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# A New Proof of the Aubry-Mather's Theorem

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**Abstract:** We present a new proof of the theorem of Aubry and Mather on the existence of quasi periodic orbits for monotone twist maps of the cylinder. The method uses Aubry's discrete setting, but works directly with sequences of irrational rotation number, avoiding to take limits of periodic orbits.

**Key words**: Twist maps, quasiperiodic orbits, Aubry-Mather sets.

## 0. Introduction

The maps that we consider here are monotone (positive) twist maps of the cylinder $\mathbb{A} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. These maps are $C^1$ diffeomorphisms that are area preserving and have "zero flux": the algebraic area enclosed between a loop and its image by the map is zero. They also satisfy a twist condition, which enables one to set up a discrete variational problem.

To understand such maps, according to Poincaré who first studied them, one should first understand periodic orbits. They can be of different homotopy type, depending on how much they turn around the circle component of $\mathbb{A}$, and the length of their period. The quotient of these two numbers gives the rotation number of the orbit which is rational. A variation on the theorem of Poincaré and Birkhoff then asserts that there exists (at least two) periodic orbits of all (prime) rational rotation number. In [G1], we proved an analog to this theorem for symplectic twist maps (called monotone maps there) of $\mathbb{T}^n \times \mathbb{R}^n$.

It was not until the late seventies that the existence of orbits of *all* rotation numbers was proved. This was done independently by Aubry [A-L] and Mather [M], with methods quite different from one another. Whereas Aubry had to take sequences of periodic orbits of a "good" ("Birkhoff") type to obtain his quasiperiodic ones, Mather worked in a certain functional space that picked up the orbits of a chosen, rational or irrational, rotation number.

The proof that we present here uses the sequence space in which Aubry's variational calculus was set, but we are able to restrict ourselves apriori to a subspace of sequences having a prescribed rotation number.

Apart from what we think is a simplification of the existing proofs ( to the exclusion of that of Angenent in [An 2]), we hope that this method may generalise to symplectic twist maps, where the main problem until now was to define what Birkhoff periodic orbits should mean in this context. Of course, we do use this notion in this paper, but some new, key steps (section 3) are valid for higher dimensional maps. We have tried to make this paper as self contained as we could, and included theorems on the energy flow that had either been stated or even proved before (section 2).

Note again that we are working here in the unbounded annulus, or cylinder. Restricting oneself to a bounded annulus is quite possible with our method, but to the price of unrewarding complications in the notation. It would also be easy to extend this method to finite compositions of twist maps *of the same sign*, by adding their generating functions.

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**1.Twist maps and their energy flow.**

Let $F$ be a diffeomorphism of $\mathbb{A} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. Denote by $\Omega = dy \wedge dx$ the canonical symplectic form on $\mathbb{A}$ (seen as the cotangent bundle of the circle), $x$ being the angular variable, $y$ the fiber variable. Then $\Omega = d\alpha$, where $\alpha = ydx$.

We write $F(x, y) = (X, Y)$. To say that F is area preserving means:

$$F^*\Omega - \Omega = dY \wedge dX - dy \wedge dx = 0 \tag{1.1}$$

(i.e., $F$ is symplectic). To say that it preserves the flux means:

$$F^*\alpha - \alpha = YdX - ydx = dS \tag{1.2}$$

for some real function $S$ on $\mathbb{A}$ (i.e., $F$ is exact symplectic). Of course (1.2) implies (1.1).

Finally, the twist condition is given by

$$\frac{\partial X}{\partial y} > 0 \tag{1.3}$$

If we work in the covering space $\mathbb{R}^2$ of $\mathbb{A}$, keeping the same notation, the twist condition implies that $\psi : (x, y) \to (x, X)$ is a diffeomorphism from $\mathbb{R}^2$ onto its image. Here, we will suppose that $\psi$ is a diffeomorphism onto $\mathbb{R}^2$ . In [G1](section 4) we gave some conditions under which this is true. The standard family, for one , satisfies these conditions. The method we expose can be reproduced in the general case, with appropriate restrictions on the set of rotation numbers, and with little gain of insight. Because $F(x, y) = (X, Y) \Rightarrow F(x + 1, y) = (X + 1, Y)$ in the covering space, we have that $\psi(x + 1, y) = (x + 1, X + 1)$. Since $\psi$ is a coordinate change, $S$ can be seen as a function from $\mathbb{R}^2$ to $\mathbb{R}$ satisfying the periodicity condition:

