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A NEW PROOF THAT ANALYTIC SETS ARE RAMSEY

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Abstract. We give a direct mathematical proof of the Mathias-Silver theorem that every analytic set is Ramsey.

§1. Introduction. Use lower case Greek letters to denote subsets of $\omega =$ the nonnegative integers. Define $\alpha < \beta$ if $(\forall x \in \alpha)(\forall y \in \beta) x < y$, $(\alpha, \beta)^{<\omega} = \{\xi \mid \alpha \subseteq \xi \subseteq \alpha \cup \beta \wedge \alpha < \xi - \alpha \wedge \xi$ is finite$\}$, $(\alpha, \beta)^{\omega} = \{\xi \mid \alpha \subseteq \xi \subseteq \alpha \cup \beta \wedge \alpha < \xi - \alpha \wedge \xi$ is infinite$\}$, $P = (\varnothing, \omega)^{\omega}$ and $Q = (\varnothing, \omega)^{<\omega}$. Regard P as a topological space endowed with a neighborhood system consisting of sets of the form $(\alpha, \xi)^{\omega}$ where $\alpha \in Q$ and $\xi \in P$. Since this is the topology we intend to use throughout most of our paper, ordinary topological words will always refer to it. On some occasions, however, we will want to speak about the relativization to P of the product topology of countably many copies of $\{0, 1\}$, each bearing the discrete topology. This is the classical topology which is relevant for the theories of recursion and definition. We prefix the word *classical* to indicate notions that are defined for this topology.

$S \subseteq P$ is called *Ramsey* if there is a $\xi \in P$ such that $(\varnothing, \xi)^{\omega} \subseteq S$ or $(\varnothing, \xi)^{\omega} \subseteq P - S$. In [1] it is shown that every classical Borel set is Ramsey and in [3], [4] it is shown that every classical analytic set is Ramsey. The former result is combinatorial, but the latter uses relatively deep metamathematical notions involving forcing. In §2 we prove the Mathias-Silver theorem on classical analytic sets using nothing more than the methods of Galvin-Prikry. It easily follows from our main result that any set with the Baire property is Ramsey, the latter closely related to the fact that any meager set is nowhere dense. A word of caution: nothing could be falser in the classical topology. Our proof should be accessible to the general mathematical reader.

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§2. The Baire property. Let $S \subseteq P$. If $\alpha \in Q$ and $\eta \in P$ we say, as in [1], η *accepts* α if $(\alpha, \eta)^{\omega} \subseteq S$ and that η *rejects* α if there is no $\xi \in (\varnothing, \eta)^{\omega}$ which accepts α. The following combinatorial results from [1] are basic to our method.

LEMMA 1 (GALVIN-PRIKRY). *There is an $\eta \in P$ which accepts or rejects each of its finite subsets.*

LEMMA 2 (GALVIN-PRIKRY). *If $\eta \in P$ accepts or rejects each of its finite subsets and η rejects $\varnothing$ then there is a $\xi \in (\varnothing, \eta)^{\omega}$ which rejects each of its finite subsets.*

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LEMMA 3. *Every open set is Ramsey.*

PROOF. Take $\eta \in P$ which by Lemma 1 accepts or rejects each of its finite subsets. If η accepts $\varnothing$ then $(\varnothing, \eta)^\omega \subseteq S$. Otherwise η rejects $\varnothing$ and by Lemma 2 we may assume that η rejects each of its finite subsets. Suppose that $\xi \in (\varnothing, \eta)^\omega \cap S$. Then there is a neighborhood $(\alpha, \beta)^\omega$ such that $\xi \in (\alpha, \beta)^\omega \subseteq S$. This implies that ξ accepts α and thus η cannot reject α, contradiction. Q.E.D.

As in [4], $S \subseteq P$ is called *completely Ramsey* if for every $\alpha \in Q$ and $\eta \in P$ there is a $\xi \in (\varnothing, \eta)^\omega$ such that $(\alpha, \xi)^\omega \subseteq S$ or $(\alpha, \xi)^\omega \subseteq P - S$.

LEMMA 4. *Every open set is completely Ramsey.*

PROOF. Suppose $\alpha \in Q$ and $\eta \in P$. Let f be a strictly increasing function mapping ω onto η and let $g(\xi) = \alpha \cup \xi$ for each $\xi \in P$. Consider any neighborhood $(\gamma, \delta)^\omega$ where without loss of generality we may assume that $\gamma < \delta$. If $f(\xi) \in (\gamma, \delta)^\omega$ then $(f^{-1}(\gamma), f^{-1}(\delta))^\omega$ is a neighborhood of ξ whose image under f is contained in $(\gamma, \delta)^\omega$. If $g(\xi) \in (\gamma, \delta)^\omega$ then $(\gamma \cap \xi, \delta)^\omega$ is a neighborhood of ξ whose image under g is contained in $(\gamma, \delta)^\omega$, the latter following from $\gamma - \xi \subseteq \alpha$. Thus f and g are continuous. $(gf)^{-1}(S)$ is open and hence, by Lemma 3, there is a $\zeta \in P$ such that $(\varnothing, \zeta)^\omega \subseteq (gf)^{-1}(S)$ or $(\varnothing, \zeta)^\omega \subseteq P - (gf)^{-1}(S)$. Setting $\xi = f(\zeta)$ we get $(\varnothing, \xi)^\omega \subseteq g^{-1}(S)$ or $(\varnothing, \xi)^\omega \subseteq P - g^{-1}(S)$ from which our result easily follows. Q.E.D.

