A new Reliable Numerical Algorithm Based on the First Kind of Bessel Functions to Solve Prandtl–Blasius Laminar Viscous Flow over a Semi-Infinite Flat Plate

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In this paper, a new numerical algorithm is introduced to solve the Blasius equation, which is a third-order nonlinear ordinary differential equation arising in the problem of two-dimensional steady state laminar viscous flow over a semi-infinite flat plate. The proposed approach is based on the first kind of Bessel functions collocation method. The first kind of Bessel function is an infinite series, defined on $\mathbb{R}$ and is convergent for any $x \in \mathbb{R}$. In this work, we solve the problem on semi-infinite domain without any domain truncation, variable transformation basis functions or transformation of the domain of the problem to a finite domain. This method reduces the solution of a nonlinear problem to the solution of a system of nonlinear algebraic equations. To illustrate the reliability of this method, we compare the numerical results of the present method with some well-known results in order to show the applicability and efficiency of our method.

Key words: Blasius Equation; Bessel Functions; Collocation Method; Semi-Infinite; Nonlinear ODE.

Mathematics Subject Classification 2000: 65L10, 65L60, 34B15

1. Introduction

Many problems arising in science and engineering are defined in unbounded domains. While the spectral approximations for ordinary differential equations (ODEs) in bounded domains have achieved great success and popularity in recent years, spectral approximations for ODEs in unbounded domains have only received limited attention. Several spectral methods for treating unbounded domains problems have been proposed by different researchers: (i) Direct approaches, using functions such as sinc, Hermite and Laguerre polynomials that are orthogonal over the unbounded domains, were investigated by Parand et al. [1], Maday et al. [2], Funaro [3], Funaro and Kavian [4], and Guo and Shen [5 – 7]. (ii) Indirect approaches, e.g. Guo [8, 9], proposed a method that proceeds by mapping the original problem in an unbounded domain to a problem in a bounded domain, and then using suitable Jacobi polynomials to approximate the resulting problems. (iii) Another class of spectral methods is based on rational approximations. For example, Christov [10] and Boyd [11] developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [12] defined a new spectral basis, named rational Chebyshev functions, on the semi-infinite interval by mapping the Chebyshev polynomials. Recently, Gou et al. [13] proposed and analyzed a set of Legendre rational functions which are mutually orthogonal in ${L_\chi}^2(0,\infty)$. (iv) A further approach consists of replacing the infinite domain with $[-L,L]$ and the semi-infinite interval with $[0,L]$ by choosing L sufficiently large. This method is named domain truncation [14].

Boyd [15] applied pseudo-spectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre, and mapped Fourier sine method. Moreover, many researchers have investigated (scrutinized) on spectral methods, for example [16 – 19].

In this paper, we attempt to introduce a new method, based on Bessel functions for solving unbounded problems. Previously, Sahin et al. [20, 21] have used the

Bessel polynomials collocation for solving the general linear Fredholm integro-differential-difference equation and systems of linear Voltera integro equations, respectively. The *Bessel polynomials* are obtained from truncated series of the first kind of *Bessel functions*, $J_0(x)$, that will be described in Section 2. But in this paper, we apply the *Bessel functions* collocation method for solving the Blasius equation which is one of the basic equations of fluid dynamics. This equation describes the velocity profile of the fluid in the boundary layer theory on a half-infinite interval [22, 23].

The remainder of this paper is organized as follows: In Section 2, we describe the Blasius equation and transform it to a nonlinear ordinary differential equation and then explain some methods previously used to solve this equation. The basic information of the Bessel function and its properties is presented in Section 3. In Section 4, we describe how to approximate the functions, as well as the collocation algorithm. In Section 5, we apply the proposed algorithm to solve the Blasius equation and demonstrate the accuracy of our method. Finally in the last Section, we give concluding remarks.

