輝度ならびに形状変化を考慮した適応的画像生成法に関する研究
A new surface model based on a fibre bundle of 1-parameter groups

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1 Introduction

In this paper, we present first a new model of surfaces called the fibre bundle model. This model represents a surface locally as a direct product of two curves: a base curve and a fibre curve. The well known generalized cylinders are trivial cases of the fibre bundle model. i.e. they are global direction product of a base curve and a fibre curve with minor modification by sweeping functions.

On the other hand, the fibre bundle model in its full generality takes still much information. Thus, we propose a fibre bundle of 1-parameter groups, i.e. using either or both fibres and base curves as 1-parameter groups. Since such curves can be completely represented by 6 invariants of theirlinear Lie algebra, the model can be described by a base curve plus 6 invariants or a base point plus 12 invariants. These invariants can also be extracted from an object using the methods developed in [4][2].

Another advantage of this model is that the shapes can be easily synthesized using calculation of elementary functions without numerical integration, thus free of numerical error.

This paper will also extend the fibre bundle model of surfaces to the fibre bundle with fibres of 1-parameter groups having Affine Lie algebras. The choice of fibres and extraction of the invariants from an object are also discussed. Finally, simulations are shown for synthesis of various objects based on the fibre bundle model.

2 1-parameter groups[5]

Definition 1. *Let X be a smooth manifold, G its transformation group. A parameterized subset $H = \{g_t \in G, t \in \mathbb{R}\}$ of G is called a 1-parameter subgroup of G if $\forall g_t, g_s \in H$, $\forall \boldsymbol{x} \in X$, $g_t g_s(\boldsymbol{x}) = g_s g_t(\boldsymbol{x}) = g_{t+s}(\boldsymbol{x})$.*

A typical example is 1-parameter Lie groups T of the matrix Lie group $GL(n, \mathbb{R})$ which is defined by the following exponential map. Here A is an $n \times n$ matrix.

$$T \ni T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \qquad t \in \mathbb{R}. \tag{1}$$

Modeling a free surface X as a smooth group manifold with $GL(n, \mathbb{R})$ as its transformation group, 1-parameter Lie subgroups of X (curves or flows on the surface) can be obtained as orbits of the 1-parameter matrix sub group:

$$\boldsymbol{x}(t) = T(t)\boldsymbol{x}_0 = e^{tA}\boldsymbol{x}_0 \qquad t \in \mathbb{R}. \tag{2}$$

Here $\boldsymbol{x}_0 \in X$ is an initial point. Since $\dot{\boldsymbol{x}}_0 = A\boldsymbol{x}_0$ and T is an addition group with respect to t, one has

$$\frac{d\boldsymbol{x}_t}{dt} := \dot{\boldsymbol{x}}_t = Ae^{tA}\boldsymbol{x}_0 = A\boldsymbol{x}_t. \tag{3}$$

Therefore

$$\dot{\boldsymbol{x}}_t = A\boldsymbol{x}_t. \tag{4}$$

i.e. the Lie algebra of tangent vector fields along the 1-parameter groups on the surface is a linear Lie algebra.

2.1 Lie algebra model of 3D objects[4]

An Lie algebra defined as (4) is called a linear algebra. More generally, an Affine Lie algebra is defined as

$$\mathcal{L}: \qquad \frac{\partial \boldsymbol{x}}{\partial t} := \dot{\boldsymbol{x}}_t = A\boldsymbol{x} + \boldsymbol{b}. \tag{5}$$

A model of 3D objects based on the linear and Affine Lie algebras of tangential or normal vector fields is discussed in [1][4]. A major advantage of this model is that these Lie algebras therefore the objects are uniquely determined by a complete set of invariants. e.g. for linear Lie algebra (4), the complete set I of invariants under Euclidean transformations is

$$I = \{\sigma_1, \sigma_2, \sigma_3, \phi_1, \phi_2, \phi_3\}$$

where $\{\sigma_i\}$ are the singular values of A, assuming the singular value decomposition of $A^R = U^T \Lambda V, \Lambda = \mathrm{diag}\sigma_i$. $\{\phi_i\}$ are the Euler angles of $VU^T \in SO_3(\mathbb{R})$.

