Open Mathematics

Research Article

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A new smoothing method for solving nonlinear complementarity problems

https://doi.org/10.1515/math-2019-0011
Received June 20, 2018; accepted January 14, 2019

Abstract: In this paper, a new improved smoothing Newton algorithm for the nonlinear complementarity problem was proposed. This method has two-fold advantages. First, compared with the classical smoothing Newton method, our proposed method needn't nonsingular of the smoothing approximation function; second, the method also inherits the advantage of the classical smoothing Newton method, it only needs to solve one linear system of equations at each iteration. Without the need of strict complementarity conditions and the assumption of P_0 property, we get the global and local quadratic convergence properties of the proposed method. Numerical experiments show that the efficiency of the proposed method.

Keywords: Nonlinear complementarity problems, Smoothing function, Smoothing method, Global convergence, Quadratic convergence

MSC: 90C33, 90C33, 65K05

1 Introduction

Consider the following nonlinear complementarity problems (denoted by NCP),

$$x \geq 0, F(x) \geq 0, x^T F(x) = 0, \tag{1.1}$$

where $F := (F_1, F_2, \dots, F_n)^T$, and $F : \Re^n \to \Re^n$ is continuously differentiable function.

NCP has been extensively studied due to its various important applications in operations research, engineering design, and economic equilibrium. Many algorithms have been developed for solving the Problem (1.1) [1–9]. Recently, smoothing method has attracted much attention because that it is a very efficient algorithm to solve NCP. It's main idea is to use a smoothing function to approximate NCP via a family of parameterized smooth equations and solve the smooth equations approximately at each iteration. By reducing the parameter to zero, we have hoped that a solution of the original problem can be found. It is obvious that smoothing functions play a very important role in smoothing methods. Up to now, a large number of smoothing functions have been proposed, Fischer-Burmeister smoothing function [10] and CHKS smoothing function [11] are the most famous ones. Based on the smoothing functions, scholars proposed a number of smoothing algorithms. In order to solve systems of nonsmooth equations, Chen et al. [7] proposed a smoothing Newton method and proved that the algorithm is globally convergent and locally supperlinearly convergent; Yang et al.[12] proposed a smoothing trust region algorithm by using trust region technique instead of line search strategy, but they had to solve complicated quadratic programming subproblems in its current version. Recently, smoothing Levenberg-Marquardt method has attracted much attention. Based on the trust region technique, a smoothing Levenberg-Marquardt method is proposed for the extended

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linear complementarity problems in[13]. By employing Fischer-Burmeister smoothing function, a smoothing Levenberg-Marquardt method is proposed for solving nonlinear complementarity problems with P_0 function in [14]. In [14], a smoothing parameter τ as an independent variable was introduced. In order to ensure the strict positivity of the smoothing parameter, a relatively complicated subproblems have to be solved at each iteration. Based on a partially smoothing function, Wan et al.[15] proposed a partially smoothing Jacobian method for solving the nonlinear complementarity problems. Like most Jacobian smoothing methods, they still have to assume the function F is a P_0-function. Under the condition that the level set of a merit function is bounded, they proved the proposed algorithm is globally convergent and superlinearly convergent.

In this paper, motivated by the above work, we propose an improved smoothing Newton method for solving the Problem (1.1). First, based on a one-parametric class of smoothing function, the Problem (1.1) can be reformulated to a system of smoothing equations, and an improved smoothing method is proposed for solving the smoothing equations. Different from the processing in [12, 14], we solve a system of linear equations instead of a quadratic programming problem for each inner iteration. Moreover, when the iteration point is close enough to the solution point of NCP, the algorithm always takes the full steps. Without strict complementarity conditions and the assumption of P_0 property, we prove that the proposed smoothing method possess the global and local quadratic convergence properties. Compared with previous smoothing methods, our method has some other good properties. Especially,

- Compared with the classical smoothing Newton method, our proposed method needn't nonsingular of the smoothing approximation function;
- Without requiring strict complementarity conditions and the assumption of P_0 property, proposed smoothing method is also proved to possess global and local quadratic convergence rate.

We introduce the following notations. Throughout this paper, all vectors are column vectors, the superscript ^{T} denotes transpose of a matrix and a vector, $\|x\|$ stands for the 2-norm of vector $x \in \Re^n$. $\Re^n_+$ ($\Re^n_{++}$) denotes the nonnegative (positive) orthant in $\Re^n$. For a continuously differentiable function $\Phi_\tau(x) : \Re^n \to \Re^n$, we denote the Jacobian of $\Phi_\tau(x)$ at $x \in \Re^n$ by $\Phi'_\tau(x)$.

The rest of paper is organized as follows. In section 2, we investigate a one-parametric class of smoothing function and discuss its properties, and recall some preliminary result used in the subsequent. The algorithm model is stated in section 3. In section 4, the global and local quadratic convergence of the new algorithm is established. Some numerical test results are reported in section 5, which show that the proposed algorithm is efficient.

2 Smoothing function and its properties

In this section, we investigate a parameter smoothing function and discuss its properties. Based on this smoothing function, the equivalent smoothing reformulation of NCP is given. Firstly, we recall a class of NCP function, which was defined in [16],

$$\varphi_\theta(a, b) := a + b - \sqrt{\theta(a-b)^2 + (1-\theta)(a^2+b^2)}, \quad \theta \in [0, 1], \forall (a, b) \in \Re^2.$$

It has the following characterizations

$$\varphi_\theta(a, b) = 0 \Leftrightarrow a \geq 0, \; b \geq 0 \text{ and } ab = 0.$$

Using the function φ_θ, we can reformulate Problem (1.1) as the following system of nonlinear equations

$$\Phi(x) = 0,$$

where $\Phi : \Re^n \to \Re^n$ is defined by

$$\Phi(x) := \begin{pmatrix} \varphi_\theta(x_1, F_1(x)) \\ \vdots \\ \varphi_\theta(x_n, F_n(x)) \end{pmatrix}. \tag{2.1}$$

Then the natural merit function $\Psi : \Re^n \to [0, +\infty)$ is defined by

$$\Psi(x) = \frac{1}{2}\|\Phi(x)\|^2.$$

Therefore, the following equivalence relation is established,

$$x^* \text{ solves Problem}(1.1) \Leftrightarrow x^* \text{ solves } \Phi(x) = 0 \Leftrightarrow x^* \text{ solves } \min_{x\in\Re^n} \Psi(x) \text{ with } \Psi(x^*) = 0.$$

As we all known, the nonsmooth equation $\Phi(x) = 0$ is very difficult to solve. In order to overcome the difficulty, Zhu et al.[22] introduced the following smoothing function for φ_θ:

$$\varphi_\theta(\tau, a, b) = a + b - \sqrt{\theta(a-b)^2 + (1-\theta)(a^2+b^2) + 2\tau^2}, \quad \forall(\tau, a, b) \in \Re^3, \tag{2.2}$$

where θ is a given constant with $\theta \in [0, 1]$. It is obvious that when $\theta = 1$, φ_θ reduces to the famous CHKS smoothing function, $\theta = 0$, φ_θ reduces to the famous Fischer-Burmeister smoothing function. As we all know, these two smoothing functions and their variants have been widely used in designing smoothing-type methods for solving mathematical programming problems, such as the nonlinear complementarity problems (NCPs) [16–22], the second-order cone complementarity problems(SOCCPs)[23–29], the second-order cone programming (SOCP) [30–39].