2. The Blasius Problem

Boundary layer flow over a flat plate is governed by the continuity and the Navier–Stokes equations. The steady, laminar, incompressible, and two-dimensional boundary layer equations for continuity and momentum can be expressed by the following boundary value problem [24–26]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ \text{(continuity)}, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \ \text{(momentum)}. \tag{2}$$

The boundary conditions for the problem of flat plate are [26]:

$$\begin{aligned} &u(x,0) = v(x,0) = 0, \ u(0,y) = U_\infty, \\ &u(x,\infty) = U_\infty. \end{aligned} \tag{3}$$

Here, u and v are the velocity components in x and y directions, ν is the kinematic viscosity, and U_∞ is a constant free stream velocity.

The goal in the problem of flat plate is the shear at the plate, $\frac{\partial u}{\partial y}(x,0)$. Subsequently, the viscous drag on the plate can be calculated easily [26]. To deal with (1) and (2), it is convenient to use the following equations to simplify them to an ordinary differential equation [25, 27]:

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}, \tag{4}$$

$$\varphi = y\sqrt{U_\infty \nu x} f(\eta), \tag{5}$$

where f is a stream function such that $u = \frac{\partial \varphi}{\partial y}$ and $v = \frac{\partial \varphi}{\partial x}$ and can be expressed as [28, 29]

$$u = \frac{\partial \varphi}{\partial y} = U_\infty f', \tag{6}$$

$$v = -\frac{\partial \varphi}{\partial x} = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}\left(\eta f' - f\right). \tag{7}$$

So, by using the above formulas, the equation of continuity is satisfied identically and the second boundary layer flow equation reduces to [25, 30]

$$f''' + \frac{1}{2} f f'' = 0, \tag{8}$$

which is Blasius's differential equation.

The boundary conditions (3) collapse to the three conditions

$$f(0) = f'(0) = 0, \ f'(\infty) = 1, \tag{9}$$

which are sufficient for the third-order equation (8). The shear at the plate is given by [26]

$$\frac{\partial u}{\partial y}(x,0) = x^{-\frac{1}{2}} f''(0). \tag{10}$$

Therefore computation of $f''(0)$ is important in this problem.

In 1908, Blasius [31] solved the above equation using a series expansion method. He found the following solution for the problem:

$$f(\eta) = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2}, \tag{11}$$

where $A_0 = A_1 = 1$ and

$$A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1}, \ k \geq 2.$$

In (11), σ denotes the unknown $f''(0)$. Blasius evaluated σ by demonstrating another approximation of

$f(\eta)$ at large η. He then obtained the numerical result $\sigma = 0.332$ by means of matching two different approximations at a proper point [27]. Later, Howarth [32] solved the Blasius problem numerically and found $f''(0) = 0.332057$. Asaithambi [33] found this number correct to nine decimal positions as 0.332057336. In 2008, Boyd [34] reported 0.33205733621519630 as $f''(0)$ in the Blasius equation.

2.1. Proposed Methods for Solving Blasius Equations

In recent years, different methods have been used to solve the Blasius equation. Liao [35, 36] applied the homotopy analysis method (HAM) to give a totally analytical solution of the Blasius equation. He [37] obtained an analytic solution which is valid in the whole region by applying variational iteration method. Yu and Chen [38] solved the equation using the differential transformation method. Lin [39] proposed the parameter iteration method and obtained an approximate analytical solution of the Blasius equation. Wang [40] employed the Adomian decomposition method (ADM) to solve the famous Blasius equation numerically. Hashim [41] corrected the numerical solution of Wang and presented an improved numerical solution using the ADM Padé approach. Cortell [42] applied the Runge-Kutta algorithm for the high-order initial value problems; the classical Blasius flat-plate equation is one of them. Wazwaz [43] applied the modified decomposition method of Adomian for the analytical treatment of the Blasius equation and combined its solution with the diagonal Padé approximation to handle the boundary condition at infinity. Later, Wazwaz [44] combined the variational iteration method with the diagonal Padé approximation and obtained the solution of Blasius equation. Abbasbandy [45] proposed Adomian's decomposition method to solve the Blasius equation and compared his results with the homotopy perturbation method (HPM) and Howarth's numerical solution [24]. The authors of [46] proposed a simple approach using the homotopy analysis method (HAM) to obtain a totally analytical solution of the viscous fluid flow over a flat plate. They showed that the homotopy perturbation method is only a special case of the homotopy analysis method. Tajvidi et al. [47] used the modified rational Legendre tau method to solve the Blasius equation. Authors of [48 – 50] solved the Blasius equation using the collocation and tau methods based on spectral methods. Recently, authors of [51] solved this equation with the two-grid quasi-linearization method.

