Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 39** 5 (2021): 73–76.
ISSN-00378712 in press
doi:10.5269/bspm.42171

A New Proof of Classical Dixon's Summation Theorem for the Series ${}_3F_2(1)$ *

Sungtae Jun, Insuk Kim* and Arjun K. Rathie

ABSTRACT: The aim of this short note is to provide a new proof of classical Dixon's summation theorem for the series ${}_3F_2(1)$.

Key Words: Dixon's summation theorem, Hypergeometric series, Generalized Hypergeometric Function.

Contents

1 Introduction **73**

2 A new proof of Dixon's summation theorem (1.4) **74**

1. Introduction

In the theory of hypergeometric and generalized hypergeometric series, classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz play a key role. Applications of the above mentioned theorems are well known now. For very interesting applications of these theorems, we refer a paper by Bailey [1].

Here we shall mention the following summation theorems that will be required in our present investigation.

Gauss summation theorem: [2,3,4]

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.1}$$

provided $\mathrm{Re}(c-a-b) > 0$.

A known result: [4]

$$ {}_2F_1\left[\begin{matrix} -k, & a+k \\ & 1+a-c \end{matrix}; 1\right] = \frac{(-1)^k (c)_k}{(1+a-c)_k}, \tag{1.2}$$

which can be obtained by (1.1).

Kummer summation theorem: [2,3,4]

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & 1+a-b \end{matrix}; -1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{1.3}$$

* Corresponding author
2010 *Mathematics Subject Classification:* 33C20.
Submitted March 27, 2018. Published June 19, 2018

The aim of this note is to provide a new proof of the following classical Dixon's summation theorem [2] for the series ${}_3F_2$ viz.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c \end{matrix}; 1\right] \\
&= \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)},
\end{aligned} \tag{1.4}
$$

provided $\mathrm{Re}(a-2b-2c) > -2$.

2. A new proof of Dixon's summation theorem (1.4)

Consider the following integral valid for $\mathrm{Re}(b) > 0$

$$
I = \int_0^\infty e^{-t}\, t^{b-1}\, {}_2F_2\left[\begin{matrix} a, & c \\ 1+a-b, & 1+a-c \end{matrix}; t\right] dt.
$$

Expressing the generalized hypergeometric function ${}_2F_2$ in series, we have

$$
I = \int_0^\infty e^{-t}\, t^{b-1} \sum_{k=0}^{\infty} \frac{(a)_k\ (c)_k\ t^k}{(1+a-b)_k\ (1+a-c)_k\ k!} dt.
$$

Changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series, we have

$$
I = \sum_{k=0}^{\infty} \frac{(a)_k\ (c)_k}{(1+a-b)_k\ (1+a-c)_k\ k!} \int_0^\infty e^{-t}\, t^{b+k-1} dt.
$$

Evaluating the gamma integral and using the result

$$
(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},
$$

we have

$$
I = \Gamma(b) \sum_{k=0}^{\infty} \frac{(a)_k\ (b)_k\ (c)_k}{(1+a-b)_k\ (1+a-c)_k\ k!}. \tag{2.1}
$$

Finally, summing up the series, we get

$$
I = \Gamma(b)\ {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c \end{matrix}; 1\right]. \tag{2.2}
$$

On the other hand, writing (2.1) in the form

$$
I = \Gamma(b) \sum_{k=0}^{\infty} \frac{(-1)^k\ (a)_k\ (b)_k}{(1+a-b)_k\ k!} \left\{ \frac{(-1)^k\ (c)_k}{(1+a-c)_k} \right\}.
$$

Using (1.2), this becomes

$$I=\Gamma(b)\sum_{k=0}^{\infty}\frac{(-1)^k\ (a)_k\ (b)_k}{(1+a-b)_k\ k!}\ {}_2F_1\left[\begin{matrix}-k, & a+k\\ 1+a-c\end{matrix};\ 1\right].$$