$$S(x + 1, X + 1) = S(x, X). \tag{1.4}$$

From (1.2), one can see that $S$ is a generating function for $F$ in the classical mechanic sense of the term:

$$y = -\partial_1 S(x, X)$$
$$Y = \partial_2 S(x, X)$$

Also, the twist condition (1.3) translates into:

$$\partial_1\partial_2 S(x, X) > 0.$$

Let $z_k = F^k(z_0) = (x_k, y_k)$. The orbit $\{z_k\}$ is completely determined by the sequence $\{x_k\}$ of $(\mathbb{R})^{\mathbb{Z}}$. Indeed, from (1.2), we deduce:

$$y_k = -\partial_1 S(x_k, x_{k+1}) = \partial_2 S(x_{k-1}, x_k)$$

This can be written:

$$\partial_1 S(x_k, x_{k+1}) + \partial_2 S(x_{k-1}, x_k) = 0.$$

This equation can be formally interpreted as:

$$\begin{aligned} &\nabla W(\mathbf{x}) = 0, \quad \text{for} \\ (1.5) \qquad &W(\mathbf{x}) = \sum_{-\infty}^{+\infty} S(x_k, x_{k+1}) \text{ and } \ \mathbf{x} \in (\mathbb{R})^{\mathbb{Z}}. \end{aligned}$$

One can think of the above construction as a discrete version of the classical mechanics one: the map $\psi$ is the analog to the Legendre transformation ($X - x$ is the discretised velocity) and equation (1.5) is a formulation of the "least action principle".

Of course, $W$ is not well defined, since the sum is in general not convergent. However, "$\nabla W$" is well defined and generates a flow on a subspace of $(\mathbb{R})^{\mathbb{Z}}$ that we call the energy flow.

More precisely, we endow $\mathbb{R}^{\mathbb{Z}}$ with the norm :

$$\|\mathbf{x}\| = \sum_{-\infty}^{+\infty} \frac{|x_k|}{2^{|k|}}$$

We let $\mathbb{X}$ be the subspace of $\mathbb{R}^{\mathbb{Z}}$ of elements of bounded norm, which is a Banach space. Giving ourselves an $\omega$ in $\mathbb{R}$, we define:

$$Y_\omega = \{\mathbf{x} \in \mathbb{R}^{\mathbb{Z}} \quad | \qquad |\mathbf{x}|_\omega = \quad \underset{k \in \mathbb{Z}}{sup}\, |x_k - k\omega| < \infty\},$$

on which one can either put the topology induced by the inclusion of $Y_\omega$ in $\mathbb{X}$ or the $l^\infty$ topology given by the metric:

$$|\mathbf{x} - \mathbf{y}|_\infty = \underset{k \in \mathbb{Z}}{sup}\, |x_k - y_k|$$

It is important to notice that elements of $Y_\omega$ have rotation number $\omega$, that is:

$$\mathbf{x} \in Y_\omega \Rightarrow \underset{|k| \to \infty}{lim} \frac{x_k}{k} = \omega,$$

and this definition coincides with the rotation number of an orbit of $F$ when $\{x_k\}$ defines the $x$ coordinates of such an orbit.

On $\mathbb{R}^{\mathbb{Z}}$, we have a $\mathbb{Z}^2$ action given by:

$$(\tau_{m,n}\mathbf{x})_k = x_{k+m} + n$$

We define $Y_\omega/\mathbb{Z} := Y_\omega/\tau_{0,1}$. This will ultimatly be the space on which we will be working.

## 2. Existence and monotonicity of the flow

In this section, we prove the existence of a $C^1$ energy flow on $Y_\omega$ which is monotone with respect to the partial order on sequences, and whose rest points correspond to orbits of $F$.