3. Bessel Functions and their Properties

In this section, we describe the first kind of Bessel functions and their properties which will be used to construct the Bessel functions collocation (BFC) method.

The Bessel equation of order n reads [52, 53]

$$x^2y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0 \quad \text{for } x \in (-\infty, \infty), \quad n \in \mathbb{R}. \tag{12}$$

A solution of this equation is [53]

$$\sum_{r=0}^{\infty} a_0 \frac{(-1)^r \Gamma(n+1)}{2^{2r} r! \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

for any value of a_0. $\Gamma(\lambda)$ is the gamma function defined as

$$\Gamma(\lambda) = \int_0^{\infty} \mathrm{e}^{-t} t^{\lambda-1} \, \mathrm{d}t \,.$$

Let us choose $a_0 = \frac{1}{2^n \Gamma(n+1)}$. Accordingly, we obtain the solution which we shall denote by $J_n(x)$ and call it the Bessel function of the first kind of order n:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}. \tag{13}$$

Series (13) is convergent for all $-\infty < x < \infty$. Now, we express some properties of the first kind of Bessel functions. Some recursive relations of derivation are as follows [53]:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n J_n(x)) = x^n J_{n-1}(x)\,,$$
$$J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)\,,$$
$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)\,.$$

The integration of the first kind of Bessel functions is defined by the relation [53]

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi - x\sin(\phi)) \, \mathrm{d}\phi \,.$$

The generating function for $J_n(x)$ is given by [53]

$$\exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} t^n J_n(x)\,.$$

The behaviour of Bessel functions for small and large arguments are [54]

$$J_n(x) \approx \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n \quad \text{for } 0 < x \ll \sqrt{n-1},$$
$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad \text{for } x \gg \left|n^2 - \frac{1}{4}\right|.$$

4. Function Approximation

Suppose that $\mathcal{H} = L^2(\Gamma)$, where $\Gamma = (-\infty, +\infty)$, let $\{J_0(x), J_1(x), ..., J_n(x)\} \subset \mathcal{H}$ be the set of Bessel functions and suppose that

$$\mathbf{J} = \operatorname{span}\{J_0(x), J_1(x), \ldots, J_n(x)\}. \tag{14}$$

Since $\mathcal{H}$ is a Hilbert space and $\mathbf{J}$ is the finite-dimensional subspace, dim $\mathbf{J} = n+1$, so $\mathbf{J}$ is a closed subspace of $\mathcal{H}$, and therefore $\mathbf{J}$ is a complete subspace of $\mathcal{H}$ [55]. Let f be an arbitrary element in $\mathcal{H}$, thus f has a unique best approximation from $\mathbf{J}$, say $\widehat{j} \in \mathbf{J}$, that is [55]

$$\exists\, \widehat{j} \in \mathbf{J};\ \forall\, j \in \mathbf{J},\ \|f - \widehat{j}\| \le \|f - j\|, \tag{15}$$

where $\|f\| = \langle f, f\rangle^{1/2}$ and $\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)g(t)\,\mathrm{d}t$.

Definition 1. (Direct sum ($\oplus$)): A vector space $\mathcal{H}$ is said to be the direct sum of two subspaces Y and Z of $\mathcal{H}$, written as [55]

$$\mathcal{H} = Y \oplus Z, \tag{16}$$

if each $h \in \mathcal{H}$ has a unique representation

$$h = y + z. \tag{17}$$

Then Z is called an algebraic complement of Y in $\mathcal{H}$ and vice versa, and Y, Z is called a complementary pair of subspaces in $\mathcal{H}$ [55].