Using the smoothing function (2.2), a smooth approximation of Φ is defined by $\Phi_\tau : \Re^n \to \Re^n$

$$\Phi_\tau(x) := \begin{pmatrix} \varphi_\theta(\tau, x_1, F_1(x)) \\ \vdots \\ \varphi_\theta(\tau, x_n, F_n(x)) \end{pmatrix}, \tag{2.3}$$

and the corresponding merit function can also be defined by

$$\Psi_\tau(x) = \frac{1}{2}\|\Phi_\tau(x)\|^2.$$

So, to solve Problem (1.1), we only need to solve $\Phi_\tau(x) = 0$ and make $\tau \downarrow 0$.

Next, we review the concept of semismooth, which was first introduced by Mifflin [40] for functions and extended to vector-valued functions by Qi and Sun [41].

Definition 2.1. *Suppose that $F : \Re^n \to \Re^n$ is locally Lipschitz function. The generalized Jacobian of F at x in the sense of Clark [42] denote by $\partial F(x)$, then, F is said to be semismooth (or strongly semismooth) at $x \in \Re^n$, if F is directionally differentiable at $x \in \Re^n$ and $F(x + h) - F(x) - Vh = o(\|h\|)$ (or=$O(\|h\|^2)$) holds for any $V \in \partial F(x + h)$.*

Lemma 2.1. *The function $\Phi_\tau(x)$ satisfies the inequality*

$$\|\Phi_{\tau_1}(x) - \Phi_{\tau_2}(x)\| \le \kappa|\tau_1 - \tau_2| \tag{2.4}$$

for all $x \in \Re^n$ and $\tau_1, \tau_2 \ge 0$, where $\kappa = \sqrt{2n}$. In particular, we have

$$\|\Phi_\tau(x) - \Phi(x)\| \le \kappa\tau$$

for all $x \in \Re^n$ and $\tau \ge 0$.

Proof. It is obvious to hold for $\tau_1 = \tau_2 = 0$. So, we suppose at least one of the perturbation parameters is positive. By simple calculation, we can have

$$\begin{aligned} |\Phi_{\tau_1,i}(x) - \Phi_{\tau_2,i}(x)| &= |\eta_\theta(\tau_1, x_i, F_i(x)) - \eta_\theta(\tau_2, x_i, F_i(x))| \\ &= \frac{2|\tau_1^2 - \tau_2^2|}{|\eta_\theta(\tau_1, x_i, F_i(x)) + \eta_\theta(\tau_2, x_i, F_i(x))|} \end{aligned}$$

$$\leq \frac{2|\tau_1^2 - \tau_2^2|}{\sqrt{2\tau_1^2} + \sqrt{2\tau_1^2}}$$
$$= \sqrt{2}|\tau_1 - \tau_2|. \tag{2.5}$$

Obviously, for any $x \in \Re^n$,

$$\|\Phi_{\tau_1}(x) - \Phi_{\tau_2}(x)\| = \sqrt{\sum_{i=1}^{n} |\Phi_{\tau_1,i}(x) - \Phi_{\tau_2,i}(x)|^2} \leq \kappa|\tau_1 - \tau_2|,$$

where $\kappa = \sqrt{2n}$. □

Similar to the proof of Lemma 2.8 in [22], we have the following lemma.

Lemma 2.2. *Let $\Phi_\tau(x)$ be defined by (2.3). Then,*

$$\lim_{\tau\downarrow 0} dist(\Phi'_\tau(x), \partial_C\Phi(x)) = 0,$$

where $\partial_C\Phi(x)$ is the C-subdifferential of $\Phi(x)$, and $\Phi'_\tau(x)$ is the Jacobian of $\Phi_\tau(x)$.

Similar to the proof of Proposition 3.4 in [8], we have the following lemma.

Lemma 2.3. *Let $x \in \Re^n$ be arbitrary but fixed. Assume that x is not a solution of NCP. Let us define the constants*

$$\gamma(x) := \max_{i\in\beta(x)}\{\|x_i e_i + F_i(x)\nabla F_i(x)\|\} \geq 0$$

and

$$\alpha(x) := \max_{i\in\beta(x)}\{x_i^2 + F_i(x)^2\} > 0,$$

where $\beta(x) := \{i|x_i = F_i(x)\}$. Let $\delta > 0$ be given, and define

$$\overline{\tau}(x,\delta) := \begin{cases} 1, & \text{if } \frac{n\gamma(x)^2}{\delta^2} - \alpha(x) \leq 0, \\ \frac{\alpha(x)^2}{2}\left(\frac{\delta^2}{n\gamma(x)^2 - \delta^2\alpha(x)}\right), & \text{otherwise.} \end{cases}$$

Then

$$dist(\Phi'_\tau(x), \partial_C\Phi(x)) \leq \delta$$

for all τ such that $0 < \tau \leq \overline{\tau}(x,\delta)$.

3 Proposed Algorithm

In this section, we establish a new smoothing method for Problem(1.1) and prove that well-definiteness of the proposed algorithm.

Algorithm 3.1.

Step 0 Given a starting point $x^0 \in \Re^n$, choose $\alpha, \sigma, \eta, \rho, \gamma \in (0,1), \delta \in (0,\infty)$, set $\beta_0 = \|\Phi(x_0)\|, \kappa = \sqrt{2n}, \tau_0 = \frac{\alpha}{2\kappa}\beta_0, \mu_0 = \|\Phi_{\tau_0}(x^0)\|, k := 0$.