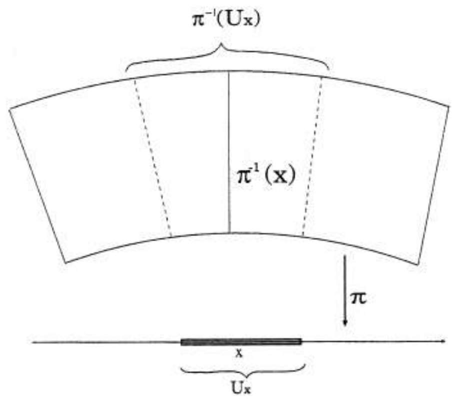

図 1　fibre bundle model of surfaces

The method to extract the complete set of invariants is discussed in [1][4] and [2][3].

3 A New Fibre Bundle Model of Surfaces

Here we show a new model of surfaces based on a mathematical concept of differential geometry and other fields called a fibre bundle.[5]

Given a curve $\boldsymbol{b} = \{\boldsymbol{b}(v), v \in \mathbb{R}\}$, a surface $F = \{F(u,v)\}$ is a fibre bundle on the base curve $\boldsymbol{b}$ if locally F is a direct product of $\boldsymbol{b}$ and another curve called a fibre curve. More specifically, there is a projection map π

$$\pi : F \longrightarrow \boldsymbol{b}$$

for any point $x \in \boldsymbol{b}$, there is a curve $\boldsymbol{f}_x = \{\boldsymbol{f}_x(u), u \in \mathbb{R}\}$ on F

$$\boldsymbol{f}_x := \pi^{-1}(x) \subset F$$

called a fibre at the base point x. For $\forall x \in \boldsymbol{b}$, there is a neighborhood of x in $\boldsymbol{b}$: $U_x \subset \boldsymbol{b}$ such that

$$\pi^{-1}(U_x) \cong U_x \times \boldsymbol{f}_x.$$

It is obviously that the fibre bundle model can represent any surface F. Indeed, e.g. one can take the base curve $\boldsymbol{b}$ as an curve on the surface, then choose fibres as a family $\{\boldsymbol{f}_v\}$ of curves which are transversal to $\boldsymbol{b}$ and sweeping through F, i.e. the angles between $\boldsymbol{f}_v$ and $\boldsymbol{b}$ at $\forall v$ are not zero. Thus, the fibre bundle can be used as a general model of surfaces. In fact, the generalized cylinders and ruled surfaces are special cases of the fibre bundle model. To see this, recall that a generalized cylinder G is defined by an axis $\boldsymbol{a}(v)$, a cross function $\boldsymbol{c}(u) = (a(u), b(u))^T$ of G on a cross plane which intersection at a particular angle with $\boldsymbol{a}$, and a sweeping functions $\{\sigma_a(v), \sigma_b(v)\}$ as

$$\boldsymbol{x}(u,v) = (\sigma_a(v)a(u), \sigma_b(v)b(u), v)^T$$

Such a generalized cylinder is in fact a fibre bundles. The base curve b can be chosen as any intersection curve of a plane along the direction of the axis. e.g.

$$\boldsymbol{b}(v) = \boldsymbol{x}(0,v) = (\sigma_a(v)a(0), \sigma_b(v)b(0), v)^T$$

and the fibre to be

$$\boldsymbol{f}_v := (\sigma_a(v)a(u), \sigma_b(v)b(u), v)^T$$

A ruled surface is defined as

$$\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{\gamma}(v)$$

where $\boldsymbol{\alpha}(v), \boldsymbol{\gamma}(v)$ are spatial curves. It includes developable surface when $\boldsymbol{\gamma}(v) = \boldsymbol{\alpha}'(v)$. A ruled surface is also a special case of fibre bundles. The base curve can be chosen as $\boldsymbol{b}(v) = \boldsymbol{\alpha}(v)$ and the fibre curve as $\boldsymbol{f}_{\boldsymbol{\alpha}(v)} = \boldsymbol{\alpha}(v) + u\boldsymbol{\gamma}(v)$.

