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A new skew logistic distribution: Properties and applications

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Abstract. Following the methodology of Azzalini, researchers have developed skew logistic distribution and studied its properties. The cumulative distribution function in their case is not explicit and therefore numerical methods are employed for estimation of parameters. In this paper, we develop a new skew logistic distribution based on the methodology of Fernández and Steel and derive its cumulative distribution function and also the characteristic function. For estimating the parameters, Method of Moments, Modified Method of Moment and Maximum likelihood estimation are used. With the help of simulation study, for different sample sizes, the parameters are estimated and their consistency was verified through Box Plot. We also proposed a regression model in which probability of occurrence of an event is derived from our proposed new skew logistic distribution. Further, proposed model fitted to a well studied lean body mass of Australian athlete data and compared with other available competing distributions.

1 Introduction

Simon's (1955) work was the first to recognize that symmetric probability distributions do not properly represent the real data. He realized the importance of skew distributions and their applications in sociology, economics, and in many biological phenomenons. Of late, researchers are searching for a class of distributions which can suitably represent the asymmetry inherent in the data. Azzalini (1985), developed a new class of density functions by introducing a shape parameter λ, such that when $\lambda = 0$, it corresponds to standard normal density. Further, Azzalini and Dalla Valle (1996), extended it to the multivariate case. Following Azzalini (1985), many researchers introduced different skew distributions like skew-Cauchy distribution (Arnold and Beaver (2000)), skew logistic distribution (Wahed and Ali (2001)), skew Student's t distribution (Jones and Faddy (2003)). Lane (2004) fitted the existing skew distributions to insurance claims data. For more applications of skew distributions see Ma and Genton (2004).

Fernández, Osiewalski and Steel (1995) developed an alternate procedure for generating skew distributions. They defined a family of ν-spherical distributions

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and as an important special case, defined a class of distributions which allow modeling skewness. Generalising this, Fernández and Steel (1998) developed skew distributions by introducing skewness into any unimodal and symmetric distribution by using inverse scaling of the probability density function on both sides of the mode. This inverse scaling does not affect the unimodality but controls the tail behaviour. They applied this procedure to obtain skew Student's t distribution, which displays both flexible tails and possible skewness each entirely controlled by separate scale parameters. Later on, Ayebo and Kozubowski (2003) studied a class of skew continuous distributions on the real line arising from symmetric exponential power laws by incorporating inverse scale factors into positive and negative orthants. In this class of distributions, skew and symmetric Laplace and normal laws are included as special cases. They also studied the main properties of skew exponential power laws and derived maximum likelihood estimates and discussed their applications to finance theory.

Recently, Huang and Chen (2007) investigated the generalized skew symmetric distributions by introducing a skew function in place of cumulative distribution function (CDF), $F(\cdot)$, where a skew function $G(\cdot)$ is Lebesgue measurable and satisfies $0 \leq G(x) \leq 1$, $G(x) + G(-x) = 1$ almost everywhere in $\Re$. Chakraborty, Hazarika and Ali (2012) introduced a skew function and developed a new skew logistic distribution.

In mathematical terms Azzalini (1985) showed that if f is a density function symmetric about 0, and G an absolutely continuous distribution function such that G' is symmetric about 0, then

$$g(x|\lambda) = 2f(x)G(\lambda x), \qquad -\infty < x < \infty \tag{1.1}$$

is a density function for any real λ. He showed that, if f is standard normal density and G its distribution function, then $g(x|\lambda)$ is a skew normal random variable with parameter λ. He studied the properties of this distribution and derived the maximum likelihood estimates.

Following Azzalini (1985), Wahed and Ali (2001) developed a skew logistic distribution by taking $f(x)$ to be logistic density function and $G(x)$ as its c.d.f. and discussed some of its important characteristics. They observed that the CDF of their skew logistic distribution with skew parameter, α, cannot be evaluated in a closed form unless $\alpha = 0$. They further observed that the mean and variance have not in closed form but are finite and can be computed numerically for different values of α.

Pewsey (2009) considered various issues of inference for the skew normal distribution and explained why the direct parametrization should not be used as a general basis of estimation.

Arellano-Valle and Azzalani (2013) commented that "a problematic aspect of this kind of formulation is that the parameters are not related to the moments or cumulants in a simple way. The location parameter does not correspond to any quantity traditionally used to quantify location such as the mean or median, neither it is related to them in a simple form, a fact which affects interpretation of this parameter."

Gupta, Chang and Huang (2002) studied the models in which $f(\cdot)$ is taken to be the p.d.f. from one of the following distributions: normal, Student's t, Cauchy, Laplace, logistic, and uniform distribution, and $G(\cdot)$ is a distribution function such that G is symmetric about 0.

Nadarajah and Kotz (2003) considered the models with $f(\cdot)$ as a normal p.d.f. with zero mean, while $G(\cdot)$ is from one of the above continuous symmetric distributions. Multivariate skew-symmetric distributions have also been studied by Gupta, Chang and Huang (2002) and Wang, Boyer and Genton (2004). The multivariate skew-Cauchy distribution and multivariate skew t-distribution are studied by Arnold and Beaver (2000), and Gupta, Chang and Huang (2002), respectively.

Arellano-Valle, Gómez and Quintana (2003) considered a general class of asymmetric univariate distribution. A key element in their construction is that the distribution can be stochastically represented as a product of two random variables.

Gupta and Kundu (2010) considered two different generalizations of the logistic distribution. They observed that the p.d.f. of the skew logistic distribution can have different shapes with both positive and negative skewness depending on the skewness parameter. Further, the distribution function, failure rate and different moments cannot be obtained in explicit form.

Chakraborty, Hazarika and Ali (2012) following Huang and Chen (2007) considered $[1 + \sin(\lambda x/2\beta)]/2$ as $G(x)$ and introduced a new skew logistic distribution and studied its properties. They obtained the CDF and the moments.

In this paper, we derive a skew logistic distribution using Fernández and Steel (1998) methodology. An advantage of our methodology is that we can obtain explicit expression for the moments and obtain the characteristics function. We also obtained the estimates of the skew parameter and scale parameter using three estimation methods viz Method of Moments, Modified Method of Moments and Maximum Likelihood Method which are found to be consistent. It may be mentioned that none of the earlier authors used the method of moments.

This paper is organized as follows. In Section 2, we give a methodology for developing skew logistic distribution. In Section 3, we derive the properties such as distribution function, survival, hazard function, cumulative generating function and rth raw moment of the distribution. In Section 4, we estimate the parameter using the Method of Moments, Modified Method of Moments and Method of Maximum likelihood estimation. In Section 5, we present an algorithm for generating a skew logistic random variable and present a simulation study, whereas data analysis comparability with the other competent models are presented in Section 6. Section 7 gives some overview of skew logistic regression. The proofs of the mathematical equations are presented in Appendix.

2 Methodology

Consider a symmetric p.d.f. $g(\cdot)$ on $\Re$, then for any $\kappa > 0$, we obtain a skew density $f(\cdot)$ as

$$f(x) = \begin{cases} cg\left(\dfrac{x}{\kappa}\right), & \text{if } x < 0, \\ cg(x\kappa), & \text{if } x \geq 0, \end{cases} \tag{2.1}$$

where c, the normalization constant can easily be obtained as $c = \frac{2\kappa}{1+\kappa^2}$.

It can be seen that when $\kappa = 1$, $f(x)$ is symmetric at $x = 0$. When $\kappa > 1$, $f(x)$ is positively skewed and when $\kappa < 1$ it is negatively skewed. Hence, κ introduces skewness into $f(x)$ both positively and negatively, see Fernández and Steel (1998).

Let g be a symmetric logistic distribution with density

$$g_X(x) = \frac{e^{-x/\beta}}{\beta \cdot (1 + e^{-x/\beta})^2}, \qquad -\infty < x < \infty, \tag{2.2}$$

where $\beta > 0$ is the scale parameter. We follow the procedure of Fernández and Steel (1998) and define:

Definition 2.1. A random variable X is said to have Skew Logistic Distribution (SLD) with scale parameter $\beta > 0$ and skew parameter $\kappa > 0$, if the density function of X is of the form

$$f(x) = \frac{2\kappa}{1+\kappa^2} \begin{cases} \dfrac{e^{-x/(\kappa\beta)}}{\beta(1 + e^{-x/(\kappa\beta)})^2}, & \text{if } x < 0, \\ \dfrac{e^{-x\kappa/\beta}}{\beta(1 + e^{-x\kappa/\beta})^2}, & \text{if } x \geq 0. \end{cases} \tag{2.3}$$

We denote the distribution of X by $\text{SLD}(\kappa, \beta)$ and write $X \sim \text{SLD}(\kappa, \beta)$.

