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A NEW REARRANGEMENT INEQUALITY FOR MULTIPLE INTEGRALS

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In this paper we state and prove an inequality (see Eq. (8) below)
for integrals on n variables, of which the case n = 2 is due to Hardy and
Littlewood. In appendices we sketch an alternate proof, as well as a proof
of an analogous inequality on discrete sums, and conjecture a further gener-
alization.

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The subject of rearrangement inequalities is discussed by Hardy, Pólya and Littlewood.[1] Briefly, f and g are equimeasurable functions from the interval $(-\infty, \infty)$ to the interval $[0, \infty)$ if the two sets $\{x \mid f(x) \geq y\}$ and $\{x \mid g(x) \geq y\}$ are equal in measure for each y; and the symmetric decreasing rearrangement of f is defined to be that function $f*$ which is equimeasurable with f and satisfies $f*(x) = f*(-x) \geq f*(x')$ when $|x| \leq |x'|$. This definition leads to various integral inequalities, of which the simplest is

$$\int F_1(x) \, F_2(x) \, dx \leq \int F_1^*(x) \, F_2^*(x) \, dx \tag{1}$$

(all integrals run from $-\infty$ to ∞ unless otherwise labeled).

The proof of (1) is fairly trivial; see sec. 10.13 of [1]. One can easily generalize it to the product of n functions. Another generalization is

$$\int F_1(x_1) \, H^*(x_1-x_2) \, F_2(x_2) \, dx_1 dx_2 \leq \int F_1^*(x_1) \, H^*(x_1-x_2) \, F_2^*(x_2) \, dx_1 dx_2 \tag{2}$$

where the same symmetric decreasing function H* appears on both sides. The inequality (2) reduces to (1) if H* is taken to be the Dirac δ-function (_i.e._ in the limit as H* becomes sharply peaked at the origin and vanishes elsewhere).

An easy proof of (2) follows the lines of sec. 10.8 of [1]. This proof can be generalized to the inequality

$$\int F_1(x_1) \, H_1^*(x_1-x_2) \ldots F_n(x_n) \, H_n^*(x_n-x_1) \, dx_1 \ldots dx_n$$
$$\leq \int F_1^*(x_1) \, H_1^*(x_1-x_2) \ldots F_n^*(x_n) \, H_n^*(x_n-x_1) \, dx_1 \ldots dx_n \tag{3}$$

as we shall show in a subsequent paper, in which (3) will be further generalized to functions on a circular domain.

Henceforth, we shall adopt the convention that

$$x_{n+1} \equiv x_1, \quad x_0 \equiv x_n, \quad H_0 \equiv H_n, \quad \text{etc.} \qquad (4)$$

so that the left side of (3), for example, can be written more compactly either as $\int \prod_1^n \left[F_i(x_i) H_i^*(x_i - x_{i+1}) dx_i \right]$ or as $\int \prod_1^n \left[H_{i-1}^*(x_{i-1} - x_i) F_i(x_i) dx_i \right]$.

A much more difficult theorem than (2) is obtained by replacing H* on the left by an arbitrary function H of which H* is the symmetric decreasing rearrangement:

$$\int F_1(x_1) H(x_1 - x_2) F_2(x_2) dx_1 dx_2 \leq \int F_1^*(x_1) H^*(x_1 - x_2) F_2^*(x_2) dx_1 dx_2. \qquad (5)$$

This inequality is proved in sec. 10.14-5 of [1], in the equivalent form

$$\int H_1(y_1) H_2(y_2) H_3(-y_1 - y_2) dy_1 dy_2 \leq \int H_1^*(y_1) H_2^*(y_2) H_3^*(-y_1 - y_2) dy_1 dy_2. \qquad (6)$$

As indicated in sec. 10.9 of [1], the proof of (6) is easily extended to the generalization

$$\int H_n\left(-\sum_1^{n-1} y_i\right) \prod_1^{n-1} \left[H_i(y_i) dy_i \right] \leq \int H_n^*\left(-\sum_1^{n-1} y_i\right) \prod_1^{n-1} \left[H_i^*(y_i) dy_i \right]. \qquad (7)$$

Our purpose in this paper is to prove a much more powerful and more difficult generalization of (5), namely

$$\int \prod_1^n \left[F_i(x_i) H_i(x_i - x_{i+1}) dx_i \right] \leq \int \prod_1^n \left[F_i^*(x_i) H_i^*(x_i - x_{i+1}) dx_i \right] \qquad (8)$$

which includes both (3) and (7) as special cases. (To obtain (7) from (8), replace $F_2, \ldots, F_n$ by constants and let $y_i \equiv x_i - x_{i+1}$ for $1 \leq i \leq n$.) Both (3) and the more general (8) have recently found application in physics, where they give rise to a class of Green's-function inequalities [2] of which certain well-known isoperimetric inequalities are special cases.

To prove (8), we first observe that it suffices to prove it in the case that each function F or H takes only the values 0 or 1; moreover, we may assume that $\left\{ x \mid f(x) = 1 \right\}$ is compact and piecewise continuous (the union of a finite number of closed intervals) where f is any F_i or H_i. (See the discussion in sec. 10.14 of [1].) Therefore (8) is a corollary of the following theorem.

Theorem I.

Let F_i ($1 \leq i \leq n$) be the characteristic function of a compact, piecewise continuous set of measure m on the real line and let F_i^* be the characteristic function of S_{m_i} where, for any m, we define

$$S_m \equiv \left\{ x \mid |x| \leq m/2 \right\} . \qquad (9)$$

Likewise let H_i ($1 \leq i \leq n$) be the characteristic function of a set of measure k_i, and H_i^* that of S_{k_i}.

Then

$$\int \prod_1^n \left[F_i(x_i) H_i(x_i - x_{i+1}) dx_i \right] \leq \int \prod_1^n \left[F_i^*(x_i) H_i^*(x_i - x_{i+1}) dx_i \right] \quad (11$$

where x_{n+1} is to be read as an alternate notation for x_1 .

Proof:

We define an operation of <u>truncation</u> as follows:
if $0 \leq a_i \leq m_i$, we let

$$F_i^{a_i}(x_i) = 1 \quad \text{if} \quad F_i(x_i = 1) \text{ and } \int_{x_i}^{\infty} F_i(x)\,dx \geq a_i$$
$$= 0 \quad \text{otherwise} ; \tag{11}$$

and if $0 \leq a_i$, $0 \leq a_{i+1}$, $a_i + a_{i+1} \leq k_i$, we let

$$H_i^{a_i \, a_{i+1}}(x_i - x_{i+1}) = 1 \quad \text{if} \quad H_i(x_i - x_{i+1}) = 1$$

$$\text{and} \int_{x_i - x_{i+1}}^{\infty} H_i(y)\,dy \geq a_i$$

$$\text{and} \int_{-\infty}^{x_i - x_{i+1}} H_i(y)\,dy \geq a_{i+1} \tag{12}$$

$$= 0 \quad \text{otherwise} .$$

The truncated function $F_i^{a_i}$ is the characteristic function of a set obtained from the set represented by F_i through the deletion of an upper portion of measure a_i . Likewise $H_i^{a_i \, a_{i+1}}$ represents a set obtained by deleting a measure a_i from the top and a_{i+1} from the bottom of the set corresponding to H_i . (See Fig. 1.)

We may similarly define $F_i^{*a_i}$ $H_i^{*a_i \, a_{i+1}}$ by truncating F_i^*, H_i^*. Alternatively we may define $F_i^{a_i *}$,

$H_i^{a_i\,a_{i+1}}*$ as characteristic functions of $S_{m_i-a_i}$, $S_{k_i-a_i-a_{i+1}}$. We note, however, that

$$F_i^{\#a_i}(x_i) = F_i^{a_i\,\#}(x_i + \tfrac{1}{2}a_i)$$

$$H_i^{\#a_i\,a_{i+1}}(x_i - x_{i+1}) = H_i^{a_i\,a_{i+1}\,\#}(x_i + \tfrac{1}{2}a_i - x_{i+1} - \tfrac{1}{2}a_{i+1}) .$$

$$(13)$$

(See Fig. 2.)

