*A New Solution Set for Tournaments and Majority Voting: Further Graph-Theoretical Approaches to the Theory of Voting**

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The Condorcet, or minimal undominated, set of proposals has been identified as a solution set for majority voting. Unfortunately, the Condorcet set may be very large and may include Pareto-inefficient proposals. An alternative solution set is proposed here: the "uncovered set," from which every other proposal in the tournament representing majority preference is reachable in no more than two steps. It is shown that this set has a number of desirable properties, and that several important voting processes lead to decisions within it. This article extends some earlier work by further applying concepts and theorems from the theory of directed graphs to characterize and analyze the structure of majority preference.

In an earlier article that appeared in this journal (Miller, 1977), I used language, concepts, and theorems from the mathematical theory of directed graphs to characterize and analyze the structure of majority preference. I derived a number of results concerning sincere, sophisticated, and cooperative voting decisions under two common majority voting procedures. These results extended the pioneering work of Black (1958) and Farquharson (1969) in developing a theory of voting. The present article carries this work further along.

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Since completing the penultimate version of this article, I have learned of two other papers that cover some of the same ground. First, Fishburn (1977) presents a comparative analysis of various Condorcet social choice functions, one of which (C_9 or "Fishburn's function") picks out from each agenda the uncovered set defined here. Second, Heiner (1977) presents a quite exhaustive analysis of "cycle equalization" methods, of which the uncovered set is a special and limiting case; moreover, Heiner's analysis does not restrict majority preference to a tournament, i.e., majority preference "ties" are allowed (cf. footnote 16 below). Neither of these papers, however, relates the uncovered set to actual voting processes.

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The earlier article showed that majority preference may be represented as a tournament, i.e., a complete asymmetric directed graph, and its results focused primarily on the Condorcet or minimal undominated set of proposals as a solution set for majority voting. For example, it was shown that sincere voting under amendment procedure and sophisticated voting under any binary procedure lead to decisions in the Condorcet set. But, unfortunately, the Condorcet set may be very large, to the point that it includes every proposal under consideration. Also, the Condorcet set may include Pareto-inefficient proposals. Thus, as a solution set for majority voting, the Condorcet set is far from satisfactory both normatively and (perhaps) empirically.

In this article, I define and analyze another solution set for tournaments and majority voting—the uncovered set. I show that this set is contained within the Condorcet set, that it is never empty, that every proposal in it is Pareto-optimal, and that several important voting processes—viz, sophisticated voting under amendment procedure, cooperative voting, and electoral competition—lead to decisions in the "uncovered set."

Tournaments and Majority Preference

I set out here basic notation and concepts. The reader is referred to my earlier article for further details and references.

A *tournament* is a complete asymmetric directed graph—that is, a finite collection V of points, together with a collection of directed lines, exactly one between each pair of points. For example, $x \to y$ for $x, y \in V$, for which we say "x dominates y." The dominance relation represented by a tournament thus is complete (i.e., for each pair of points, at least one dominates the other), asymmetric (i.e., for each pair of points, no more than one dominates the other), and irreflexive (i.e., no point dominates itself).

We introduce the following notation: $D(x) \subset V$ is the set of points dominated by x, which we call the *dominion* of x (cf. Shepsle, 1974, p. 510; and $F(x) \subset V$ is the set of points that dominate x.

The following relationships follow immediately from the completeness, asymmetry, and irreflexivity of the dominance relation represented by a tournament:

LEMMA 1: In a tournament,

(a) for any $x \in V$, the family of sets $[F(x), \{x\}, D(x)]$ is a partition in V, i.e., every point in V belongs to exactly one of these sets; and

(b) if $x \neq y$, then $F(x) \neq F(y)$ and $D(x) \neq D(y)$.

The *score* $s(x)$ of a point x is the cardinality of its dominion, i.e., $|D(x)|$—in words, the number of points that x dominates.

Let $R(x) \subseteq V$ be the set of all points reachable from x via some path of domination, e.g., $y \in R(x)$ if $x \to z \to v \to y$; and let $R_k(x)$ be the set of all points reachable from x via some path of domination of length no greater than k, i.e., including no more than k directed lines—thus, in the preceding example, $y \in R_3(x)$. By convention, reachability is reflexive, i.e., $x \in R_k(x)$; thus, in particular, $R_1(x) = D(x) \cup \{x\}$.

The following point is self-evident:

LEMMA 2: For all $x \in V$ and all $h < k$, $R_h(x) \subseteq R_k(x)$.

Consider a set V of m proposals and a set N of n voters, where n is odd and each voter has complete, transitive, and strict preferences over the proposals in V. Let $x \to y$ mean "proposal x is majority-preferred to proposal y." Then, for any preference profile (i.e., collection of preference orderings, one for each voter), the system of majority preference over the proposals in V is a tournament. Less obviously, the converse is also true (McGarvey, 1953; cf. Miller, 1977, pp. 774–775):

> *McGarvey's Theorem*: Given any tournament, there is some finite strict preference profile such that, for every pair of points/proposals x and y such that $x \to y$, x is majority-preferred to y.

Let $P(V) \subseteq V$ designate the set of Pareto-optimal proposals, i.e., x belongs to $P(V)$ if and only if there is no y that is unanimously preferred to x.

A *Condorcet proposal* x is one that is majority-preferred to every other proposal y. Since majority preference is complete, there is at most one Condorcet proposal. But since any tournament can represent majority preference, and many tournaments have cycles such that there is no point that dominates every other point, it follows that a Condorcet proposal does not always exist.

The above point, of course, has long been recognized as the "paradox of voting." Majority preference does not in general order proposals; thus, majority voting can lead to quite arbitrary decisions. More recently it has been recognized that, even if majority preference does not prescribe a unique solution (i.e., even if there is no Condorcet proposal), it may narrow down the set of possible or desirable decisions in the following manner (cf. Miller, 1977, p. 775):

DEFINITION 1: A *minimal undominated* (or *Condorcet*) set $V^* \subseteq V$

of points in a tournament is a nonempty set of points such that
(a) no point in V^* is dominated by any point not in V^*; and
(b) no proper subset of V^* meets condition (a).

In my earlier article (1977, Theorems 3, 6, and 7) the following points concerning V^* were demonstrated:

Theorem A: Every tournament has exactly one set V^*.

Of course, if there is a Condorcet proposal x, then $V^* = \{x\}$.

Theorem B: If and only if every point in a tournament is dominated, there is a cycle including precisely the points in V^*.

Theorem C: In a tournament, the following statements are equivalent:
(a) there is a complete path of domination beginning with x;
(b) every other point is reachable from x, i.e., $R(x) = V$; and
(c) x belongs to V^*.

It was also shown that many, though not all, binary majority voting processes lead to a decision in V^*.

There are at least two significant problems with using V^* as a solution set for majority voting, however. First, V^* may be very large. Indeed, we may have $V^* = V$, so that the set of possible or desirable decisions is not narrowed down at all. Given a finite set of discrete proposals, V^* may be small or large (relative to V), depending on the nature of the preference profile. Given a continuous proposal space of two or more dimensions—in the manner of spatial models of voting and electoral competition but technically beyond the scope of the present analysis—and given conventional assumptions concerning individual preferences, McKelvey (1976, 1977; also see Schofield, 1978 and Cohen, 1979) has shown that majority rule fails totally if it fails at all and V^* includes all points in the proposal space.[1]

Secondly, V^* may include Pareto-inefficient proposals. By now this point has been frequently noted for finite V (see Miller, 1977, p. 793, for some references; also cf. Campbell, 1977, p. 89; Ferejohn and Grether, 1977, p. 27; and Richelson, 1978, pp. 344–345). It is also implied by McKelvey's result for an infinite proposal space. Since it can be shown that, under many voting processes, any proposal in V^* can become the voting decision, given appropriate manipulation of the voting order (Miller, 1977, Propositions 2, 4, and 6), it follows that majority voting can lead to a de-

[1] Many years ago, Ward (1961) showed that, when proposals pertain to allocation and voters have individualistic preferences, V^* includes all, or virtually all, feasible proposals.

cision x even though there is another proposal y that is unanimously preferred to x.