Definition 2. Let $\mathcal{H}$ be an Hilbert space and Y be any closed subspace of $\mathcal{H}$. $Y^{\perp}$ is defined the *orthogonal complement* [55]:

$$Y^{\perp} = \{z \in \mathcal{H} |\ z \perp Y\}. \tag{18}$$

Lemma 1. *Let Y be any closed subspace of a Hilbert space $\mathcal{H}$. Then*

$$\mathcal{H} = Y \oplus Y^{\perp}. \tag{19}$$

Proof. See [55].

Now, by using (14) and (15), we can say

$$\mathcal{H} = \mathbf{J} \oplus Z, \tag{20}$$

where $Z = \mathbf{J}^{\perp}$, so that for each $x \in \mathcal{H}$,

$$h = j + z, \tag{21}$$

where $z = h - j \perp j$, hence $\langle h - j, j\rangle = 0$. We have $j \in \mathbf{J}$, therefore [55]

$$j = \sum_{k=0}^{n} a_k J_k(x), \tag{22}$$

and $h - j \perp j$ gives the n conditions

$$\langle J_m(x), h - j\rangle = \left\langle J_m(x), h - \sum_{k=0}^{n} a_k J_k(x)\right\rangle = 0, \tag{23}$$

that is

$$\langle J_m(x), h\rangle = \sum_{k=0}^{n} \bar{a}_k \langle J_m(x), J_k(x)\rangle, \quad m = 0, 1, \ldots, n. \tag{24}$$

This is a nonhomogeneous system of $n+1$ linear equations with $n+1$ unknown coefficients $\{\bar{a}_k\}_{k=0}^{n}$. The determinant of the coefficients is [55]

$$G(J_0(x), J_1(x), \ldots, J_n(x)) = \begin{vmatrix} \langle J_0(x), J_0(x)\rangle & \langle J_0(x), J_1(x)\rangle & \ldots & \langle J_0(x), J_n(x)\rangle \\ \langle J_1(x), J_0(x)\rangle & \langle J_1(x), J_1(x)\rangle & \ldots & \langle J_1(x), J_n(x)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle J_n(x), J_0(x)\rangle & \langle J_n(x), J_1(x)\rangle & \ldots & \langle J_n(x), J_n(x)\rangle \end{vmatrix}. \tag{25}$$

Since $\mathbf{J}$ exists and is unique, that system has a unique solution. Hence $G(J_0(x), J_1(x), \ldots, J_n(x))$ must be different from 0. The determinant $G(J_0(x), J_1(x), \ldots, J_n(x))$ is called the Gram determinant of $J_0(x), J_1(x), \ldots, J_n(x)$. □

Theorem 1. *Suppose that $\mathcal{H}$ is a Hilbert space and Y a closed subspace of $\mathcal{H}$ such that* dim$Y < \infty$, *and $\{y_1, y_2, \ldots, y_n\}$ is any basis for Y. Let h be an arbitrary element in $\mathcal{H}$ and y_0 be the unique best approximation to h from Y. Then [55]*

$$\|h - y_0\|^2 = \frac{G(h, y_1, y_2, \ldots, y_n)}{G(y_1, y_2, \ldots, y_n)}, \tag{26}$$

Proof. See [55]. □

4.1. Collocation Algorithm

To obtain the spectral coefficients $\{a_i\}_{i=0}^{n}$ in series (22) and an approach of $u(x)$ via (22), we employ the collocation algorithm. In this algorithm, to solve equation $Lu(x) = f(x)$, where L is the operator of the differential or integral equation, we do:

BEGIN

1. Construct the following series from (22)

$$u_N(x) = \sum_{i=0}^{N} a_i J_i(x)\,.$$

2. Insert the constructed series of Step 1 into equation $Lu = f(x)$.
3. Construct the residual function as

$$\mathrm{Res}(x; a_0, a_1, \ldots, a_n) = Lu_N(x) - f(x). \qquad (27)$$

   Now we have $N+1$ unknown $\{a_n\}_{n=0}^{N}$. To obtain this unknown coefficients, we need $N+1$ equations.
4. By choice of $N+1$ points x_i, $i = 0, 1, \ldots, N$, as $N+1$ in the domain of the problem as collocation points and substituting them in $\mathrm{Res}(x; a_0, a_1, \ldots, a_n) = Lu_N(x) - f(x)$, we construct a system containing $N+1$ equations.
5. Solve this system of equations by Newton's method and gain the a_n, $n = 0, 1, \ldots, N$.