**Proposition 1:** *Suppose that the generating function $S$ is $C^2$. The infinite system of O.D.E's:*

$$-\nabla W(\mathbf{x})_k = \dot{x}_k = -[\partial_1 S(x_k, x_{k+1}) + \partial_2 S(x_{k-1}, x_k)] \tag{1.6}$$

*defines a $C^1$ local flow $\zeta^t$ on $Y_\omega$, for both topologies on $Y_\omega$. The rest points of $\zeta^t$ on $Y_\omega$ correspond to orbits of the map $F$ with rotation number $\omega$. Furthermore, when $S$ has a bounded second derivative, the system defines a $C^1$ flow on $\mathbb{X}$.*

*proof:* $Y_\omega$ is a Banach manifold diffeomorphic to $l^\infty$: $x_k \to x_k + k\omega$ gives the diffeomorphism from $l^\infty \cong Y_0$ to $Y_\omega$, with obvious inverse. We want to show that the map $\mathbf{x} \to \nabla W(\mathbf{x})$ is a (locally) Lipschitz vector field on $Y_\omega$, and that it has one order of differentiability less than that of $S$. For this we prove the following lemma:

**Lemma 1:** *Let $\mathbf{x} \in Y_\omega$. Then the set $\{(x_k, x_{k+1})\}_{k\in\mathbb{Z}}/\mathbb{Z}$ is bounded. Hence $S$ and all its (existing ) derivatives are bounded on this set.*

*proof:* We want to show that $(x_k - E(k\omega), x_{k+1} - E(k\omega))$ is bounded in $\mathbb{R}^2$, uniformaly in $k$, where $E$ is the integer part function. But:

$$|x_k - E(k\omega)| \leq |x_k - k\omega| + |k\omega - E(k\omega)| \leq |\mathbf{x}|_\omega + 1.$$

Since $|E(k\omega) - E((k+1)\omega)| \leq |\omega|$, the same method shows:

$$|x_{k+1} - E(k\omega)| \leq |\mathbf{x}|_\omega + 1 + |\omega|$$

We conclude the proof of Lemma 1 reminding the reader that $S$ is periodic (see (1.4)). $\square$

Coming back to the proof of the proposition, we notice that lemma 1 shows, among other things that, when $\mathbf{x}$ is in $Y_\omega$, $\nabla W(\mathbf{x})$ is in $l^\infty$, which is exactly the tangent space to $Y_\omega$ since the latter is an affine manifold modeled on $l^\infty$.

To show that $\nabla W$ is Lipschitz in the norm $\| \quad \|$, we calculate:

$$\begin{aligned}
&\|\nabla W(\mathbf{x}) - \nabla W(\mathbf{y})\| \\
&\quad = \sum_{-\infty}^{\infty} \frac{|\nabla W(\mathbf{x})_k - \nabla W(\mathbf{y})_k|}{2^{|k|}} \\
&\quad \leq \underset{k\in\mathbb{Z}, t\in[0,1]}{sup} |\nabla^2 S((1-t)x_k + ty_k, (1-t)x_{k+1} + ty_{k+1})| \sum_{-\infty}^{\infty} \frac{|(x_k - y_k, x_{k+1} - y_{k+1})|}{2^{|k|}} \\
&\quad \leq 2K\|\mathbf{x} - \mathbf{y}\|,
\end{aligned}$$

where $K$ exists since by lemma 1 the second derivative of $S$ is estimated over a bounded set in the case where both $\mathbf{x}$ and $\mathbf{y}$ are in $Y_\omega$. In the case where $\mathbf{x}$ and $\mathbf{y}$ are only assumed to be in $\mathbb{X}$, and the second derivative is bounded, the previous computation shows both that $\nabla W$ is in $\mathbb{X}$ (set $\mathbf{y} = 0$ in the computation) and that it is Lipschitz.