The passage from F to F^a is equivalent to the passage from F to F_1 in sec. 10.15 of [1], except that Hardy et al. arrange to let a have only integer values. The difference is a matter of taste. By making a vary continuously, we approach more closely the method of Riesz (sec 10.16 of [1]) except that he truncates symmetrically by deleting a set of measure t simultaneously from top and bottom. (See footnote, p. 286 of [1].) In this matter we have no choice; it will be essential that in forming the double truncation H^{ab} we do not allow the parameters a and b to interfere with one another, and this is achieved by letting them act at opposite ends as specified by (12).

To give the reader a clear idea of how our truncation is related to that of Hardy et al., let us paraphrase their proof (sec 10.15 of [1]) of Theorem I in the case n = 2. First we drop the factor H_2 from (10) because it is inessential in this case, having the same argument as H_1 . Then we point out that (10) must hold if $m_1 + k_1 \le m_2$, because the right side then takes the value $m_1 k_1$, and the left side cannot be more.

Now suppose that $m_1 + k_1 > m_2$, and let

$$I(a) = \int F_1^a(x_1) \, H_1^{a0}(x_1 - x_2) \, F_2(x_2) \, dx_1 \, dx_2 \, ,$$

$$^*I(a) = \int F_1^{*a}(x_1) \, H_1^{*a0}(x_1 - x_2) \, F_2^*(x_2) \, dx_1 \, dx_2 \, ,$$

$$A = \min\left(m_1, \, k_1, \, \tfrac{1}{2}(m_1 + k_1 - m_2)\right).$$

We already know that $I(A) \leq {}^*I(A)$ —note the use of (13)— and we must prove $I(0) \leq {}^*I(0)$. Thus it suffices to show that $-\dfrac{dI}{da} \leq -\dfrac{d^*I}{da}$ for $0 \leq a < A$. This is done by showing on the one hand that

$$-\frac{d}{da} \int F_1^a(x_1) \, H_1^{a0}(x_1 - x_2) \, dx_1 \leq 1 \qquad \text{for all } x_2$$

(compare Eq. (10.15.16) of [1]) so that $-\dfrac{dI}{da} \leq m_2$; and on the other hand that

$$-\frac{d}{da} \int F_1^{*a}(x_1) \, H_1^{*a0}(x_1 - x_2) \, dx_1 = 1 \quad \text{for } |x_2| \leq \tfrac{1}{2}(m_1 - a) + \tfrac{1}{2}(k_1 - a) \, ,$$

so that $-\dfrac{d^*I}{da} = m_2$ exactly (see Eq. (10.15.13) of [1]). The proof of Riesz (sec. 10-16 of [1]) is essentially the same.

It is possible to extend this proof to $n > 2$; see Appendix I. However, the proof we shall present is based instead on the following variation of the above argument for $n = 2$.

Without loss of generality we may suppose that $m_1 \geq m_2$. If $m_1 > m_2$, we define $I(a)$, $^*I(a)$ as above, but let $A = \min(k_1, \, m_1 - m_2)$. As above, we can prove $-\dfrac{dI}{da} \leq -\dfrac{d^*I}{da}$ for $0 \leq a < A$, and so it suffices to show that $I(A) \leq {}^*I(A)$. This is trivial if $A = k_1$, and if $A = m_1 - m_2$ it reduces to the original theorem with $m_1 = m_2$.

We treat the case $m_1 = m_2$ by defining

$$I(a,b) = \int F_1^{a}(x_1) H_1^{ab}(x_1 - x_2) F_2^{b}(x_2) \, dx_1 dx_2 ,$$

$$^*I(a,b) = \int F_1^{*a}(x_1) H_1^{*ab}(x_1 - x_2) F_2^{*b}(x_2) \, dx_1 dx_2 ,$$

$$B = \min(m_2, \tfrac{1}{2} k_1) .$$

Then $\quad I(B,B) = 0 = {}^*I(B,B) \qquad$, so that it

suffices to prove $\quad -\frac{d}{db} I(b,b) \leq -\frac{d}{db} {}^*I(b,b) \qquad$ for

$0 \leq b \leq B$. By a repetition of earlier arguments we

find that $\quad -\frac{\partial}{\partial a} I(a,b) \leq -\frac{\partial}{\partial a} {}^*I(a,b) \quad$ if $\ a \leq b$,

and $\quad -\frac{\partial}{\partial b} I(a,b) \leq -\frac{\partial}{\partial b} {}^*I(a,b) \quad$ if $\ b \leq a$.

Setting $a = b$ and adding, we obtain the desired inequality.

The advantage of this longer argument is that it

avoids reduction to the inequality $\ I(A) \leq {}^*I(A) \quad$ for

$A = \frac{1}{2}(m_1 + k_1 - m_2) \qquad$ – the case $\mu = 0$ in sec. 10.15

of [1]. In Appendix I we indicate how this step can be

generalized to $n > 2$, if one prefers the first argument.

We now return to the proof of Theorem I for arbitrary

n. We begin by defining

$$I(a_1 \ldots a_n) = \int \prod_1^n \left[F_i^{a_i}(x_i) H_i^{a_i a_{i+1}}(x_i - x_{i+1}) \, dx_i \right] \quad (14)$$

$$^*I(a_1 \ldots a_n) = \int \prod_1^n \left[F_i^{a_i *}(x_i) H_i^{a_i a_{i+1} *}(x_i - x_{i+1}) \, dx_i \right]$$

$$\qquad\qquad (15)$$

$$= \int \prod_1^n \left[F_i^{* a_i}(x_i) H_i^{* a_i a_{i+1}}(x_i - x_{i+1}) \, dx_i \right]$$

the two lines of (15) being equivalent because of (13).

Let $0 \leqslant j < n$. We shall obtain (10) as a special case
(j = 0) of the following theorem.

Theorem IA.

Suppose that

and
$$K_i > 0, \quad \text{for} \quad i = 1, \ldots, j \qquad (16)$$

Then
$$m_n \leqslant m_i, \quad \text{for} \quad i = 1, \ldots, n-1 . \qquad (17)$$

$$\prod_1^j \left(-\frac{\partial}{\partial a_i}\right) I(a_1 \ldots a_n)_0 \leqslant \prod_1^j \left(-\frac{\partial}{\partial a_i}\right)^* I(a_1 \ldots a_n)_0 \qquad (18)$$

where the final subscript 0 means that all the a_i are set
equal to 0 after differentiation.

The proof of Theorem IA will proceed by double induction.
Let $n = \ell + j$. We shall prove, first, that Theorem IA holds
for $\ell = 1$ and arbitrary j; and second, that if it holds for
$\ell - 1$ (and any j) then it holds for ℓ (and any j). In the
latter part, we shall use induction on j for fixed ℓ, start-
ing with j = 0.

After proving Theorem IA, we shall derive Theorem I from
it. So as to make the logical structure of the proof readily
visible, we shall defer the demonstration of certain state-
ments to notes following the main argument.

The case $\ell = 1$ is treated by showing (Note 1) that the
function

$$h(x_n) = \prod_1^{n-1} \left(-\frac{\partial}{\partial a_i}\right) \left\{ \int H_n^{a_n a_1}(x_n - x_1) \right.$$

$$\left. \prod_1^{n-1} \left[F_i^{a_i}(x_i) H_i^{a_i a_{i+1}}(x_i - x_{i+1}) dx_i \right] \right\}_0 \qquad (19)$$

takes on only 0 and 1 as values, whereas (Note 2) the corresponding function

$$*h(x_n) = \prod_1^{n-1}\left(-\frac{\partial}{\partial a_i}\right)\left\{\int H_n^{*a_n a_1}(x_n - x_1)\right.$$
$$\left.\prod_1^{n-1}\left[F_i^{*a_i}(x_i) H_i^{*a_i a_{i+1}}(x_i - x_{i+1}) dx_i\right]\right\}_0 \quad (20)$$

is the characteristic function of $S_{\bar{m}}$ where

$$\bar{m} \geq \min\{m_i\}. \quad (21)$$

Writing (18) in the form

$$\int F_n(x_n) h_n(x_n) dx_n \leq \int F_n^*(x_n) *h_n(x_n) dx_n \quad (22)$$

we see from Note 1 that the left side cannot exceed $\int F_n(x_n) dx_n = m_n$, and from Note 2 that the right side is $\min(\bar{m}, m_n)$ which by (21) and (17) is m_n. This verifies (22), and hence (18) when $\ell = 1$.