END.

In Step 5, it is worth to note that solving a system of nonlinear equations even by applying Newton's method is very common. The main difficulty with such a system is to find out how we can choose an initial guess to handle the method; in other words, how many solutions the system of nonlinear equations admits. We think the best way to discover the proper initial guess (or initial guesses) is to solve the system analytically for a very small N (by means of symbolic software programs such as Mathematica or Maple) and, then, we can find proper initial guesses and, particularly, the multiplicity of solutions of such system. This action has been done by starting from proper initial guesses with the maximum number of ten iterations.

The collocation method has been used increasingly for solving differential or integro equations [1, 48, 49]. Also it is very useful in providing highly accurate solutions for differential equations. This method is easy to implement and yields the desired accuracy. This method reduces the solution of a nonlinear problem to the solution of a system of nonlinear algebraic equations. The important concerns of the collocation approach are the selection of the basis functions and collocation points. The basis functions have three different properties: easy computation, rapid convergence, and completeness, which means that any solution can be represented to arbitrary high accuracy by taking the truncation N to be sufficiently large. We used the first kind of Bessel functions as the basis functions and roots of shifted (rational) Chebyshev function into the domain of $u(x)$ as collocation points. For more explanations about Chebyshev functions, see [14].

5. Solving the Blasius Equation

In this section, we apply the Bessel functions collocation (BFC) algorithm described in Section 4 to solve the Blasius equation (8), subject to the conditions (9). Since for any n, $J_n(\eta)$ are differentiable at the point $\eta = 0$, we can simply satisfy the conditions of Blasius equation (9). Now, since $J_0(0) = 1$ and $\frac{\mathrm{d}}{\mathrm{d}\eta} J_1(0) = 1$, we discard both of them. Add $\frac{\eta^2}{\eta+\ell}$ to the series (22), where ℓ is constant. Thus we approximate $f(\eta)$ in (8) with boundary conditions (9) via $\hat{f}_N$ as follows:

$$\hat{f}_N(\eta) = \frac{\eta^2}{\eta+\ell} + \sum_{k=2}^{n+2} a_k\, J_k(\eta)\,, \qquad (28)$$

hence $\hat{f}_N(0) = 0$, $\frac{\mathrm{d}}{\mathrm{d}\eta}\hat{f}_N(0) = 0$, and $\frac{\mathrm{d}}{\mathrm{d}\eta}\hat{f}_N(\eta) = 1$ when η tends to infinity, therefore, the third boundary condition (9) is satisfied. By constructing $\mathrm{Res}(\eta)$ via (8) as

$$\mathrm{Res}(\eta) = \frac{\mathrm{d}^3}{\mathrm{d}\eta^3}\hat{f}_N(\eta) + \frac{1}{2}\hat{f}_N(\eta)\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\hat{f}_N(\eta)\,, \qquad (29)$$

we can apply the BFC algorithm of Section 4 to obtain the spectral coefficients as follows:

BEGIN

1. Constructing the series (28).
2. Inserting the constructed series of Step 1 into equation (8).

Table 1. Comparison of the convergence rate and value of $f''(0)$ between BFC method and results of Liao [36] and Parand and Taghavi [48].