These considerations lead us to ask whether it may be possible to identify some kind of solution set for majority voting that is less inclusive than V^* and assures Pareto optimality. (For some suggestions, see Schwartz, 1977, p. 1008.) Presumably, however, no solution set is of great interest unless it can be shown that certain voting processes actually lead to it. It is clear that the decision of most binary majority voting processes depends only on the majority preference tournament, and not on such other factors as the size of certain majorities or the number of first-place preferences for any proposal. It seems desirable, therefore, to restrict the search for a new solution set to sets determined—like the Condorcet proposal and the Condorcet set—by the majority preference tournament alone, not by the underlying preference profile. In any case, this is the approach taken in the present article.

The Covering Relation

To identify this more satisfactory solution set, we must look not only at pairs of proposals and at the majority preference relationships between them, but also at triples of proposals and at majority preference paths of two steps. We shall also exploit the assumption that individual preferences are transitive—a highly orthodox assumption that is used surprisingly rarely in reaching voting theory results—because, as McGarvey's Theorem tells us, individual transitivity in no way restricts the structure of majority preference tournaments.

The following analysis is based on the concept below:

DEFINITION 2: Point x *covers* point y if and only if the dominion of x contains the dominion of y, i.e., $D(y) \subseteq D(x)$.

That is, every point dominated by y is also dominated by x.

The following relationship exists between covering and domination in a tournament:

LEMMA 3: In a tournament, covering is a transitive subrelation of domination.

That is, x covers y only if x dominates y (subrelation) and, if x covers y and y covers z, then x covers z (transitive).

Suppose x covers y, i.e., $D(y) \subseteq D(x)$. By Lemma 1 (b), the containment must be strict, i.e., $D(y) \subset D(x)$. Thus y cannot dominate x, and x must dominate y.

Suppose x covers y and y covers z, i.e., $D(z) \subset D(y) \subset D(x)$. By the transitivity of set inclusion, $D(z) \subset D(x)$, so x covers z.

LEMMA 4: In a tournament, the following statements are equivalent:
(a) x covers y, i.e., $D(y) \subset D(x)$;
(b) $F(x) \cap D(y) = \emptyset$;
(c) $F(x) \subset F(y)$;
(d) $D(x) \cup F(y) = V$; and
(e) $x \notin R_2(y)$.

In words, these statements say: (a) every point dominated by y is also dominated by x; (b) no point both dominates x and is dominated by y; (c) every point that dominates x also dominates y; (d) every point either is dominated by x or dominates y (or both); and (e) there is no path of one or two steps from y to x—that is, y does not dominate x and there is no third point that both dominates x and is dominated by y.

If x covers y, by definition $D(y) \subset D(x)$; $F(x) \cap D(x)$ is always empty, so $F(x) \cap D(y) = \emptyset$. So (a) implies (b).

If $F(x) \cap D(y) = \emptyset$, then $F(x) \subseteq V - D(y) = F(y) \cup \{y\}$. Since x covers y, also x dominates y, i.e., $y \in F(x)$, so $F(x) \subseteq F(y)$. In a tournament $F(x) \neq F(y)$, thus $F(x) \subset F(y)$. So (b) implies (c).

If $F(x) \subset F(y)$, then $V = [V - F(x)] \cup F(y) = D(x) \cup \{x\} \cup F(y)$. But x dominates y, so $x \in F(y)$. Thus $D(x) \cup F(y) = V$. So (c) implies (d).

If $D(x) \cup F(y) = V$, then $V - F(y) = D(y) \cup \{y\} \subseteq D(x)$; thus $D(y) \subset D(x)$. So (d) implies (a).

We now have this cycle of implication: (a) implies (b) implies (c) implies (d) implies (a). Since the relation of logical implication is transitive, the reverse implications must hold as well. Thus (a), (b), (c), and (d) are equivalent. Finally, it is obvious that (b) and (e) are equivalent—they say literally the same thing in different notation. Thus all five statements are equivalent in a tournament.

Finally, we have the following:

LEMMA 5: In a tournament, if x covers y, then $s(x) > s(y)$.

Since $D(y) \subset D(x)$, $|D(x)| > |D(y)|$. Not that the converse of this lemma does not hold.

V^{**}: The Uncovered Set

In this article, we focus on the following set of points in a tournament:

DEFINITION 3: In a tournament, a point x belongs to the *uncovered set* $V^{**} \subseteq V$ of points if and only if there is no other point y such that $D(x) \subset D(y)$.

By the equivalence of statements (a) and (e) in Lemma 4—and thus also of their negations—we have:

LEMMA 6: In a tournament, a point x belongs to V^{**} if and only if $R_2(x) = V$.

In words, a point x is uncovered if and only if every other point in V is reachable from x via a path of domination of no more than two steps. In terms of majority preference, a proposal x is uncovered if and only if, for every other proposal y, either x is majority-preferred to y or there is some third proposal z such that x is majority-preferred to z and z is majority-preferred to y.

Thus we have $V^{**} = \{x \mid R_2(x) = V\}$—that is, V^{**} is composed of precisely those points from which every other point is reachable in no more than two steps.

By the equivalence of statements (b) and (c) in Theorem C, we have this parallel equivalence for the Condorcet set: $V^* = \{x \mid R(x) = V\}$.

Finally, let us define a set of points as follows: $V^{***} = \{x \mid F(x) = \emptyset\}$, i.e., V^{***} is the set of undominated points or the "core." In a tournament, it must be that $V^{***} = \{x \mid D(x) \cup \{x\} = V\}$, which is equivalent to saying $V^{***} = \{x \mid R_1(x) = V\}$. Of course, if there is a Condorcet proposal x, then $V^{***} = \{x\}$; otherwise, $V^{***} = \emptyset$.

By virtue of these relationships for V^*, V^{**}, and V^{***} and of Lemma 2, we have:

THEOREM 1: $V^{***} \subseteq V^{**} \subseteq V^*$.

In words, all uncovered points also belong to the Condorcet set. Moreover, if there is a Condorcet proposal x, then $V^{***} = V^{**} = V^* = \{x\}$.

Thus, whatever its other merits or demerits, V^{**} is no more inclusive than V^*. We may ask whether V^{**} is ever properly contained in V^*. We can easily see that it may be so contained by examining the tournament in Figure 1: $V^* = \{x, y, z, v\}$; however, y covers z (while there is no other covering relationship) and thus $V^{**} = \{x, y, v\}$. On the other hand, if V^* is, for example, a cyclic triple of points, then clearly $V^{**} = V^*$.

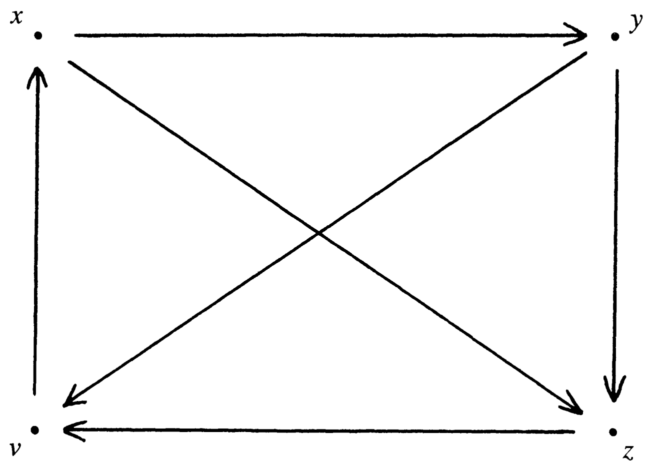


FIGURE 1
A Tournament with Four Points in a Cycle

Of course, if the containment relationship $V^{**} \subseteq V^{*}$ is a point in favor of V^{**}, it is also true that $V^{***} \subseteq V^{**}$, so in this respect V^{***} is better still. The trouble is, of course, that V^{***} may be empty. Might the same be true of V^{**}, or can we show that at least one point must be uncovered? One way to show the latter is the following:

Landau's Theorem: In a tournament, if x is a point with maximum score, i.e., if $s(x) \geq s(y)$ for all y in V, then $R_2(x) = V$.