Now suppose that $\ell > 1$ and that Theorem IA holds when ℓ is reduced by 1.

Define

$$g(a) = \prod_1^j\left(-\frac{\partial}{\partial a_i}\right) I(a_1 \ldots a_n)_{0,a} \quad (23)$$

$$*g(a) = \prod_1^j\left(-\frac{\partial}{\partial a_i}\right) *I(a_1 \ldots a_n)_{0,a} \quad (24)$$

where the final subscripts mean that we set $a_{j+1} = a$ and all other $a_i = 0$ after differentiation. Then (18) may be written

$$g(0) \leq *g(0). \quad (25)$$

Define

$$A = \min(k_{j+1}, k_j, m_{j+1} - m_n) . \qquad (26)$$

(We read k_0 as k_n according to (4).) Then for any a satisfying $0 \leq a < A$, we see that (16) and (17) are still satisfied if m_{j+1}, k_j, k_{j+1} are all diminished by a.

It follows that

$$-\frac{d}{da} g(a) \leq -\frac{d}{da} {}^*g(a) \qquad (27)$$

for all $0 \leq a < A$, as an application of Theorem IA with $\ell \to \ell + 1$. Therefore, to prove (25) it suffices to prove

$$g(A) \leq {}^*g(A) . \qquad (28)$$

We distinguish three cases.

Case 1: $A = k_{j+1}$, or $j = 0$ and $A = k_0 (\equiv k_n)$. Here (28) is obviously true since both sides vanish.

Case 2: $j > 0$ and $A = k_j < k_{j+1}$. This is really the most difficult case. Although the integrand in $g(a)$ contains the factor $H_j^{a_j a}$ which vanishes for $a_j = 0$ and $a = A$, $g(A)$ itself does not vanish because the operator $-\frac{\partial}{\partial a_j}$ is to be applied before letting $a \to A-$. However, we show in Note 3 that (28) may be transformed, in this case, into an application of Theorem IA with $\ell \to \ell$ and $j \to j-1$, by eliminating the variable x_j. Thus eventually we arrive at $j = 0$, when Case 2 can no longer occur.

Case 3: $A = m_{j+1} - m_n < \min(k_j, k_{j+1})$. Here (28) may be regarded as an example of (25) with m_{j+1}, k_j, k_{j+1} all diminished by A. Conditions (16) and (17) remain

satisfied, but there is now the additional equality

$$m_{j+1} = m_n \; . \tag{29}$$

It therefore suffices, in order to complete the proof of (25), to prove it in the presence of (29).

We cannot proceed now according to (23) and (24), because if we truncate any further with respect to the variable x_{j+1} (thus diminishing m_{j+1}) we shall destroy the validity of (17), which is needed in proving the case $\ell = 1$ on which our induction is based. Instead, we truncate simultaneously with respect to x_{j+1} and x_n, maintaining (29), and hence (17), throughout. (Since $\ell > 1$, x_{j+1} and x_n must be distinct.) Thus we shall reduce (25) to a statement analogous to (28) which will be proved true in every case.

Assuming (29), we define

$$g(a,b) = \prod_1^j \left(-\frac{\partial}{\partial a_i}\right) I(a_1 \cdots a_n)_{0,a,b} \tag{30}$$

$$^{*}g(a,b) = \prod_1^j \left(-\frac{\partial}{\partial a_i}\right) {}^{*}I(a_1 \cdots a_n)_{0,a,b} \tag{31}$$

where we are to set $a_{j+1} = a$, $a_n = b$, all other $a_i = 0$ after differentiation. Our real intention is to put $a = b$, but we keep the two parameters distinct in order to define partial derivatives. We may replace (25), then, by

$$g(0,0) \leq {}^{*}g(0,0) \; . \tag{32}$$

Let us assume for the moment that $j > 0$ and $\ell > 2$. We define

$$B = \min(k_{j+1}, k_{n+1}, k_j, k_n), \qquad (33)$$

and note that if $0 \le a \le b < B$, conditions (16) and (17) will still hold if m_{j+1}, k_j, k_{j+1} are all diminished by a, and m_n, k_n, k_{n-1} are all diminished by b. Using the notation $g_1 = \partial g / \partial a$, $g_2 = \partial g / \partial b$, and similarly for *g, we obtain

$$-g_1(a,b) \le -{}^*g_1(a,b) \qquad (34)$$

for $0 \le a \le b < B$ by induction on ℓ in analogy with (27). By a renumbering (Note 4) of the indices $1 \ldots n$, we also obtain

$$-g_2(a,b) \le -{}^*g_2(a,b) \qquad (35)$$

for $0 \le b \le a < B$ and the sum of (34) and (35) for $a = b$ givees

$$-\frac{d}{db} g(b,b) \le -\frac{d}{db}{}^*g(b,b) \qquad (36)$$

for $0 \le b < B$. To prove (32), therefore, it suffices to prove

$$g(B,B) \le {}^+g(B,B) \qquad (37)$$

We distinguish three cases of (37), analogous to those of (28).

Case 1: $B = k_{j+1}$ or k_{n+1}. Then $g(B,B) = 0 = {}^*g(B,B)$.

Case 2a: $B = k_j \leqslant \min(k_{j+1}, k_{n-1})$. This is handled by induction on j according to Note 3.

Case 2b: $B = k_n < \min(k_{j+1}, k_{n-1})$. This is reduced to Case 2a by renumbering the indices according to Note 4.

Case 3: $B = m_n$. Here $g(B, B) = 0 = {}^*g(B, B)$ as in Case 1.

If $\ell = 2$ or j = 0, the proof of (32) given above must be modified slightly (Note 5).

The proof of Theorem IA is now complete. To obtain Theorem I, we cyclically permute the indices 1...n so as to satisfy (17). If (16) holds, we can apply Theorem IA to obtain (18), which for j = 0 reduces to (10). If (16) fails, (10) follows anyway since both sides vanish. But (10) is invariant under the cyclic permutation of indices; therefore it still holds under the original numbering. This proves Theorem I.

Note 1. We study the right side of (19) by means of the following lemma, which we state in two parts.

Lemma IA. Let $i < n$, and let $g(x_i)$ be the characteristic function of some compact, piecewise continuous set of measure p . Let $g^{a_i}(x_i)$ be the truncation of g according to (11), for $0 \leq a_i < p$. Define

$$G(x_{i+1}) = -\frac{d}{da_i}\left[\int g^{a_i}(x_i)\, F_i^{a_i}(x_i)\, H_i^{a_i, 0}(x_i - x_{i+1})\, dx_i\right]_0 \qquad (38)$$

where the final subscript means that we set $a_i = 0$ after differentiation. Then $G(x_{i+1})$ takes only the values 0 and 1 .

Lemma IB. Moreover, let $G^a(x_{i+1})$ be the truncation of G according to (11), and let $^aG(x_{i+1})$ be defined by

$$^aG(x_{i+1}) = - \frac{d}{da_i} \left[\int g^{a_i}(x_i) \, F_i^{a_i}(x_i) \, H^{a_i \, a_{i+1}}(x_i - x_{i+1}) dx_i \right]_{0,a} \quad (39)$$

where we set $a_i = 0$, $a_{i+1} = a$. Then either (Case 1) $^aG(x_{i+1}) = G(x_{i+1})$ for all x_{i+1} and all a from 0 to k_i, or (Case 2) $^aG(x_{i+1}) = G^a(x_{i+1})$ for all x_{i+1} and all a from 0 to k_i.