3 Properties of SLD(κ, β) distribution

Let $X \sim \text{SLD}(\kappa, \beta)$ with density defined in (2.3), then c.d.f. of X is given by (see Appendix A.1)

$$F_X(x) = \begin{cases} \dfrac{2\kappa^2}{1+\kappa^2} \dfrac{1}{(1 + e^{-x/(\kappa\beta)})}, & \text{if } x < 0, \\ \dfrac{\kappa^2}{1+\kappa^2} + \dfrac{2}{1+\kappa^2}\left(\dfrac{1}{1 + e^{-x\kappa/\beta}} - \dfrac{1}{2}\right), & \text{if } x \geq 0. \end{cases} \tag{3.1}$$

The Survival and hazard functions are

$$S_X(x) = \begin{cases} 1 - \dfrac{2\kappa^2}{1+\kappa^2} \dfrac{1}{(1 + e^{-x/(\kappa\beta)})}, & \text{if } x < 0, \\ \dfrac{2}{1+\kappa^2}\left(\dfrac{e^{-x\kappa/\beta}}{1 + e^{-x\kappa/\beta}}\right), & \text{if } x \geq 0 \end{cases} \tag{3.2}$$

and

$$H_X(x) = \begin{cases} \dfrac{2\kappa e^{-x/(\kappa\beta)}}{\beta((1-\kappa^2)+(1+\kappa^2)e^{-x/(\kappa\beta)})(1+e^{-x/(\kappa\beta)})}, & \text{if } x < 0, \\ \dfrac{\kappa}{\beta}\dfrac{1}{(1+e^{-x\kappa/\beta})}, & \text{if } x \geq 0. \end{cases} \tag{3.3}$$

Remark 3.1.

1. If $X_{\kappa,\beta}$ follows skew logistic distribution, then $X_{1,\beta}$ is logistic distribution.
2. If $X \sim \mathrm{SLD}(\kappa, \beta)$, then $Y = -X \sim \mathrm{SLD}(\frac{1}{\kappa}, \beta)$.
3. Let $u \sim U(0, 1)$. Choose κ such that $\frac{\kappa^2}{1+\kappa^2} = u_0$, $u_0 \in (0, 1)$ and a scale parameter β. Then

$$X = \begin{cases} \beta\kappa \log\left(\dfrac{u}{2u_0 - u}\right), & \text{if } 0 < u < u_0, \\ \dfrac{\beta}{\kappa}\log\left(\dfrac{u_0 + \kappa^2(u-u_0)}{(1-u)\kappa^2}\right), & \text{if } u_0 \leq u < 1 \end{cases}$$

follows skew logistic distribution with parameter (κ, β).

4. Let $X \sim \mathrm{SLD}(\kappa, \beta)$. The distribution of random variable $X|X > 0$ follows half-logistic distribution with parameter (β/κ) and $X|X < 0$ it follows half-logistic distribution with parameter $(\beta\kappa)$.
5. In the case of skew logistic distribution derived in Jones and Faddy (2003), the skew parameter, α changes the shape of the curve. In our proposed distribution as the skew parameter κ changes, the shape of the curve changes and accommodates more right tail as shown below.

Proposition 3.1. *Consider a random variable* $X_1 \sim \mathrm{SLD}(\kappa, \beta)$ *and* $X_2 \sim \mathrm{SL}_\lambda$ *defined by* Jones and Faddy (2003). *Then following condition holds*:

(i) *For* $\lambda > 0$ *and* $0 < \kappa < 1$, X_1 *has thicker tail than* X_2.
(ii) *For* $\lambda > 0$ *and* $\kappa > 1$, X_1 *has thinner tail than* X_2.
(iii) *For* $\lambda < 0$, $\kappa < 1$ *the* X_1 *has thicker tail than* X_2.
(iv) *For* $\lambda < 0$, $\kappa > 1$ *and* $\lambda + \kappa - 1 > (<) 0$ *then* X_1 *has thinner* (*thicker*) *tail than* X_2.

Proof. The tail behaviour of two distributions can be compared by taking the limiting ratio (LR) of their density (see Tse (2009), p. 60). Faster the ratio approaches to zero (infinity) thinner (thicker) will be the tail of numerator density compared with the denominator density. The limiting ratio of densities of random variables

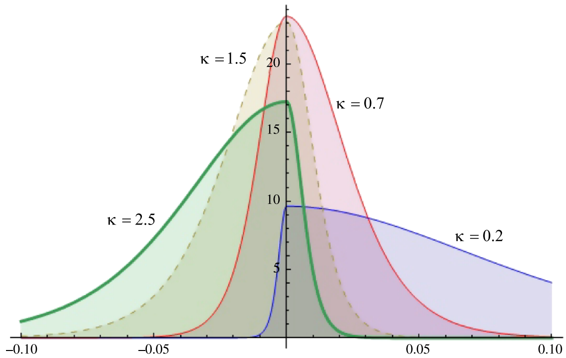

Figure 1 *PDF of SLD for different value of* κ.

$X_1 \sim \text{SLD}(\kappa, \beta)$ and $X_2 \sim \text{SL}_\lambda$ is defined as

$$\text{LR} = \lim_{x\to\infty} \frac{f_{X_1}(x)}{f_{X_2}(x)} = \lim_{x\to\infty} \frac{((2\kappa)/(1+\kappa^2))(e^{-\kappa x/\beta}/(\beta(1+e^{-\kappa x/\beta})^2))}{(2e^{-x/\beta})/(\beta(1+e^{-x/\beta})^2(1+e^{-\lambda x/\beta}))}$$

$$= \lim_{x\to\infty}\left(\frac{\kappa}{1+\kappa^2}\right)\left(\frac{1+e^{-x/\beta}}{1+e^{-\kappa x/\beta}}\right)^2\left(\frac{1+e^{-\lambda x/\beta}}{e^{-x(1-\kappa)/\beta}}\right).$$

(i) If $\lambda > 0$ and $0 < \kappa < 1$, then LR $\to \infty$ which means $f_{X_1}(x)$ has thicker tail than $f_{X_2}(x)$.

(ii) For $\lambda > 0$ and $\kappa > 1$, then LR $= 0$ which implies $f_{X_1}(x)$ has thinner tail than $f_{X_2}(x)$.

(iii) For $\lambda < 0$ and $\kappa < 1$, LR $\to \infty$ which implies $f_{X_1}(x)$ has thicker tail than $f_{X_2}(x)$.

(iv) For $\lambda < 0, \kappa > 1$ and $\lambda + \kappa - 1 > (<)\, 0$, LR $= 0$ (∞) which implies $f_{X_1}(x)$ has thinner (thicker) tail than $f_{X_2}(x)$. □

See Appendix A.2 for detailed proof of above results.

The changes in the shape of the distribution for different values of κ are illustrated in Figure 1. As the scale parameter does not change the shape, β is fixed at $\beta = 0.01$, say. It can be seen from Figure 1 that for $\kappa < 1$, the distribution is right skewed and for $\kappa > 1$ it is left skewed. When κ tends to zero it tends to uniform distribution. Moreover in Figures 2 and 3, density plot for different scale parameters were shown.

Beside scale and skew parameter we can also introduce a location parameter (θ) thereby generalizing our proposed skew logistic model $Y = X + \theta$, where $X \sim \text{SLD}(\kappa, \beta)$. Thus, the density of Y can be written as

$$f(x) = \frac{2\kappa}{1+\kappa^2}\begin{cases} \dfrac{e^{-(x-\theta)/(\kappa\beta)}}{\beta(1+e^{-(x-\theta)/(\kappa\beta)})^2}, & \text{if } x < \theta, \\ \dfrac{e^{-\kappa(x-\theta)/\beta}}{\beta(1+e^{-\kappa(x-\theta)/\beta})^2}, & \text{if } x \geq \theta, \end{cases}$$

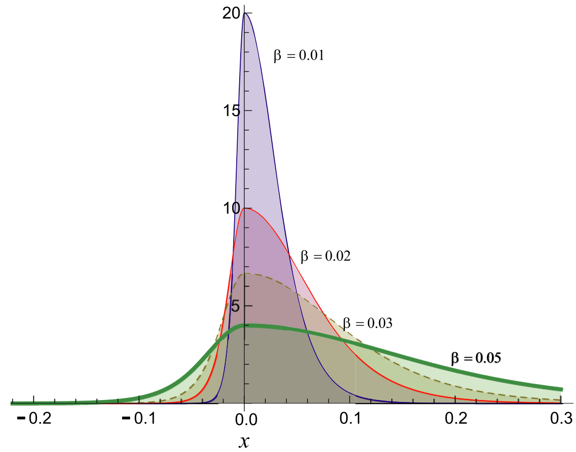

Figure 2 *PDF of SLD for different value of* β *with* $\kappa = 0.5$.