To show the differentiability, just notice that the obvious candidate for the derivatives of $\nabla W$ is " $HessW$", a tridiagonal, matrix operator, which is bounded, again because of the lemma 1 or the assumption (see more on $HessW$ in the proof of proposition 2). This, and the theorems of existence of O.D.E.'s on Banach spaces conclude the proof of the proposition for the first norm. To prove that $\nabla W$ is Lipschitz for the $l^\infty$ topology, one would proceed in a similar fashion. We leave it to the reader to check this since we will not use the $l^\infty$ norm anymore in this paper. We will not need that $\zeta^t$ is a flow on $\mathbb{X}$ either, but want to keep it for the record. □

One of the key properties of the flow $\zeta^t$ is that it is *strictly monotone*. This important fact, which holds in both $Y_\omega$ and $\mathbb{X}$ (when the second derivative is bounded in the latter), was first used in [An1]. The following proof , due to Angenent, was already given in [G2], Lemma 1.22, and we include it for the convenience of the reader. We remind the reader that $\mathbb{R}^{\mathbb{Z}}$ is endowed with a partial order given by $\mathbf{x} < \mathbf{y} \Leftrightarrow x_k \leq y_k, \forall\ k$ and $\mathbf{x} \neq \mathbf{y}$.

To say that $\zeta^t$ is monotone means that $\mathbf{x} < \mathbf{y} \Rightarrow \zeta^t(\mathbf{x}) < \zeta^t(\mathbf{y}),\ \forall t > 0$. It is *strictly* monotone if $\mathbf{x} < \mathbf{y} \Rightarrow \zeta^t(\mathbf{x})_k < \zeta^t(\mathbf{y})_k, \forall k \in \mathbb{Z}, t > 0$.

**Proposition 2:** *The flow $\zeta^t$ is strictly monotone where it is defined.*

*proof:* We let the reader show that if the operator solution of the linearised equation:

$$\dot{u} = -HessW(\mathbf{x}(t))u(t) \tag{2.2}$$

is strictly positive, then the flow is strictly monotone. One can check that $-HessW(\mathbf{x}(t))$ is an infinite tridiagonal matrix with positive off diagonal terms $-\partial_1\partial_2 S(x_k, x_{k+1})$. The diagonal terms $\partial_1\partial_1 S(x_k, x_{k+1}) + \partial_2\partial_2 S(x_{k-1}, x_k)$ are uniformaly bounded , either by assumption, or on a compact time interval along an orbit, by lemma 1. In all cases, for any $T > 0$ for which $\mathbf{x}(t)$ is defined when $0 \leq t \leq T$, we can find a positive $\lambda$ such that:

$$B(t) = -HessW(\mathbf{x}(t)) + \lambda Id$$

is a positive matrix with strictly positive off diagonal terms. If $u(t)$ is solution of the equation (2.2) then $e^{\lambda t}u(t)$ is solution of :

$$\dot{v}(t) = B(t)v(t), \tag{2.3}$$

and hence the strict positivity of the solution operator for (2.2) is equivalent to that of (2.3). Looking at the integral equation:

$$v(t) = v(0) + \int_0^t B(s)v(s)ds,$$

one sees that Picard's iteration will give positive solutions for positive vector $v(0)$. This will imply, assuming that $v_k(0) > 0, v_l(0) \geq 0$, for $l \neq k$:

$$v_{k+1}(t) \geq v_{k+1}(0) + \int_0^t B_{k,k+1}(s)v_k(s)ds > 0$$

The same holding for $v_{k-1}$. By induction, $v_k(t) > 0, \forall k \in \mathbb{Z}$ and the operator solution is strictly positive. This finishes the proof of proposition 2. $\square$

**3. Lyapunov functions for non rest points.**

Let $W_N = \sum_{-N}^{N} S(x_k, x_{k+1})$ and $\frac{d}{dt}W_N(x)$ its (well defined) derivative along the flow at $t = 0$. We want to prove the following :

**Lemma 2**: *Let $x \in Y_\omega$ be not a rest point, then*

$$\exists N_0 \text{ such that } \forall N > N_0, \quad \frac{d}{dt}W_N(x) < 0$$

*and the same holds in a neighborhood of $x$ for the same $N_0$, in either topologies. It also holds for the flow $\zeta^t$ induced by equivariance on $Y_\omega/\mathbb{Z}$ .*