To prove Lemma IA, let q_1, q_2, q_3 be the highest numbers satisfying $g(q_1) = F_i(q_2) = H_i(q_3) = 1$. Let $q = q(x_{i+1}) = \min(q_1, q_2, x_{i+1} + q_3) = \min(q', x_{i+1} + q_3)$ where $q' = \min(q_1, q_2)$. Then for any particular x_{i+1}, the set $\{ x_i \mid g(x_i) \, F_i(x_i) \, H_i(x_i - x_{i+1}) = 1 \}$ has as highest member a number $r \leq q$. The effect on this set of infinitesimal truncation performed simultaneously on g, F_i, H_i is to lower q and thereby delete a single infinitesimal portion of the set, if r = q; or to leave the set unchanged, if r < q. In the former event $G(x_{i+1}) = 1$; in the latter event, $G(x_{i+1}) = 0$.

To prove Lemma IB, we observe that

$$\{ x_{i+1} \mid G(x_{i+1}) = 1 \} = \{ x_{i+1} \mid g(q) \, F_i(q) \, H_i(q - x_{i+1}) = 1 \}$$
$$= T_1 \cup T_2 \quad (40)$$

where

$$T_1 = \left\{ x_{i+1} \,\middle|\, H_i(q' - x_{i+1}) = 1 \,\&\, g(q') F_i(q') = 1 \right\} \qquad (41)$$

$$T_2 = \left\{ x_{i+1} \,\middle|\, q \leq q' \,\&\, g(q) F_i(q) = 1 \right\} \qquad (42)$$

The sets T_1 and T_2 are restricted to $x_{i+1} \geq q' - q_3$ and $x_{i+1} \leq q' - q_3$ respectively, so that they intersect at most at one point. We note also that q', and hence $g(q') F_i(q')$ is independent of x_{i+1}.

Now suppose (Case 1) that $g(q') F_i(q') = 0$. (See Fig. 3.) Then T_1 is empty and from (40) and (42) we find

$$G(x_{i+1}) = g(x_{i+1} + q_3) F_i(x_{i+1} + q_3) \qquad (43)$$

for all x_{i+1}. But if H_i had been truncated from below as in (39), the value of q_3 would not be altered; i.e., the sets $\left\{ y \,\middle|\, H_i^{0a}(y) = 1 \right\}$ and $\left\{ y \,\middle|\, H_i(y) = 1 \right\}$ have the same highest member. Therefore, in this case, ${}^{a}G$ is the same function as G, for $0 \leq a < k_i$.

Suppose instead (Case 2) that $g(q') F_i(q') = 1$. (See Fig. 4.) Then the only way that ${}^{a}G$ can differ from G is through the replacement of H_i by H_i^{0a} in (41). Since x_{i+1} occurs with a negative sign in the argument, the effect of this replacement is just to delete a set of measure

a from the <u>top</u> of T_1 . But T_1 lies "above" T_2 , so that this is the same as truncating G from above. Thus ${}^{a}G$ and G^{a} are the same function for $0 \leq a < k_i$. This completes the proof.

Lemma IA corresponds to the kernel of the Hardy=Littlewood-Riesz proof (see footnote c, p. 284 of [1]). Lemma IB is required for $n > 2$, as we shall see in the following argument.

For $1 \leq i \leq n$, let us define

$$h_i(x_i, z) = \prod_1^{i-1}\left(-\frac{\partial}{\partial a_s}\right)\left\{\int H_n^{a_n a_1}(z-x_1)\prod_1^{i-1}\left[F_s^{a_s}(x_s) H_s^{a_s a_{s+1}}(x_s - x_{s+1}) dx_s\right]\right\}_0 \tag{44}$$

with the conventions of (4) and (18). We note that $h_n(x_n, x_n)$ is identical to $h(x_n)$ as defined by (19). Our aim is to prove that this function takes only the values 0 and 1. To this end, we assert that for each $i < n$, $h_i(x_i, z)$ takes only the values 0 and 1. Moreover, if $h_i^{a_i}$ is the truncation of h_i according to (11), regarded as a function of x_i with fixed z, and ${}^{a}h_i$ is the result of setting $a_i = a$ instead of 0 in (44), then ${}^{a}h_i$ is identical, for fixed z, either to h_i or to h_i^{a}, the choice depending on z but not on a.

We prove these statements by induction on i. Since $h_1(x_1, z) = H_n(z - x_1)$, ${}^{a}h_1(x_1, z) = H_n^{0a}(z - x_1)$,the proof is immediate for i = 1. Now suppose the assertions true of h_i , $i < n-1$; we must prove them for h_{i+1} . From (44) we obtain

$$h_{i+1}(x_{i+1}, \mathfrak{z}) = -\frac{\partial}{\partial a_i}\left[\int^{a_i} h_i(x_i, \mathfrak{z}) F_i^{a_i}(x_i) H_i^{a_i, 0}(x_i - x_{i+1}) dx_i\right]_0 \quad (45)$$

and

$$^{a_i}h_{i+1}(x_{i+1}, \mathfrak{z}) = -\frac{\partial}{\partial a_i}\left[\int^{a_i} h_i(x_i, \mathfrak{z}) F_i^{a_i}(x_i) H_i^{a_i, a_{i+1}}(x_i - x_{i+1}) dx_i\right]_{0, a} \quad (46)$$

with the convention of (39). If z is such that $^{a_i}h_i$ and $h_i^{a_i}$ are identical, then these two equations are identical to (38) and (39) when we set $g = h_i$, $G = h_{i+1}$ and suppress the argument z. If z is such that $^{a_i}h_i$ and h_i are identical, we can still recover (38) and (39) by setting $G = h_{i+1}$ and defining $g(x_i) = 1$ if $h_i(x_i, \mathfrak{z}) = 1$ or $q_2 \leq x_i \leq q_2 + k_i$, where q_2 is the highest number such that $F_i(q_2) = 1$. For then we have $g^{a_i} F_i^{a_i} = g F_i^{a_i} = h_i F_i^{a_i} = {}^{a_i}h_i F_i^{a_i}$ for $0 \leq a_i \leq k_i$. In either case, the desired properties of h_{i+1} follow from Lemma I.

Now taking $i = n - 1$, we apply only Lemma IA to show from (45) that h_{i+1} takes the values 0 and 1 alone. This is the desired result needed to treat (19).

We may observe that the value of z affects the form of the successive $h_i(x_i, \mathfrak{z})$ in a complicated way through its effect on the choices between Case 1 and Case 2 in the successive applications of Lemma IB. Therefore, it is very difficult to say anything about the set $\{x_n | h_n(x_n, x_n) = 1\}$. This point is what makes (8) so much harder to prove than (7), and it is for this that the condition (17) is vital.

Note 2. We study the right side of (20) by means of the following lemma.

Lemma IIA . Let g_1^* , g_2^* be the characteristic functions of S_{p_1} , S_{p_2} as defined by (9), and let g_1^{*a} , g_2^{*a} be their truncations for $0 \quad a$ $0 \le a < \min(p_1 , p_2)$, in analogy with (11). Then if $1 \le i < n$, the function

$$G(x_{i+1}, x_n) = -\frac{d}{da_i}\left[\int g_1^{*a_i}(x_i - x_n)\, g_2^{*a_i}(x_i)\, H_i^{*a_i 0}(x_i - x_{i+1})\, dx_i\right]_0 \quad (47)$$

can be expressed in the form

$$G(x_{i+1}, x_n) = f_{12}^*(x_n)\, f_{13}^*(x_{i+1} - x_n)\, f_{23}^*(x_{i+1}) \quad (48)$$

where f_{12}^* , f_{13}^* , f_{23}^* are the characteristic functions of $S_{p_1 + p_2}$, $S_{p_1+k_i}$, $S_{p_2+k_i}$ respectively.

Lemma IIB. Moreover, if

$$^a G(x_{i+1}, x_n) = -\frac{d}{da_i}\left[\int g_1^{*a_i}(x_i - x_n)\, g_2^{*a_i}(x_i)\, H_i^{a_i a_{i+1}}(x_i - x_{i+1})\, dx_i\right]_{0,a} \quad (49)$$

with the convention of (39), then

$$^a G(x_{i+1}, x_n) = f_{12}^*(x_n)\, f_{13}^{*a}(x_{i+1} - x_n)\, f_{23}^{*a}(x_{i+1})$$

where f_{13}^{*a} , f_{23}^{*a} are the truncations of f_{13}^* , f_{23}^* according to (11).