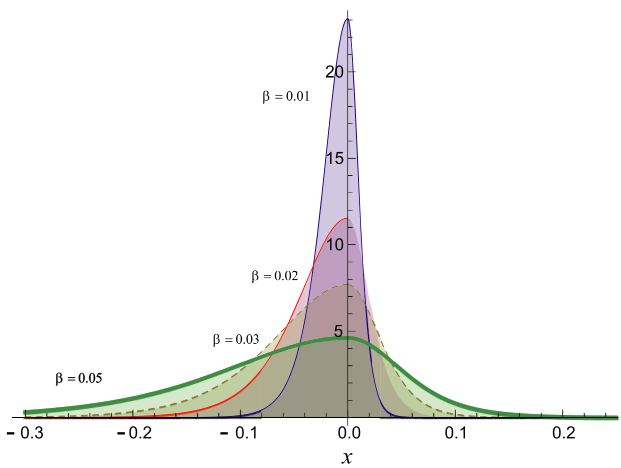

Figure 3 *PDF of SLD for different value of* β *with* $\kappa = 1.5$.

we express the above location family of skew logistic distribution as $Y \sim \text{SLD}(\theta, \kappa, \beta)$. Further, irrespective of κ, β the mode(y_0) of the random variable Y is at θ.

Proposition 3.2. *If* $Y \sim \text{SLD}(\theta, \kappa, \beta)$, *then*

$$\mathbb{E}\big((Y-\theta)^r\big) = \frac{2\kappa}{(1+\kappa^2)\beta} \sum_{j=0}^{\infty} \binom{-2}{j} \frac{\beta^{r+1}\Gamma(r+1)}{(j+1)^{r+1}} \Big(\frac{1}{\kappa^{r+1}} + (-1)^r \kappa^{r+1}\Big).$$

3.1 Characteristic function (C.F.)

Using Taylor series expansion (see Appendix A.4) for $(1+z)^{-1}$ in (2.3), we get

$$f(x) = \begin{cases} \dfrac{2\kappa}{(1+\kappa^2)\beta} \displaystyle\sum_{j=0}^{\infty} \binom{-2}{j} e^{x/(\beta\kappa)(j+1)}, & \text{if } x < 0, \\ \dfrac{2\kappa}{(1+\kappa^2)\beta} \displaystyle\sum_{j=0}^{\infty} \binom{-2}{j} e^{-\kappa x/\beta(j+1)}, & \text{if } x \geq 0. \end{cases} \tag{3.4}$$

The moment generating function of X is $\mathbb{M}_X(t) = \int_{-\infty}^{\infty} e^{xt} f(x)\, dx$. Therefore,

$$\begin{aligned} \mathbb{M}_X(t) &= \frac{2\kappa}{(1+\kappa^2)\beta} \\ &\quad \times \sum_{j=0}^{\infty} \binom{-2}{j} \left(\int_{-\infty}^{0} e^{xt} e^{x/(\beta\kappa)(j+1)}\, dx + \int_{0}^{\infty} e^{xt} e^{-x\kappa/\beta(j+1)}\, dx \right) \\ &= \frac{2\kappa}{(1+\kappa^2)\beta} \sum_{j=0}^{\infty} \binom{-2}{j} \left(\frac{1}{(t+(j+1)/(\beta\kappa))} + \frac{1}{(\kappa(j+1)/\beta - t)} \right), \end{aligned} \tag{3.5}$$

$$\mathbb{M}_X(t) = \frac{2\kappa}{(1+\kappa^2)} \sum_{j=0}^{\infty} \binom{-2}{j} \left(\frac{\kappa}{\beta t\kappa + j + 1} + \frac{1}{\kappa(j+1) - t\beta} \right).$$

Thus, the characteristic function of X takes the form

$$\phi_X(t) = \frac{2\kappa}{(1+\kappa^2)} \sum_{j=0}^{\infty} \binom{-2}{j} \left(\frac{\kappa}{i\beta t\kappa + j + 1} + \frac{1}{\kappa(j+1) - it\beta} \right). \tag{3.6}$$

3.2 rth moment of skew logistic distribution

From (3.4), the rth raw moment is

$$\begin{aligned} \mathbb{E}(X^r) &= \frac{2\kappa}{(1+\kappa^2)\beta} \\ &\quad \times \sum_{j=0}^{\infty} \binom{-2}{j} \left(\int_{-\infty}^{0} x^r e^{x/(\beta\kappa)(j+1)}\, dx + \int_{0}^{\infty} x^r e^{-x\kappa/\beta(j+1)}\, dx \right) \\ &= \frac{2\kappa}{(1+\kappa^2)\beta} \sum_{j=0}^{\infty} \binom{-2}{j} \left(\frac{(-1)^r \beta^{r+1}\kappa^{r+1}\Gamma(r+1)}{(j+1)^{r+1}} + \frac{\beta^{r+1}\Gamma(r+1)}{\kappa^{r+1}(j+1)^{r+1}} \right), \end{aligned}$$

$$\mathbb{E}(X^r) = \frac{2\kappa}{(1+\kappa^2)\beta} \sum_{j=0}^{\infty} \binom{-2}{j} \frac{\beta^{r+1}\Gamma(r+1)}{(j+1)^{r+1}} \left(\frac{1}{\kappa^{r+1}} + (-1)^r \kappa^{r+1} \right).$$

Hence, first four raw moments are given as

$$\mathbb{E}(X) = 2\beta\left(\frac{1-\kappa^2}{\kappa}\right)\log_e 2, \tag{3.7}$$

$$\mathbb{E}(X^2) = \frac{\kappa\beta^2}{1+\kappa^2}\left(\kappa^3 + \frac{1}{\kappa^3}\right)\frac{\pi^2}{3}, \tag{3.8}$$

$$\mathbb{E}(X^3) = \frac{9\kappa\beta^3\varsigma(3)}{(1+\kappa^2)}\left(\frac{1}{\kappa^4} - \kappa^4\right) \qquad \text{where } \varsigma(s) = \sum_{j=0}^{\infty}\frac{1}{(j+1)^s}, \tag{3.9}$$

$$\mathbb{E}(X^4) = \frac{7\kappa\pi^4\beta^4}{15(1+\kappa^2)}\left(\frac{1}{\kappa^5} + \kappa^5\right), \tag{3.10}$$

$$\mathbb{V}(X) = \frac{\kappa\beta^2}{1+\kappa^2}\left(\kappa^3 + \frac{1}{\kappa^3}\right)\frac{\pi^2}{3} - \left(2\beta\left(\frac{1-\kappa^2}{\kappa}\right)\log_e 2\right)^2. \tag{3.11}$$

It is observed from the Figure 4 that the mean is a decreasing function of κ. The skewness is a decreasing function of κ, and it varies from 1.540 to -1.540 as κ

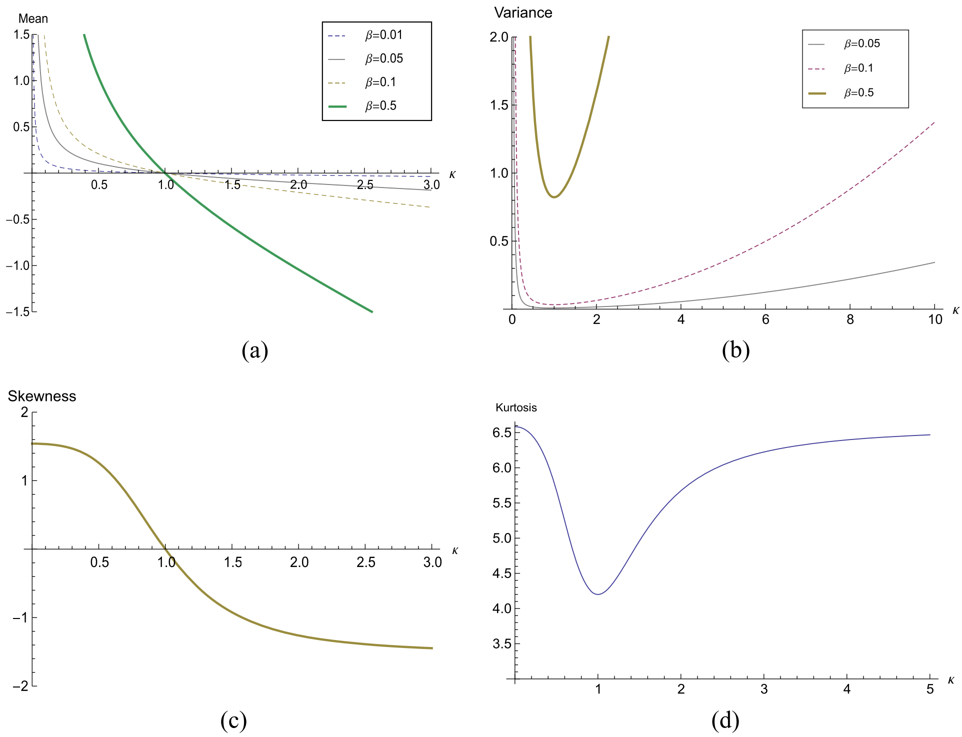

Figure 4 *The variation of the four measures*: (a) $\mathbb{E}(X)$, (b) $\mathbb{V}(X)$, (c) Skewness(X) *and* (d) Kurtosis(X).

