q_{\alpha}\right)$ is a ( -2 )-curve. Thus the contraction of $M_{1}$ (resp. $M_{2}$ ) starts with $A^{\prime}$ (resp. $B^{\prime}$ ) and ends with the components meeting $E\left(\alpha, q_{\alpha}\right)$.
(2) In the case $l_{1}>1$ we can show the assertion (2) by the same argument as in Lemma 3.7 (3). Moreover, by the same argument there, the contraction of $M_{2}$ if $\alpha$ is odd (resp. $M_{1}$ if $\alpha$ is even) starts with $B^{\prime}$ (resp. $A^{\prime}$ ) and ends with the contraction of $E\left(\alpha, q_{\alpha}-1\right)$. The last assertion (3) is thus proved.

Lemma 3.10. With the notations and assumptions as in Lemma 3.9, the following assertions hold:
(1)

$$
\left(A^{2}\right)=\left\{\begin{array}{lll}
q_{1} & \text { if } & \alpha>1, \\
q_{1}-l_{1} & \text { if } & \alpha=1
\end{array}\right.
$$

(2) $\left(B^{2}\right)=0$.
(3) If $\alpha>1$ (resp. if $\alpha=1$ ) then the dual graphs of $L_{1}$ and $L_{2}$ are given as in Figure 9 with $(-2)$ components counted $\left(q_{\alpha}-l_{1}-1\right)$-times instead of $\left(q_{\alpha}-2\right)$-times $\left(\right.$ resp. $\left.L_{1}=L_{2}=\emptyset\right)$.

Proof. We can prove the assertions from Lemma 3.9 by the same fashion as we proved Lemma 3.8 from Lemma 3.7.


Figure 11

As we determined the dual graph of the boundary $D$ from Lemma 3.8 in the previous case, we can also do the same in the present case from Lemma 3.10. Namely, the graph of $D$ is of type (b) if $\alpha=1$, of type (c) if $\alpha=2$ and of type (e) or (f) if $\alpha>2$ in Figure 1.

We next consider the case where the process $\sigma$ to resolve Bs $\Lambda$ consists of a single Euclidean transformation $\sigma_{1}$. The configuration of $\operatorname{Supp}\left(\sigma^{-1}(D)\right)$ is then given as in Figure 11, where $E\left(\alpha, q_{\alpha}\right)$ is a cross-section of $\widetilde{\Lambda}$ and the dual graph of $E_{1}+E\left(\alpha, q_{\alpha}\right)+E_{2}$ is the same as given in Figure 2. The component $A^{\prime}$ (resp. $B^{\prime}$ ) is the proper transform of $A$ (resp. $B$ ). We then prove the following result:

Lemma 3.11. Let the notations and assumptions be the same as above. Then one of $M_{1}:=L_{1}+A^{\prime}+E_{1}$ and $M_{2}:=L_{2}+B^{\prime}+E_{2}$ (see Figure 11) supports $F_{\infty}$ and the other is contractible to a smooth point, starting with the contraction of $A^{\prime}$ or $B^{\prime}$.

Proof. Supp $\left(\sigma^{-1}(D)\right)-E\left(\alpha, q_{\alpha}\right)$ consists of two connected components $M_{1}$ and $M_{2}$. Since the fibration $\varphi$ is parametrized by the affine line $\mathbf{A}^{1}$, one of $M_{1}$ and $M_{2}$ supports the member $F_{\infty}$ of $\widetilde{\Lambda}$ lying outside $\mathbf{A}^{2}$ and the other is contained in a member $F$ of $\widetilde{\Lambda}$ different from $F_{\infty}$. Since all the fibers of $\varphi$ are isomorphic to $\mathbf{A}^{1}$ scheme-theoretically (see the argument at the beginning of this section), it follows that $C_{0}:=F \cap \mathbf{A}^{2} \cong \mathbf{A}^{1}$ scheme-theoretically. Hence the closure $C$ of $C_{0}$ in $\widetilde{V}$ is contained in $F$ with multiplicity one. So, the fiber $F$ can be reduced to a smooth fiber which is the image of $C$. Thus one of $M_{1}$ and $M_{2}$ supports $F_{\infty}$ and the other is contracted to a smooth point. Note that by the minimality of $D$ and the construction of an Euclidean transformation, every component of $M_{1}$ and $M_{2}$ other than $A^{\prime}$ and $B^{\prime}$ has self-intersection number less than or equal to -2 . Hence the contraction of $M_{1}$ or $M_{2}$ starts with $A^{\prime}$ or $B^{\prime}$.