4 Fibre bundle model with fibres of 1-parameter groups

Below, we choose the fibre curves in the fibre bundle model as 1-parameter groups. As a result, we obtain a fibre bundle of 1-parameter groups.

Let the base curve be $\boldsymbol{b} = \{\boldsymbol{b}(v), v \in \mathbb{R}\}$, the fibre curve a 1-parameter Lie group $\boldsymbol{g}_v = \{\boldsymbol{g}_v(u) = e^{uA}\boldsymbol{b}(v), u \in \mathbb{R}\}$. The surface is defined as

$$F := \{\boldsymbol{x}(u,v) = e^{uA}\boldsymbol{b}(v) \quad u, v \in \mathbb{R}\}$$

The points $\boldsymbol{b}(v)$ on the base curve $\boldsymbol{b}$ are initial points for the integral flow of 1-parameter groups $\boldsymbol{g}_v$. In fact, the base curve needs not to be in a form of a parameterized curve $\boldsymbol{b}(v)$.

Thus, the Lie algebra of this fibre bundle is a linear Lie algebra, as follows.

$$\mathcal{L} : \quad \frac{\partial \boldsymbol{x}}{\partial u} := \dot{\boldsymbol{x}}_u = Ae^{Au}\boldsymbol{b}_v = A\boldsymbol{x}$$

$$\mathcal{L} : \quad \dot{\boldsymbol{x}}_u = A\boldsymbol{x}$$

Shifting the origin by $\{\boldsymbol{c}(v)\}$, one obtains a fibre bundle with an Affine Lie algebra.

$$F = \{\boldsymbol{x}(u,v) = e^{Au}(\boldsymbol{b}(v) - \boldsymbol{c}(v)), u, v \in \mathbb{R}\}$$

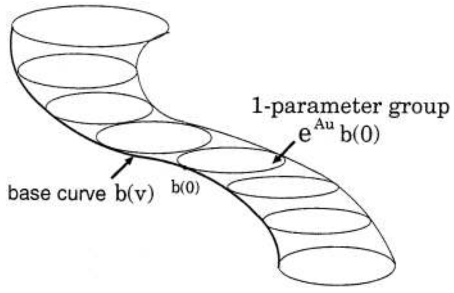

図 2 fibre bundle of 1-parameter groups

This is in fact a special case of the fibre bundle

$$F := \{\boldsymbol{x}(u, v) = e^{Au}\boldsymbol{b}(v) + \boldsymbol{d}(v), \quad u, v \in \mathbb{R}\}.$$

Its Lie algebra is an Affine Lie algebra as follows.

$$\mathcal{L}: \quad \dot{\boldsymbol{x}}_u = Ae^{Au}\boldsymbol{b}(v) = A(\boldsymbol{x} - \boldsymbol{d}(v)) = A\boldsymbol{x} - A\boldsymbol{d}(v)$$

$$\mathcal{L}: \quad \dot{\boldsymbol{x}}_u = A\boldsymbol{x} + \boldsymbol{\delta}(v), \qquad \boldsymbol{\delta}(v) = -A\boldsymbol{d}(v).$$

The information to describe the fibre bundle model is the base curve and the six invariants of the linear Lie algebra, i.e. of the matrix A.

A more general model is to use both the base curves and the fibre curves as 1-parameter groups:

$$\boldsymbol{x}(u, v) = e^{uA+vB}\boldsymbol{x}_0 + \boldsymbol{d}$$

The Lie algebra consists of the following two linear algebras.

$$\dot{\boldsymbol{x}}_u = A\boldsymbol{x}, \quad \dot{\boldsymbol{x}}_v = B\boldsymbol{x},$$

In this case, the information to describe the fibre bundle model is a base point $\boldsymbol{x}(0, 0) = \boldsymbol{b}(0)$ and twelve invariants of matrices A and B.