varies from 0 to ∞. It is interesting to note that the range of achievable skewness obtained by our proposed distribution is the same to that of Wahed and Ali (2001) and is higher than the range of achievable skewness for skew normal distribution which is $(-\frac{\sqrt{2/\pi}(4/\pi-1)}{(1-2/\pi)^{3/2}}, \frac{\sqrt{2/\pi}(4/\pi-1)}{(1-2/\pi)^{3/2}}) \approx (-0.9953, 0.9953)$. Therefore, it is clear that the proposed skew logistic distribution will be more flexible than the skew normal distribution for data analysis purpose (see Gupta and Kundu (2010)). The kurtosis of our proposed model attains a maximum value of 6.58370 as κ tends to either 0 or ∞.

3.3 Quantile functions

The quantile function of order γ, x_γ, can be derived by inverting the c.d.f. given in (3.1),

$$x_\gamma = \begin{cases} \beta\kappa \log\left(\dfrac{\gamma}{2u_0 - \gamma}\right), & \text{if } 0 < \gamma < u_0, \\ \dfrac{\beta}{\kappa} \log\left(\dfrac{u_0 + \kappa^2(\gamma - u_0)}{(1-\gamma)\kappa^2}\right), & \text{if } u_0 \le \gamma < 1. \end{cases}$$

By choosing $\gamma = 0.5$, the median of the SLD(κ, β) can be given as

$$\text{median}(X) = \begin{cases} \dfrac{\beta}{\kappa} \log\left(\dfrac{3-\kappa^2}{1+\kappa^2}\right), & \text{if } \kappa < 1, \\ \beta\kappa \log\left(\dfrac{1+\kappa^2}{3\kappa^2 - 1}\right), & \text{if } \kappa > 1. \end{cases}$$

4 Estimation of parameters

4.1 Method of Moments

Let the first two sample moments be m_1 and m_2. Equating them with the first two raw moments of skew logistic distribution (3.7) and (3.8) and eliminating beta, we obtain

$$\begin{aligned} \frac{m_2}{m_1^2} &= \frac{(2\tilde{\kappa})/(1+\tilde{\kappa}^2)(\tilde{\kappa}^3 + 1/\tilde{\kappa}^3)\pi^2/6}{(2((1-\tilde{\kappa}^2)/\tilde{\kappa})\log_e 2)^2}, \\ \frac{m_2}{m_1^2} &= \frac{(2\tilde{\kappa})/(1+\tilde{\kappa}^2)(\tilde{\kappa}^3 + 1/\tilde{\kappa}^3)\pi^2/6}{4((1-\tilde{\kappa}^2)/\tilde{\kappa})^2(\log_e 2)^2}, \\ \frac{m_2}{m_1^2} &= \frac{(\tilde{\kappa}^6 + 1)\pi^2}{12(1-\tilde{\kappa}^2)^2(1+\tilde{\kappa}^2)(\log_e 2)^2} \end{aligned} \tag{4.1}$$

putting $q = \frac{12(\log_e 2)^2}{\pi^2}$, equation (3.1) reduces to

$$(1-\tilde{\kappa}^2)^2(1+\tilde{\kappa}^2)q\frac{m_2}{m_1^2} = \tilde{\kappa}^6 + 1$$

which gives

$$\left(\frac{m_2 q}{m_1^2}-1\right)\tilde{\kappa}^6-\frac{m_2 q}{m_1^2}\tilde{\kappa}^4-\frac{m_2 q}{m_1^2}\tilde{\kappa}^2+\left(\frac{m_2 q}{m_1^2}-1\right)=0. \tag{4.2}$$

The roots of the above sixth order polynomial are shown in the Appendix A.3. It can be observed that of the six roots, two are imaginary, two are negative and the remaining two are reciprocal to each other. These two roots are

$$\sqrt{\frac{1}{2}+\frac{b}{2a}-\frac{\sqrt{-3a^2+2ab+b^2}}{2a}} \tag{4.3}$$

and

$$\sqrt{\frac{1}{2}+\frac{b}{2a}+\frac{\sqrt{-3a^2+2ab+b^2}}{2a}}, \tag{4.4}$$

where $a=(\frac{m_2 q}{m_1^2}-1)$ and $b=\frac{m_2 q}{m_1^2}$.

One of the above roots estimates κ and the other will estimate the reciprocal of κ. Which of the above roots estimate κ or (inverse of κ) depends upon "a." It can be shown that

1. $b-a=1$,
2. if $a<0$, then (4.3) estimates κ,
3. if $a>0$, then (4.4) estimates inverse of κ.

By substituting the estimated value of $\tilde{\kappa}$'s in (3.7), we obtain $\tilde{\beta}$, the estimate of scale parameter β.

4.2 Modified Method of Moments

Assume $x_i^+ = x_i \cdot \mathbb{I}_{x_i \geq 0}$ and $x_i^- = x_i \cdot \mathbb{I}_{x_i<0}$, where $\mathbb{I}_{(\cdot)}$ is an indicator function. Then,

$$\begin{aligned}\mathbb{E}(X|X\geq 0)&=\sum_{i=1}^{n} x_i^+/s^+,\\ \mathbb{E}(X|X<0)&=\sum_{i=1}^{n} x_i^-/s^-,\end{aligned} \tag{4.5}$$

where $s^+=\sum_{i=1}^n \mathbb{I}_{(x_i\geq 0)}$ and $s^-=\sum_{i=1}^n \mathbb{I}_{(x_i<0)}$, also from Remark 3.1(4) in Section 3, we know $X|X\geq 0 \sim$ half-Logistic$(\frac{\beta}{\kappa})$ and $X|X<0\sim$ half-Logistic$(\beta\kappa)$ therefore

$$\begin{aligned}\mathbb{E}(X|X<0)&=-2\beta\kappa^2\log_e(2),\\ \mathbb{E}(X|X\geq 0)&=\frac{2\beta}{\kappa^2}\log_e(2).\end{aligned} \tag{4.6}$$

Solving equations (4.5) and (4.6), we obtain

$$\breve{\kappa} = \left(-\frac{\sum_{i=1}^{n}(x_i^-/s^-)}{\sum_{i=1}^{n}(x_i^+/s^+)}\right)^{1/4},$$
$$\breve{\beta} = \frac{1}{2\log_e 2}\left(-\sum_{i=1}^{n}\left(\frac{x_i^-}{s^-}\right)\cdot\sum_{i=1}^{n}\left(\frac{x_i^+}{s^+}\right)\right)^{1/2}. \tag{4.7}$$

It is important to note that the above method of estimation hinges upon the requirement that the conditional expectation equal with the corresponding conditional sample means, where the conditioning events are $X \geq 0$ and $X < 0$. Special care needs to be taken while applying this method, because κ and β cannot be estimated if the samples contain only negative values or only positive values.

4.3 Maximum likelihood estimation

Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ be a random sample of size n from SLD(κ, β). Then, the likelihood L

$$L = \left(\frac{2\kappa}{\beta(1+\kappa^2)}\right)^n \prod_{i=1}^{n} \frac{e^{-x_i^-/(\kappa\beta)}}{(1+e^{-x_i^-/(\kappa\beta)})^2} \cdot \frac{e^{-\kappa x_i^+/\beta}}{(1+e^{-\kappa x_i^+/\beta})^2}.$$

Taking the logarithms

$$l = n\log(\kappa) - n\log\big(1+\kappa^2\big) - n\log\beta + 2n\log 2 - \frac{\sum_{i=1}^{n} x_i^-}{\kappa\beta} - \frac{\kappa\sum_{i=1}^{n} x_i^+}{\beta}$$
$$- 2\sum_{i=1}^{n}\log\big(1+e^{-x_i^-/(\kappa\beta)}\big) - 2\sum_{i=1}^{n}\log\big(1+e^{-\kappa x_i^+/\beta}\big). \tag{4.8}$$