N	ℓ	Present method	[36]	[48]
10	4.2	0.33205788991564789	0.32992	0.33205823
15	4.1	0.33205733482212538	0.33164	0.33205724
20	4.2	0.33205733621516374	0.33204	0.33205732
25	3.87	0.33205733621519542	0.33206	0.33205733
Boyd [34]		0.33205733621519630		

Table 2. Comparison between values of the obtained $f(\eta)$ by the present method with $N = 25$, $\ell = 3.87$, and results of [32, 42, 48]; representation of the residual function.

η	Present method	[32]	[42]	[48]	Residual
0.0	0.000000000000	0.00000	0.00000	0.0000000	0
1.0	0.165571725847	0.16557	0.16557	0.1655724	$1.283465 \cdot 10^{-10}$
2.0	0.650024370165	0.65003	0.65003	0.6500351	$7.399958 \cdot 10^{-10}$
3.0	1.396808231342	1.39682	1.39682	1.3968223	$2.021690 \cdot 10^{-11}$
4.0	2.305746419194	2.30576	2.30576	2.3057618	$6.908179 \cdot 10^{-10}$
5.0	3.283273666139	3.28329	3.28330	3.2832910	$6.520093 \cdot 10^{-10}$
6.0	4.279620923737	4.27964	4.27965	4.2796435	$1.027459 \cdot 10^{-10}$
7.0	5.279238812489	5.27926	5.27927	5.2792684	$3.391439 \cdot 10^{-10}$
8.0	6.279213433045	6.27923	6.27923	6.2792336	$5.706506 \cdot 10^{-09}$

3. Constructing the residual function (29):
   Now we have $N+1$ unknown $\{a_n\}_{n=0}^{N}$. To obtain these unknown coefficients, we need $N+1$ equations.
4. Choosing roots of order $N+1$ rational Chebyshev functions as $N+1$ collocation points.
5. Substituting the collocation points in $\mathrm{Res}(\eta; a_0, a_1, \ldots, a_n) = 0$, and constructing a system containing $N+1$ equations.
6. Solving this system of equations by Newton's method and gaining the a_n, $n = 0, 1, \ldots, N$.

END.

Now, we have an approximation function $\hat{f}_N(\eta)$ for $f(\eta)$.

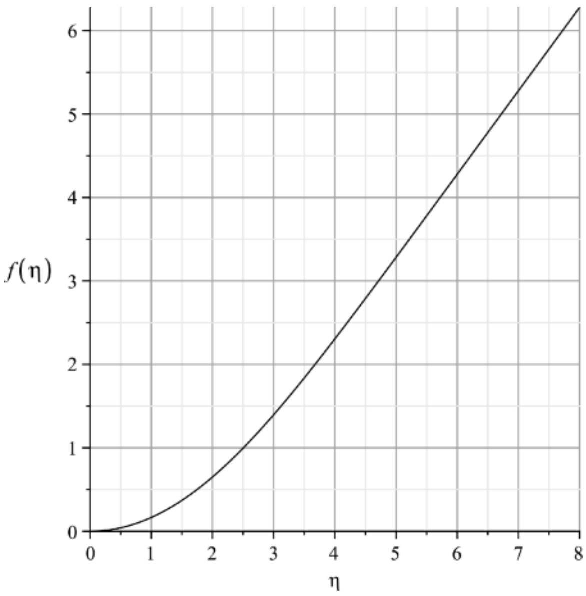

Fig. 1. Graph of the approximation function for $f(\eta)$ of Blasius equation solution for $N = 25$, $\ell = 3.87$.

As mentioned before, in the Blasius equation, $f''(0)$ plays an important role, and Asaithambi [33] found the value of this point to fourteen decimal positions as 0.33205733621519. In Table 1, we represent values of $f''(0)$ by different orders of approximation in the BFC method and compare the convergency of them with the results of Liao [36] and Parand and Taghavi [48]. Taking this table into account, the proposed method leads to a more accurate solution with high convergence by increasing N. The values of f, f', and f'' of the Blasius equation obtained by the BFC method with $N = 25$ are

Table 3. Comparison between values of obtained $f'(\eta)$ by the present method with $N = 25$, $\ell = 3.87$, and results of [32, 42, 48].