In words, every other point is reachable in one or two steps from a point with maximum score (for the proof, see Harary et al., 1965, p. 293, or Landau, 1953, p. 148). Since a point with maximum score must exist, it follows from Lemma 6 that:

COROLLARY 1: In a tournament, V^{**} is never empty, i.e., an uncovered point always exists.

We can also reach this result directly, however, by simply recalling that the covering relationship is transitive (Lemma 3) and is therefore acyclic; it follows (cf. Theorem 1 in Miller, 1977) that some point is uncovered. And we can go further:

THEOREM 2: In a tournament, if V^{***} is empty, then V^{**} includes at least three points.

Let x be a point with maximum score. By Landau's Theorem $R_2(x) = V$. Since V^{***} is empty, $F(x)$ is not empty. Among the (one or more) points in $F(x)$, let y be one with maximum score. Suppose there is some point v not in $R_2(y)$; then v must belong to $F(x)$ and, by Lemma 4, v covers y, so by Lemma 5 $s(v) > s(y)$, contradicting the choice of y as a point in $F(x)$ with maximum score. So we must have $R_2(y) = V$, and y belongs to V^{**}. By the same argument, some point w has maximum score among the points in $F(y)$ and $R_2(w) = V$; thus w belongs to V^{**}. Moreover, since w dominates y and y dominates x, x and w are distinct points, and we have three distinct uncovered points (cf. Harary et al., 1965, p. 294).

However, while V^{**} may be less inclusive than V^*, it does not escape the cycling problem. Of course, since $V^{**} \subseteq V^*$, the points in V^{**} are included in the cycle which we know (by Theorem B) includes all points in V^*. The next question is whether the subtournament including *only* the points in V^{**} and the directed lines between these points has a complete cycle. This question can be answered in the affirmative.

THEOREM 3: In a tournament, if V^{***} is empty, there is a cycle including precisely the points in V^{**}.

Let the points be labelled so that $V^* = \{v_1, v_2, \ldots, v_r\}$ and $v_1 \to v_2 \to \ldots \to v_r \to v_1$. (By Theorem B, such a labelling is always possible.) Of course, if $V^{**} = V^*$, the result is immediate by Theorem B. Otherwise (i.e., if some point in V^* is covered), we proceed by showing that, upon the removal of any such covered point, either there is a cycle including precisely the remaining points or, failing that, there is some additional covered point. Repeated application of this result proves the theorem.

Suppose that $V^{**} \subset V^*$ and, in particular, that v_h in V^* is covered, necessarily by some other point in V^*. If $v_{h-1} \to v_{h+1}$ (which would be true if v_h were covered by v_{h-1}), we immediately have the required cycle $v_1 \to \ldots \to v_{h-1} \to v_{h+1} \to \ldots \to v_r \to v_1$. Otherwise, v_h must be covered by some v_k, where $h+2 \leq k \leq h-2$ (in modulo m^*,[2] where m^* *is* the number of points in V^*); then $v_k \to v_{h+1}$, and we have the cycle $v_k \to$

[2] That is, $k + 1 = 1$ if $k = m^*$ and $k - 1 = m^*$ if $k = 1$. (Arithmetic on the hours of the day, for example, is done in modulo 12, or modulo 24 in the military.)

$v_{h+1} \rightarrow \ldots \rightarrow v_k$. Now, suppose there is some v_g where $h+2 \leq g \leq k$ (in modulo m^*). If, for any such point v_p, we have $v_p \rightarrow v_k$, then we have $\rightarrow v_{h-1}$ intersecting the cycle $v_k \rightarrow v_{h+1} \rightarrow \ldots \rightarrow v_k$. By Lemma 2 in Miller (1977), if two cycles intersect, there is another cycle including precisely the points in their union—in this case the required cycle including precisely the points in $V^* - \{v_h\}$.

Suppose, however, that there is no such point v_g and that v_{h-1} dominates (in addition to v_h) only some point(s) v_p such that $k+2 \leq p \leq h-3$ (in modulo m^*). If, for any such point v_p, we have $v_p \rightarrow v_k$, then we have three cycles—$v_{h-1} \rightarrow v_p \rightarrow \ldots \rightarrow v_{h-1}$; $v_p \rightarrow v_k \rightarrow \ldots \rightarrow v_p$; and $v_k \rightarrow v_{h+1} \rightarrow \ldots \rightarrow v_k$—that pairwise intersect. So, again by Lemma 2 in Miller (1977), there is a cycle including precisely the points in their union, i.e., $V^* - \{v_h\}$.

Finally, if, for all such points v_p, we have $v_k \rightarrow v_p$, or if v_{k+1} is the only point (other than v_h) that v_{h-1} dominates, or if v_{h-1} dominates no point other than v_h, it follows immediately that v_k covers v_{h-1}.

Thus, if some point v_h in V^* is covered, either there is a cycle including precisely the points in $V^* - \{v_h\}$, or, failing that, v_{h-1} is covered by the same point v_k that covers v_h. So, if $V^{**} = V^* - \{v_h\}$, i.e., if only one point in V^* is covered, then the theorem is proved. Otherwise (i.e., if there are additional covered points in V^*), we must consider two cases: there is a cycle including precisely the points in $V^* - \{v_h\}$ or there is not. In the latter case v_{h-1} is covered by the same point v_k that covers v_h.

Case 1. Let v_q be an additional covered point. Necessarily, v_q is covered by some point in V^*. It may be covered by v_h, but since v_h is itself covered, and since covering is transitive, v_q must in any case be covered by some point in $V^* - \{v_h\}$, which, by supposition, includes a complete cycle. Thus we may proceed as above and show that either there is a cycle including precisely the points in $V^* - \{v_h, v_q\}$ or, failing that, that v_{q-1} is covered by the same point that covers v_q.

Case 2. If $v_{h-2} \rightarrow v_{h+1}$, we immediately have the required cycle $v_1 \rightarrow \ldots \rightarrow v_{h-2} \rightarrow v_{h+1} \rightarrow \ldots \rightarrow v_r \rightarrow v_1$. Otherwise, since v_k covers v_{h-1}, we can proceed as above and show that either there is a cycle including precisely the points in $V^* - \{v_{h-1}, v_h\}$ or, failing that, that v_{h-2} is covered by the same point v_K that covers both v_{h-1} and v_h.

In either case, if only two points in V^* are covered, the theorem is proved. Otherwise, we proceed as before and show that, upon removal of an additional covered point, we are left with a cycle or, failing that, that there is yet another covered point. By Theorem 2, there are at least three uncovered points, so this argument must terminate after some number of repetitions, leaving us with a cycle including precisely the uncovered points.

The Size of V^{**}

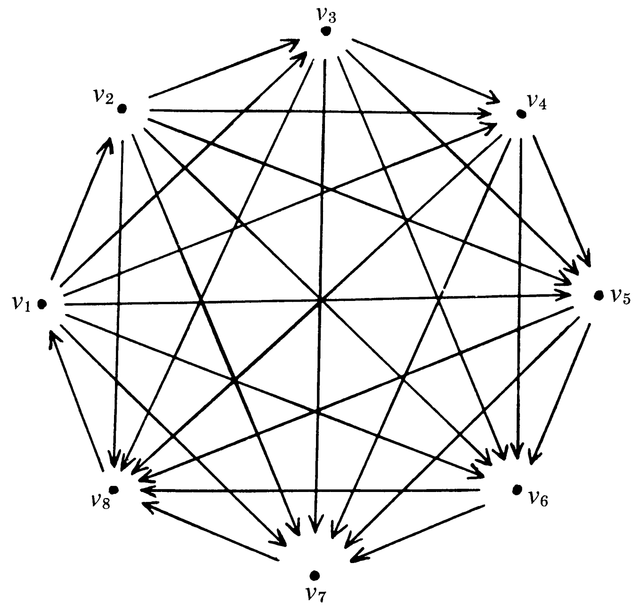
Let m^* designate the number of points in the Condorcet set V^* and m^{**} the number of points in the uncovered set V^{**}. If there is a Condorcet point, then of course $m^{**} = m^* = 1$. If $m^* = 3$, then $m^{**} = 3$. And since—apart from the labelling of the points—the diagram in Figure 1 represents the only tournament with four points in a cycle, if $m^* = 4$, then $m^{**} = 3$. From the results in the previous section we know that generally, in the absence of a Condorcet point, $3 \leq m^{**} \leq m^*$. But, for $m^* \geq 5$, what determines the size of m^{**} within these bounds?