Proof of lemma. For fixed x_{i+1} and x_n , we define

$$q_1^\pm = x_n \pm \tfrac{1}{2} p_1 \, , \quad q_2^\pm = \pm \tfrac{1}{2} p_2 \, , \quad q_3^\pm = x_{i+1} \pm \tfrac{1}{2} k_i \, .$$

Then the integrand in (47) is the characteristic function of the set $\{x_i (q^- \le x_i \le q^+\}$ where $q^+ = \min(q_1^+, q_2^+, q_3^+)$

and $q^- = \max(q_1^-, q_2^-, q_3^-)$. The effect of an infinitesimal truncation is to lower q_+. It follows that $G(x_{i+1}, x_n)$ takes the value 1 or 0 according as $q^+ \geq q^-$ or not.

Now, the condition $q^+ \geq q^-$ is equivalent to the six conditions $q_1^+ \geq q_2^-$, $q_2^+ \geq q_1^-$, $q_2^+ \geq q_3^-$, $q_3^+ \geq q_2^-$, $q_3^+ \geq q_1^-$, $q_1^+ \geq q_3^-$ which are in turn equivalent to the three conditions

$$|x_n| \leq \tfrac{1}{2}(p_1 + p_2), \quad |x_{i+1} - x_n| \leq \tfrac{1}{2}(p_1 + k_i), \quad |x_{i+1}| \leq \tfrac{1}{2}(p_2 + k_i). \tag{50}$$

Equation (48) follows immediately. This proves Lemma IIA.

Lemma IIB can be reduced, by the substitution $x_{i+1} \rightarrow x_{i+1} - \tfrac{1}{2}a$ and the use of (13), to an application of Lemma IIA with $k_i \rightarrow k_i - a$.

To treat the right side of (20), let us represent its integrand by a graph containing points $P_0, P_1, \ldots, P_n$ joined by lines of fictitious "length" according to the following rule. To each factor $F_i^{*a_i}$ in (20) there corresponds a line of "length" m_i joining P_0 to P_i; to each factor $H_i^{a_i \, a_{i+1}}(x_i - x_{i+1})$ there corresponds a line of "length" k_i joining P_i to P_{i+1} . (See Fig. 5.)

We now integrate over x_1 , apply the operator $-\dfrac{\partial}{\partial a_1}$, and set $a_1 = 0$. Lemma II tells us that the result corresponds to a new graph obtained by deleting the point P_1 and its associated lines, and connecting the pairs $(P_n, P_0), (P_n, P_2), (P_0, P_2)$ by new lines of "length" $k_n + m_1, k_n + k_1, m_1 + k_1$. It is easily seen that this operation does not change the "length" of the shortest path between any pair of points other than P_1 .

The new graph has two lines connecting P_0 to P_2. They represent two factors $f_{23}^{*a_2}(x_2)$ and $F_2^{\#a_2}(x_2)$, of which the one corresponding to the "longer" line is superfluous. Therefore, the "longer" line can be removed. This operation also leaves unchanged the shortest path between any two points in the graph.

The above process can be repeated so as to eliminate the points $P_2, \ldots, P_{n-1}$. At the ith step we integrate over x_i, operate with $-\frac{\partial}{\partial a_i}$ and set $a_i = 0$. By Lemma II the graphical result is to delete P_i and replace the lines joining P_i to its neighbors by new lines joining each pair of such neighbors with "length" equal to that of the path which formerly linked them indirectly through P_i. Superfluous lines are then removed.

The end result of the algebraic operations is $^{*}h(x_n)$ as given by (20); that of the graphical operations is a graph consisting of P_0 and P_n joined by a line of some "length" $\bar{m}$. Therefore $^{*}h(x_n)$ is the characteristic function of $S_{\bar{m}}$. This $\bar{m}$ is equal to the "length" of the shortest path from P_0 to P_n in the original graph, since this "length" was not altered by any of the graphical changes. But any such path included a step from P_0 to some P_i having "length" m_i. Consequently (21) holds.

Note 3.

From the role played by (28) in the proof of (25), it is clear that g(a) and *g(A) must be understood as the limits of g(a) and *g(a) as $a \longrightarrow A$ from below. Letting $0 \leqq a < A$, we can write g(a), since $\ell > 1$ and $j > 0$, in the form

$$g(a) = \int dx_{j+1} \, dx_n \, Q(a, x_{j+1}, x_n) \, {}^a h_{j+1}(x_{j+1}, x_n) \quad (51)$$

where

$$Q(a, x_{j+1}, x_n) = \int F_{j+1}^a(x_{j+1}) \, H_{j+1}^{a0}(x_{j+1} - x_{j+2}) \cdots F_n(x_n) \, dx_{j+2} \cdots dx_{n-1}$$

$$\leq F_n(x_n) \, m_{j+2} \cdots m_{n-1} \quad (52)$$

and ${}^a h_{j+1}(x_{j+1}, x_n)$ is defined by (44) with $a_i \to a$.

By referring to Note 1, we see that ${}^a h_{j+1}(x_{j+1}, x_n)$ takes on only the values 0 and 1, and that, except at one value of x_{j+1}, it is the sum of two terms. The first term (corresponding to T_1 in the proof of Lemma IB) is either $H_j^{0a}(q'(x_n) - x_{j+1})$ or 0, depending on the value of x_n; here $q'(x_n)$ is q' as defined in the proof of Lemma IA for $i = j$, $g(x_j) = h_j(x_j, x_n)$. The second term (corresponding to T_2) is $h_j(x_{j+1} + S, x_n) \, F_j(x_{j+1} + S)$, where S is the least upper bound of the set represented by $H_j^{0a}(y)$.

Applying (52), we see that the contribution of the first term to (51) cannot be greater than

$$\int dx_{j+1} \, dx_n \, F_n(x_n) \, m_{j+2} \cdots m_{n-1} H_j^{0a}(q'(x_n) - x_{j+1}) = (k_j - a) \prod_{j+2}^{n} m_n \quad (53)$$

which approaches zero as $a \to k_j$. Therefore, in the case $A = k_j$, we can ignore this term in computing $g(A)$, and we have

$$g(A) = \lim_{a \to A-} \int dx_{j+1} \, dx_n \, Q(a, x_{j+1}, x_n) \, h_j(x_{j+1} + S, x_n) \, F_j(x_{j+1} + S)$$

$$= \prod_1^{j-1} \left(-\frac{\partial}{\partial a_i} \right) J(a_1 \cdots a_{j-1})_0 \quad (54)$$

in the notation of (18), where

$$J(a_1 \dots a_{j-1}) = \int \prod_1^{j-2} \left[F_i^{a_i}(x_i) \, H_i^{a_i a_{i+1}}(x_i - x_{i+1}) \, dx_i \right]$$

$$\cdot F_{j-1}^{a_{j-1}}(x_{j-1}) \, H_{j-1}^{a_{j-1} 0}(x_{j-1} - x_{j+1} - S) \, F_j(x_{j+1} + S) \, F_{j+1}^A(x_{j+1})$$

$$H_{j+1}^{A0}(x_{j+1} - x_{j+2}) \, dx_{j-1} \, dx_{j+1}$$

$$\cdot \prod_{j+2}^{n-1} \left[F_i(x_i) \, H_i(x_i - x_{i+1}) \, dx_i \right] F_n(x_n) \, H_n^{0 a_1}(x_n - x_1) \, dx_n \qquad (55)$$

if $j > 1$. For $j = 1$ we have the simpler formula

$$g(A) = \int H_n(x_n - x_2 - S) \, F_1(x_2 + S) \, F_2^A(x_2) \, H_2^{A0}(x_2 - x_3) \, dx_2$$

$$\cdot \prod_3^{n-1} \left[F_i(x_i) \, H_i(x_i - x_{i+1}) \, dx_i \right] F_n(x_n) \, dx_n \qquad (56)$$