LEMMA 5. *The complement of a completely Ramsey set is completely Ramsey.*

LEMMA 6. *If S is nowhere dense then for any $\alpha \in Q$ and $\eta \in P$ there is a $\xi \in (\varnothing, \eta)^\omega$ such that $(\alpha, \xi)^\omega \subseteq P - S$.*

PROOF. By Lemmas 4 and 5 the closure $\bar{S}$ of S is completely Ramsey. Then there is a $\xi \in (\varnothing, \eta)^\omega$ such that $(\alpha, \xi)^\omega \subseteq \bar{S}$ or $(\alpha, \xi)^\omega \subseteq P - \bar{S} \subseteq P - S$. Since $\bar{S}$ is nowhere dense the former case cannot occur. Q.E.D.

LEMMA 7. *If S is meager then for any $\alpha \in Q$ and $\eta \in P$ there is a $\xi \in (\varnothing, \eta)^\omega$ such that $(\alpha, \xi)^\omega \subseteq P - S$.*

PROOF. Let S_n be a sequence of nowhere dense sets whose union is S. Let $\alpha_0 = \alpha$ and choose $\eta_0 \in (\varnothing, \eta)^\omega$ so that $(\alpha_0, \eta_0)^\omega \subseteq P - S_0$ and $\alpha_0 < \eta_0$. Suppose we have defined α_n and η_n with $\alpha_n < \eta_n$. Let a_n be the least element of η_n. Set $\alpha_{n+1} = \alpha_n \cup \{a_n\}$ and choose $\eta_{n+1} \in (\varnothing, \eta_n - \{a_n\})^\omega$ so that for each $\alpha_0 \subseteq \gamma \subseteq \alpha_{n+1}$ we have $(\gamma, \eta_{n+1})^\omega \subseteq P - S_{n+1}$. Then $\xi = \bigcup \alpha_n$ will do. Q.E.D.

COROLLARY 8. *Every meager set is completely Ramsey and is nowhere dense.*

THEOREM 9. *Every set with the Baire property is completely Ramsey.*

PROOF. Let $\alpha \in Q$ and $\eta \in P$. Any set S with the Baire property can be expressed as $S = S_0 \Delta S_1$ where S_0 is open, S_1 is meager and Δ is symmetric difference. By Lemma 7 we can choose $\zeta \in (\varnothing, \eta)^\omega$ so that $(\alpha, \zeta)^\omega \subseteq P - S_1$ and then by Lemma 4 there is a $\xi \in (\varnothing, \zeta)^\omega$ such that $(\alpha, \xi)^\omega \subseteq S_0$ or $(\alpha, \xi)^\omega \subseteq P - S_0$. In the former case $(\alpha, \xi)^\omega \subseteq S$ and in the latter $(\alpha, \xi)^\omega \subseteq P - S$. Q.E.D.

A *Souslin system* is a class of closed sets that are indexed by finite sequences of nonnegative integers. A *Souslin set* is one which can be expressed in the form $\bigcup_{f \in \omega^\omega} \bigcap_{n \in \omega} S_{f/n}$ where S_e is a Souslin system, f/n is the restriction of f to the predecessors of n, and ω^ω is the set of all functions mapping ω into ω.

LEMMA 10. *Every Souslin set is completely Ramsey.*

PROOF. It is an ancient result that the Baire property is preserved under the Souslin operation. We do not know who first proved this theorem, but for some bibliography and a proof see p. 94 of [2]. Since closed sets have the Baire property, Lemma 9 gives our result. Q.E.D.

COROLLARY 11 (MATHIAS-SILVER). *Every classical analytic set is completely Ramsey.*

PROOF. Every classical analytic set is a classical Souslin set (cf. [2, p. 482]). Since classical closed sets are also closed in our topology, Lemma 10 applies directly. Q.E.D.

Postscript. F. Galvin has made the following observations. It immediately follows by definition that if S is completely Ramsey then S minus its interior is nowhere dense. Thus every completely Ramsey set has the Baire property. This is converse to Theorem 9. By the proof of Lemma 10 the class of completely Ramsey sets is closed under the Souslin operation. Then Lemma 5 implies that all sets in the classical Lusin hierarchy are completely Ramsey (the classical Lusin hierarchy is the closure under complementation and the Souslin operation of the classical analytic sets).

REFERENCES

[1] F. GALVIN and K. PRIKRY, *Borel sets and Ramsey's theorem*, this JOURNAL, vol. 38 (1973), pp. 193–198.

[2] K. KUROTOWSKI, ***Topology,*** vol. I, Academic Press, New York, 1966.

[3] A. R. D. MATHIAS, *On a generalization of Ramsey's theorem*, ***Notices of the American Mathematical Society,*** vol. 15 (1968), p. 931. Abstract 68T-E19.

[4] J. SILVER, *Every analytic set is Ramsey*, this JOURNAL, vol. 35 (1970), pp. 60–64.

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