The size of V^{**} relative to V^* depends largely on the degree of intransitivity within V^*, which in turn may be measured by the proportion of all triples of points in V^* that are cyclic, e.g., $x, y, z \in V^*$, such that $x \to y \to z \to x$ or $x \to z \to y \to x$. With some complications, the pattern is this: as intransitivity in V^* declines, m^{**} approaches or equals 3; as it increases, m^{**} approaches or equals m^*.

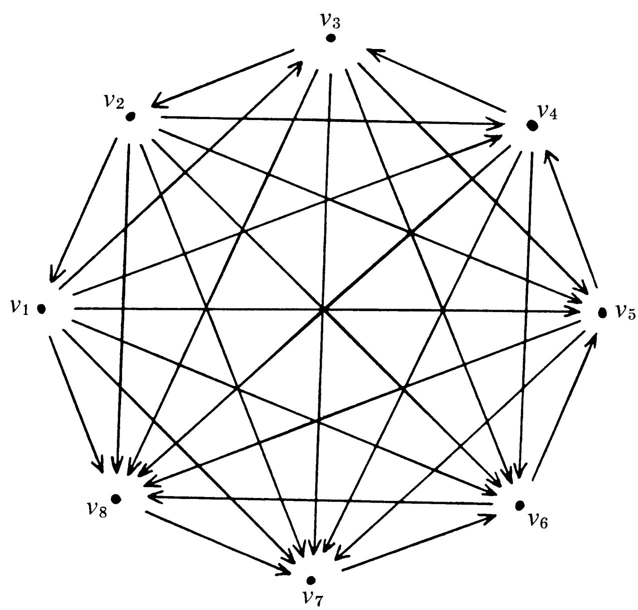
A strong tournament is one in which there is a cycle including all points. It can be shown that every strong tournament with m^* points has at least $m^*—2$ cyclic triples, and that there are strong tournaments with m^* points that have exactly $m^*—2$ cyclic triples (cf. Harary et al., 1965, pp. 306–307). Furthermore, minimally intransitive strong tournaments with m^* points, i.e., strong tournaments with exactly $m^*—2$ cyclic triples, can be constructed in at least two distinct ways. In both constructions, we start with a fully transitive tournament, i.e., a strict ordering. Let the points be labelled $v_1, v_2, \ldots, v_{m^*}$ so that v_h dominates v_k if and only if $h < k$. (In such a fully transitive tournament, covering and domination are equivalent.)

Type I. Reverse $v_1 \to v_{m^*}$ to $v_{m^*} \to v_1$. The resulting tournament is obviously strong and includes precisely $m^*—2$ cyclic triples: viz, $v_1 \to v_k \to v_m \to v_1$ for all k such that $2 \leq k \leq m^*—1$. In this case, v_h covers v_k if and only if $2 \leq h < k \leq m^*—1$; thus $V^{**} = \{v_1, v_2, v_{m^*}\}$ and is of minimum size (cf. Figure 2(a) for $m^* = 8$).

Type II. For each $k \neq m^*$, reverse $v_k \to v_{k+1}$ to $v_{k+1} \to v_k$. The resulting tournament is obviously strong. It also includes precisely $m^*—2$ cyclic triples: viz, $v_{k-1} \to v_{k+1} \to v_K \to v_{K-1}$ for all $2 \leq k \leq m^*—1$. And v_1 covers v_k for all $4 \leq k \leq m^*$ (and generally v_h covers v_k for all $4 \leq h + 3 \leq k \leq m^*$); thus $V^{**} = \{v_1, v_2, v_3\}$ and is of minimum size (cf. Figure 2(b) for $m^* = 8$).



(a) Type I



(b) Type II

FIGURE 2
Minimally Intransitive Strong Tournaments

Let $s^*(x) = | D(x) \cap V^* |$, i.e., the number of points in V^* that x dominates. Then, if m^* is odd, the maximum number of cyclic triples in V^* occurs when all points in V^* have the same score, i.e., when $s^*(x) = \frac{m^*-1}{2}$ for all x in V^*. Then, by Lemma 5, no point can be covered, so $V^{**} = V^*$ and is of maximum size. If m^* is even, matters are less clear. Maximum intransitivity occurs when all points in V^* have scores that are as equal as possible, i.e., when $s^*(x) = \frac{m^*}{2}$ for half the points x in V^* and $s^*(x) = \frac{m^*-2}{2}$ for the other half. If $m^* = 4$, as we have seen, there is only one structurally distinct strong tournament, for which $m^{**} = 3$. If $m^* = 6$, at least two structurally distinct maximally intransitive strong tournaments exist. For one of these, $m^{**} = 3$; for the other $m^{**} = 6$. Most likely, the same contingent pattern holds for larger even values of m^*.

V^{**} and Pareto Optimality

We now consider a fundamental relationship between covering and preference profiles.

LEMMA 7: If x is unanimously preferred to y, then x covers y.

Suppose that x is unanimously preferred to y. Obviously, x dominates y. If y dominates nothing, then x covers y trivially. Otherwise, consider any z belonging to $D(y)$. Let S designate the set of voters who prefer y to z; necessarily, S is of majority size, and all members of S prefer x to y. By the transitivity of individual preference, all members of S prefer x to z. Thus, x dominates z. And since this is true for any z in $D(y)$, $D(y) \subset D(x)$, and x covers y.

As an immediate consequence of Lemma 7, we have:

THEOREM 4: $V^{**} \subseteq P(V)$.

In words, every proposal in V^{**} is Pareto-optimal. In conjunction with Theorem 1, we have $V^{**} \subseteq [V^* \cap P(V)]$. It is not true, however, that $V^{**} = [V^* \cap P(V)]$—that is, the converse of Lemma 7 does not hold. For example, in the tournament represented in Figure 1, while y *may* be unanimously preferred to z, and while there can be *no other* unanimous preference, y *need not* be unanimously preferred to z. Indeed, in any majority preference tournament with any number of covering relationships, there may be no unanimous preferences and all proposals may be Pareto-optimal.

V^{**} and Sophisticated Voting

Two voting procedures have received particular attention in recent literature (e.g., Farquharson, 1969; Miller, 1977). Under *amendment procedure*, two proposals are paired for a majority vote, the defeated proposal being eliminated as a possible decision; the surviving proposal is then paired with a third at the second vote, and so forth. The proposal that survives the final or $(m-1)$th vote is the voting decision. Under *successive procedure*, the first proposal is voted up or down on a majority basis; if voted up, it is the decision and voting terminates; otherwise, the second proposal is voted up or down, and so forth. If the first $m-1$ proposals are voted down, the remaining proposal is the voting decision.

In my previous article, (Miller, 1977, Propositions 2, 4, and 6) I showed that, for sincere voting under either successive or amendment procedure, and also for sophisticated voting under successive procedure, any proposal in V^* could be made the voting decision, given an appropriate order of voting.[3] I also showed that, given sophisticated voting under amendment procedure, there might be proposals in V^* which could not be made the voting decision under any order of voting. Here I show that these include all covered proposals. Hence we have:

THEOREM 5: Under amendment procedure, the sophisticated voting decision belongs to V^{**}.