In either (55) or (56) the product $F_j(x_{j+1} + S) \, F_{j+1}^A(x_{j+1})$ is the characteristic function of a set of measure $\leq \bar{m} = \min(m_j, m_{j+1} - A)$. Let us replace this product by $\bar{F}(x_{j+1})$, the characteristic function of some set having measure $\bar{m}$ and including the original set. We may call the result of this replacement $\bar{g}(A)$; then (using the fact that $h_j(x_{j+1} + S, x_n) \geq 0$) we have

$$g(A) \leq \bar{g}(A) \qquad (57)$$

and also, from (17) and (26),

$$m_n \leq \bar{m} \, . \tag{58}$$

We treat $*g$ (a) by applying Lemma II to eliminate the variable x_j . In the graphical language of Note 2, this leaves us (in the limit $a \rightarrow A = k_j$) with new lines $P_{j-1} P_{j+1}$, $P_{j+1} P_0$, $P_0 P_{j-1}$ having "lengths" k_{j-1} , m_j , $k_{j-1} + m_j$. The third of these lines is made superfluous by the first two, since the condition $|x_{j-1}| \leq \frac{1}{2}(k_{j-1} + m_j)$ is implied by $|x_{j-1} - x_{j+1}| \leq \frac{1}{2} k_{j-1}$ and $|x_{j+1}| \leq \frac{1}{2} m_j$. Also the "longer" of the two lines now joining P_{j+1} to P_0 may be dropped, leaving one line of "length" $\bar{m}$ as defined above. (See Fig. 6.) With a suitable use of (13) for the variable x_{j+1} , we now have

$$*g (A) = \prod_{1}^{j-1} \left(- \frac{\partial}{\partial a_i}\right) *J(a_1 \cdots a_{j-1})_0 \tag{59}$$

where

$$*J(a_1 \cdots a_{j-1}) = \int \prod_{1}^{j-2} \left[F_i^{*a_i}(x_i) \, H_i^{*a_i \, a_{i+1}}(x_i - x_{i+1}) \, dx_i \right]$$

$$\cdot F_{j-1}^{*a_{j-1}}(x_{j-1}) \, H_{j-1}^{*a_{j-1} \, 0}(x_{j-1} - x_{j+1}) \, \bar{F}_j^*(x_{j+1}) \, H_{j+1}^{A0*}(x_{j+1} - x_{j+2}) \, dx_{j-1} \, dx_{j+1}$$

$$\cdot \prod_{j+2}^{n-1} \left[F_i^*(x_i) \, H_i^*(x_i - x_{i+1}) \, dx_i \right] F_n^*(x_n) \, H_n^{*0a_1}(x_n - x_1) \, dx_n \tag{60}$$

if $j > 1$; and for $j = 1$ we have

$$*g(A) = \int H_n^*(x_n - x_2) \, \bar{F}^*(x_2) \, H_2^{AO*}(x_2 - x_3) \, dx_2$$
$$\cdot \prod_3^{n-1} \left[F_i^*(x_i) \, H_i^*(x_i - x_{i+1}) \, dx_i \right] F_n^*(x_n) \, dx_n \qquad (61)$$

where $\bar{F}^*$ is the characteristic function of $S_{\bar{m}}$.

We may now apply Theorem IA with $\ell \to \ell$, $j \to j-1$, $m_{j+1} \to \bar{m}$, $k_{j+1} \to k_{j+1} - k_j$, obtaining

$$\bar{g}(A) \leq {}^*g(A) . \qquad (62)$$

Note that (16) is satisfied because by hypothesis $k_j < k_{j+1}$, and (17) is satisfied because of (58).

Combining (57) with (62), we have (28).

The proof of (37) when $B = k_j < \min(k_{j+1}, k_{n-1})$ is quite similar, although some of the formulas are different. For example, in (55) the factors $H_{n-1}(x_{n-1} - x_n) \, F_n(x_n) \, H_n^{Oa_1}(x_n - x_1)$ are replaced by $H_{n-1}^{OB}(x_{n-1} - x_n) \, \bar{F}_n^B(x_n) \, H_n^{Ba_1}(x_n - x_1)$. Of course, $a \to b$ and $A \to B$ throughout. The most important change is that we no longer have (58) but only

$$m_n - k_j \leq \bar{m} \qquad (63)$$

This is sufficient to give (17) in the final application of Theorem IA with $j \to j-1$, as this application now

involves the substitutions $m_n \to m_n - k_j$, $k_n - k_j$, $k_{n-1} \to k_{n-1} - k_j$
in addition to those previously listed.*

A snag arises if $k_n = k_j$ since (16) is not satisfied.
This may be handled as follows. If $j = 1$, the left side of
(37) vanishes because both terms now have a form like (53).
If $j > 1$, we may eliminate both x_1 and x_j , reducing (37) to
an application of Theorem IA with $j \to j-2$.
Note 4.

By renumbering the indices in the order $j, \cdots, 1, n, \cdots, j+1$,
we transform (35) into (34). The various k's and m's are
thereby altered, but (17) is unaffected because of (29).

By the same transformation, the task of proving (37)
when $B = k_n$ (Case 2b) is reduced to that of proving it when
$B = k_j$ (Case 2a).
Note 5.

If $\ell = 2$, the indices $j+1$ and $n-1$ are identical, and
therefore (30) involves a factor $H_{n-1}^{ab}(x_{n-1} - x_n)$. Similarly
(31). Hence we must replace (33), still assuming that
$j > 0$, by

$$B = \min\left(\tfrac{1}{2} k_{n-1}, k_j, k_n, m_n\right) \qquad (64)$$

We observe that conditions (16) and (17) hold if m_{j+1}, m_n, k_j, k_n
are diminished by b and k_{n-1} is diminished by 2b, where

*Strictly we should also have substituted $i \to i-1$ for all
$i > j$; for clarity's sake we omitted to do so.

$0 \leqslant b < B$. The proof of (36) now proceeds as before.

In Case 1 of (37), we observe that both sides vanish if

$B = \frac{1}{2} k_{n-1}$. The proof in the other cases is unaffected

except for inessential changes in the formulas of Note 3.

If j = 0, (30) involves a factor $H_n^{ba}(x_n - x_0)$.

Therefore we replace (33) by

$$B = \min(k_1, k_{n-1}, \tfrac{1}{2} k_n, m_n) \qquad (65)$$

if $\ell > 2$, or by

$$B = \min(\tfrac{1}{2} k_1, \tfrac{1}{2} k_2, m_2) \qquad (66)$$

if $\ell = 2$ (in this case n = 2). The proof of (36) is again

unaltered. Both sides of (37) now vanish in all cases, so

that Note 3 is not needed.

Appendix I.

It is possible to modify this proof so as to follow

more closely the argument of [1] for n = 2. In Theorem IA,

one should replace (17) by

$$m_n \leqslant \min\left(m_j + \sum_0^{j-1} k_i, m_j + \sum_j^{n-1} k_i\right), \quad j = 1, \cdots, n-1 \qquad (67)$$

which is equivalent to the assertion that the shortest path

from P_0 to P_n, in the graphs of Fig. 5, is of length not

less than m_n. Thus the conclusion of Note 2 - particularly

Eq. (21) - still holds good.

We may now replace (26) by

$$A = \min\left[k_{j+1}, k_j, \tfrac{1}{2}\left(m_{j+1} - m_n + \sum_0^j k_i\right), \tfrac{1}{2}\left(m_{j+1} - m_n + \sum_{j+1}^{n-1} k_i\right)\right] \quad (68)$$

without disturbing the proof of (27). The proof of (28) is thus made easier, in that Case 3 becomes

$$A = \tfrac{1}{2}\left[m_{j+1} - m_n + \min\left(\sum_0^j k_i, \sum_{j+1}^{n-1} k_i\right)\right] \quad (69)$$

and (29) is replaced by

$$m_n = m_{j+1} + \min\left(\sum_0^j k_i, \sum_{j+1}^{n-1} k_i\right) \quad (70)$$

To prove (25) in the presence of (70), we first replace $F_n(x_n)$ and $F_n{}^*(x_n)$ by 1 for all x_n; this cannot increase $*g(0)$ because of (70). Then we renumber the indices as in Note 4, and proceed as in the text following (25) except that we now are entitled to put

$$A = \min(k_{j+1}, k_j) \quad (71)$$

Since Case 3 is now ruled out, the proof is complete, except for details the reader may supply.