For each case stated in Lemma 3.11 we can easily determine the dual graph of the boundary $D$ since we know the dual graphs of $A^{\prime}+E_{1}$ and $B^{\prime}+E_{2}$ concretely (see Figure 11). Namely, we have the following:

Lemma 3.12. With the notations and assumptions as in Lemma 3.11, the following assertions hold:
(1)

$$
\left(A^{2}\right)= \begin{cases}q_{1}-1 & \text { if } \alpha=1 \text { and } M_{1} \text { is contractible } \\ q_{1} & \text { otherwise }\end{cases}
$$

(2) $\left(B^{2}\right)=0$.
$\alpha$ : odd

$\alpha$ : even


Figure 12
(3) Suppose that $\alpha>1$. If $M_{1}$ (resp. $M_{2}$ ) supports $F_{\infty}$ and $M_{2}$ (resp. $M_{1}$ ) is contractible, the dual graphs of $L_{1}$ and $L_{2}$ are given as in Figure 12 (resp. Figure 13), where the linear chains in the dotted frames might be empty, $n$ is an arbitrary non-negative integer and where the $-\left(1+q_{2}\right)$-curve in $L_{1}$ and the $-\left(1+q_{1}\right)$-curve in $L_{2}$ meet $A$ and $B$, respectively.
(4) Suppose that $\alpha=1$. If $M_{1}$ (resp. $M_{2}$ ) supports $F_{\infty}$ and $M_{2}$ (resp. $M_{1}$ ) is contractible, the dual graph of $D$ is given by (i) (resp. (ii)) in Figure 14, where the linear chains in the dotted frames might be empty and $n$ is an arbitrary non-negative integer.

Case II-(2)-(i'). $i\left(G \cdot B ; p_{0}\right)>i\left(G \cdot A ; p_{0}\right)=d_{1}$; see the argument after the proof of Lemma 3.4 for the notations.

This case can be treated in the same fashion as in Case II-(2)-(i) and consequently all possible weighted dual graphs of the boundary $D$ are the same as those obtained in Case II-(2)-(i).

Case II-(2)-(ii). $i\left(G \cdot A ; p_{0}\right)=i\left(G \cdot B ; p_{0}\right)=d_{1}$.
Then the process $\sigma$ to resolve the base point $\mathrm{Bs} \Lambda=\left\{p_{0}\right\}$ starts with an EM-transformation. Suppose that $\sigma$ consists only of a single blowing-up. This case can be treated in the same fashion as in Lemmas 3.11 and 3.12 with $E_{1}$ and $E_{2}$ empty. As a consequence, the dual graph of $D$ is given by 1) in Figure 14 with $q_{1}=1$, that is, of type (b) or (c) in Figure 1 with $m=1$. In the subsequent we exclude this case. Then the same argument as in Lemma 3.2 shows that the following result holds:

Lemma 3.13. The process $\sigma$ ends with an EM-transformation.