Step 1 Find a solution $d^k \in \Re^n$ of the linear system

$$\left(\Phi'_{\tau_k}(x^k)^T\Phi'_{\tau_k}(x^k) + \mu_k I\right) d = -\Phi'_{\tau_k}(x^k)^T\Phi_{\tau_k}(x^k) \tag{3.1}$$

Step 2 If the condition

$$\|\Phi_{\tau_k}(x^k + d^k)\| \leq \gamma\|\Phi_{\tau_k}(x^k)\| \tag{3.2}$$

is satisfied, then set $x^{k+1} := x^k + d^k$(we call this 'fast step') and go to Step 4.
Step 3 set $\lambda_k = \rho^{m_k}$ and $x^{k+1} = x^k + \lambda_k d^k$, where m_k be the smallest nonnegative integer m such that

$$\Psi_{\tau_k}(x^k + \rho^m d^k) - \Psi_{\tau_k}(x^k) \le \sigma\rho^m \nabla\Psi_{\tau_k}(x^k)^T d^k. \tag{3.3}$$

Step 4 If $\nabla\Psi(x^{k+1}) = 0$, stop.
Step 5 If

$$\|\Phi(x^{k+1})\| \le \max\left\{\eta\beta_k, \alpha^{-1}\|\Phi(x^{k+1}) - \Phi_{\tau_k}(x^{k+1})\|\right\}, \tag{3.4}$$

then set

$$\beta_{k+1} := \|\Phi(x^{k+1})\|$$

and choose τ_{k+1} satisfied

$$0 < \tau_{k+1} \le \min\left\{\left(\frac{\alpha}{2\kappa}\beta_{k+1}\right)^2, \frac{\tau_k}{2}, \overline{\tau}(x^{k+1}, \delta\beta_{k+1})\right\}, \tag{3.5}$$

where $\overline{\tau}(\cdot,\cdot)$ is defined in Lemma 2.4; otherwise, let $\beta_{k+1} := \beta_k$ and $\tau_{k+1} := \tau_k$.
Step 6 Set $\mu_{k+1} = \|\Phi_{\tau_{k+1}}(x^{k+1})\|$ and $k := k + 1$. Go to Step 1.
Remark Differently from the algorithm in [7, 15], our proposed algorithm is well-defined without the assumption that smoothing function $\Phi_{\tau_k}(x^k)$ is nonsingular.

For proving that the algorithm 3.1 is well-defined and has the global convergence property, we assume that Algorithm 3.1 does not terminate in a finite number of iterations.
Define the index set

$$K := \{0\} \cup \left\{k| \ \|\Phi(x^k)\| \le \max\{\eta\beta_{k-1}, \alpha^{-1}\|\Phi(x^k) - \Phi_{\tau_{k-1}}(x^k)\|\}, k = 1, 2, \dots\right\} \tag{3.6}$$

It follows from Lemma 2.1 and Step 5 of the Algorithm 3.1 that

$$\|\Phi(x^k) - \Phi_{\tau_k}(x^k)\| < \alpha\|\Phi(x^k)\|, \tag{3.7}$$

which indicates $\|\Phi_{\tau_k}(x^k)\| \neq 0$ for all $k \in K$. By the definition $\overline{\tau}(\cdot,\cdot)$ in Lemma 2.3 and the updating rule (3.5), it is not difficult to find that

$$dist(\Phi'_{\tau_k}(x^k), \partial_C\Phi(x^k)) \le \delta\|\Phi(x^k)\| \tag{3.8}$$

holds, for any $k \in K$ with $k \ge 1$.

The following proposition shows the well-definiteness of the proposed algorithm.

Proposition 3.1. *Suppose that x_k is the sequence generated by Algorithm 3.1. Then Algorithm 3.1 is well-defined.*

Proof. (1) If $d^k = 0$. Obviously, there exists a finite nonnegative m_k such that (3.3) always holds.

(2) If $d^k \neq 0$. From the above analysis, we know that $\mu_k = \|\Phi_{\tau_k}(x^k)\|$ is positive, then $\Phi'_{\tau_k}(x^k)^T\Phi'_{\tau_k}(x^k)+\mu_k I$ is always positive definite. Since, $\nabla\Psi_{\tau_k}(x^k) = \Phi'_{\tau_k}(x^k)^T\Phi_{\tau_k}(x^k)$. By the construction of Algorithm 3.1, we have

$$\begin{aligned}\nabla\Psi_{\tau_k}(x^k)^T d^k &= -(d^k)^T(\Phi'_{\tau_k}(x^k)^T\Phi'_{\tau_k}(x^k) + \mu_k I)(d^k)\\ &< 0,\end{aligned} \tag{3.9}$$

moreover

$$\begin{aligned}\Psi_{\tau_k}(x^k + \lambda d^k) - \Psi_{\tau_k}(x^k) &= \lambda\nabla\Psi_{\tau_k}(x^k)^T d^k + o(\lambda)\\ &= \sigma\lambda\nabla\Psi_{\tau_k}(x^k)^T d^k + (1-\sigma)\lambda\nabla\Psi_{\tau_k}(x^k)^T d^k + o(\lambda).\end{aligned} \tag{3.10}$$

It follows from (3.9) and $\sigma \in (0, 1)$ that

$$(1-\sigma)\lambda\nabla\Psi_{\tau_k}(x^k)^T d^k < 0. \tag{3.11}$$

By (3.10) and (3.11), we can obtain the following inequality

$$\Psi_{\tau_k}(x^k + \lambda d^k) - \Psi_{\tau_k}(x^k) \le \sigma\lambda \nabla \Psi_{\tau_k}(x^k)^T d^k.$$

So, there exists a finite nonnegative integer m_k such that

$$\Psi_{\tau_k}(x^k + \rho^{m_k} d^k) - \Psi_{\tau_k}(x^k) \le \sigma\rho^{m_k} \nabla \Psi_{\tau_k}(x^k)^T d^k.$$

Therefore, Algorithm 3.1 is well-defined. □

4 Convergence of the proposed algorithm

In this section, we will give the global and local quadratic convergence of the proposed algorithm.