I have demonstrated (1977, pp. 790–791), that the proposal last in the voting order is the sophisticated voting decision if and only if it is the Condorcet proposal. I further proved that proposal x, if not last in the voting order, can be the sophisticated voting decision only if it meets a series of increasingly fine conditions:

Condition 1: Proposal x must belong to $F(z)$, where z is the last proposal in the voting order.

If several proposals meet Condition 1, i.e., if $|F(z)| \geq 2$, we consider:

Condition 2: Proposal x must belong to $[F(z)]^*$, where z is the last proposal in the voting order.[4]

[3] Sincere voting was defined as in Farquharson (1969, p. 18); sophisticated voting was defined in the "tree" or multistage sense suggested in Miller (1973) and presented formally in Niemi et al. (1974) and McKelvey and Niemi (1978).

[4] For any $V' \subset V$, there is a subtournament including only the points in V' and the directed lines between them. $[V']^*$ designates the Condorcet set and $[V']^{**}$ the uncovered set within this subtournament.

If several proposals meet Condition 2, i.e., if $|[F(z)]^*| \geq 2$, we consider:

Condition 3: Proposal x must belong to $\{[F(z)]^* \cap F(v)\}^*$, where v is the proposal in $[F(z)]^*$ last in the voting order.

If several proposals meet Condition 3, we must identify the proposal in $\{[F(z)]^* \cap F(v)\}^*$ last in the voting order, and so forth. It was further shown that successive application of these conditions must at some stage result in a one-element set, which identifies the sophisticated voting decision.

Now, suppose that proposal x is covered, say by proposal y. Since each condition above requires that certain proposals be in the dominion of x—viz, z, some other proposal in $F(z)$, v, etc.—it follows that if x meets a given condition, y meets the same condition, since $D(x) \subset D(y)$. Thus, when we reach the one-element set that is the sophisticated voting decision, it cannot be x.[5]

This result has implications for the efficacy of agenda control, as discussed by McKelvey (1976, 1977). As mentioned earlier, McKelvey and others have shown that, given a continuous multidimensional policy space and standard (e.g., Euclidean) assumptions about preferences, if V^{***} is empty, then $V^* = V = R^m$, i.e., V^* includes all points in the m-dimensional space. In words (McKelvey, 1976, p. 472), "if there is no equilibrium outcome [i.e., no Condorcet proposal], then the intransitivities extend to the whole policy space in such a way that all points are in the same cycle set." In particular, it is possible to find a majority preference path $x \rightarrow \ldots \rightarrow y$ between any two points x and y in the space, even if x is located at the periphery of or entirely outside the distribution of voters' ideal points while y is centrally located in that distribution. A simple two-dimensional, seven-voter Euclidean example is shown in Figure 3 (adapted from McKelvey, 1976, p. 479).[6]

McKelvey further observes (1976, pp. 480–481; some emphasis de-

[5] I conjecture that, for any proposal x in V^{**}, there is some order of voting such that x is the sophisticated voting decision. I have not been able to construct any counter-example to this assertion, but neither have I been able to devise a general proof or to state a rule for identifying the appropriate voting order (in the manner of Propositions 2, 4, and 6 in Miller, 1977).

[6] Each of the seven numbered points represents a voter's ideal point. Each voter prefers a point closer to his ideal to a point more distant from his ideal (thus all indifference curves are circles). The points labelled v_1, v_2, etc., represent particular proposals.

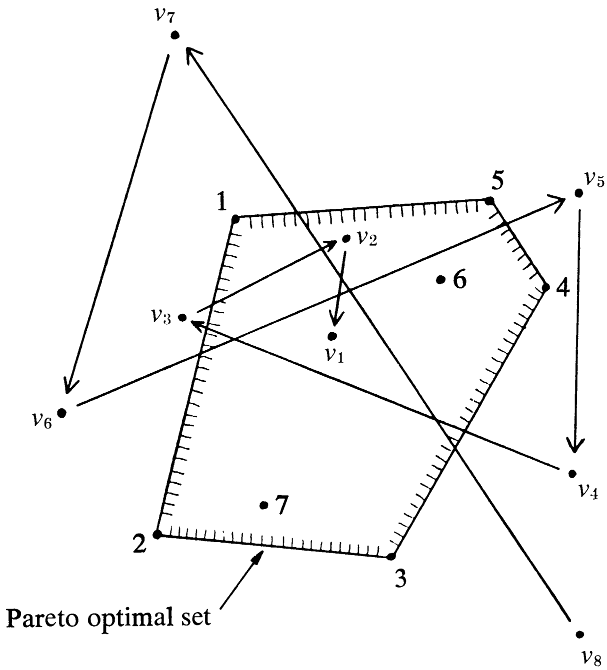


FIGURE 3
A Two-Dimensional Policy Space

leted): "When there is the possibility of control of the agenda [i.e., of the motions proposed and the order of voting], either exogenously or by some member of the voting body, the existence of a single cycle set [i.e., $V^* = V$] would be of considerable importance. . . . [For] if any one voter, say the 'Chairman,' has complete control over the agenda . . . , he can construct an agenda which will arrive at any point in the space, in particular at his ideal point." However, this "result depends on other voters voting sincerely and without collusion. If the other voters see what is occurring and know what agenda is being used they might, even without collusion, vote against their preferences at some stage (i.e., vote sophisticatedly) in order to outwit our clever Chairman." Theorem 5 confirms this last conjecture. Let us look at the example in Figure 3 a bit more closely.

McKelvey's notion of manipulation of the voting decision by agenda control is most applicable where amendment procedure is in use and voting is sincere. The "Chairman" puts v_1 and v_2 up for a vote, which v_2 wins; then v_2 and v_3, which v_3 wins; and so forth. Voting follows precisely the reverse path of domination shown in the diagram (i.e., it moves against the direction of the arrows).[7] But, by Theorem 5, if voting is sophisticated, the efficacy of agenda control may be considerably constrained, depending on the size of V^{**}. Examination of the diagram in Figure 3 shows that the majority preference tournament for $V = \{v_1, \ldots, v_8\}$ is that depicted before in Figure 2(b), a Type II minimally intransitive strong tournament, and that $V^{**} = \{v_1, v_2, v_3\}$.[8] Thus, by Theorem 5, no matter what order of voting the "Chairman" uses, if amendment procedure is in use and voting is sophisticated, the "Chairman" cannot manipulate the voting decision to be anything other than v_1, v_2, or v_3.[9]

Of course, we have decisively demonstrated constraint on manipulation by agenda control due to sophisticated voting only for the particular example depicted in Figure 3. But it seems clear that the example is quite typical. In any reverse path of domination leading outward from the central neighborhood of the distribution of ideal points, each majority preference relationship is effected by a relatively small majority (in Figure 3, 4 out of 7

[7] By Proposition 4 in Miller (1977), the "Chairman" can achieve the same result (with the same voting order) under successive procedure if voting is sincere: v_1 is voted down since a majority of voters like something later in the agenda better; likewise v_2; and so forth. And, by Proposition 6 in Miller (1977), even if voting is sophisticated, the "Chairman" can still achieve the same result under successive procedure (though he must reverse the voting order).

[8] An important point should be noted. Given that any point in the policy space is a feasible proposal, the Pareto-optimal set of proposals is the "convex hull" of voter ideal points (outlined in the diagram). At the same time, $V^{**} = \{v_1, v_2, v_3\}$ is not contained within this convex hull (specifically, v_3 lies outside it). Does this contradict Theorem 4? No, because $V^{**} = \{v_1, v_2, v_3\}$ is defined relative to $V = \{v_1, \ldots, v_8\}$ not relative to $V = R^2$. Relative to $V = \{v_1, \ldots, v_8\}$, v_3 is Pareto-optimal; relative to $V = R^2$, v_3 is Pareto-dominated (by nearby points in the direction of, or just inside, the convex hull, where no points in $V = \{v_1, \ldots, v_8\}$ lie).