*proof:* By chain rule:

$$\begin{aligned}
&\tfrac{d}{dt}W_N(x) \\
&= -\left[\sum_{-N}^{N} \partial_1 S(x_k, x_{k+1}) \nabla W(x)_k + \partial_2 S(x_k, x_{k+1}) \nabla W(x)_{k+1}\right] \\
&= -\left[\sum_{-N}^{N} \partial_1 S(x_k, x_{k+1}) \nabla W(x)_k + \sum_{-N+1}^{N+1} \partial_2 S(x_{k-1}, x_k) \nabla W(x)_k\right] \\
&= -\partial_1 S(x_{-N}, x_{-N+1}) \nabla W(x)_{-N} - \partial_2 S(x_N, x_{N+1}) \nabla W(x)_{N+1} - \sum_{-N+1}^{N} \big(\nabla W(x)_k\big)^2
\end{aligned}$$

We want to show that, for an $x$ which is not a rest point in $Y_\omega$, the first 2 terms of this expression are negligeable compared to the third for all large $N$'s. As we have seen in Lemma 1, we can assume

$$\big|\partial_i S(x_k, x_{k+1})\big| < a, \text{ for } i = 1 \text{ or } 2 \text{ and } \big|\nabla W(x)_k\big| < 2a,$$

for some constant $a$ depending on $x$ *but not on* $k$. Therefore, to show that our expression is negative, it suffices to show that:

$$\sum_{-N+1}^{N} \big(\nabla W(x)_k\big)^2 > a\big(\big|\nabla W(x)_{-N}\big| + \big|\nabla W(x)_{N+1}\big|\big).$$

There are two cases:

(i) $\sum_{-\infty}^{+\infty} \big(\nabla W(x)_k\big)^2 = \infty$. Then for all $N$ greater than some $N_0$,

$$\sum_{-N+1}^{N-1} \big(\nabla W(x)_k\big)^2 > 4a^2 > a\big(\big|\nabla W(x)_{-N}\big| + \big|\nabla W(x)_{N+1}\big|\big),$$

which proves the lemma in this case.

(ii) $\sum_{-\infty}^{+\infty}\big(\nabla W(x)_k\big)^2 = K < \infty$. Then we must have $lim_{|k|\to\infty}\nabla W(x)_k = 0$ and, for all $N$ greater than some $N_0$,

$$\sum_{-N+1}^{N}\big(\nabla W(x)_k\big)^2 > K/2 > a\big(\big|\nabla W(x)_{-N}\big| + \big|\nabla W(x)_{N+1}\big|\big).$$

and the lemma is proven for the case $\mathbf{x} \in Y_\omega$. All the terms in the computation above being invariant under $\mathbf{x} \to \tau_{0,1}\mathbf{x}$, the lemma is also proven for $\mathbf{x}$ in $Y_\omega/\mathbb{Z}$. □

The next lemma will show how to use the "Lyapunov" functions $W_N$ to find rest points for the flow.

**Lemma 3:** *Any compact invariant set for the flow $\zeta^t$ ( in $Y_\omega$ or $Y_\omega/\mathbb{Z}$) must contain a rest point.*

*proof:* Let $C$ be compact invariant for the flow $\zeta^t$. Note that in particular, the flow is defined for all time on $C$. Suppose there are no rest points in $C$. Then we can cover $C$ with open sets on each one of which $\frac{d}{dt}W_{N_i}$ is negative for some $N_i$ (and hence for all $N > N_i$). Taking a finite subcovering and $M = supN_i$ on this covering, we find that $W_M$ is strictly decreasing on all of $C$, i.e., it is a Lyapunov function on the compact $C$. $C$ being compact and $S$ being $C^2$, there is an $a > 0$ such that $sup_{x\in C}\frac{d}{dt}W_M(\mathbf{x}(t)) < -a$. But this implies that, for $\mathbf{x}(0) \in C$, $lim_{t\to\infty}W_M(\mathbf{x}(t)) = -\infty$, a contradiction to the compactness of $C$. □