Theorem 4.1. *Suppose that $\{x^k\}$ is a sequence generated by Algorithm 3.1. If there exists at least an accumulation point in the sequence $\{x^k\}$, then the index set K defined by (3.6) is infinite, and*

$$\lim_{k\to\infty} \tau_k = 0, \ \lim_{k\to\infty} \Phi_{\tau_k}(x^k) = 0, \ \lim_{k\to\infty} \Phi(x^k) = 0.$$

Proof. Assume that x^* is an arbitrary accumulation point of the sequence $\{x^k\}$, and $\{x^k\}_{k\in K}$ be a subsequence converging to x^*. We first show that the index set K is infinite. By contradiction, we assume that the set K is finite. Let $\widehat{k}$ be the largest number in K, then for any $k \ge \widehat{k}, we have \tau_k = \tau_{\widehat{k}}$ and $\beta_k = \beta_{\widehat{k}}$.
Denote

$$\widehat{\tau} = \tau_{\widehat{k}}, \ \widehat{\beta} = \beta_{\widehat{k}}, \ q(x) = \Phi(x) - \Phi_{\widehat{\tau}}(x),$$

then for all $k \ge \widehat{k}$, we have

$$\|\Phi(x^k)\| > \max\left\{\eta\widehat{\beta}, \frac{\|q(x^k)\|}{\alpha}\right\} \tag{4.1}$$

and

$$\Phi(x^k) = \Phi_{\widehat{\tau}}(x^k) + q(x^k). \tag{4.2}$$

In the following paragraphs, we will show that the following equation holds.

$$\nabla \Psi_{\widehat{\tau}}(x^*) = 0. \tag{4.3}$$

Now, we consider two cases for the sequence $\{\mu_k\}$:
case 1: If $\mu_k \to 0 (k \in K)$, then, we can obtain $\|\Phi_{\widehat{\tau}}(x^k)\| \to 0 (k \in K)$. It follows from the boundness of $\{\|\Phi'_{\widehat{\tau}}(x^k)\|\}$ that

$$\nabla \Psi_{\widehat{\tau}}(x^k) = \|\Phi'_{\widehat{\tau}}(x^k)\|\|\Phi_{\widehat{\tau}}(x^k)\| \to 0 \ (k \in K), \tag{4.4}$$

which implies $\nabla \Psi_{\widehat{\tau}}(x^*) = 0$.
case 2: In this case, there exists a constant ξ such that $\mu_k \ge \xi > 0$ for all $k \in K$. Next, we show that $\nabla \Psi_{\widehat{\tau}}(x^*) = 0$. Suppose to the contrary that $\nabla \Psi_{\widehat{\tau}}(x^*) \ne 0$.

It follows from (3.1), we have

$$\|d^k\| \ge \frac{\left\|\Phi'_{\widehat{\tau}}(x^k)^T \Phi_{\widehat{\tau}}(x^k)\right\|}{\left\|(\Phi'_{\widehat{\tau}}(x^k)^T \Phi'_{\widehat{\tau}}(x^k) + \mu_k I\right\|}, \tag{4.5}$$

and

$$\begin{aligned} \|d^k\| &= \|(\Phi'_{\widehat{\tau}}(x^k)^T \Phi'_{\widehat{\tau}}(x^k) + \mu_k I)^{-1} \Phi'_{\widehat{\tau}}(x^k)^T \Phi_{\widehat{\tau}}(x^k)\| \\ &\le \|(\Phi'_{\widehat{\tau}}(x^k)^T \Phi'_{\widehat{\tau}}(x^k) + \mu_k I)^{-1}\|\|\Phi'_{\widehat{\tau}}(x^k)^T \Phi_{\widehat{\tau}}(x^k)\| \\ &\le \frac{\|\Phi'_{\widehat{\tau}}(x^k)^T \Phi_{\widehat{\tau}}(x^k)\|}{\xi}. \end{aligned} \tag{4.6}$$

From the continuity of $\Phi_{\widehat{\tau}}(x^k)$ and the upper semicontinuity of the generalized Jacobian, we deduce that for a constant $\delta > 0$,

$$\|\Phi'_{\widehat{\tau}}(x^k)^T \Phi'_{\widehat{\tau}}(x^k) + \mu_k I\| \leq \delta \text{ for all } k \in K. \tag{4.7}$$

We now note that

$$0 < m \leq \|d^k\|, \text{ for some positive } m. \tag{4.8}$$

In fact, if $\{\|d^k\|\}_K \to 0$, by (4.5) and (4.7), we can deduce $\{\|\Phi'_{\widehat{\tau}}(x^k)^T \Phi_{\widehat{\tau}}(x^k)\|\}_K \to 0$, thus contradicting the assumption $\nabla \Psi_{\widehat{\tau}}(x^\star) \neq 0$. On the other hand, we denote the bound of $\{\|\Phi'_{\widehat{\tau}}(x^k)^T \Phi_{\widehat{\tau}}(x^k)\|\}$ as M, it follows from (4.6) and (4.8) that

$$0 < m \leq \|d^k\| \leq \frac{M}{\xi}. \tag{4.9}$$

Since $\mu_k = \|\Phi_{\widehat{\tau}}(x^k)\| > 0$, for all $k \in K$, $\Phi'_{\widehat{\tau}}(x^k)^T \Phi_{\widehat{\tau}}(x^k) + \mu_k I$ is positive definite, this together with (3.1) and (4.9), implies that

$$\nabla \Psi_{\widehat{\tau}}(x^k)^T d^k = -(d^k)^T \left(\Phi'_{\widehat{\tau}}(x^k)^T \Phi_{\widehat{\tau}}(x^k) + \mu_k I \right) d^k < 0. \tag{4.10}$$

Next, we consider two cases for the sequence $\{\inf_k \lambda_k\}$:
(1) If $\inf_{k \in K} \lambda_k = \lambda^\star > 0$ for all $k \geq \widehat{k}$. By (3.3) and (4.10), we can obtain that

$$\begin{aligned} \Psi_{\widehat{\tau}}(x^{k+1}) - \Psi_{\widehat{\tau}}(x^k) &\leq \sigma \lambda_k \nabla \Psi_{\widehat{\tau}}(x^k)^T d^k \\ &\leq \sigma \lambda^\star \nabla \Psi_{\widehat{\tau}}(x^k)^T d^k \\ &< 0. \end{aligned} \tag{4.11}$$

Using $\Psi_{\widehat{\tau}}(x^{k+1}) - \Psi_{\widehat{\tau}}(x^k) \to 0$ yields that

$$\{\nabla \Psi_{\widehat{\tau}}(x^k)^T d^k\}_K \to 0. \tag{4.12}$$

By (3.1), we have

$$\nabla \Psi_{\widehat{\tau}}(x^k)^T d^k = -\nabla \Psi_{\widehat{\tau}}(x^k)^T \left(\Phi'_{\widehat{\tau}}(x^k)^T \Phi'_{\widehat{\tau}}(x^k) + \mu_k I \right)^{-1} \nabla \Psi_{\widehat{\tau}}(x^k) \ \forall k. \tag{4.13}$$