η	Present method	[32]	[42]	[48]
0.0	0.000000000000	0.00000	0.00000	0.0000000
1.0	0.329780031374	0.32979	0.32978	0.3297963
2.0	0.629765736721	0.62977	0.62977	0.6297763
3.0	0.846044443911	0.84605	0.84605	0.8460595
4.0	0.955518230061	0.95552	0.95552	0.9555236
5.0	0.991541900402	0.99155	0.99155	0.9915546
6.0	0.998972872665	0.99898	0.99898	0.9989817
7.0	0.999921604384	0.99992	0.99993	0.9999236
8.0	0.999996274776	1.00000	1.00000	1.0000000

Table 4. Comparison between values of the obtained $f''(\eta)$ by the present method with $N = 25$, $\ell = 3.87$, and results of [32, 42, 48].

η	Present method	[32]	[42]	[48]
0.0	0.332057336215	0.33206	0.33206	0.3320571
1.0	0.323007116854	0.32301	0.32301	0.3230136
2.0	0.266751545647	0.26675	0.26675	0.2667557
3.0	0.161360319408	0.16136	0.16136	0.1613637
4.0	0.064234120940	0.06424	0.06423	0.0642411
5.0	0.015906798564	0.01591	0.01591	0.0159134
6.0	0.002402039794	0.00240	0.00240	0.0024016
7.0	0.000220169901	0.00022	0.00022	0.0002234
8.0	0.000012241135	0.00001	0.00001	0.0000100

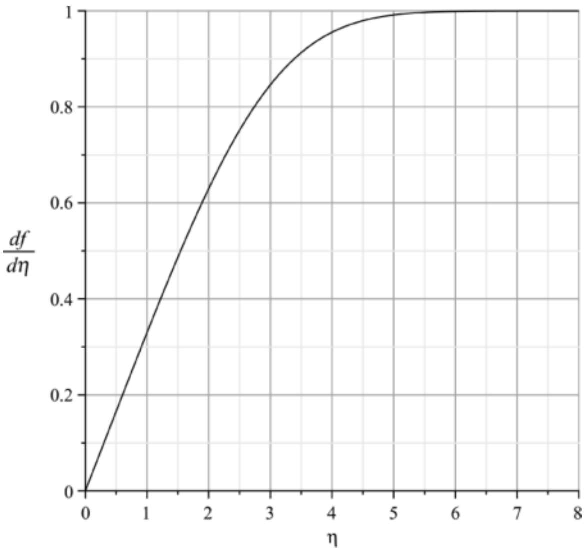

Fig. 2. Graph of the approximation of $\frac{d}{d\eta}f$ of Blasius equation solution for $N = 25,\ \ell = 3.87$.

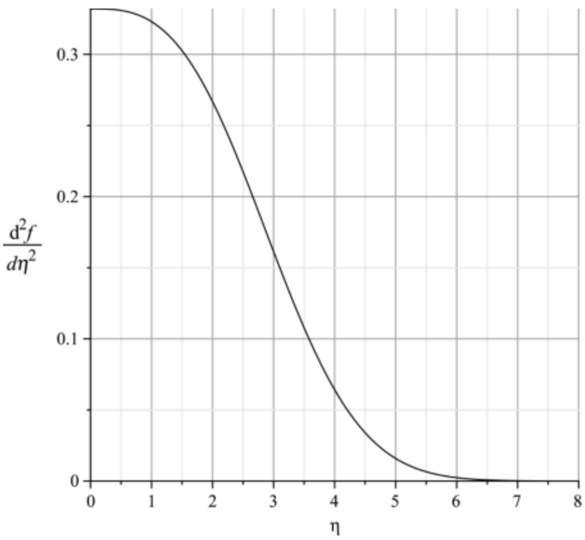

Fig. 3. Graph of the approximation of $\frac{d^2}{d\eta^2}f$ of Blasius equation solution for $N = 25,\ \ell = 3.87$.