## 4. Aubry-Mather's theorem.

We first state the theorem:

**Theorem:** *Let $F$ be a twist map of $\mathbb{A}$, then $F$ has orbits of all rotation number $\omega$ in $\mathbb{R}$. These orbits can be chosen to be ordered like orbits under a rigid rotation. When $\omega$ is irrational, the $x$ coordinates of the orbit may either be dense in $\mathbb{S}^1$, dense in a Cantor set, or lie in the gaps of a Cantor set contained in its closure.*

*proof:* In the following definition of Birkhoff sequences, we follow [A2]. Remember that:

$$(\tau_{m,n}\mathbf{x})_k = x_{k+m} + n$$

We define the set $B$ of Birkhoff sequences to be those $\mathbf{x}$ in $\mathbb{R}^{\mathbb{Z}}$ such that $\tau_{m,n}\mathbf{x} \geq \mathbf{x}$ or $\tau_{m,n}\mathbf{x} \leq \mathbf{x}$. This is a closed set in $\mathbb{X}$. All its elements have a rotation number. In fact one can show that $B \subset \bigcup_{\omega\in\mathbb{R}} Y_\omega$ (see Lemma 4.11 in [G2]). Since $\zeta^t$ is monotone, $B$ is invariant by the flow. Now let $\mathbf{x}$ be a Birkhoff sequence of rotation number $\omega$. One can prove:

$$x_0 + k\omega \leq x_k < x_0 + k\omega + 1$$

Moding out by $\tau_{0,1}$ (translations) the set of such sequences is equivalent to choosing $x_0 \in \mathbb{S}^1$. Then one sees that one obtains a set diffeomorphic to $\mathbb{S}^1 \times [\{k\omega\}, \{k\omega\} + 1]$, the latter term being an order interval in $\mathbb{R}^{\mathbb{Z}}$, diffeomorphic to a hilbert cube, hence compact: the topology induced by the norm $\| \quad \|$ is equivalent to the product topology *on this set.* We call this space $B_\omega$. We have $B_\omega = (Y_\omega \bigcap B)/\mathbb{Z}$ is a compact, positively invariant subset of $Y_\omega/\mathbb{Z}$ for the quotient flow, with the norm induced by $\| \ \|$ on the quotient.

The flow $\zeta^t$ is defined for all positive $t$ when $x(0)$ is in $B_\omega$. The $\omega$-limit set of $B_\omega$ is compact, (positively and negatively) invariant by the quotient flow and hence from the previous section it must contain a rest point. By Proposition 1, this sequence corresponds to an orbit of the map with rotation number $\omega$. Now to be a Birkhoff sequence is equivalent to be an orbit of a circle homeomorphism and hence the topological structure of the orbit is given by classical theorems on these homeomorphisms. This finishes the proof of the theorem. $\square$

**Erratum for "A new proof of Aubry-Mather's theorem"**

Sinisa Slijepcevic pointed out an error in Section 3 of this paper. I present here a modified version of this Section. It is greatly inspired by the paper "Aubry-Mather Theory for Functions on Lattices" by Koch, de la Llave and Radin.

Denote by $Y_\omega^K = \{\mathbf{x} \in Y_\omega; |x_k - x_0 - k\omega| < K\}$.

**Lemma 2** *Let $C \subset Y_\omega^K$ be a compact invariant set for both the flow $\zeta^t$ and the shift map $\sigma = \tau_{0,1}$. Then $C$ must contain a rest point for the flow.*

*Proof.* Assume, by contradiction, that there are no rest points in $C$. We will show that, for some large enough $N$, the truncated energy function $W_N = \sum_{-N}^{N} S(x_k, x_{k+1})$ is a strict Lyapunov function for the flow $\zeta^t$ on $C$. More precisely, we find a real $a > 0$ such that $\frac{d}{dt} W_N(\mathbf{x}) < -a$ for all $\mathbf{x}$ in $C$. This immediately yields a contradiction since on one hand $W_N$ decreases to $-\infty$ on any orbit in $C$, on the other hand, the continuous $W_N$ is bounded on the compact $K$.