Since $\partial \Phi_{\widehat{\tau}}(x)$ is a nonempty compact set for any $x \in \Re^n$, $\{\Phi'_{\widehat{\tau}}\}_K$ is bounded. Without loss of generality, assume that $\{\Phi'_{\widehat{\tau}}\}_K \to \Phi_\star$. Considering that the set-valued mapping $x \to \partial \Phi_{\widehat{\tau}}(x)$ is closed and $\{x^k\}_K \to x^\star$, we have $\Phi_\star \in \partial \Phi_{\widehat{\tau}}(x^\star)$. In addition, since $\Phi_{\widehat{\tau}}(x^\star) \neq 0$, we have $\mu_k \to \mu_\star$ with $\mu_\star = \|\Phi_{\widehat{\tau}}(x^\star)\| > 0$. Thus, $\{\Phi'_{\widehat{\tau}}(x^k)^T \Phi'_{\widehat{\tau}}(x^k) + \mu_k I\}_K \to \Phi_\star^T \Phi_\star + \mu_\star I \succ 0$. This together with (4.13) and the continuity of $\nabla \Psi_{\widehat{\tau}}$, implies that $\nabla \Psi_{\widehat{\tau}}(x^k)^T d^k$ has a nonzero limit as $k \to +\infty$, which contradicts (4.12). Hence $\nabla \Psi_{\widehat{\tau}}(x^\star) = 0$.
(2) If $\inf_{k \in K} \lambda_k = 0$. In this case, without loss of generality, we assume that $\{\lambda_k\}_{k \in K} \to 0$. By (4.9), there exists a subsequence $\{d^k\}_{k \in K} \to \overline{d} \neq 0$.

From the line search rule (3.3), it is clear that for all $k \geq \widehat{k}$,

$$\frac{\Psi_{\widehat{\tau}}(x^k + \rho^{m_k - 1} d^k) - \Psi_{\widehat{\tau}}(x^k)}{\rho^{m_k - 1}} > \sigma \nabla \Psi_{\widehat{\tau}}(x^k)^T d^k. \tag{4.14}$$

On K, by taking the limit $k \to \infty$, we obtain from (4.14),

$$\nabla \Psi_{\widehat{\tau}}(x^\star)^T \overline{d} \geq \sigma \nabla \Psi_{\widehat{\tau}}(x^\star)^T \overline{d}. \tag{4.15}$$

Since, $\sigma \in (0, 1)$, then, by (4.15) we have $\nabla \Psi_{\widehat{\tau}}(x^\star)^T \overline{d} \geq 0$. On the other hand, $\overline{d}$ is the solution of $(\Phi_\star^T \Phi_\star + \mu_\star I)\overline{d} = -\nabla \Psi_{\widehat{\tau}}(x^\star)$, which implies $\nabla \Psi_{\widehat{\tau}}(x^\star)^T \overline{d} < 0$, (since $(\Phi_\star^T \Phi_\star + \mu_\star I) \succ 0$.) Thus, we get a contradiction. Then, $\nabla \Psi_{\widehat{\tau}}(x^\star) = 0$.

To sum up by case (1) and case (2), we have $\nabla \Psi_{\widehat{\tau}}(x^\star) = 0$.

On the other hand, in view of Lemma 2.1 and (4.3), we have $\Psi_{\widehat{\tau}}(x^k) \to \Psi_{\widehat{\tau}}(x^\star) = 0$, then, there exists $\widetilde{k} \geq \widehat{k}$ such that for all $k \in L$ with $k \geq \widetilde{k}$,

$$\|\Phi_{\widehat{\tau}}(x^k)\| \leq (1 - \alpha)\eta \widehat{\beta}.$$

In view of (4.1) and (4.2), we deduce that for all $k \in K$ with $k \geq \widetilde{k}$,

$$\begin{aligned}\|\Phi_{\widehat{\tau}}(x^k)\| &\leq (1-\alpha)\|\Phi(x^k)\| \\ &\leq (1-\alpha)(\|\Phi_{\widehat{\tau}}(x^k)\| + \|q(x^k)\|),\end{aligned}$$

i.e.,

$$\|\Phi_{\widehat{\tau}}(x^k)\| \leq (1-\frac{1}{\alpha})\|q(x^k)\|.$$

Therefore,

$$\begin{aligned}\|\Phi(x^k)\| &\leq \|\Phi_{\widehat{\tau}}(x^k)\| + \|q(x^k)\| \\ &< \frac{\|q(x^k)\|}{\alpha}\end{aligned}$$

which in contradiction to (4.1). Hence the set K is infinite.

By the updating rule of τ_k, we have $\{\tau_k\} \to 0$. Similar to the proof of Proposition 4.2 in [8], we deduce that

$$\|\Phi(x^k)\| \leq r^j(1+\alpha)\|\Phi(x^0)\|, \ \ as\ k_j \leq k < k_{j+1}. \tag{4.16}$$

where $r = \max\{\frac{1}{2}, \eta\}$.

Since the set K is infinite, it follows from (3.8) and (4.16) that

$$\lim_{k\to\infty} \Phi_{\tau_k}(x^k) = 0, \ \lim_{k\to\infty} \Phi(x^k) = 0.$$

□

As a consequence of Theorem 4.1, we can obtain the following global convergence result.

Theorem 4.2. *Let $\{x^k\}$ be a sequence generated by Algorithm 3.1, then every accumulation point of the sequence $\{x^k\}$ is a solution of the Problem(1.1).*

In order to obtain the local convergent result, we introduce the following lemma.

Lemma 4.1. *Let $\{x^k\}_{k\in K}$ converge to x^*. If $V \in \partial_C\Phi(x^*)$ is nonsingular, then there exist two constants $M_1 > 0$, $M_2 > 0$ and $\widehat{k}$ such that for all $k \in K$ with $k \geq \widehat{k}$,*

$$\|\Phi'_{\tau_k}(x^k)\| \leq M_1, \ \|(\Phi'_{\tau_k}(x^k)^T\Phi'_{\tau_k}(x^k))^{-1}\| \leq M_2. \tag{4.17}$$

Proof. By the condition, we know that the matrix V is nonsingular. Notice that $\partial_C\Phi(x)$ is compact for all $x \in \Re^n$. Therefore, there exists $V_k \in \partial_C\Phi(x^k)$ such that

$$dist(\Phi'_{\tau_k}(x^k), \partial_C\Phi(x^k)) = \|\Phi'_{\tau_k}(x^k) - V_k\|.$$

Combining with (3.8), we can deduce that for all $k \in K$,

$$\|\Phi'_{\tau_k}(x^k) - V_k\| \leq \delta\beta_k = \delta\|\Phi(x^k)\|.$$

By theorem 4.1, we have $\{\|\Phi(x^k)\|\} \to 0$. This together with the compactness and the upper semicontinuity of $\partial_C\Phi(x^*)$, we obtain from the above inequality that the matrices $\Phi'_{\tau_k}(x^k)$ and $\Phi'_{\tau_k}(x^k)^T\Phi'_{\tau_k}(x^k)$ are nonsingular and there exist $M_1 > 0$, $M_2 > 0$ such that (4.17) holds. □

The following lemma can be seen in Ref. [44].