5 Synthesis by inverse Laplacian transformation

Another advantage for the surface model using fibre bundle of 1- parameter groups is that shape synthesis needs only calculation of elementary functions therefore without numerical errors of integration. In fact, the integration can be obtained using Laplacian transformation as follows.

For a curve with fixed v

$$\boldsymbol{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))^T$$

define the Laplacian transformation with respect to u as $\boldsymbol{X}(p, v)$:

$$\boldsymbol{X}(p, v) = (X_1(p, v), X_2(p, v), X_3(p, v))^T.$$

Since the fibre has a linear Lie algebra $\dot{\boldsymbol{x}}_u = A\boldsymbol{x}$

$$p\boldsymbol{X}(p, v) - \boldsymbol{x}(0_+, v) = A\boldsymbol{X}(p, v),$$
$$(pI - A)\boldsymbol{X}(p, v) = \boldsymbol{x}(0_+, v)$$

The fibre curve can be obtained by the inverse Laplacian transformation

$$\boldsymbol{X}(p, v) = (pI - A)^{-1}\boldsymbol{x}(0_+, v)$$

where $\boldsymbol{x}(0_+, v)$ is the initial point on the base curve.

Here $(pI - A)^{-1}$ is a 3×3 matrix with entries of rational functions in p. The denominators are polynomials with degrees no more than three. By the inverse Laplacian transformation of partial expansions, the fibre can be expressed by elementary functions such as trigonometric functions and exponentials. In other words, the shape can be synthesized fast and error free without numerical integration.

For fibres with Affine Lie algebras

$$\dot{\boldsymbol{x}}_u = A\boldsymbol{x} + \boldsymbol{d}_v,$$
$$(pI - A)\boldsymbol{X}(p, v) = \boldsymbol{x}(0_+, v) + \frac{1}{p}\boldsymbol{d}_v$$

Thus, the fibres can be obtained from the inverse Laplacian transformation of

$$\boldsymbol{X}(p, v) = (pI - A)^{-1}\boldsymbol{x}(0_+, v) + \frac{1}{p}(pI - A)^{-1}\boldsymbol{d}_v$$

again using elementary functions.

In the model with both the base curve and the fibres generated by 1-parameter groups

$$\boldsymbol{x}(u, v) = e^{uA+vB}\boldsymbol{x}_0 + \boldsymbol{d},$$
$$\frac{\partial \boldsymbol{x}}{\partial u} = A\boldsymbol{x}, \quad \frac{\partial \boldsymbol{x}}{\partial v} = B\boldsymbol{x}.$$

The tangent plane is spanned by these two basis.

$$\boldsymbol{t_x} = a\frac{\partial \boldsymbol{x}}{\partial u} + b\frac{\partial \boldsymbol{x}}{\partial b} = (aA + bB)\boldsymbol{x}$$

The surface can be easily generated by a double Laplacian transform

$$\boldsymbol{X}(p, v) = (pI - A)^{-1}(\boldsymbol{x}(0, v) + \frac{1}{p}\boldsymbol{d})$$
$$\boldsymbol{X}(u, q) = (qI - B)^{-1}(\boldsymbol{x}(u, 0) + \frac{1}{q}\boldsymbol{d}).$$

6 Simulations

Below, simulation results of shape synthesis using the proposed models are shown together with the invariants of representation matrix A.

図 3　Fibre (0.1742 0.7561 1.8236 4.3758 1.5539 0.0000)

図 4　Fibre (0.0000 -0.2517 -1.5708 3,2787 0.8660 0.0000)

図 5　Base curve(3,4641 2.0000 0.0000 0.9553 0.0000 1.5708) fibre(5.4772 1.2247 0.0000 -0.4243 0.9612 -1.7850)

図 6　Base curve (1.0000 1.0000 0.0000 0.0000 0.0000 1.5708) fibre (1.7321 1.77321 0.0000 0.0000 0.0000 -1.5708)

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