Figure 13


Figure 14


Figure 15

We write $\sigma$ as

$$
\sigma=\tau_{1} \cdot \sigma_{1} \cdots \tau_{n-1} \cdot \sigma_{n-1} \cdot \tau_{n} \quad \text { with } \quad n \geq 1
$$

where $\sigma_{j}$ (resp. $\tau_{j}$ ) is the $j$-th Euclidean transformation (resp. EM-transformation) in $\sigma$. Let the notations $\mathcal{D}_{j}=\left\{p_{0}^{(j)}, l_{0}^{(j)}, d_{0}^{(j)}, d_{1}^{(j)}\right\}, d_{2}^{(j)}, \ldots, d_{\alpha_{j}}^{(j)}, q_{1}^{(j)}, \ldots$, $q_{\alpha_{j}}^{(j)}, l_{j}, E^{(j)}(s, t)$ and $E^{(j)}(k)$ be the same as those defined after the proof of Lemma 3.2. With these notations, the same assertions as in Lemma 3.3 hold by the same argument there. The last component $E^{(n)}\left(l_{n}\right)$ is a cross-section of $\widetilde{\Lambda}$ and $\operatorname{Supp}\left(\sigma^{-1}(D)\right)-E^{(n)}\left(l_{n}\right)$ supports the member $F_{\infty}$ of $\widetilde{\Lambda}$ lying outside $\mathbf{A}^{2}$. The dual graph of $F_{\infty}$ is given as in Figure 15, where the notations $S(l)$ and $T(l)$ are those defined in Figure 7 and $A^{\prime}, B^{\prime}$ are the proper transforms on $\widetilde{V}$ of $A, B$, respectively.

The part $T_{n}$ is a linear chain supported by the (-2)-curves $E^{(n)}(1), \ldots, E^{(n)}\left(l_{n}-\right.$ $1)$. We assume for the moment that $n \geq 2$. Then we have the following result:

Lemma 3.14. With the notations as above, suppose that $n \geq 2$. Then $E^{(1)}\left(l_{1}\right)$, which is the rightmost component of $T\left(l_{1}\right)$ in Figure 15, becomes a $(-1)$-curve after the leftside of $E^{(1)}\left(l_{1}\right)$ in Figure 15 is contracted to a smooth point.

Proof. Assume the contrary that $E^{(1)}\left(l_{1}\right)$ becomes a $(-1)$-curve before the leftside of it is contracted. Then we contract the components in the linear chain between $E^{(1)}\left(l_{1}\right)$ and the last exceptional component $C_{1}$ from $\sigma_{1}$, which is the rightmost component of $S\left(l_{2}\right)$. Then the image of $C_{1}$ is a $(-1)$-curve meeting three or more other fiber components. This is a contradiction to Lemma 2.1 (4).

By Lemma 3.14 we can determine the dual graph of $D$. Namely, we have the following:

Lemma 3.15. With the notations and assumptions as above, the dual graph of $D$ is given by $(b)$ in Figure 1 with $m=-l_{1}+1$.

Proof. Since $E^{(1)}(1)$, which is the leftmost component of $T\left(l_{1}\right)$, meets three other fiber components and the rightside of $E^{(1)}(1)$ in Figure 15 contains no ( -1 )-curve, at least one of $A^{\prime}+L_{1}$ and $B^{\prime}+L_{2}$ is contractible (see Remark 2.2 ). Let us suppose that $A^{\prime}+L_{1}$ is contractible. Then $A^{\prime}$ is a $(-1)$-curve and
$L_{1}$ is empty. Assume the contrary that $L_{1}$ is not empty. In the case $l_{1}=1$ (resp. $l_{1}>1$ ), when we contract $A^{\prime}+L_{1}$ the image of $E^{(1)}(1)$ has self-intersection number greater than or equal to -1 (resp. 0). This contradicts Lemma 3.14, so $L_{1}$ is empty. We contract the components $A^{\prime}, E^{(1)}(1), \ldots, E^{(1)}\left(l_{1}-1\right)$ in this order. Then the images of $B^{\prime}$ and $E^{(1)}\left(l_{1}\right)$ have self-intersection number $\left(B^{\prime 2}\right)+l_{1}-1$ and -2 , respectively. Lemma 3.14 then implies that $\left(B^{\prime 2}\right)+l_{1}-1=$ -1 and $L_{2}$ is empty. Noting that $\left(A^{\prime 2}\right)=\left(A^{2}\right)-1$ and $\left(B^{\prime 2}\right)=\left(B^{2}\right)-1$, we proved the assertion. In the case where $B^{\prime}+L_{2}$ is contracted instead of $A^{\prime}+L_{1}$, we can also prove the assertion by the same fashion.