The advantage of this version is that it bypasses the "simultaneous truncation" leading to Eqs. (30)-(37). The price paid is the more elaborate form of (67) as opposed to (17). However, the difficulties dealt with in Note 3 are unchanged.

Appendix II.

An analogue of Theorem I for discrete sets may be stated as follows:

<u>Theorem II</u>

Let $m_1, \cdots, m_n, k_1, \cdots, k_n$ be non-negative integers, satisfying

$$m_i + m_{i+1} + k_i \equiv 1 \pmod 2 \qquad (72)$$

for $1 \leq i \leq n$. (As before we understand m_{n+1} to mean m_1 , etc.) For each $i = 1, \ldots, n$ let ϕ_i be a set of m_i odd (even) integers, m being even (odd), and let ψ_i be a set of k_i odd (even) integers, k_i being even (odd). Let ϕ_i^* be the set $\{ m_i - 1, m_i - 3, \cdots, -m_i + 1 \}$, and let ψ_i^* be the set $\{ k_i - 1, k_i - 3, \cdots, -k_i + 1 \}$. Let I be the number of n-tuplets $\{ x_1, \cdots, x_n \}$ satisfying

$$x_i \, \varepsilon \, \phi_i \, , \; x_i - x_{i+1} \, \varepsilon \, \psi_i \qquad (73)$$

for each i; and let $^*I$ be the number of n-tuplets satisfying

$$x_i \, \varepsilon \, \phi_i^* \, , \; x_i - x_{i+1} \, \varepsilon \, \psi_i^* \qquad (74)$$

for each i. Then

$$I \leq {}^*I \qquad (75)$$

(The restriction to odd or even integers is a device to allow symmetrization of both even and odd sets. Condition (72) is needed to make the sets match. Doing without this condition would entail complications we prefer not to discuss.)

We sketch a proof of this theorem, parallel to the one given in the text for Theorem I. First, an operation of truncation must be defined, by which, for any non-negative integer a and any set ϕ of at least a integers, the set ϕ^a is obtained by deleting the top a members of ϕ; and for non-negative a, b and set Ψ of at least a+b integers, Ψ^{ab} is obtained by deleting the top a and bottom b members of Ψ. This leads to definitions of $I(a_1, \ldots, a_n)$ and $*I(a_1, \ldots, a_n)$ like (14) and (15). Then we define the operator Δ_a such that for any function f(a) we have

$$\Delta_a f(a) = f(a) - f(a+1) \; . \tag{76}$$

We can now state the analogue to Eqs. (16)-(18):

Theorem IIA

Suppose that

$$k_i > 0 \quad \text{for} \quad i = j \text{ or } n \tag{77}$$

$$k_i > 1 \quad \text{for} \quad i = 1, \ldots, j-1 \tag{78}$$

$$1 \leq m_n \leq m_i \quad \text{for} \quad i = 1, \ldots, n-1 . \tag{79}$$

Then

$$\prod_1^j \Delta_{a_i} I(a_1, \ldots, a_n)_0 \leq \prod_1^j \Delta_{a_i} *I(a_1, \ldots, a_n)_0 \; . \tag{80}$$

(Note that (77) and (78) are required to ensure that the operations in (80) are defined.)

We shall merely indicate how the proof of this theorem differs from that of Theorem IA.

The reasoning of Note 1 is unimpaired. The sets T_1 and T_2 do not overlap at all.

The reasoning of Note 2 is unchanged except that in Lemma IIA we obtain $p_1 + p_2 - 1$, $p_1 + k_i - 1$, $p_2 + k_i - 1$ instead of $p_1 + p_2$, $p_1 + k_i$, $p_2 + k_i$. Therefore, to maintain the invariance of "shortest path length" under the graphical operations, we must take the "length" of a line to be 1 <u>less</u> than the number of elements in the corresponding set. Conditions (77)-(79) prevent the occurrence of negative "lengths", since $j \equiv n - \ell = n - 1$ in this part of the proof.

Equation (27) is replaced by

$$\Delta_a \, g^{(a)} \le \Delta_a^* \, g^{(a)} \tag{81}$$

which follows by induction on ℓ provided that a is small enough to satisfy

$$k_{j+1} - a > 0, \quad k_j - a > 1, \quad m_{j+1} - a \ge m_n . \tag{82}$$

But we need only prove (81) for $a \le A$; therefore, instead of (26), we have

$$A = \min \left(k_{j+1}, \, k_j - 1, \, m_{j+1} - m_n + 1 \right) . \tag{83}$$

This leads, as before, to three cases in the proof of (28).

In Case 1 ($A = k_{j+1}$) both sides of (28) vanish.

In Case 2 ($A = k_j - 1$) the operator Δ_{a_j} can be dropped from both sides of (28), since the set ψ_j^{IA} is empty. Then, since ψ_i^{OA} has only one member, the variable x_j can be eliminated and we arrive at formulas like (55) and (59) of Note 3. The arguments of Note 3 are otherwise unchanged.

In Case 3 ($A = m_{j+1} - m_n + 1$) we are faced with the problem of proving (25) with the condition

$$m_{j+1} = m_n - 1 \tag{84}$$

which violates (79). Actually, this condition makes the proof easier. We introduce an additional induction on m_n. If $m_n = 1$, the proof is completed for Case 3 by noting that (84) makes both sides of (25) vanish. If $m_n > 1$, we treat Case 3 by renumbering the indices according to Note 4; condition (79) is thus restored and (25) becomes an instance of the theorem with $m_n \to m_n - 1$. Note that m_n remains fixed throughout the double induction on ℓ and j elsewhere in the proof.

The simultaneous truncation described in Eqs. (30)-(37) is thus bypassed in the discrete version of the theorem without the need to introduce the more complicated condition (67).

By setting j = 0 in Theorem IIA, we obtain Theorem II except for the case where some m_i or k_i vanishes; but in this case we have $I = {}^*I = 0$.

Appendix III.

We conjecture the following generalization of (8).

Let $\{ u_{i\gamma} \}$ be an array of real numbers, where

$$i = 1, \cdots, n \qquad (85)$$

$$\gamma = 1, \cdots, \Gamma \qquad (86)$$

Let $H_1, \cdots, H_\Gamma$ be functions from the interval $(-\infty, \infty)$ to the interval $[0, \infty)$. Let $H_1^*, \cdots, H_\Gamma^*$ be their symmetric decreasing rearrangements. Then

$$\int \prod_i (dx_i) \prod_\gamma H_\gamma \left(\sum_i u_{i\gamma} x_i \right) \leq \int \prod_i (dx_i) \prod_\gamma H_\gamma^* \left(\sum_i u_{i\gamma} x_i \right). \quad (87)$$

To obtain Theorem I, set $\Gamma = 2n$, $u_{ij} = \delta_{i\gamma} + \delta_{i, \gamma-n} - \delta_{i-1, \gamma-n} - \delta_{i1} \delta_{\gamma\Gamma}$.

We have been unable to prove even the following two special cases of (87):

$$\int \prod_i \left[H_i(x_i - x_{i+1}) \, H_{i+n}(x_i + x_{i+1}) \, dx_i \right]$$

$$\leq \int \prod_i \left[H_i^*(x_i - x_{i+1}) \, H_{i+n}^*(x_i + x_{i+1}) \, dx_i \right] \quad (88)$$

and

$$\int \prod_{i=1}^{n-1} \left[dx_i \prod_{j=i+1}^{n} H_{ij}(x_i - x_j) \right] \le \int \prod_{i=1}^{n-1} \left[dx_i \prod_{j=i+1}^{n} H_{ij}^*(x_i - x_j) \right]. \tag{5.4}$$

The difficulty arises in Lemma IB, which breaks down when the same function is truncated twice from the same end.

FIGURE CAPTIONS

Figure 1.   Examples of truncation.

Each example depicts a subset of the real line. If F is the characteristic function of the set $\left[r_1, r_2\right] \cup \left[r_3, r_4\right] \cup \left[r_5, r_6\right]$ depicted by (a), then F^a is the characteristic function of the set depicted by (b), if $a < r_6 - r_5$, or by (c), if $r_6 - r_5 < a < r_6 - r_5 + r_4 - r_3$. Likewise, if H corresponds to (d), then H^{ab} corresponds to (e) if $b > r_2 - r_1$ and $a + b < r_4 - r_3 + r_2 - r_1$. Note that a affects only the "top" of example (e) and (b) only the "bottom".