Lemma 4.2. *Assume that $A, B \in \Re^{n\times n}$ and A is nonsingular. If $\|A^{-1}B\| < 1$, then $A + B$ is nonsingular and satisfies*

$$\|(A+B)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}B\|}.$$

From Lemmas 4.1 and 4.2, we can prove that the following lemmas hold.

Lemma 4.3. *Assume that the conditions of Lemma 4.1 hold. Then, for* $\mu_k \le \frac{1}{2M_2}$,

$$\|(\Phi'_{\tau_k}(x^k)^T \Phi'_{\tau_k}(x^k) + \mu_k I)^{-1}\| \le 2M_2. \tag{4.18}$$

Theorem 4.3. *Assume that* $\{x^k\}$ *is a sequence generated by Algorithm 3.1. If* $x^\star$ *is an accumulation point of the sequence* $\{x^k\}$, *and for all* $V \in \partial_C \Phi(x^\star)$ *are nonsinglar. Then the sequence* $\{x^k\}$ *converges to* $x^\star$ *quadratically.*

Proof. It follows from Theorem 4.1 that $x^\star$ is a solution of $\Phi(x) = 0$. Notice that $\partial_B \Phi(x^\star) \subseteq \partial_C \Phi(x^\star)$. From Proposition 2.5 in [45], it follows that there exists a neighbourhood of $x^\star$ such that $x^\star$ is the unique solution.

Since the sequence $\{x^k\}$ has an accumulation point $x^\star$, there exists a subsequence $\{x^k\}_{k\in K}$ that converges to $x^\star$. By Lemma 4.1,there exist $M_1 > 0$, $M_2 > 0$ and $k \in K$,

$$\|\Phi'_{\tau_k}(x^k)\| \le M_1, \ \|(\Phi'_{\tau_k}(x^k)^T \Phi'_{\tau_k}(x^k))^{-1}\| \le M_2.$$

Further, from Lemma 2.1, (3.5) and the Lipschitz continuity of $\Phi(x)$, we deduce that

$$\beta_k = \Phi(x^k) = O(\|x^k - x^\star\|), \tag{4.19}$$

$$\tau_k = O(\|x^k - x^\star\|^2), \tag{4.20}$$

$$\mu_k = \|\Phi_{\tau_k}(x^k)\| \le \kappa\tau_k + \|\Phi(x^k)\| = O(\|x^k - x^\star\|). \tag{4.21}$$

Then, it follows from (4.21) and Lemmas 4.2, 4.3 that

$$\|(\Phi'_{\tau_k}(x^k)^T \Phi'_{\tau_k}(x^k) + \mu_k I)^{-1}\| \le 2M_2.$$

Therefore, for all $k > \widehat{k}$ and $k \in K$, we have

$$\begin{aligned}
\|x^k + d^k - x^\star\| &= \| - (\Phi'_{\tau_k}(x^k)^T \Phi'_{\tau_k}(x^k) + \mu_k I)^{-1} \Phi'_{\tau_k}(x^k)^T \Phi_{\tau_k}(x^k) + x^k - x^\star\| \\
&\le \|(\Phi'_{\tau_k}(x^k)^T \Phi'_{\tau_k}(x^k) + \mu_k I)^{-1}\| \\
&\quad \|\Phi'_{\tau_k}(x^k)^T \Phi_{\tau_k}(x^k) - (\Phi'_{\tau_k}(x^k)^T \Phi'_{\tau_k}(x^k) + \mu_k I)(x^k - x^\star)\| \\
&\le \|(\Phi'_{\tau_k}(x^k)^T \Phi'_{\tau_k}(x^k) + \mu_k I)^{-1}\| \\
&\quad \left\{\|\Phi'_{\tau_k}(x^k)^T (\Phi_{\tau_k}(x^k) - \Phi'_{\tau_k}(x^k)(x^k - x^\star))\| + \mu_k \|x^k - x^\star\|\right\} \\
&\le 2M_1 M_2 \left\{\|\Phi'_{\tau_k}(x^k)^T (\Phi_{\tau_k}(x^k) - \Phi'_{\tau_k}(x^k)(x^k - x^\star))\| + \mu_k \|x^k - x^\star\|\right\} \\
&\le 2M_1 M_2 \left\{\mu_k \|x^k - x^\star\| + \|(\Phi_{\tau_k}(x^k) - \Phi(x^k)) \right. \\
&\quad \left. -(\Phi'_{\tau_k}(x^k) - V_k)(x^k - x^\star) + \Phi(x^k) - \Phi(x^\star) - V_k(x^k - x^\star)\|\right\} \\
&\le 2M_1 M_2 \left\{\|\Phi_{\tau_k}(x^k) - \Phi(x^k)\| + \delta\beta_k \|x^k - x^\star\| \right. \\
&\quad \left. + \|\Phi(x^k) - \Phi(x^\star) - V_k(x^k - x^\star)\| + \mu_k \|x^k - x^\star\|\right\} \\
&\le 2M_1 M_2 \left\{\kappa\tau_k + \delta\beta_k \|x^k - x^\star\| + \mu_k \|x^k - x^\star\| \right. \\
&\quad \left. + \|\Phi(x^k) - \Phi(x^\star) - V_k(x^k - x^\star)\|\right\}
\end{aligned} \tag{4.22}$$

By Proposition 2.3 in [16], we known that $\Phi(x)$ is strongly semismooth, combining with Theorem 3.2 in [41], we have

$$\begin{aligned}
\|\Phi(x^k) - \Phi(x^\star) - V_k(x^k - x^\star)\| &\le \Sigma_{i=1}^n \|\Phi_i(x^k) - \Phi_i(x^\star) - V_k^i(x^k - x^\star)\| \\
&= O\left(\|x^k - x^\star\|^2\right) \ \ as\ k \to \infty, \ k \in K,
\end{aligned} \tag{4.23}$$

where V_k^i denotes the i-th row of V_k. Hence, From (4.19)-(4.23), we can deduce

$$\|x^k + d^k - x^\star\| = O(\|x^k - x^\star\|^2) \ \ as\ k \to \infty,\ k \in K. \tag{4.24}$$

The above equation together with (4.21) indicate that

$$\|\Phi_{\tau_{k+1}}(x^k + d^k)\| = O(\|x^k + d^k - x^\star\|) = O(\|x^k - x^\star\|^2) \ \ as\ k \to \infty,\ k \in K. \tag{4.25}$$