Finally we consider the case where the process $\sigma$ consists of a single EMtransformation $\tau_{1}$. By assumption before Lemma 3.13, the length $l_{1}$ of $\tau_{1}$ is greater than 1 . The last component $E^{(1)}\left(l_{1}\right)$ is a cross-section of $\widetilde{\Lambda}$ and the fiber $F_{\infty}$ is supported by

$$
J:=A^{\prime}+L_{1}+B^{\prime}+L_{2}+E^{(1)}(1)+\cdots+E^{(1)}\left(l_{1}-1\right) .
$$

The configuration of $F_{\infty}$ is described as in Figure 15 with the rightside of $T\left(l_{1}\right)$ excluded.

We prove the following result:
Lemma 3.16. With the notations and assumptions as above, the dual graph of the boundary $D$ is (b) in Figure 1 with $m=-l_{1}+2$.

Proof. The proof is somewhat similar to the one of Lemma 3.15. Note that $E^{(1)}(1)$ is a $(-2)$-curve meeting three other fiber components or two fiber components and a cross-section. In order to obtain a smooth fiber from $F_{\infty}$, one of $A^{\prime}$ and $B^{\prime}$, say $A^{\prime}$, is a $(-1)$-curve and $L_{1}$ is accordingly empty. We contract $A^{\prime}, E^{(1)}(1), \ldots, E^{(1)}\left(l_{1}-1\right)$ in this order. Let $B^{\prime \prime}$ be the image of $B^{\prime}$ after this contraction. Since $B^{\prime \prime}$ meets a cross-section, we can reduce $B^{\prime \prime}+L_{2}$ to a smooth fiber retaining $B^{\prime \prime}$ as a final component. So, if $L_{2}$ is not empty it contains a $(-1)$-curve. But this is impossible by the minimality of $D$. Thus we know that $L_{2}$ is empty and $B^{\prime \prime}$ itself is a smooth fiber. Noting that $\left(A^{\prime 2}\right)=\left(A^{2}\right)-1$ and $\left(B^{\prime \prime 2}\right)=\left(B^{2}\right)+l_{1}-2$, we proved the assertion.

We thus obtain that for any m.n.c. of $\mathbf{A}^{2}$ the dual graph of the boundary is given by the one of (a)-(f) in Figure 1.

## 4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. So, suppose that a smooth affine surface $X$ has a normal compactification $X \hookrightarrow V$ such that each irreducible component of the boundary divisor $D:=V-X$ is rational and the weighted dual graph of $D$ is the same as that of the boundary of a suitable normal compactification of the affine plane $\mathbf{A}^{2}$. Then $D$ can be brought to a smooth rational curve $\bar{D}$ with the self-intersection number 1 on a smooth projective
surface $\bar{V}$ after suitable blowing-ups and blowing-downs with centers outside $X$, because any boundary divisor with respect to the normal compactification of an affine plane $\mathbf{A}^{2}$ can be brought to a line on a projective plane $\mathbf{P}^{2}$ via suitable blowing-ups and blowing-downs with centers outside $\mathbf{A}^{2}$. Now we prepare the following lemma by which we know that $\bar{V}$ is a rational surface:

Lemma 4.1. Let $W$ be a smooth projective surface. Suppose that there exists a smooth rational curve $C$ on $W$ with positive self-intersection number. Then $W$ is a rational surface.