Figure 2.   Truncation and rearrangement.

Examples (a), (b), (c), (d), (e) depict the functions F, F^a, F^*, F^{*a}, and F^{a*} respectively. Note that (d) and (e) are related by Eq. (13).

Figure 3.   Case 1 of Lemma IB.

Examples (a) – (e) depict $g(x_i)$, $F_i(x_i)$, $g(x_i)F_i(x_i)$, $H_i(y)$ and $G(x_{i+i})$. Here $q_1 > q_2$ and hence $q' = q_2$. Since $g(q') = 0$, a simultaneous infinitesimal truncation of (a) and (b) from above will have no effect on (c). Therefore G vanishes unless the point $x_{i+i} + q_3$ lies in (c). Thus (e) is the same as (c), displaced by q_3. Clearly (e) is unaffected by a truncation of (d) from below, since q_3 is unchanged.

Figure 4.   Case 2 of Lemma IB.

Same five examples as in Figure 3. Here $g(q') = 1$, so that a truncation of (a) and (b) also truncates (c). Therefore (e) consists of T_1, for which

$q' - x_{i+i}$ lies in (d), as well as T_2, for which $x_{i+i} + q_3$ lies in (c). Truncation of (d) at $y = t$ results in truncation of (e) at $x_i + 1 = q' - t$.
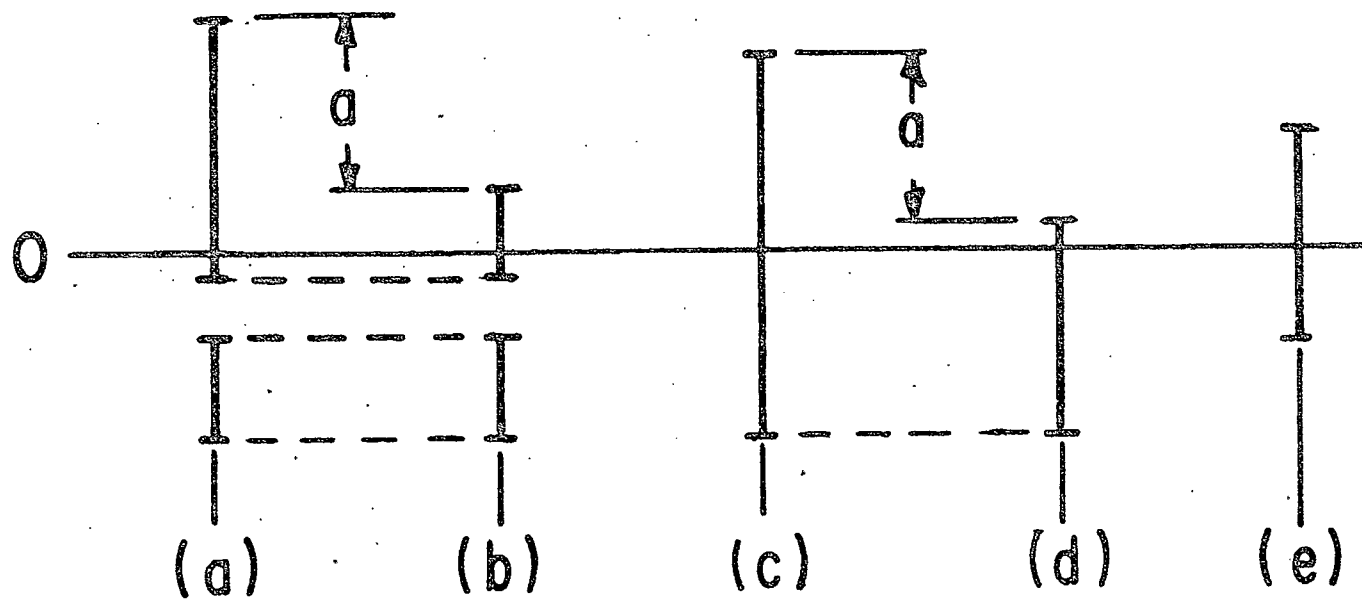
Figure 5. The first five graphs described in Note 2.

Example (a) shows the original graph (points P_4 to P_{n-1} not shown). In (b), the variable x, has been eliminated by the use of Lemma II. In (c), the superfluous line $P_0 P_2$ has been dropped. (We write $\overline{m}_2$ for min $(m_2, m_1 + k_1)$.) In (d) we have eliminated x_2, and in (e) we have again dropped superfluous lines, putting $\overline{m}_3 = $ min $(m_3, \overline{m}_2 + k_2)$, $\overline{m}_n = $ min $(k_n + m_1 > k_n + k_1 + \overline{m}_2)$. Note that if any graph con-tains a path from P_0 to P_n with total length L, then all the other graphs contain some such path with length $L \leq L$.

Figure 6. Graphical treatment of the right side of (28) according to Note 3. The original graph for $*g(a)$ is shown in (a). In (b) we have used Lemma II to eliminate the variable x_i. In (c) we have replaced a by $A = k_i$. In (d) we have dropped superfluous lines; $\overline{m} = $ min $(m_i, m_{i+1} - A)$.
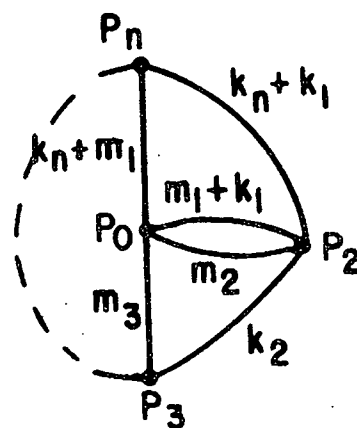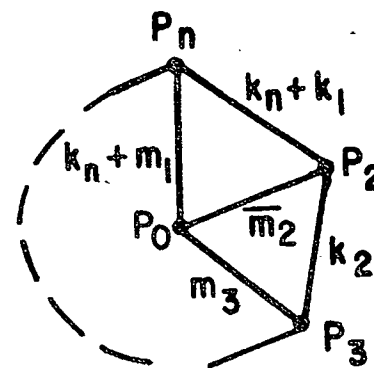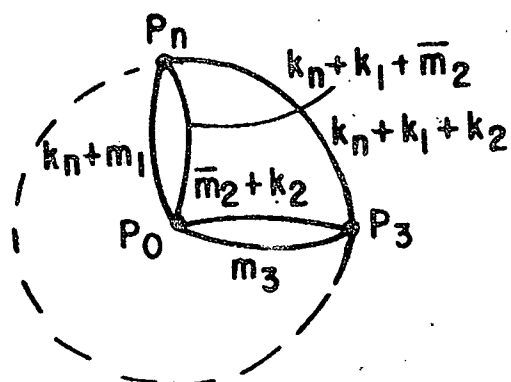
r_6

r_5 ⫶ $r_6 - a$

r_4

r_3 $r_4 - [a - (r_6 - r_5)]$

r_4

$r_4 - a$

r_2 $r_3 + [b - (r_2 - r_1)]$

r_3

r_1

r_2

r_1
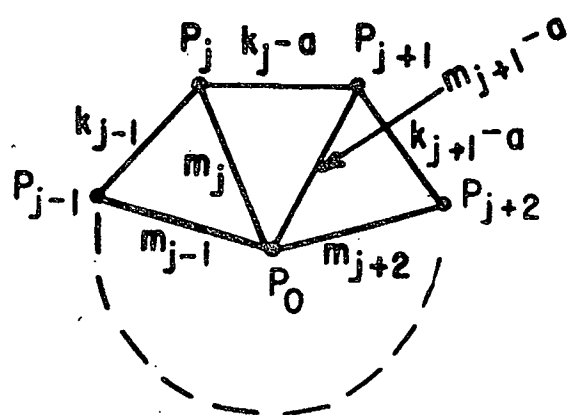
(a) (b) (c) (d) (e)

O

a

a

(a) (b) (c) (d) (e)

(a)   (b)   (c)

(e)   (f)

(a)

(b)

(c)

(d)