It follows from triangular inequality and (3.8) that

$$\begin{aligned}\|\Phi_{\tau_k}(x^k)\| &= \|\Phi(x^k) - (\Phi(x^k) - \Phi_{\tau_k}(x^k))\| \\ &\geq \|\Phi(x^k)\| - \alpha\|\Phi(x^k)\| \\ &= (1-\alpha)\|\Phi(x^k)\|. \end{aligned} \tag{4.26}$$

The above equations (4.25) and (4.26) imply that

$$\begin{aligned}\frac{\|\Phi_{\tau_{k+1}}(x^k + d^k)\|}{\|\Phi_{\tau_k}(x^k)\|} &\leq \frac{\|\Phi_{\tau_{k+1}}(x^k + d^k)\|}{(1-\alpha)\|\Phi(x^k)\|} \\ &= \frac{O\left(\|x^k - x^\star\|^2\right)}{(1-\alpha)O\left(\|x^k - x^\star\|\right)} \\ &\to 0 \quad (\text{as } k \to \infty\ k \in K),\end{aligned}$$

which means that there exists $\bar{k} \geq \hat{k}$ such that $\bar{k} \in K$ and for any $k \geq \bar{k}$,

$$\|\Phi_{\tau_{k+1}}(x^k + d^k)\| \leq \gamma\|\Phi_{\tau_k}(x^k)\|.$$

That is, $x^{k+1} = x^k + d^k$ for all $k \geq \bar{k}$ and $k \in K$. This together with (4.24) implies that $\{x^k\}$ converges to $x^\star$ quadratically.

□

5 Numerical experiments

In this section, we implement Algorithm 3.1 on some typical test problems for two purposes: one is to see the numerical behavior of Algorithm 3.1; and the other is to investigate the behavior of these test problems for different $\theta \in [0, 1]$. All the codes are finished in MATLAB 7.8 and done using a PC with Intel (R) Core (TM) i3-3240 CPU @ 3.40 GHz and RAM of 4 GB.

The parameters used in Algorithm 3.1 are as follows:

$$\alpha = 0.95,\ \sigma = 0.01,\ \eta = 0.9,\ \rho = 0.8,\ \gamma = 0.9,\ \delta = 30.$$

We use

$$\|\nabla\Psi(x^k)\| \leq 10^{-6}$$

as the stop rule.

In the tables of experimental results, ST denotes the starting point; DIM denotes the dimension of the problem; θ denotes the values of θ; IT denotes the number of iteration; Fast denotes the number of 'full step' taken during the iteration; B denotes the number of backtracking steps; τ denotes the value of τ at the final iteration.

From Tables 1-3, It is easy to see that not all the best numerical results based on the proposed smoothing algorithm occur in the case of $\theta = 1$ (in this case, the smoothing function is the famous CHKS smoothing function) or $\theta = 0$ (in this case, the smoothing function is the famous Fischer-Burmeister smoothing function). On the other hand, from the column Fast, it can be seen that, in general, the number of the iteration in which 'fast step' are accepted occupied almost all the iterations. From Table 2, we can see that $\theta = 1$ seems to be more suitable to find a non-degenerate solution. We also investigate the behavior of our proposed algorithm for solving large scale linear complementarity problems with $\theta = 1$. The numerical results are listed in Tables 4 and 5. From Tables 4 and 5, we can see that our proposed method is efficient for solving large scale linear complementarity problems. From all the test results, we can see that our algorithm is promising.

Example 5.1. *Mathiesen Problem. This test problem was used by Jiang and Qi [46] with four variables, which was also tested by Pang and Gabriel [47]. Let*

$$\begin{aligned} &F_1(x) = -x_2 + x_3 + x_4, \\ &F_2(x) = x_1 - \alpha(b_2x_3 + b_3x_4)/x_2, \\ &F_3(x) = b_2 - x_1 - (1-\alpha)(b_2x_3 + b_3x_4)/x_3, \\ &F_4(x) = b_3 - x_1. \\ &\text{where } \alpha = 0.75,\ b_2 = 1,\ b_3 = 2. \end{aligned}$$

We test this problem by using different starting points. The test results are listed in Table 1.

Table 1: Numerical behavior for Example 5.1

ST	θ	IT	τ	Fast	$\|\nabla\Psi(x^k)\|$	B
$(-2,-2,-2,-2)^T$	0	14	2.8e-12	6	4.4e-7	16
	0.25	12	3.2e-17	7	2.2e-7	11
	0.5	13	9.3e-16	6	4.5e-7	14
	0.75	8	9.2e-21	8	1.4e-9	0
	1	12	2.2e-14	11	5.7e-7	2
$(1,4,1,4)^T$	0	19	4.5e-25	13	1.4e-9	6
	0.25	17	2.2e-17	12	1.3e-12	5
	0.5	15	9.5e-22	10	1.1e-8	5
	0.75	15	2.3e-20	10	2.3e-12	5
	1	22	4.6e-20	8	2.3e-10	14
$(3,3,3,3)^T$	0	14	4.1e-23	14	8.9e-11	0
	0.25	12	4.1e-17	12	5.1e-7	0
	0.5	11	1.1e-17	11	6.5e-8	0
	0.75	11	2.3e-20	11	2.3e-10	0
	1	14	9.7e-24	8	7.8e-11	6

Example 5.2. *Kojima-Shindo Problem. This problem was tested by Pang and Gabriel in [47], which was also tested by Mangasarian and Solodov [48], and Kanzow [49] with four variables. Let*

$$\begin{aligned} &F_1(x) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6, \\ &F_2(x) = 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2, \\ &F_3(x) = 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9, \\ &F_4(x) = x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3. \end{aligned}$$

This example has one degenerate solution $(\frac{\sqrt{6}}{2}, 0, 0, \frac{1}{2})^T$ *and one nondegenerate solution* $(1, 0, 3, 0)^T$. *We summarize the results in Table 2 by using different starting points.*

Table 2: Numerical behavior for Example 5.2

ST	θ	IT	τ	Fast	$\|\nabla\Psi(x^k)\|$	B	solution
$(6, 6, 6, 6)^T$	0	21	3.7e-17	16	3.2e-7	6	Degenerate
	0.25	21	2.8e-28	17	1.8e-12	6	Degenerate
	0.5	16	3.3e-22	16	5.2e-9	0	Degenerate
	0.75	15	7.5e-24	14	2.3e-10	2	Degenerate
	1	23	7.7e-21	22	2.8e-9	6	Nondegenerate
$(1, 2, 3, 4)^T$	0	12	3.4e-32	12	4.9e-14	0	Degenerate
	0.25	11	6.6e-32	10	4.4e-14	2	Degenerate
	0.5	11	5.7e-27	10	1.3e-11	2	Degenerate
	0.75	11	1.1e-17	9	3.5e-7	6	Degenerate
	1	21	8.7e-26	20	3.4e-11	6	Nondegenerate
$(2, -3, -3, 2)^T$	0	13	1.6e-24	13	3.2e-11	0	Degenerate
	0.25	12	2.8e-16	11	5.1e-7	2	Degenerate
	0.5	11	1.3e-22	10	3.1e-9	2	Degenerate
	0.75	11	1.4e-28	10	2.8e-12	3	Degenerate
	1	25	1.5e-25	21	1.3e-11	9	Nondegenerate