Proof. Since $\left(C \cdot K_{W}\right)=-\left(C^{2}\right)-2<0$ and $\left(C^{2}\right)>0$, it is easy to see that $W$ has negative Kodaira dimension, i.e., $W$ is ruled. Suppose that $W$ has the positive irregularity $q(W)>0$. Then there is a $\mathbf{P}^{1}$-fibration

$$
f: W \rightarrow B
$$

where $B$ is a smooth projective curve with genus $q(W)$. Since $\left(C^{2}\right)>0$, the curve $C$ is not contained in a fiber of $f$. Hence $C$ is a quasi-section of $f$. But this is impossible by Lüroth theorem because $C$ is rational. Thus we have that $q(W)=0$ and consequently $W$ is rational.

We blow-up $\bar{V}$ at a point of $\bar{D}$. Let $V^{\prime}$ be the resulting surface, let $D^{\prime}$ be the proper transform of $\bar{D}$ on $V^{\prime}$ and let $E^{\prime}$ be the exceptional curve. Since $D^{\prime} \cong \mathbf{P}^{1}$ and $\left(D^{\prime 2}\right)=0$, we have $\mathcal{O}_{D^{\prime}}\left(D^{\prime}\right) \cong \mathcal{O}_{\mathbf{P}^{1}}$. Consider an exact sequence:

$$
0 \rightarrow \mathcal{O}_{V^{\prime}} \rightarrow \mathcal{O}_{V^{\prime}}\left(D^{\prime}\right) \rightarrow \mathcal{O}_{\mathbf{P}^{1}} \rightarrow 0
$$

Noting that $H^{1}\left(V^{\prime}, \mathcal{O}_{V^{\prime}}\right)=0$ because of the rationality of $V^{\prime}$ (see Lemma 4.1), the induced cohomology exact sequence implies that $h^{0}\left(V^{\prime}, \mathcal{O}_{V^{\prime}}\left(D^{\prime}\right)\right)=2$. The linear pencil $\left|D^{\prime}\right|$ defines a $\mathbf{P}^{1}$-fibration $h: V^{\prime} \rightarrow \mathbf{P}^{1}$ and $E^{\prime}$ is a cross-section of $h$. All the fibers of $h$ are irreducible since $X=V^{\prime}-\left(D^{\prime} \cup E^{\prime}\right)$ is affine and $E^{\prime}$ is a cross-section of $h$. Thus we know that $\left.h\right|_{X}$ is an $\mathbf{A}^{1}$-bundle over $\mathbf{A}^{1}$. Hence $X \cong \mathbf{A}^{1} \times \mathbf{A}^{1} \cong \mathbf{A}^{2}$. This completes the proof of Theorem 1.2.

Remark 4.2. In the statement of Theorem 1.2, if we replace the affine plane $\mathbf{A}^{2}$ by another $\mathbf{Q}$-homology plane then a similar result does not hold in general. For instance, H. Flenner-M. Zaidenberg constructed a family $\left(V_{t}, D_{t}\right) ; t$ $\in \mathbf{A}^{1}$, of smooth projective surfaces $V_{t}$ and divisors $D_{t}$ on $V_{t}$ such that, for $t \neq t^{\prime}$, the affine surfaces $X_{t}:=V_{t}-D_{t}$ and $X_{t^{\prime}}:=V_{t^{\prime}}-D_{t^{\prime}}$ are non-isomorphic Q-homology planes, whereas the boundaries $D_{t}, D_{t^{\prime}}$ are isomorphic with weight (cf. [3, Example 4.16]).

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Added in proof: Even if we replace, in the statement of Theorem 1.2, $X$; a smooth affine surface by $X$ : a normal affine surface, we can show that $X$ is isomorphic to the affine plane $\mathbf{A}^{2}$. For the details see T. Kishiomto, Abhyankar-Sathaye Embedding problem in dimension Three, preprint.


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