An important question for future research is whether V^{**} is definable relative to the infinite set $V = R^m$ and, if so, what are its location and size. It appears that V^{**} would be a relatively small subset of $P(V)$, centrally located in the distribution of ideal points, and that it would shrink in size as the number and diversity of ideal points increase. V^{**} further appears to be related to the set of partial medians discussed by Hoyer and Mayer (1974), McKelvey and Ordeshook (1976), and others.

[9] It may be worthwhile to recall two prerequisites for sophisticated voting. First, the voters must be aware of the "division tree" (cf. Miller, 1977, p. 784) or "agenda" (McKelvey's term), i.e., they must know what proposals will be voted on under what procedure and in what order, before any voting takes place. Second, voters must know each other's preferences. (Actually, sophisticated voters need know only the majority preference tournament, while, as McKelvey [1976, p. 481] observes, the manipulative "Chairman" must have complete knowledge of the preferences of each individual voter.)

in each case), many members of each majority are nearly indifferent (in the sense that v_{h+1} is barely closer to their ideal points than v_h), and adjacent majorities in this sequence are nearly disjoint (e.g., {1, 4, 5, 6}, {1, 2, 3, 7}, {3, 4, 5, 6}, etc., in the present example). In more political language, bare, apathetic, and shifting majorities repeatedly triumph over large and intense minorities. After a few such triumphs (probably only two) a large majority "coalition of intense minorities" would prefer to go back to the starting point (e.g., the majority {3, 4, 5, 6, 7} would like to move from v_3 back to v_1 and all but voter 7 intensely desire this). Thus the sequence of proposals required for manipulation by agenda control typically generates a Type II maximally intransitive strong majority preference tournament (or something close to it), which has a V^{**} of minimum size (or close to minimum size). Furthermore, it seems evident that, as the number of voters increases, and thus also the density and diversity of their ideal points, the early steps in a manipulative sequence leading outwards from the central neighborhood of the distribution must be very short. Then the few proposals in V^{**} would all lie very close to the central starting point.

V^{**} and Cooperative Voting

A sophisticated strategy is a best voting strategy under non-cooperative conditions. But if voters can cooperate—that is, communicate and make binding agreements (or contracts) implemented through coalitions—further strategic opportunities arise. A *decisive coalition* is a set of voters who, acting in concert, have the power to impose any decision on the voting body. A *majoritarian procedure* makes any majority coalition decisive. The cooperative voting decision depends on the outcome of preplay negotiations which result in members of some decisive coalition agreeing to make some proposal the voting decision; once such an agreement is struck, voting itself is only a formality to be played out (e.g., by members of the decisive coalition voting as a bloc at each division).

Of course, the majority preference tournament influences the course of these preplay negotiations (if it is majority coalitions that are decisive). In general, we may suppose that any tentative contract among members of one coalition to make any proposal y the decision may be upset or preempted by another tentative contract among members of another coalition to make some proposal x in $F(y)$ the decision. This process of recontracting continues until an agreement is reached that cannot be upset, i.e., an agreement on some x such that $F(x)$ is empty (the Condorcet proposal) or, in the absence of a Condorcet proposal, until the process is broken off essentially arbitrarily (e.g., under the constraint of time). Thus, the cooperative voting decision depends only on the preferences of the voters and on the fact that the procedure is majoritarian. Whether the procedure is amendment or successive, and the order in which proposals are voted on, are irrelevant under fully cooperative conditions.

We may speak then of a "recontracting trajectory" as a sequence of proposals beginning at some arbitrary starting point and then following some reverse path of domination. The question is: where may such a trajectory end up? In other words, what proposals may be the voting decision if voters can cooperate fully? And to answer this, we must answer a preliminary question: is every reverse path of domination (from some arbitrary starting point) a plausible recontracting trajectory, or are some such trajectories more plausible than others?

In my earlier article (1977, Proposition 13), through implicitly accepting that all reverse paths of domination are equally plausible recontracting trajectories, I appealed to the relationship $V^* = \{x \mid R(x) = V\}$ to argue that: (1) every recontracting trajectory that does not begin with V^* leads into V^*, and (2) once in V^*, no recontracting trajectory leads out of V^*. Point (2) is clearly true, but (1) is more questionable, for a reverse path of domination that begins outside of V^* *can* remain outside indefinitely, if $V - V^*$ includes some cycle around which the path follows indefinitely. Thus, in order to assure that every recontracting trajectory leads into V^*, we must argue that not all reverse paths of domination are equally plausible trajectories. And once we argue this, we may be led to the conclusion that all plausible trajectories lead into, and remain within, some subset of V^*—for example, V^{**}.

Suppose there is a tentative contract on proposal z. The next contract in the recontracting trajectory must be on a proposal in $F(z)$. (Of course, if $F(z)$ is empty, the trajectory terminates and z is the cooperative decision.) If $F(z) = \{x\}$, i.e., if only one proposal dominates z, then the course of the trajectory is clear. Suppose, however, that $F(z) = \{x, y\}$, i.e., that two reverse paths of domination branch apart at z. Is there reason to predict that the recontracting trajectory will go one way rather than the other?

With respect to the question of what proposal the next contract will be on, the voters are partitioned into six sets (some of which may be empty) according to their preferences regarding x, y, and z. Let us label them as follows:

	S_1	S_2	S_3	S_4	S_5	S_6
first preference	x	x	y	y	z	z
second preference	y	z	x	z	x	y
third preference	z	y	z	x	y	x

A coalition contracting on y would include voters in the union $S_1 \cup S_3 \cup S_4$, and a coalition contracting on x would include voters in the union $S_1 \cup S_3 \cup S_2$.

By assumption, both x and y are majority-preferred to z and x is majority-preferred to y. Every pair of majorities must intersect, so, in particular, the intersection $(S_1 \cup S_2 \cup S_5) \cap (S_1 \cup S_3 \cup S_4)$ is not empty; thus the set S_1 of voters cannot be empty.

We now consider three cases concerning the size of S_1.

Case 1: $| S_1 | \geq \frac{n+1}{2}$. In this case, the members of S_1 alone can determine the next step in the recontracting trajectory, and clearly they prefer to move to x rather than y.

Case 2: $| S_1 \cup S_2 | \geq \frac{n+1}{2}$ (includes Case 1 as a special case). In this case, S_1 holds the balance of power between the two rival and disjoint "proto-coalitions" (the term is loosely adapted from Riker, 1962, pp. 104ff.), $S_3 \cup S_4$ and S_2, that are striving for contracts on y and x respectively. Clearly, the members of S_1 prefer to join S_2 in a contract on x.

Case 3: $| S_1 \cup S_2 | \leq \frac{n-1}{2}$ (the complement of Case 2; hence these cases are exhaustive). In this case, a contract on x requires the support of voters in S_3. These voters of course prefer such a contract to the status quo (i.e., to a contract on z), but they prefer a contract on y still more, which voters in S_1 also prefer to the status quo. So there is an apparent bargaining relationship between voters in S_1 and S_3: they have a common interest tract. But, in fact, the situation is asymmetric and decisively favors the voters in S_1. For S_1 can say to S_3: "Look, if you refuse to join us (and S_2) in a contract on x and try to compel us to join you (and S_4) in a contract on y, we can get S_5 to join us (and S_2) in a contract on x, since they prefer x to y."

(Since $|S_1 \cup S_2| \leq \frac{n-1}{2}$, necessarily the set S_5 is not empty.) The voters in S_3 have no similar counter-argument and so are compelled to participate in a (tentative) contract on x. Note that this argument does not rest on the relative sizes of the two sets S_1 and S_3; in both this case and Case 2, S_3 may be larger than S_1.