Example 5.3. *HS 66 Problem. This test problem is the NCP reformulation for 66th problem in the book of Hock and Schittkowski [50]: Let*

$$\begin{aligned}
F_1(x) &= -0.8 + x_4 e^{x_1} + x_6,\\
F_2(x) &= -x_4 + x_5 e^{x_2} + x_7,\\
F_3(x) &= -0.2 - x_5 + x_8,\\
F_4(x) &= x_2 - e^{x_1},\\
F_5(x) &= x_3 - e^{x_2},\\
F_6(x) &= 100 - x_1,\\
F_7(x) &= 100 - x_2,\\
F_8(x) &= 10 - x_3.
\end{aligned}$$

We summarize the test results in Table 3 by using different starting points.

Table 3: Numerical behavior for Example 5.3

ST	θ	IT	τ	Fast	$\|\nabla\Psi(x^k)\|$	B
$(-1,-1,-1,-1,-1,-1,-1,-1)^T$	0	25	4.7e-18	25	1.1e-8	0
	0.25	22	3.2e-15	22	4.1e-7	0
	0.5	21	5.8e-20	21	2.5e-9	0
	0.75	20	1.5e-19	20	5.7e-9	0
	1	24	1.6e-15	20	7.1e-7	13
$(-1,-1,-1,-1,1,1,1,1)^T$	0	26	7.4e-17	26	9.4e-8	0
	0.25	23	1.4e-17	23	2.9e-8	0
	0.5	21	1.9e-17	21	4.5e-8	0
	0.75	21	1.7e-19	20	5.8e-9	4
	1	24	4.1e-17	21	1.3e-7	17
$(0,0,0,0,0,0,0,0)^T$	0	23	8.7e-16	23	1.4e-7	0
	0.25	20	1.7e-17	20	9.1e-7	0
	0.5	19	5.7e-20	19	2.5e-9	0
	0.75	18	5.3e-21	18	1.3e-9	0
	1	19	7.7e-18	18	4.8e-8	1

Example 5.4. *This problem is from Geiger and Kanzow [51], which was also tested by Kanzow [11]: Let $F(x) = Mx + q$, where*

$$M = \begin{pmatrix} 4 & -1 & 0 & \dots & 0 & 0 \\ -1 & 4 & -1 & \dots & 0 & 0 \\ 0 & -1 & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & -1 \\ 0 & 0 & 0 & \dots & -1 & 4 \end{pmatrix}, \quad q = (-1,-1,\dots,-1)^T.$$

We test this problem by using different starting points. The test results are listed in Table 4.

Example 5.5. *This example was used by Ahn [52]. Let $F(x) = Mx + q$, where*

$$M = \begin{pmatrix} 4 & -2 & 0 & \dots & 0 & 0 \\ 1 & 4 & -2 & \dots & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & -2 \\ 0 & 0 & 0 & \dots & 1 & 4 \end{pmatrix}, \quad q = (-1,-1,\dots,-1)^T.$$

The test results for Example 5.5 are listed in Table 5 by using different starting points.

Table 4: Numerical results for Example 5.4 with $\theta = 1$

ST	DIM	IT	τ	Fast	$\|\nabla\Psi(x^k)\|$	B
$(-1, \cdots, -1)^T$	500	15	1.1e-23	14	1.2e-9	1
	1000	19	7.5e-26	15	1.5e-10	4
	2000	24	2.6e-22	15	1.2e-8	9
	3000	28	6.2e-22	15	2.2e-8	13
$(0, \cdots, 0)^T$	500	8	4.3e-19	8	2.4e-7	0
	1000	10	1.1e-26	10	5.6e-11	0
	2000	12	1.5e-27	12	2.9e-11	0
	3000	13	4.2e-21	13	5.8e-8	0
$(1, \cdots, 1)^T$	500	9	3.9e-29	9	2.4e-12	0
	1000	10	2.2e-21	10	2.5e-8	0
	2000	12	8.5e-22	12	2.2e-8	0
	3000	14	9.1e-28	14	2.8e-11	0

Table 5: Numerical results for Example 5.5 with $\theta = 1$

ST	DIM	IT	τ	Fast	$\|\nabla\Psi(x^k)\|$	B
$(-1, \cdots, -1)^T$	500	11	1.1e-21	11	1.9e-8	0
	1000	14	1.2e-32	13	8.7e-14	1
	2000	17	1.1e-26	15	1.1e-10	2
	3000	19	1.9e-20	15	3.2e-8	4
$(0, \cdots, 0)^T$	500	6	9.5e-22	6	1.7e-8	0
	1000	7	9.2e-25	7	7.6e-10	0
	2000	8	1.9e-24	8	1.6e-9	0
	3000	9	6.5e-29	9	1.1e-11	0
$(1, \cdots, 1)^T$	500	12	3.3e-33	12	3.9e-14	0
	1000	15	2.9e-33	9	4.6e-14	6
	2000	19	2.4e-30	8	2.0e-12	11
	3000	21	1.8e-29	7	6.1e-12	14

6 Conclusion

In this paper, we have presented a new improved smoothing algorithm for the NCP. Compared with the classical smoothing method, our proposed method needn't nonsingular of the smoothing approximation function. We have established the global convergence and the local quadratic convergence for the developed algorithm without strict complementarity conditions and the assumption of P_0 property. Numerical results have showed that the new algorithm works very well.

Competing interests
The authors declare that they have no competing interests.

Acknowledgement: This work was supported by the Training Program of the Major Research Plan of National Science Foundation of China under Grant(91746104), by the National Nature Science Foundation of China(Grant Nos.61101208, 11326186) Qindao Postdoctoral Science Foundation(Grant No.2016114), A Project of Shandong Province Higher Educational Science and Technology Program(Grant No. J17KA166), Joint Innovative Center for Safe and Effective Mining Technology and Equipment of Coal Resources, Shandong Province of China and SDUST Research Fund(Grant No.2014TDJH102).

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