Differentiating the log-likelihood defined in (4.8) partially w.r.t. κ and β, we get

$$\frac{\partial l}{\partial \kappa} = \frac{n}{\kappa} - \frac{2\kappa n}{1+\kappa^2} + \frac{\sum_{i=1}^{n} x_i^-}{\kappa^2\beta} - \frac{\sum_{i=1}^{n} x_i^+}{\beta}$$
$$- \frac{2}{\kappa^2\beta}\sum_{i=1}^{n}\left(\frac{x_i^- e^{-x_i^-/(\kappa\beta)}}{1+e^{-x_i^-/(\kappa\beta)}}\right) + \frac{2}{\beta}\sum_{i=1}^{n}\left(\frac{x_i^+ e^{-\kappa x_i^+/\beta}}{1+e^{-\kappa x_i^+/\beta}}\right),$$
$$\frac{\partial l}{\partial \beta} = -\frac{n}{\beta} + \frac{\sum_{i=1}^{n} x_i^-}{\kappa\beta^2} + \frac{\kappa\sum_{i=1}^{n} x_i^+}{\beta^2}$$
$$- \frac{2}{\kappa\beta^2}\sum_{i=1}^{n}\left(\frac{x_i^- e^{-x_i^-/(\kappa\beta)}}{1+e^{-x_i^-/(\kappa\beta)}}\right) - \frac{2\kappa}{\beta^2}\sum_{i=1}^{n}\left(\frac{x_i^+ e^{-\kappa x_i^+/\beta}}{1+e^{-\kappa x_i^+/\beta}}\right).$$

The maximum likelihood estimates of κ and β can be obtained by solving the above two log-likelihood equations using numerical methods by searching for global maximum on the log-likelihood surface.

5 Simulation study

For given values of parameters κ (>0) and β (>0), a simulation algorithm for generating *SLD* random variables consists of the following steps:

Step 1. Compute $u_0 = \frac{\kappa^2}{1+\kappa^2}$.
Step 2. Generate uniform random variable u from $U(0,1)$.
Step 3. If $0 < u \leq u_0$, then $x = -\beta\kappa \log_e(\frac{2\kappa^2}{u(1+\kappa^2)} - 1)$.
Step 4. If $u_0 < u < 1$, then $x = -\frac{\beta}{\kappa} \log(\frac{(1-u)(1+\kappa^2)}{(1-\kappa^2)+u(1+\kappa^2)})$.

In order to assess the estimation methods under several scenarios for different values of κ, β, a simulation study is being carried out using nlm() optimization package in *R*-language. The values for κ and β are $\kappa_0 = 0.3, 0.7, 1.2, 1.5$ and $\beta_0 = 0.4, 0.9$ and 1.5. Samples are drawn with sizes $n = 25$, 50 and 100. For each sample size, $N = 1000$ replicas, that is, simulated values for the SLD distribution were generated." The point estimates were obtained with the method of moments (MM), modified method of moments (MMM), and method of maximum likelihood (MLE) and are presented in Table 1. For these, bias and mean square error(mse), defined as

$$\text{bias}(\hat{\Theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Theta}_i - \Theta_0) \quad \text{and} \quad \text{mse}(\hat{\Theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Theta}_i - \Theta_0)^2,$$

where $\hat{\Theta}_i$ is the estimated value of $\Theta = (\kappa, \beta)$ obtained in ith replicas and Θ_0 is true value of parameter, are calculated and presented in Table 1.

It can be observed that, the MM estimator is negatively biased for both κ and β under each scenario. It can also be noted that MM gives the worst estimates for larger value of κ say 1.2 and 1.5 and for any value of β. The MMM seems to be better estimate as the bias is reduced in absolute value. Moreover, for $\kappa < 1$ MMM estimator is positively biased and for $\kappa > 1$ bias is negative. The MLE of both κ and β are biased.

In Figure 5, Box Plots of estimated values of κ for different $N = 1000$ have been presented for different scenarios and it can be clearly seen that *MLE* performs relatively better for any choice of κ and β compared to MM and MMM estimator.

Further, in Figure 6, consistency of the estimator were shown by considering different sample sized, decreasing inter quartile range of boxplots gives an evidence that all three estimators for both κ and β are consistent. One can also infer that MLE perform better than the others in almost all scenarios.

6 Data analysis

In this section, we try to illustrate the better performance of the above proposed model by fitting the distribution to a data set on lean body mass of Australian athletes see Cook and Weisberg (1994), see Table 2. Further, we compare our model

Table 1 *Bias and MSE of $\hat{\kappa}$ and $\hat{\beta}$ obtained from different estimation methods*

			Method of Moments				Modified Method of Moments				Maximum likelihood methods			
κ	β	n	bias($\hat{\kappa}$)	mse($\hat{\kappa}$)	bias($\hat{\beta}$)	mse($\hat{\beta}$)	bias($\breve{\kappa}$)	mse($\breve{\kappa}$)	bias($\breve{\beta}$)	mse($\breve{\beta}$)	bias($\tilde{\kappa}$)	mse($\tilde{\kappa}$)	bias($\tilde{\beta}$)	mse($\tilde{\beta}$)
0.3	0.4	25	−0.07537	0.03893	−0.09018	0.08383	0.22833	0.05421	−0.02102	0.01574	−0.04705	0.01576	−0.07089	0.02980
		50	−0.05870	0.02879	−0.06581	0.06035	0.23654	0.06080	−0.00969	0.01014	−0.01505	0.00387	−0.02190	0.00767
		100	−0.04344	−0.04344	−0.04812	0.04071	0.24253	0.06099	−0.00509	0.00452	**−0.00812**	**0.00146**	**−0.01152**	**0.00309**
0.7	0.4	25	−0.05234	0.03469	−0.03364	0.01501	0.13334	0.02447	−0.00753	0.00526	−0.00458	0.01216	−0.01413	0.00484
		50	−0.00837	0.00885	−0.00890	0.00462	0.13553	0.02133	−0.00425	0.00248	−0.00096	0.00492	−0.00684	0.00233
		100	−0.00008	0.00287	0.00037	0.00199	0.13760	0.02046	0.00033	0.00130	**0.00087**	**0.00238**	**−0.00136**	**0.00123**
1.2	0.4	25	−0.63371	0.52567	−0.66997	0.48709	−0.09590	0.01916	−0.00688	0.00522	0.02408	0.02903	−0.01291	0.00488
		50	−0.59883	0.48099	−0.69598	0.51573	−0.10466	0.01569	−0.00175	0.00252	0.00500	0.01342	−0.00469	0.00241
		100	−0.51710	−0.51710	−0.73199	0.55873	−0.10356	0.01276	−0.00146	0.00115	**0.00332**	**0.00592**	**−0.00299**	**0.00112**
1.5	0.4	25	−0.86272	0.76472	−0.77947	0.61907	−0.26276	0.08303	−0.00284	0.00603	0.04532	0.05855	−0.01003	0.00544
		50	−0.83996	0.71216	−0.79391	0.63456	−0.27128	0.08034	−0.00135	0.00263	0.02063	0.02640	−0.00563	0.00238
		100	−0.83195	0.69494	−0.79750	0.63800	−0.27360	0.07806	−0.00208	0.00128	**0.00202**	**0.01105**	**−0.00366**	**0.00120**
0.3	0.9	25	−0.08452	0.04049	−0.22253	0.43697	0.22630	0.05368	−0.05186	0.08035	−0.05073	0.01605	−0.16832	0.15122
		50	−0.05981	0.02766	−0.15679	0.29109	0.23141	0.05774	−0.04209	0.04892	−0.02050	0.00471	−0.06991	0.04650
		100	−0.03480	0.01787	−0.08722	0.19063	0.24436	0.06165	−0.00942	0.01986	**−0.00483**	**0.00139**	**−0.01928**	**−0.01928**
0.7	0.9	25	−0.03966	0.02488	−0.05718	0.06516	0.13222	0.02360	−0.01602	0.02946	−0.00862	0.01046	−0.03084	0.02703
		50	−0.00898	0.00804	−0.02216	0.02358	0.13732	0.02179	−0.00728	0.01396	−0.00100	0.00477	−0.01549	0.01266
		100	−0.00396	0.00283	−0.00372	0.00886	0.13549	0.13549	−0.00262	−0.00262	**−0.00261**	**0.00231**	**−0.00569**	**0.00613**
1.2	0.9	25	−0.65236	0.55441	−1.48091	2.39788	−0.10064	0.01972	−0.00967	0.02414	0.01751	0.03057	−0.02330	0.02306
		50	−0.59720	0.48046	−1.56549	2.61240	−0.10293	0.01520	0.00019	0.01241	0.00136	0.01300	−0.00609	0.01184
		100	−0.49982	0.33331	−1.66773	2.88536	−0.10255	−0.10255	−0.00100	0.00563	**0.00596**	**0.00639**	**−0.00470**	**0.00530**
1.5	0.9	25	−0.85439	0.74755	−1.75819	3.14147	−0.25881	0.08212	−0.01702	0.02778	0.04399	0.05862	−0.02961	0.02521
		50	−0.84381	0.71845	−1.78011	3.19042	−0.26391	0.07599	−0.01411	0.01325	0.03137	0.02594	−0.02120	0.01230
		100	−0.83593	0.70167	−1.79685	3.23908	−0.27449	0.07862	−0.00216	0.00686	**0.01094**	**0.01224**	**−0.00681**	**0.00633**

