A NEW PROOF OF A THEOREM OF SOLOMON

M. D. ATKINSON

The main purpose of this note is to give another proof and a different view of a theorem of Solomon [3] on Coxeter groups (finite groups of symmetries of $\boldsymbol{R}^n$ generated by reflections). A secondary purpose is to give more publicity to the fascinating special case of symmetric groups. This case was handled by Etienne whose recent paper [2] kindled my interest in this subject; I owe to Michelle Wachs my awareness of Solomon's earlier and more general theorem. The new proof is shorter and apparently more elementary than the proof of [3] (and the proof in the appendix to [3] due to Tits); nor does it reduce to Etienne's proof in the special case of the symmetric group.

We begin by recalling some of the terminology of Coxeter groups. By definition they are finite groups of symmetries of n-dimensional real Euclidean space generated by those elements which are reflections in hyperplanes. Each reflecting hyperplane has two unit vectors r and $-r$ orthogonal to it called *roots* and the set of roots has a reasonably canonical partition into *positive* roots and *negative* roots. The set of positive roots has a distinguished subset called the set of *fundamental* roots and the associated set of fundamental reflections (the *Coxeter generators*) also generates the group; with respect to this generating set the group has a very simple set of defining relations. Suppose that $\{\tau_1, \tau_2, \ldots, \tau_m\}$ is a set of Coxeter generators for a Coxeter group G associated with the set of fundamental roots $\{r_1, r_2, \ldots, r_m\}$. For any $g \in G$ let $\lambda(g)$ denote the length of a minimal word in the generators which represents g. It is known [1, 6.1.1] that $\lambda(\tau_i g) = \lambda(g) \pm 1$, with the positive sign if $r_i g$ is a positive root and the negative sign if $r_i g$ is a negative root. If S is any subset of $\{1, 2, \ldots, m\}$ the *signature class* associated with S is the set

$$\begin{aligned} &\{g \in G \mid \lambda(\tau_i g) > \lambda(g) \text{ if and only if } i \in S\} \\ &= \{g \in G \mid \lambda(\tau_i g) = \lambda(g) + 1 \text{ if and only if } i \in S\} \\ &= \{g \in G \mid r_i g \text{ is a positive root if and only if } i \in S\}. \end{aligned}$$

SOLOMON'S THEOREM. *Let A, B, C be three signature classes (not all necessarily distinct),*
and let $\gamma \in C$. Then the cardinality of

$$\{(\alpha, \beta) \mid \alpha \in A, \beta \in B \text{ with } \alpha\beta = \gamma\}$$

depends on A, B, C alone and not on the representative γ.

The theorem is more striking when stated for the special case that G is the symmetric group on the set $\{1, 2, \ldots, n\}$ which has the set of $n-1$ transpositions

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$(i, i+1)$ as Coxeter generators. It is easy to verify that, in this case, $\lambda(g)$ is the number of inversions in the permutation g and therefore that $\lambda(\tau_i g) > \lambda(g)$ if and only if $ig < (i+1)g$. Consequently the signature of a permutation σ of $1, 2, \ldots, n$ is the subset S of $\{1, 2, \ldots, n-1\}$ defined by

$$i \in S \text{ if and only if } i\sigma < (i+1)\sigma.$$

Thus, in this case, the signature of σ is the 'up-down profile' of the permutation and the theorem states that the number of solutions (α, β) of $\alpha\beta = \gamma$, with α of a fixed profile, β of another fixed profile depends only on the profile of γ.

LEMMA 1. *Let $\gamma \in G$, and let τ be any Coxeter generator of G. Then γ and $\gamma\tau$ have the same signature if and only if $\gamma\tau\gamma^{-1}$ is not a Coxeter generator of G.*

Proof. Suppose that $\gamma\tau\gamma^{-1} = \sigma$ is a Coxeter generator with associated fundamental root r_σ. then $r_\sigma \gamma = -r_\sigma \sigma\gamma = -r_\sigma \gamma\tau$ and so one of $r_\sigma \gamma$ and $r_\sigma \gamma\tau$ is a positive root and the other is negative. Hence γ and $\gamma\tau$ have different signatures.