shown in Tables 2 – 4, respectively, and are compared with some established results [32, 42, 48]. The plots of the Blasius functions $f(\eta)$, $f'(\eta)$, and $f''(\eta)$, are shown in Figures 1 – 3. Furthermore, in order to assess the rate of accuracy of our method, we plot the logarithmic graph of the residual function for solving Blasius equation by $N = 25$ in Figure 4. Also, to show the

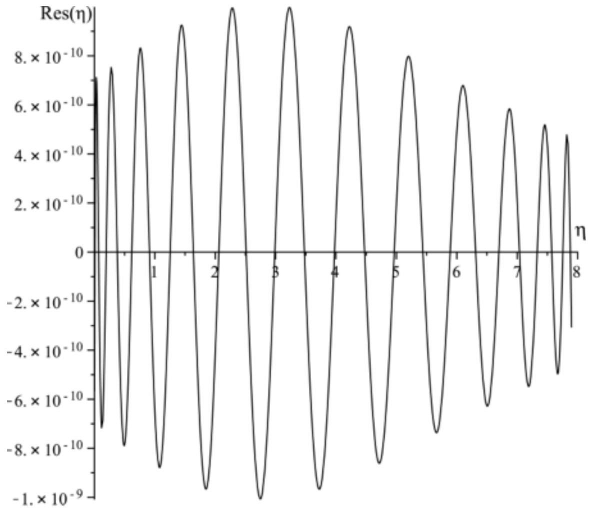

Fig. 4. Logarithmic graph of the residual function of BFC method to solve Blasius equation for $N = 25,\ \ell = 3.87$.

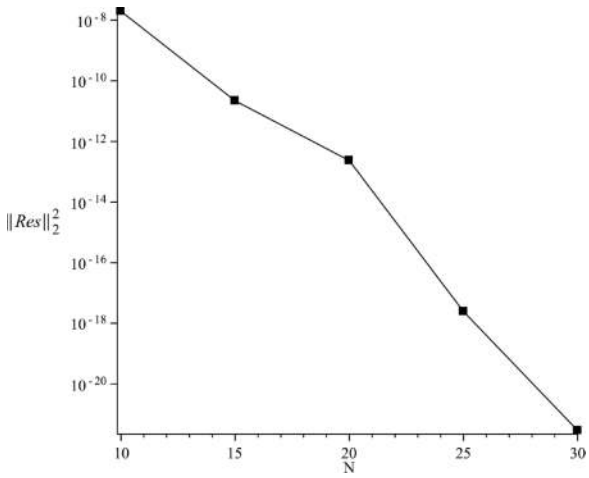

Fig. 5. Logarithmic graph of $\|\mathrm{Res}(\eta)\|_2^2$ of BFC method to solve Blasius equation for $\ell = 3.87$ and several N.

convergence of proposed method, we present the graph of $\|\mathrm{Res}\|_2^2$ for several N in Figure 5. This graph shows that, by increasing N (number of collocation points) the residual function tends to zero, and shows the convergence rate of the proposed method.

6. Conclusions

In this paper, we have applied Bessel functions collocation method for solving the Blasius equation which occurs in the study of laminar boundary-layer problem for Newtonian fluids. In the Blasius equation, the second derivative at zero is an important point of the function, so we have computed $f''(0)$ and have compared it with other methods. The results of the comparison in Table 1 show that the convergence rate of our method is better than of other methods. For $N = 25$, we achieve fourteen decimal positions for $f''(0)$, whereas in other methods eight decimal positions are obtained for $N = 30$. In Table 2-4, we present the values of $f(\eta)$, $f'(\eta)$, and $f''(\eta)$ more accurate. Also, the accuracy and convergence of this method is confirmed by the figures of the residual function and $\|\text{Res}(\eta)\|_2^2$ of this method. Figure 5 shows that by increasing N the error tends to zero. The given results suggest that using the present approach leads to acceptable results in comparison with different approximation methods. Finally, we note that the proposed method can be applied to a large class of nonlinear and linear partial differential equations [56–58], ordinary differential equations [59], and integral equations [60, 61] in finite or infinite domain.

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