What can we say now in the general case, i.e., where there is a tentative contract on z and $F(z)$ contains any number of proposals? Proposal z is at the moment the status quo in that it will become the voting decision unless the present tentative contract is upset by another one. A new contract must be on some proposal in $F(z)$. But which one, if there are several? Suppose some proposal y in $F(z)$ is suggested. If $F(y) \cap F(z)$ is empty (i.e., if nothing dominates y as well as z), then the next contract might be on y. But if $F(y) \cap F(z)$ is not empty (i.e., if there is some proposal that dominates y as well as z), then by the argument made earlier y would surely be rejected in favor of some proposal x in $F(y) \cap F(z)$. Then, if $F(x) \cap F(y) \cap F(z)$ is empty, the next contract might be on x. But, if $F(x) \cap F(y) \cap F(z)$ is not empty, then by the argument made earlier x would surely be rejected in favor of some proposal w in $F(x) \cap F(y) \cap F(z)$. And so forth. Since the number of proposals is finite, this sequence of intersections must at some stage identify a single proposal that is a possible next contract. And all possible such sequences (i.e., those considering *all* y in $F(z)$, *all* x in $F(y) \cap F(z)$, etc.) identify all possible next contracts. We designate this set of proposals, that is, the proposals that may follow z in a recontracting trajectory, $C(z)$. Obviously, $C(z) \subseteq F(z)$.

Consider the subtournament including just the points in $F(z)$ and the directed lines between them. Let $[F(z)]^{**}$ designate the uncovered set of proposals within this subtournament, so that for every x in $[F(z)]^{**}$ and for every y in $F(z)$, either $x \to y$ or there is some v in $F(z)$ such that $x \to v \to y$.

Now we show the following:

LEMMA 8: For all z in V, $C(z) \subseteq F^{**}(z)$.[10]

We consider any x in $C(z)$ and show that x also belongs to $[F(z)]^{**}$. Since x belongs to $C(z)$, it follows that: (1) $\{x\} = F(z)$, in which case it follows trivially that x belongs to $[F(z)]^{**}$; or (2) there is some y in $F(z)$ such that $\{x\} = F(y) \cap F(z)$, in which case $x \to y$; thus, for any v in $F(z) - \{x, y\}$, $x \to y \to v$—otherwise v would belong to $F(y) \cap F(z)$—so x belongs to $[F(z)]^{**}$; or (3) there is some y in $F(z)$ and some w in $F(y) \cap F(z)$ such that $\{x\} = F(w) \cap F(y) \cap F(z)$, in which case $x \to y$, $x \to w$; thus, for any v in $F(z) - \{x, y, w\}$, either $x \to w \to v$ or $x \to y \to v$ or both—otherwise v would belong to $F(w) \cap F(y) \cap F(z)$—so x belongs to $[F(z)]^{**}$; and so forth.

Next we show the following:

LEMMA 9: For all z in V, $[F(z)]^{**} \subseteq V^{**}$.

Consider any x in $[F(z)]^{**}$ and suppose, contrary to the lemma, that x does not belong to V^{**}, i.e., that there is some y that covers x in the full tournament. Since x belongs to $F(z)$, so does y. But if y covers x, by Lemma 4 (the equivalence of [a] and [e]), there is no v in V such that $x \to v \to y$. So there is certainly no v in $F(z) \subset V$ such that $x \to v \to y$, contradicting the supposition that x belongs to $[F(z)]^{**}$. Thus x must belong to V^{**}.

[10] I conjecture, but have not yet been able to prove, that $C(z) = [F(z)]^{**}$.

So, finally, we can state the following (which "refines" Proposition 13 in Miller [1977]):

THEOREM 6: Every recontracting trajectory cycles entirely within V^{**}, i.e., under any majoritarian voting procedure, the cooperative voting decision belongs to V^{**}.[11]

Let the recontracting trajectory begin arbitrarily with any z in V. As an immediate consequence of Lemmas 8 and 9, the second and all subsequent proposals in the trajectory belong to V^{**}.

As a corollary, cooperative voting under any majoritarian procedure assures Pareto optimality.[12] Thus, the Riker-Brams (1973) "paradox of vote trading" is exorcised, if we can assume full cooperation.[13]

V^{**} and Electoral Competition

We now consider briefly the relationship between V^{**} and electoral competition between two power-oriented political parties, modelled as a symmetric, strictly competitive game in the manner deriving from Downs (1957). Each party must commit itself to a platform. Policy-oriented voters, with strict preferences over the set V of all possible platforms, then choose between the two parties on the basis of their platform commitments, and the party receiving a majority of votes wins the election. In a two-party contest, each voter has a dominant (and, moreover, obvious) strategy: to vote sincerely for the party that proposes the preferable platform. (We may suppose that voters flip mental coins if both parties offer the same platform.)

If the objective of each party, A and B, is to win the election, this model generates a two-party, strictly competitive game with three outcomes: "A wins," "draw" (if both parties select the same platform), and "B wins."

[11] I would further conjecture, especially if the conjecture in the previous footnote turns out to be true, that every recontracting trajectory cycles entirely within V^{**u}, as defined in the following section.

[12] Actually, the assertion that the cooperative voting decision belongs to $P(V)$ can be justified more persuasively than the assertion that it belongs to V^{**}. First, if both x and y belong to $F(z)$ and if y is Pareto-dominated by x, the most persuasive Case 1 of the argument made on p. 87 applies. Secondly, while a final contract is binding on the members of a winning decisive coalition, its terms can presumably be modified with the consent of all the members of that coalition (cf. Schwartz, 1977, pp. 1008–1009; the argument made in the text implies Schwartz's "condition [XC_4]" but not vice versa). And, of course, if a proposal is Pareto-inefficient for the whole group, it is also Pareto-inefficient for any coalition.

[13] Riker and Brams' example, justifying their result, assumes in effect that vote trading is piecemeal and less than fully cooperative.

Furthermore, this game is symmetric in two senses: (1) the two parties have the same strategy sets, and (2) if a given strategy pair gives the outcome "*A* wins," the reversed strategy pair gives the outcome "*B* wins," and vice versa.

If V^{***} is not empty, the dominant or "straightforward" strategy (in the sense of Farquharson, 1969, p. 30) of both parties is to select the Condorcet platform. But if there is no Condorcet platform, whatever platform it offers, a party may be defeated; and, since the game is symmetric, each party faces a strategic dilemma, much like a Farquharson type of voter who lacks a straightforward strategy. However, as Farquharson argues, even if a player has no clearly "best" strategy, he may have some clearly "bad" or "inadmissible" strategies that can be eliminated from consideration. Let us define a *contingency* for a player as a set of strategies, one for each other player. In this two-party game, then, a contingency for one party is simply a strategy for the other party. Then, following Farquharson (1969, p. 28), we say:

> DEFINITION 4: A strategy is *initially admissible* if and only if no other strategy gives at least as good an outcome in every contingency and a better outcome in at least one contingency.

By symmetry, of course, a strategy that is admissible for one party is likewise admissible for the other.

Now we can state the following:

> THEOREM 7: An electoral strategy is initially admissible if and only if it commits a party to a platform in V^{**}.

Let us consider the efficacy of two arbitrarily selected electoral strategies, respectively committing party *A* to platforms *x* and *y*. In choosing between these two strategies, party *A* must consider, in general, six classes of contingencies, as outlined in the table below (where "+1" indicates "*A* wins," "0" indicates "draw," and "–1" indicates "*B* wins"). Suppose, without loss of generality, that *x* is majority-preferred to *y*.

Strategy for Party *A*	Contingencies (Strategy for Party *B*)					
	x	y	$z \epsilon D(x) \cap D(y)$	$z \epsilon D(x) \cap F(y)$	$z \epsilon F(x) \cap D(y)$	$z \epsilon F(x) \cap F(y)$
x	0	+1	+1	+1	–1	–1
y	–1	0	+1	–1	+1	–1

In general, party A does not have a clearcut choice. Strategy x is better in some contingencies, strategy y in others, and the two do equally well (or badly) in the remaining contingencies. And, clearly, no matter what the nature of the electorate's majority preference tournament over the platforms in V, strategy y cannot give an outcome that will be at least as good as strategy x's outcome in every contingency, because it fails to do so in the event B selects x. So the question comes down to this: what conditions on the majority preference tournament assure that strategy x gives at least as good an outcome as strategy y in every contingency? A necessary condition, obviously, is that the intersection $F(x) \cap D(y)$ be empty, so that no such contingency can occur. Furthermore, examination of the remaining classes of contingencies shows that this condition is sufficient. And by Lemma 4 (equivalence of [a] and [b]), the requirement that $F(x) \cap D(y)$ be empty is equivalent to the requirement that x cover y. Finally, strategy x always gives a better outcome than strategy y in the event that B selects y. Thus, an electoral strategy is initially admissible if and only if it commits a party to a platform in V^{**}.