Conversely, suppose that γ and $\gamma\tau$ have different signatures. So, for some fundamental root r_σ and associated Coxeter generator σ, one of $r_\sigma \gamma$ and $r_\sigma \gamma\tau$ is positive and the other is negative. However, according to [**1**, 4.1.9], the only positive root mapped to a negative root by τ is the root r_τ itself. Thus, $r_\sigma \gamma = \pm r_\tau$ and, by [**1**, 4.1.1], $\tau = \gamma^{-1}\sigma\gamma$ as required.

LEMMA 2. *Suppose that $\gamma \in G$, and $\sigma_1, \sigma_2, \ldots, \sigma_k$ are each Coxeter generators. Suppose also that γ and $\gamma\sigma_1\sigma_2 \ldots \sigma_k$ have the same signature. Then either*

(i) *$\gamma\sigma_1\sigma_2 \ldots \sigma_{k-1}$ also has this signature, or*

(ii) *for some $i < k$, $\sigma_i \sigma_{i+1} \ldots \sigma_{k-1} = \sigma_{i+1}\sigma_{i+2} \ldots \sigma_k$.*

Proof. We shall suppose that (i) does not hold and derive (ii). There exists a fundamental root r_τ and Coxeter generator τ such that the roots $r_\tau \gamma$ and $r_\tau \gamma\sigma_1 \ldots \sigma_k$ are of the same sign (say, positive without loss of generality) but the root $r_\tau \gamma\sigma_1 \ldots \sigma_{k-1}$ is of the other sign (therefore negative). In particular, using [**1**, 4.1.9] again,

$$r_k = -r_\tau \gamma\sigma_1 \ldots \sigma_{k-1}. \tag{1}$$

The root sequence $r_\tau \gamma, r_\tau \gamma\sigma_1, \ldots, r_\tau \gamma\sigma_1\sigma_2 \ldots \sigma_{k-1}$ begins with a positive root and ends with a negative root. Suppose that $r_\tau \gamma\sigma_1 \ldots \sigma_i$ is the first negative root in this sequence. Then $r_\tau \gamma\sigma_1 \ldots \sigma_{i-1}$ is a positive root and $r_\tau \gamma\sigma_1 \ldots \sigma_{i-1}\sigma_i$ is a negative root. Therefore, using [**1**, 4.1.9] once more, $r_\tau \gamma\sigma_1 \ldots \sigma_{i-1}$ is the root which corresponds to the reflection σ_i, that is,

$$r_i = r_\tau \gamma\sigma_1 \ldots \sigma_{i-1}. \tag{2}$$

From (1) and (2) we have $r_i \sigma_i \ldots \sigma_{k-1} = -r_k$ and so, from [**1**, 4.1.1], condition (ii) holds.

COROLLARY. *If γ and δ each have the same signature, there is a sequence of group elements $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_k = \delta$ which all have the same signature and where, for each j, $\gamma_{j+1} = \gamma_j \sigma_{j+1}$ for some Coxeter generator σ_{j+1}.*

Proof. Let $\gamma^{-1}\delta = \sigma_1\sigma_2 \ldots \sigma_k$ be a minimal word for $\gamma^{-1}\delta$. We apply Lemma 2 to γ and $\delta = \gamma\sigma_1\sigma_2 \ldots \sigma_k$ which have the same signature. By the minimality of the word $\sigma_1 \sigma_2 \ldots \sigma_k$ conclusion (ii) cannot hold and therefore $\gamma\sigma_1 \sigma_2 \ldots \sigma_{k-1}$ also has this signature. Since $\sigma_1 \sigma_2 \ldots \sigma_{k-1}$ is also a minimal word we can repeat the argument.