Table 1 (*Continued*)

			Method of Moments				Modified Method of Moments				Maximum likelihood methods			
κ	β	n	bias($\hat{\kappa}$)	mse($\hat{\kappa}$)	bias($\hat{\beta}$)	mse($\hat{\beta}$)	bias($\breve{\kappa}$)	mse($\breve{\kappa}$)	bias($\breve{\beta}$)	mse($\breve{\beta}$)	bias($\tilde{\kappa}$)	mse($\tilde{\kappa}$)	bias($\tilde{\beta}$)	mse($\tilde{\beta}$)
0.3	1.5	25	−0.06234	0.03730	−0.25415	1.18623	0.22241	0.05183	−0.11074	0.22305	−0.04942	0.01523	−0.27647	0.40789
		50	−0.06081	0.02946	−0.26503	0.87792	0.23600	0.05958	−0.05014	0.12853	−0.01624	0.00405	−0.09365	0.11399
		100	0.24332	0.06130	−0.01747	0.05988	−0.00585	−0.00585	−0.03492	−0.03492	**−0.00585**	**0.00151**	**−0.03492**	**0.04389**
0.7	1.5	25	−0.03640	0.02740	−0.09817	0.17732	0.13635	0.02504	−0.02987	0.07908	−0.00224	0.01058	−0.05173	0.07200
		50	−0.01278	0.00973	−0.03010	0.07173	0.13492	0.02114	−0.00954	0.03706	−0.00317	0.00500	−0.02189	0.03426
		100	−0.00137	0.00295	−0.00917	0.02613	0.13717	0.02033	−0.00927	0.01741	**−0.00053**	**0.00238**	**−0.01540**	**0.01605**
1.2	1.5	25	−0.64767	0.54841	−2.48045	6.71985	−0.09632	0.01973	−0.01430	0.07535	0.02068	0.03373	−0.03592	0.07025
		50	−0.58757	0.46249	−2.63028	7.36345	−0.10110	0.01479	0.00132	0.00132	0.00912	0.01430	−0.01034	0.03117
		100	−0.49016	0.31794	−2.79495	8.08152	−0.10101	0.01219	−0.00857	0.01625	**0.00870**	**0.00601**	**−0.01326**	**0.01565**
1.5	1.5	25	−0.85835	0.756505	−2.90868	8.60468	−0.26409	0.084725	−0.02880	0.08134	0.04305	0.06793	−0.05600	0.07312
		50	−0.83692	0.70695	−2.96876	8.87763	−0.27037	0.07952	−0.01619	0.03898	0.01367	0.02466	−0.02975	0.03667
		100	−0.83517	0.70052	−2.99611	9.00591	−0.27123	0.07670	−0.00462	0.01801	**0.01014**	**0.01240**	**−0.00956**	**0.01678**

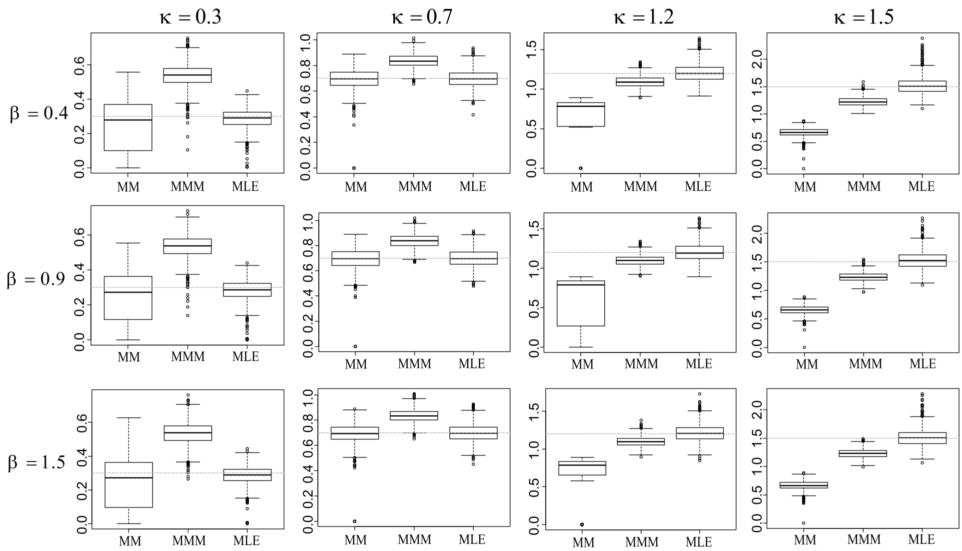

Figure 5 *Box plot for various κ and β under different methods of estimation.*

with other available skew distribution like general skew logistic GSL$(\mu, \sigma, \lambda, \alpha)$ proposed by Asgharzadeh et al. (2013), the skew logistic distribution SL(μ, σ, λ) by Wahed and Ali (2001), skew normal distribution SN(μ, σ, λ) proposed by Azzalini (1985), and skew-t distribution ST$(\mu, \sigma, \lambda, \alpha)$ by Azzalini and Capitanio (2003). The goodness of fit for above mentioned model are tested through log likelihoods, the Kolmogorov–Smirnov (KS) statistics. The following Table 3 presents the parameter of the model their log likelihoods, the Kolmogorov–Smirnov (KS) statistics with corresponding p-values. It can be seen from the Table 3 that log likelihood values of our model is closed to SL and GSL but higher than other models. But it may be interesting to note that the KS statistics of our model is lowest compare to other model and with highest p-value this confirms the better performance of our model.

From Table 3, loglikelihood value for SLD(κ, β) is closed to SL and GSL whereas higher than the other models but the KS statistics for SLD(κ, β) is lowest compared to all other distribution and having largest p-value.

7 Skew logistic regression

In many real situations, the response variable is qualitative in nature instead of quantitative which can be measure by Indicator variable 1(0). In statistical literature, logistic regression proves to be an effective tool for modelling situations for prediction of probability of occurrence of an event, in which several predictor variables either numerical or categorical can be used. Extensive application of Logistic regression can be found in medical and social science.