Expressing ${}_2F_1$ as a series, we have after some simplification

$$I=\Gamma(b)\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(-1)^k\ (a)_k\ (b)_k\ (-k)_m\ (a+k)_m}{(1+a-b)_k\ (1+a-c)_m\ k!\ m!}.$$

Using the identities

$$(a)_k(a+k)_m=(a)_{k+m}\quad \text{and}\quad (-k)_m=\frac{(-1)^m\ k!}{(k-m)!},$$

we have, after some calculation

$$I=\Gamma(b)\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{(-1)^{k+m}\ (a)_{k+m}\ (b)_k}{(1+a-b)_k\ (1+a-c)_m\ m!\ (k-m)!}.$$

Now, using a known result [4, p.57, Equ.(2)]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k),$$

we have

$$I=\Gamma(b)\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^k\ (a)_{k+2m}\ (b)_{k+m}}{(1+a-b)_{k+m}\ (1+a-c)_m\ m!\ k!}.$$

Using the identities

$$(a)_{k+2m}=(a)_{2m}(a+2m)_k\quad \text{and}\quad (b)_{k+m}=(b)_m(b+m)_k,$$

we have, after some simplification

$$I=\Gamma(b)\sum_{m=0}^{\infty}\frac{(a)_{2m}\ (b)_m}{(1+a-b)_m\ (1+a-c)_m\ m!}\times\sum_{k=0}^{\infty}\frac{(-1)^k(a+2m)_k\ (b+m)_k}{(1+a-b+m)_k\ k!}.$$

Summing up the inner series, we have

$$I=\Gamma(b)\sum_{m=0}^{\infty}\frac{(a)_{2m}\ (b)_m}{(1+a-b)_m\ (1+a-c)_m m!}\times{}_2F_1\left[\begin{matrix}a+2m, & b+m\\ 1+a-b+m\end{matrix};-1\right].$$

Now using Kummer's summation theorem (1.3) and then applying the identity

$$(a)_{2m}=2^{2m}\left(\frac{1}{2}a\right)_m\left(\frac{1}{2}a+\frac{1}{2}\right)_m,$$

we get after some simplification

$$I = \frac{\Gamma(b)\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}a)_m\ (b)_m}{(1+a-c)_m\ m!}.$$

Summing up the series, we get

$$I = \frac{\Gamma(b)\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}\ {}_2F_1\left[\begin{matrix}\frac{1}{2}a, & b \\ 1+a-c\end{matrix};\ 1\right].$$

Applying Gauss summation theorem (1.1), we finally have

$$I = \frac{\Gamma(b)\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}. \tag{2.3}$$

Therefore, equating (2.2) and (2.3), we get the desired Dixon's summation theorem (1.4).

This completes our new proof of Dixon's summation theorem for the series ${}_3F_2(1)$.

References

1. Bailey, W.N., *Products of generalized Hypergeometric Series*, Proc. London Math. Soc., (2), **28**, 242-254 (1928).
2. Bailey, W.N., *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, (1935).
3. Prudnikov, A.P., Brychkov, Yu.A. and Marichev, O.I., *Integrals and Series*, vol. 3 : More Special Functions, Gordon and Breach Science Publishers, (1986).
4. Rainville, E.D., *Special Functions*, The Macmillan Company, New York, (1960) ; Reprinted by Chelsea Publishing Company, Bronx, New York, (1971).

Sungtae Jun,
General Education Institute,
Konkuk University, Chungju 380-701,
Republic of Korea.
E-mail address: `sjun@kku.ac.kr`

and

Insuk Kim,*
Department of Mathematics Education,
Wonkwang University, Iksan 570-749,
Republic of Korea.
E-mail address: `iki@wku.ac.kr`

and

Arjun K. Rathie,
Department of Mathematics,
Vedant College of Engineering and Technology (Rajasthan Technical University),
Bundi 323021, Rajasthan,
India.
E-mail address: `arjunkumarrathie@gmail.com`