LEMMA 3. *Let α, β, γ be such that $\alpha\beta = \gamma$. Let τ be any Coxeter generator such that γ and $\gamma\tau$ have the same signature. Then exactly one of the following holds:*

(i) *β and $\beta\tau$ have the same signature,*

(ii) *$\beta\tau = \sigma\beta$ for some Coxeter generator σ, and α and $\alpha\sigma$ have the same signature.*

Proof. By Lemma 1 condition (i) fails precisely when $\beta\tau = \sigma\beta$ for some Coxeter generator σ. When this happens α and $\alpha\sigma$ do indeed have the same signature; for otherwise we would have $\alpha\sigma = \rho\alpha$ for some Coxeter generator ρ and then $\rho\gamma = \rho\alpha\beta = \alpha\sigma\beta = \alpha\beta\tau = \gamma\tau$ and this contradicts that γ and $\gamma\tau$ have the same signature.

If $\gamma \in G$, and A, B are signature classes, let $N_{AB\gamma} = \{(\alpha, \beta) \mid \alpha \in A, \beta \in B \text{ and } \alpha\beta = \gamma\}$ and let $n_{AB\gamma} = |N_{AB\gamma}|$.

LEMMA 4. *If γ and $\gamma^* = \gamma\tau$ have the same signature, where τ is a Coxeter generator, then $n_{AB\gamma} = n_{AB\gamma^*}$.*

Proof. $N_{AB\gamma}$ can be partitioned into subsets N_i defined as

$$N_0 = \{(\alpha, \beta) \in N_{AB\gamma} \mid \beta\tau \neq \tau_r \beta \text{ for any Coxeter generator } \tau_r\},$$
$$N_r = \{(\alpha, \beta) \in N_{AB\gamma} \mid \beta\tau = \tau_r \beta \text{ for some Coxeter generator } \tau_r\}.$$

Define corresponding sets N_i^* as follows:

$$N_0^* = \{(\alpha, \beta\tau) \mid (\alpha, \beta) \in N_0\},$$
$$N_r^* = \{(\alpha\tau_r, \beta) \mid (\alpha, \beta) \in N_r\}.$$

Obviously $|N_i| = |N_i^*|$ and we shall prove the lemma by showing that these latter sets form a partition of $N_{AB\gamma^*}$.

Observe first of all that, because of Lemmas 1 and 3, each N_i^* is contained in $N_{AB\gamma^*}$. Next we show that the sets N_i^* are disjoint. It is obvious that the sets N_i^* for $i \geqslant 1$ are disjoint and if $(\alpha, \beta\tau) = (\theta\tau_r, \phi) \in N_0^* \cap N_r^*$ (so that $(\alpha, \beta) \in N_0$ and $(\theta, \phi) \in N_r$) then $\alpha = \theta\tau_r$, $\beta\tau = \phi$, and $\phi\tau = \tau_r\phi$ and it follows that $\tau_r\beta = \beta\tau$ which contradicts $(\alpha, \beta) \in N_0$.

Finally we shall show that the disjoint union of the N_i^* is the whole of $N_{AB\gamma^*}$. Let $(\alpha^*, \beta^*) \in N_{AB\gamma^*}$ and let $\beta^*\tau = \beta$. If $\beta\tau \neq \tau_r\beta$ for any Coxeter generator τ_r then $(\alpha^*, \beta) \in N_0$ and $(\alpha^*, \beta^*) \in N_0^*$. However if $\beta\tau = \tau_r\beta$ then also $\beta^*\tau = \tau_r\beta^*$ and, by Lemma 3, α^* and $\alpha^*\tau_r = \alpha$ have the same signature; thus $(\alpha^*, \beta^*) = (\alpha\tau_r, \beta^*) \in N_r^*$ as required.

Solomon's Theorem is now a direct consequence of Lemma 4 and the Corollary to Lemma 2 which together show that $n_{AB\gamma}$ depends only on the signature class of γ and not on any representative γ of the class.

References

1. C. T. BENSON and L. C. GROVE, *Finite reflection groups* (Bogden and Quigley, New York/Belmont, 1971).
2. G. ETIENNE, 'Linear extensions of finite posets and a conjecture of G. Kreweras on permutations', *Discrete Math.* **52** (1984) 107–112.
3. L. SOLOMON, 'A Mackey formula in the group ring of a Coxeter group', *J. Algebra* **41** (1976) 255–268.

School of Computer Science
Carleton University
Ottawa, Canada K1S 5B6