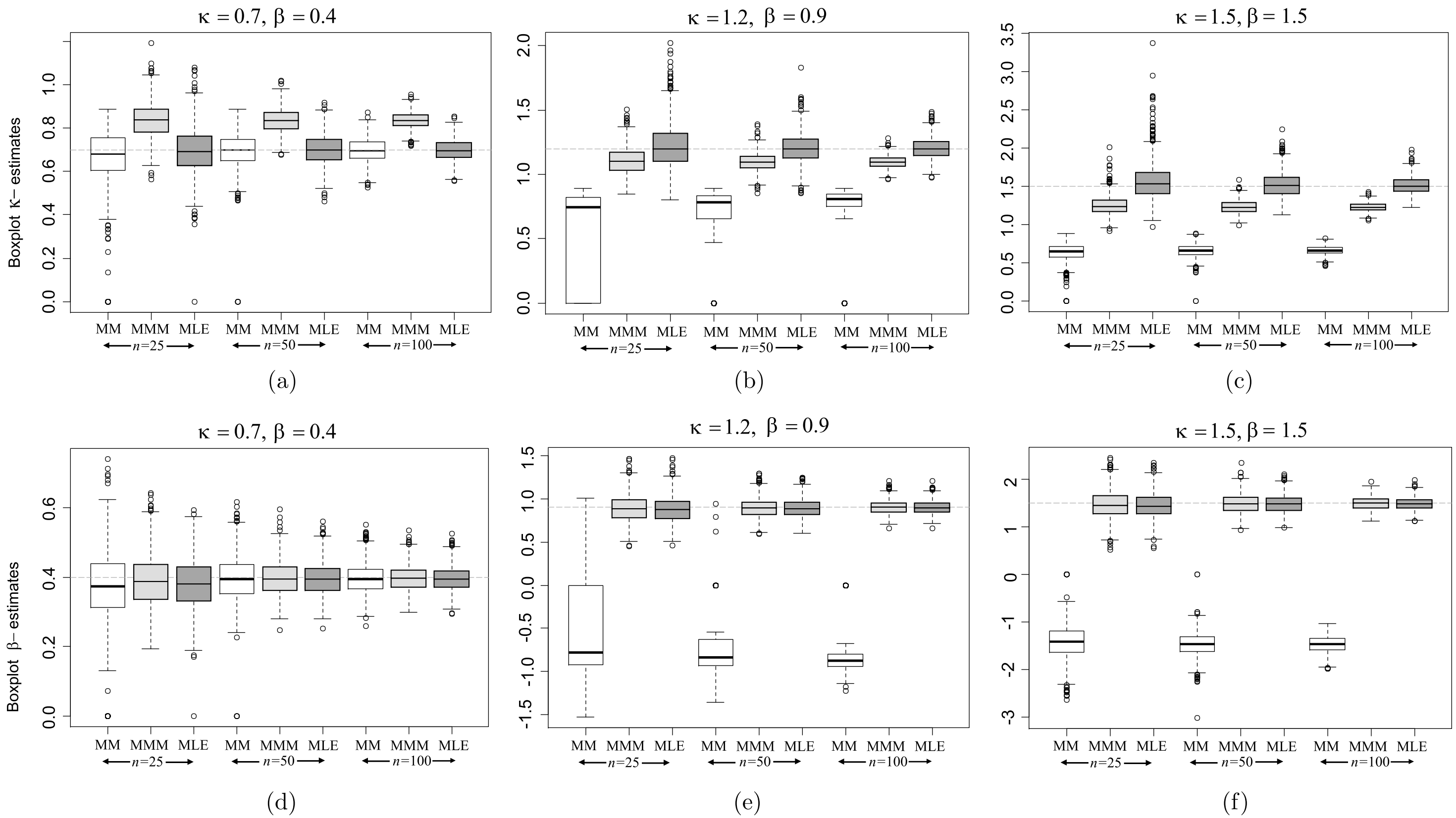

Figure 6 *Box plot for representing consistency of estimators for various sample size n for different κ and β.*

Table 2

63.32	58.55	55.36	57.18	53.2	53.77	60.17	48.33	54.57	53.42
68.53	61.85	48.32	66.24	57.92	56.52	54.78	56.31	62.96	56.68
62.39	63.05	56.05	53.65	65.45	64.62	60.05	56.48	41.54	52.78
52.72	61.29	59.59	61.7	62.46	53.14	47.09	53.44	48.78	56.05
56.45	53.11	54.41	55.97	51.62	58.27	57.28	57.3	54.18	42.96
54.46	57.2	54.38	57.58	61.46	53.46	54.11	55.35	55.39	52.23
59.33	61.63	63.39	60.22	55.73	48.57	51.99	51.17	57.54	68.86
63.04	63.03	66.85	59.89	72.98	45.23	55.06	46.96	53.54	47.57
54.63	46.31	49.13	53.71	53.11	46.12	53.41	51.48	53.2	56.58
56.01	46.52	51.75	42.15	48.76	41.93	42.95	38.3	34.36	39.03

Table 3 *Log-likelihoods, Kolmogorov–Smirnov statistics and corresponding p-values*

Distributions	Parameters	LL	KS	p-values
SLD	$\hat{\theta} = 56.05$, $\hat{\kappa} = 1.11190$, $\hat{\beta} = 3.781194$	−333.715	**0.059**	**0.961**
SL	$\hat{\mu} = 57.148$, $\hat{\sigma} = 3.990$, $\hat{\lambda} = -0.389$	−333.557	0.071	0.712
GSL	$\hat{\mu} = 55.356$, $\hat{\sigma} = 0.671$, $\hat{\lambda} = -0.057$, $\hat{\alpha} = 0.133$	−333.265	0.069	0.715
Logistic	$\hat{\mu} = 55.101$, $\hat{\sigma} = 0.389$	−334.013	0.072	0.658
Skew normal	$\hat{\mu} = 54.895$, $\hat{\sigma} = 6.887$, $\hat{\lambda} = 8.706 \times 10^{-6}$	−334.865	0.080	0.642
Skew-t	$\hat{\mu} = 59.085$, $\hat{\sigma} = 7.233$, $\hat{\lambda} = -0.903$, $\hat{\alpha} = 9.924$	−333.738	0.072	0.711

If $Y_i \sim \text{Bin}(n_i, p_i)$, where the numbers of Bernoulli trials n_i are known and the probabilities of success p_i are unknown. Logistic regression proposes that for each i, there is a set of predictor variables $x_{1,i}, \ldots, x_{k,i}$ such that

$$\log\left(\frac{p_i}{1-p_i}\right) = \alpha_0 + \alpha_1 x_{1i} + \cdots + \alpha_k x_{ki} \tag{7.1}$$

or equivalently

$$p_i = \frac{1}{1 + e^{\alpha_0 + \alpha_1 x_{1i} + \cdots + \alpha_k x_{ki}}}, \tag{7.2}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k$ are regression coefficients. For our skew logistic distribution the p_i's are

$$p_i = \frac{2\kappa}{1+\kappa^2} \begin{cases} \displaystyle\int_{-\infty}^{z_i} \frac{e^{-t/(\kappa\beta)}\, dt}{\beta(1 + e^{-t/(\kappa\beta)})^2}, & \text{if } z_i < 0, \\ \displaystyle\frac{\kappa}{2} + \frac{1}{\kappa} \int_0^{z_i} \frac{e^{-\kappa t/\beta}\, dt}{\beta(1 + e^{-\kappa t/\beta})^2}, & \text{if } z_i \geq 0. \end{cases} \tag{7.3}$$

Note that (7.1) and (7.2) are contained as particular cases of (7.3) if $\kappa = 0$. The integral in (7.3) can be expressed similarly as in (3.1)

$$p_i = \begin{cases} \dfrac{2\kappa^2}{1+\kappa^2} \dfrac{1}{(1+e^{-z_i/(\kappa\beta)})}, & \text{if } z_i < 0, \\ \dfrac{\kappa^2}{1+\kappa^2} + \dfrac{2}{1+\kappa^2}\left(\dfrac{1}{1+e^{-z_i\kappa/\beta}} - \dfrac{1}{2}\right), & \text{if } z_i \geq 0, \end{cases} \tag{7.4}$$

where $z_i = \alpha_0 + \alpha_1 x_{1i} + \cdots + \alpha_k x_{ki}$. Further (7.3) is a general case over standard logistic regression as it contains one more parameter κ.

8 Further comments

As suggested by one of the referees, we can explore the possibility of our proposed model to item response data, such data are asymmetric in nature and therefore one can find application in medical science, finance, insurance etc.

Bazan, Branco and Bolfarine (2006) proposed a asymmetric function using the skew normal distribution (Azzalini (1985)) and consequently a new item characteristics curve (ICC). They introduced a skewness parameter associated to the item which can be interpreted as penalised parameter. The skew link proposed by Bazan, Branco and Bolfarine (2006) use the skew normal family of distribution.

In the context of binary regression, symmetric links do not always provide good fits for some data sets. This is specifically true when the probability of a given response approaches zero at a faster rate then it approaches to one see Chen, Dey and Shao (1999). For such cases Samejima (1997) indicated the necessity of considering departure from normal assumption in developing psychometric theories and methodologies indicating that asymmetric ICC are more appropriate for modelling human item response behaviour. Samejima (1997, 2000) derived a skew logistic IRT model, and Logistic Positive Exponent (LPE) family.

Bolfarine and Bazan (2010) presented a Bayesian estimation procedure for the two skew logistic IRT models using Markov Chain Monte Carlo Methodology (MCMC).

It may be noted that our model proposed distribution applies to those variables which are continuous. However one can explore deriving the new IRT model using our skew logistic distribution for IRT model as a link function.

Appendix

A.1 Cumulative distribution function

Proof of c.d.f. (3.1). For $x < 0$,

$$F_X(x) = \frac{2\kappa}{1+\kappa^2} \int_{-\infty}^{x} \frac{e^{-t/(\kappa\beta)}\,dt}{\beta(1+e^{-t/(\kappa\beta)})^2}.$$

Assuming $e^{-t/(\kappa\beta)} = p$ and $t = -\beta\kappa \log_e p$

$$\begin{aligned}
F_X(x) &= \frac{2\kappa}{(1+\kappa^2)} \int_{\infty}^{e^{-x/\kappa\beta}} \frac{p}{\beta(1+p)^2} \left(\frac{-\kappa\beta}{p}\right) dp \\
&= \frac{2\kappa^2}{1+\kappa^2} \int_{e^{-x/\kappa\beta}}^{\infty} \frac{1}{(1+p)^2}\, dp \\
&= \frac{2\kappa^2}{1+\kappa^2} \frac{1}{(1+e^{-x/(\kappa\beta)})}
\end{aligned}$$

as $x \to -\infty$, $F_X(x) \to 0$.