To show that $W_N$ is a Lyapunov function for some $N$, we start with:

**Lemma 3***Let $C$ be as in Lemma 2. Suppose that there are no rest points in $C$. Then, there exist a real $\epsilon_0 > 0$, a positive integer $L_0$ such that, for all $\mathbf{x} \in C$*

$$N \geq L_0 \Rightarrow \sum_{j}^{j+N} (\nabla W(\mathbf{x})_k)^2 > \epsilon_0.$$

*Proof.* Suppose by contradiction that there exist $j_n, N_n$ such that

$$\sum_{j_n}^{j_n+N_n} \left(\nabla W(\mathbf{x}^{(n)})_k\right)^2 \to 0 \tag{2.1}$$

. Let $m(n) = -j_n - E(N_n/2)$ where $E$ is the integer part function, and let $\mathbf{x}'^{(n)} = \sigma^{m(n)}\mathbf{x}^{(n)}$. This new sequence $\mathbf{x}'^{(}n)$ is still in $C$, and satisfies (2.1). By compactness of $C$, a subsequence of it converges pointwise (i.e. in the product topology) to some $\mathbf{x}^\infty$ in

$C$. Clearly, $\nabla W(\mathbf{x}^\infty)_k = lim_{n\to\infty} \nabla W(\mathbf{x}^{(n)})_k = 0$ for all $k$ and $\mathbf{x}^\infty$ is a rest point, a contradiction. *Q.E.D*

We now show that $W_N$ is a strict Lyapunov function. By chain rule:

(2.2)

$$
\begin{aligned}
\frac{d}{dt}W_N(x) &= -\left[\sum_{-N}^{N} \partial_1 S(x_k, x_{k+1})\nabla W(x)_k + \partial_2 S(x_k, x_{k+1})\nabla W(x)_{k+1}\right] \\
&= -\left[\sum_{-N}^{N} \partial_1 S(x_k, x_{k+1})\nabla W(x)_k + \sum_{-N+1}^{N+1} \partial_2 S(x_{k-1}, x_k)\nabla W(x)_k\right] \\
&= -\partial_1 S(x_{-N}, x_{-N+1})\nabla W(x)_{-N} - \partial_2 S(x_N, x_{N+1})\nabla W(x)_{N+1} - \sum_{-N+1}^{N} \big(\nabla W(x)_k\big)^2
\end{aligned}
$$

For all $\mathbf{x}$ in $Y_\omega^K$, we have $|x_k - x_{k-1}| < K$ and hence $S$, its partial derivatives and thus $\nabla W_k$ are all bounded on that set. In particular, we can find some $M(K)$ such that

$$
|-\partial_1 S(x_{-N}, x_{-N+1})\nabla W(\mathbf{x})_{-N} - \partial_2 S(x_N, x_{N+1})\nabla W(\mathbf{x})_{N+1}| < M
$$

for all $\mathbf{x}$ in $Y_\omega^K$ and all integer $k$. We claim that for $N > (M+2)L_0/(2\epsilon_0)$ (where $L_0, \epsilon_0$ are as in Lemma 2), $W_N$ is a Lyapunov function. Indeed, $N \geq (p+1)L_0$ where $p > M/\epsilon_0$ and we can split the sum $\sum_{-N+1}^{N}\big(\nabla W(x)_k\big)^2$ into $p$ sums of length greater than $L_0$. By Lemma 2, each of these sums must be greater than $\epsilon_0$, and thus the total sum must be greater than $M + 2\epsilon_0$, making the expression in (2.2) less than $-2\epsilon_0$. *Q.E.D.*

The end of the proof of the Aubry-Mather theorem proceeds as before. One only needs to note that the $\omega$-limit set of $B_\omega$ is included in $Y_\omega^2$ and is $\sigma$ invariant since $B_\omega$ itself has these properties.

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