Thus "rational" parties will restrict themselves to platforms in V^{**}, and the electoral decision will be likewise restricted. By Theorem 4, therefore, electoral competition between rational parties assures Pareto optimality.

The question arises whether admissible electoral strategies can be further restricted. Again, Farquharson (1969, pp. 38–40) suggests how further restriction might occur. For example, if party A rationally restricts itself to initially admissible strategies, it can figure that its equally rational opposition B will do the same. Thus, in choosing among its initially admissible strategies, party A might take into account only those contingencies that could occur given that its opposition will choose only initially admissible strategies. Formally (and following Farquharson):

DEFINITION 5: A strategy is *secondarily*—or generally *k-arily—admissible* if and only if no other strategy gives at least as good an outcome in every contingency composed entirely of initially—or generally $(k-1)$-arily—admissible strategies and gives a better outcome in at least one such contingency.

So finally:

DEFINITION 6: A strategy is *ultimately admissible* if and only if it is k-arily admissible for some k and all k-arily admissible strategies of all players are also $(k+1)$-arily admissible.

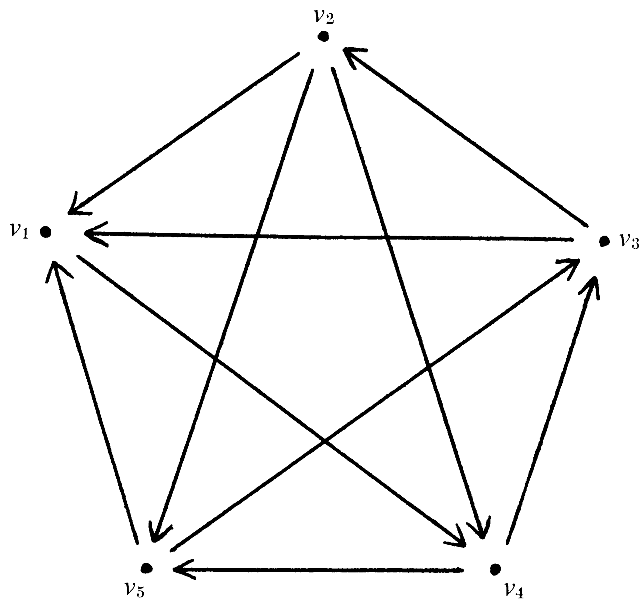


FIGURE 4
A Tournament in which $V^{**2} \neq V^{**}$

Farquharson (1969, p. 42) shows that, for voters with strict preferences under any binary voting procedure, this reduction of strategy sets by successively removing inadmissible strategies of higher orders continues until the reduction is complete—not necessarily until each voter has but one ultimately admissible strategy, but until every combination of ultimately admissible strategies, one for each player, gives the same outcome (i.e., leads to the adoption of the same proposal). How far may this reduction continue in the case of electoral competition?

First, if $V^* = V^{**} = V^{***} = \{x\}$, of course reduction is complete after just one stage. But what if V^{***} is empty?

Certainly, reduction may continue beyond the initial stage. The tournament in Figure 4 provides an illustration. We see that $V^{**} = \{v_2,v_3,v_4,v_5\}$, so only the strategies committing a party to these proposals are initially admissible. Thus, in identifying their secondarily inadmissible strategies, the parties are concerned only with the subtournament which includes the points in V^{**} and the directed lines between them. But within this subtournament, v_5 is covered by v_4, i.e., $[D(v_5) \cap V^{**}] \subset [D(v_4) \cap V^{**}]$. (In the full tournament, v_4 is reachable from v_5 in two steps, but only via v_1.)

So only the strategies committing a party to v_2, v_3, and v_4 are secondarily admissible.

In general, then, we identify the set of ultimately admissible electoral strategies by examining the electorate's majority preference tournament and identifying the set V^{**}. We then look at the subtournament which includes only the points in V^{**}, and identify the uncovered set—call it $[V^{**}]^{**}$ or V^{**2}—within V^{**}. We continue in like manner until no further reduction can occur, i.e., until we find some k such that $V^{**k} = V^{**(k+1)}$. The final set—call it V^{**u}—indicates that electoral decisions may occur given that both parties adopt ultimately admissible strategies. Obviously, $V^{**u} \subseteq V^{**}$.[14] In sum:

THEOREM 7′: An electoral strategy is k-arily admissible if and only if it commits a party to a platform in V^{**k}.

Theorem 7, of course, is a corollary of Theorem 7′ for the special case of $k = 1$. And likewise:

COROLLARY 2: An electoral strategy is ultimately admissible if and only if it commits a party to a platform in V^{**u}.

On the other hand, if V^{***} is empty, reduction of the electoral game can never be complete, since after each stage k, including the ultimate stage u, V^{**k} (and V^{**u}) includes at least three platforms (by Theorem 2). Moreover, the parties never reach a (pure strategy) equilibrium since, by Theorem 3, there is a cycle including precisely the points in V^{**k} (and V^{**u}). Nevertheless, V^{**u} may be a very small subset of V, and electoral competition, even in the absence of a "majority rule equilibrium" (i.e., a Condorcet platform), may be a good deal more determinate than some analyses have suggested.[15]

[14] Incidentally, we cannot strengthen Theorem 5 by replacing V^{**} with V^{**u}. For example, given the majority preference tournament shown in Figure 4, v_5 is the sophisticated voting decision under amendment procedure if v_1 is voted on last and v_3 is voted on after v_2 and v_5. But, with regard to cooperative voting, see footnote 12 above.

[15] Most analyses of electoral competition take place within a spatial context. Therefore, the general question raised in footnote 8 concerning the existence, location, and characteristics of V^{**} (or V^{**u}) relative to $V = R^m$ becomes especially relevant. Put otherwise, the relationship between V^{**} in the finite proposal set case and McKelvey-Ordeshook (1976) "admissible sets" in the infinite case needs to be clarified.

Conclusion

We have noted that the Condorcet set V^* suffers from certain disadvantages as a solution set for tournaments and majority voting. We have defined the uncovered set V^{**} as an alternative solution set, and have shown that V^{**} has certain desirable properties and that decisions in V^{**} result from certain important types of voting processes.

Several avenues for subsequent research are clearly open. First, the entire discussion can be extended—with some appropriate reinterpretation—to nonmajoritarian voting procedures and collective decision-making processes, provided that domination is complete and asymmetric, i.e., that it generates a tournament. Secondly, it is possible to generalize our concepts and results to cover the broad class of cases in which domination fails to be complete, asymmetric, or both.[16] Third, the relevance of V^{**} for spatial models of voting and electoral competition needs to be systematically investigated (cf. footnotes 8 and 15). Finally, just as V^* has an analogue in axiomatic social choice theory in the transitive closure rule, V^{**} has an analogue in the social choice framework, the properties of which can be investigated.[17]

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[16] If majority preference is complete but not asymmetric, i.e., if majority preference "ties" can occur, then the appropriate generalization is to define "x covers y" as $D(y) \subseteq D(x)$ *and* $F(x) \subseteq F(y)$ (the two relationships are no longer equivalent). This, in turn, is equivalent to saying that x is uncovered if and only if there is a path from x to each other y in V that has no more than two steps and that includes at least one strict majority preference relationship. The uncovered set, when so defined, is a nonempty subset of Pareto-optimal proposals (cf. Heiner [1977] and Miller [1979]). Whether such voting processes as sophisticated voting under amendment procedure, cooperative voting, and electoral competition lead to decisions in the uncovered set, more broadly defined in this way, depends in part on how we suppose individuals vote when indifferent and how ties are broken.

[17] Cf. Miller (1979).

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