For $x > 0$,

$$F_X(x) = 1 - \frac{2\kappa}{1+\kappa^2} \int_x^{\infty} \frac{e^{-t\kappa/\beta}}{\beta(1+e^{-t\kappa/\beta})^2}\, dt.$$

Assuming $e^{-t\kappa/\beta} = u \Rightarrow t = -\frac{\beta \log u}{\kappa}$

$$\begin{aligned}
F_X(x) &= 1 - \int_{e^{-x\kappa/\beta}}^{0} \frac{2\kappa u}{(1+\kappa^2)\beta(1+u)^2} \left(\frac{-\beta}{\kappa}\right) \frac{1}{u}\, du \\
&= 1 - \int_0^{e^{-xk/\beta}} \frac{2}{(1+\kappa^2)(1+u)^2}\, du \\
&= 1 - \frac{2}{1+\kappa^2} \left(-\frac{1}{1+u}\right)_0^{e^{-xk/\beta}} \\
&= 1 - \frac{2}{1+\kappa^2} \left(1 - \frac{1}{1+e^{-x\kappa/\beta}}\right) \\
&= \frac{\kappa^2}{1+\kappa^2} + \frac{2}{1+\kappa^2} \left(\frac{1}{1+e^{-x\kappa/\beta}} - \frac{1}{2}\right)
\end{aligned}$$

as $x \to \infty$, $F_X(x) \to 1$.

Further, it is right continuous and a non-decreasing function satisfying the conditions (i) $\lim_{x\to-\infty} F_X(x) = 0$ and (ii) $\lim_{x\to\infty} F_X(x) = 1$. Therefore, $F_X(x)$ is a cumulative distribution function. □

A.2 Limiting ratio

(i) For $\lambda > 0, 0 < \kappa < 1$

$$\lim_{x\to\infty} (1 + e^{-\lambda x/\beta}) = \lim_{x\to\infty} (1 + e^{-\kappa x/\beta}) = 1$$

and

$$\lim_{x\to\infty} (e^{-(1-\kappa)x/\beta}) = 0.$$

Hence,

$$\mathrm{LR} = \lim_{x\to\infty}\left(\frac{\kappa}{1+\kappa^2}\right)\left(\frac{1+e^{-x/\beta}}{1+e^{-\kappa x/\beta}}\right)^2\left(\frac{1+e^{-\lambda x/\beta}}{e^{-x(1-\kappa)/\beta}}\right) \to \infty, \qquad \text{as } x\to\infty.$$

Thus, for $\lambda > 0$ and $0 < \kappa < 1$, X_1 has thicker tail than X_2.

(ii) For $\lambda > 0, \kappa > 1$

$$\lim_{x\to\infty}(1+e^{-\lambda x/\beta}) = 0, \qquad \lim_{x\to\infty}(1+e^{-\kappa x/\beta}) = 1$$

and

$$\lim_{x\to\infty}(e^{-(1-\kappa)x/\beta}) = \infty,$$

$$\mathrm{LR} = \lim_{x\to\infty}\left(\frac{\kappa}{1+\kappa^2}\right)\left(\frac{1+e^{-x/\beta}}{1+e^{-\kappa x/\beta}}\right)^2\left(\frac{1+e^{-\lambda x/\beta}}{e^{-x(1-\kappa)/\beta}}\right) = 0 \qquad \text{as } x\to\infty.$$

Thus, for $\lambda > 0, \kappa > 1$, X_1 has thinner tail than X_2.

(iii) For $\lambda < 0, \kappa < 1$

$$\lim_{x\to\infty}(1+e^{-\lambda x/\beta}) = \lim_{x\to\infty}(1+e^{-\kappa x/\beta}) = \infty$$

and

$$\lim_{x\to\infty}(e^{-(1-\kappa)x/\beta}) = 0,$$

$$\mathrm{LR} = \lim_{x\to\infty}\left(\frac{\kappa}{1+\kappa^2}\right)\left(\frac{1+e^{-x/\beta}}{1+e^{-\kappa x/\beta}}\right)^2\left(\frac{1+e^{-\lambda x/\beta}}{e^{-x(1-\kappa)/\beta}}\right) \to \infty \qquad \text{as } x\to\infty.$$

Thus, for $\lambda < 0, \kappa < 1$, X_1 has thicker tail than X_2.

(iv) For $\lambda < 0, \kappa < 1$ and $\lambda+\kappa-1 > (<)\, 0$

$$\lim_{x\to\infty}(1+e^{-\lambda x/\beta}) = \lim_{x\to\infty}(1+e^{-\kappa x/\beta}) = \infty$$

and

$$\lim_{x\to\infty}(e^{-(\lambda+\kappa-1)x/\beta}) = 0\ (\infty),$$

$$\mathrm{LR} = \lim_{x\to\infty}\left(\frac{\kappa}{1+\kappa^2}\right)\left(\frac{1+e^{-x/\beta}}{1+e^{-\kappa x/\beta}}\right)^2\left(\frac{1+e^{-\lambda x/\beta}}{e^{-x(1-\kappa)/\beta}}\right) \to \infty\ (0) \qquad \text{as } x\to\infty.$$

Thus, $\lambda < 0, \kappa < 1$ and $\lambda+\kappa-1 > (<)\, 0$, X_1 has thinner (thicker) tail than X_2.

A.3 The roots of the polynomial

From (4.2),

$$\left(\frac{m_2 q}{m_1^2} - 1\right)\tilde{\kappa}^6 - \frac{m_2 q}{m_1^2}\tilde{\kappa}^4 - \frac{m_2 q}{m_1^2}\tilde{\kappa}^2 + \left(\frac{m_2 q}{m_1^2} - 1\right) = 0$$

assuming $a = (\frac{m_2 q}{m_1^2} - 1)$, $b = \frac{m_2 q}{m_1^2}$ and $\tilde{\kappa}^2 = y$, we have

$$\begin{aligned}
& ay^3 - by^2 - by + a = 0 \\
& \implies \quad a\big(y^3 + 1\big) - by(y+1) = 0 \\
& \implies \quad a(y+1)\big(y^2 - y + 1\big) - by(y+1) = 0 \\
& \implies \quad (y+1)\big(a\big(y^2 - y + 1\big) - by\big) = 0 \\
& \implies \quad (y+1)\big(ay^2 - (a+b)y - a\big) = 0 \\
& \implies \quad y = -1, \qquad \big(ay^2 - (a+b)y - a\big) = 0 \\
& \implies \quad y = -1, \qquad y = \frac{a + b \pm \sqrt{(a+b)^2 - 4a^2}}{2a} \\
& \implies \quad x = \pm i, \qquad x = \pm\sqrt{\frac{1}{2} + \frac{b}{2a} + \frac{\sqrt{-3a^2 + b^2 + 2ab}}{2a}}, \\
& \qquad\qquad x = \pm\sqrt{\frac{1}{2} + \frac{b}{2a} - \frac{\sqrt{-3a^2 + b^2 + 2ab}}{2a}}.
\end{aligned}$$

Hence two roots are imaginary and two are negative.

A.4 Taylor series expansion

We know from series expansion of

$$(1+z)^{-2} = \sum_{j=0}^{\infty} \binom{-2}{j} z^j, \qquad 0 < z < 1.$$

For $x < 0$, $0 < e^{x/(\beta\kappa)} < 1$, therefore

$$\big(1 + e^{x/(\beta\kappa)}\big)^{-2} = \sum_{j=0}^{\infty} \binom{-2}{j} e^{xj/(\beta\kappa)}.$$

Similarly, for $x > 0$, $0 < e^{-\kappa x/\beta} < 1$, hence

$$\big(1 + e^{-\kappa x/\beta}\big)^{-2} = \sum_{j=0}^{\infty} \binom{-2}{j} e^{-\kappa x j/\beta}.$$

By substituting the above expression in (2.3), we get

$$f(x) = \begin{cases} \dfrac{2\kappa}{(1+\kappa^2)\beta} \displaystyle\sum_{j=0}^{\infty} \binom{-2}{j} e^{x/(\beta\kappa)(j+1)}, & \text{if } x < 0, \\[2ex] \dfrac{2\kappa}{(1+\kappa^2)\beta} \displaystyle\sum_{j=0}^{\infty} \binom{-2}{j} e^{-\kappa x/\beta(j+1)}, & \text{if } x \geq 0. \end{cases}$$

Further, using the above expansion the c.d.f. in (3.1) can be rewritten as

$$F_X(x) = \begin{cases} \dfrac{2\kappa}{1+\kappa^2} \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \binom{-1}{r} \frac{1}{s!} \left(-\frac{xr}{\kappa\beta}\right)^s, & \text{if } x < 0, \\ \dfrac{\kappa^2-1}{\kappa^2+1} + \dfrac{2}{1+\kappa^2} \displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \binom{-1}{r} \frac{1}{s!} \left(-\frac{\kappa x r}{\beta}\right)^s, & \text{if } x \geq 0. \end{cases}$$

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