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A NEW SYSTEM OF VARIATIONAL INCLUSIONS WITH (H,η)-MONOTONE OPERATORS

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In this paper, We introduce and study a new system of variational inclusions involving (H,η)-monotone operators in Hilbert spaces. By using the resolvent operator method associated with (H,η)-monotone operators, we prove the existence and uniqueness of solutions and the convergence of some new three-step iterative algorithms for this system of variational inclusions and its special cases. The results in this paper extends and improves some results in the literature.

1. INTRODUCTION

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimisation and control, economics and transportation equilibrium, engineering science. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied.

Recently, some new and interesting problems, which are called to be system of variational inequality problems were introduced and studied. Pang [17], Cohen and Chaplais [8], Bianchi [5] and Ansari and Yao [4] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. Ansari, Schaible and Yao [3] introduced and studied a system of vector equilibrium problems and a system of vector variational inequalities by a fixed point theorem. Allevi, Gnudi and Konnov [2] considered a system of generalised vector variational inequalities and established some existence results with relative pseudomonotonicity. Kassay and Kolumbán [13] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay, Kolumbán and Páles [14] studied Minty and Stampacchia variational inequality systems

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with the help of the Kakutani–Fan–Glicksberg fixed point theorem. Peng [**18, 19**] introduced a system of quasi-variational inequality problems and proved its existence theorem by maximal element theorems. Verma [**21, 22, 24, 20, 23**] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of system of variational inequalities in Hilbert spaces. Kim and Kim [**16**] introduced a new system of generalised nonlinear quasi-variational inequalities and obtained some existence and uniqueness results for solution of this system of generalised nonlinear quasi-variational inequalities in Hilbert spaces. Cho, Fang and Huang [**7**] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. They proved some existence and uniqueness theorems for solutions of the system of nonlinear variational inequalities.

As generalisations of system of variational inequalities, Agarwal, Cho and Huang [**1**] introduced a system of generalised nonlinear mixed quasi-variational inclusions and investigated the sensitivity analysis of solutions for this system in Hilbert spaces. Kazmi and Bhat [**15**] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. Fang and Huang [**9**], Verma [**25**], Fang, Huang and Thompson [**11**] introduced and studied a new system of variational inclusions involving H-monotone operators, A-monotone operators and (H,η)-monotone operators, respectively.

Inspired and motivated by the results in [**4, 13, 14, 18, 19, 21, 22, 24, 20, 23, 16, 7, 17, 8, 5, 3, 2, 1, 15, 9, 25, 11**], the purpose of this paper is to introduce and study a new system of variational inclusions with (H,η)-monotone operators, which contains the mathematical models in [**21, 22, 24, 20, 23, 16, 7, 9, 11**] as special cases. By using the resolvent technique for the (H,η)-monotone operators, we prove the existence of solutions for this system of set-valued variational inclusions and its special cases. We also prove the convergence of some new three-step iterative algorithms approximating the solution for this system of variational inclusions and its special cases. The results in this paper extends and improves some results in [**21, 22, 24, 20, 23, 16, 7, 9, 11**].

2. Preliminaries

We suppose that $\mathcal{H}$ is a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let us recall some definitions needed later.

Definition 2.1. ([**11, 12**]) Let $\eta : \mathcal{H}\times\mathcal{H}\longrightarrow\mathcal{H}$ and $H : \mathcal{H}\longrightarrow\mathcal{H}$ be two single-valued operators and $M : \mathcal{H}\longrightarrow 2^{\mathcal{H}}$ be a set-valued operator. M is said to be

(i) η-monotone if,

$$\langle x-y, \eta(u,v)\rangle \geqslant 0, \forall u,v\in\mathcal{H}, x\in Mu, y\in Mv.$$

(ii) (H,η)-monotone if M is η-monotone and $(H+\lambda M)(\mathcal{H})=\mathcal{H}$, for all $\lambda>0$.

REMARK 2.1. (1) If $\eta(u,v) = u - v$, then the definition of η-monotonicity is that of monotonicity and the definition of (H,η)-monotonicity becomes that of H-monotonicity in [**10**]. It is easy to know that if $H = I$ (the identity map on $\mathcal{H}$), then the definition of (I,η)-monotone operators is that of maximal η-monotone operators and the definition of I-monotone operators is that of maximal monotone operators. Hence, the class of (H,η)-monotone operators provides a unifying frameworks for classes of maximal monotone operators, maximal η-monotone operators, H-monotone operators. For more details about the above definitions, please refer [**10, 9, 11, 12, 6**] and the references thèrein.

DEFINITION 2.2: ([**10, 12**]) Let $H, g : \mathcal{H} \longrightarrow \mathcal{H}$, $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be three single-valued operators. g is said to be

(i) monotone if
$$\langle gu - gv, u - v\rangle \geqslant 0, \forall u, v \in \mathcal{H};$$

(ii) strictly monotone if g is monotone and
$$\langle gu - gv, u - v\rangle = 0$$
if and only if $u = v$;

(iii) strongly monotone if there exists a constant $r > 0$ such that
$$\langle gu - gv, u - v\rangle \geqslant r\|u - v\|^2, \forall u, v \in \mathcal{H}.$$

(iv) Lipschitz continuous if there exists a constant $s > 0$ such that
$$\|g(u) - g(v)\| \leqslant s\|u - v\|, \forall u, v \in \mathcal{H}.$$

(v) strongly monotone with respect to H if there exists a constant $\gamma > 0$ such that
$$\langle gu - gv, Hu - Hv\rangle \geqslant \gamma\|u - v\|^2, \forall u, v \in \mathcal{H}.$$

(vi) η-monotone if
$$\big\langle gu - gv, \eta(u,v)\big\rangle \geqslant 0, \forall u, v \in \mathcal{H};$$

(vii) strictly η-monotone if g is η-monotone and
$$\big\langle gu - gv, \eta(u,v)\big\rangle = 0$$
if and only if $u = v$;

(viii) strongly η-monotone if there exists a constant $r > 0$ such that
$$\big\langle gu - gv, \eta(u,v)\big\rangle \geqslant r\|u - v\|^2, \forall u, v \in \mathcal{H}.$$

DEFINITION 2.3: ([**12**]) Let $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be a single-valued operator, then for all $u, v \in \mathcal{H}$, $\eta(.,.)$ is said to be

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(i) monotone, if

$$\langle \eta(u,v), u-v \rangle \geqslant 0;$$

(ii) Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\eta(u,v) \leqslant \tau \|u-v\|.$$

DEFINITION 2.4: ([11]) Let $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be a single-valued operator, $H : \mathcal{H} \longrightarrow \mathcal{H}$ be a strongly η-monotone operator and $M : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be an (H,η)-monotone operator. Then the resolvent operator $R_{M,\lambda}^{H,\eta} : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$R_{M,\lambda}^{H,\eta}(x) = (H + \lambda M)^{-1}(x), \forall x \in \mathcal{H}.$$

DEFINITION 2.5: Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be three Hilbert spaces, $g_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ and $N_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \longrightarrow \mathcal{H}_1$ be two single-valued mappings.

(i) $N_1(\cdot,\cdot,\cdot)$ is said to be Lipschitz continuous in the first argument if there exists a constant $\xi > 0$ such that

$$\|N_1(u,s,t) - N_1(v,s,t)\| \leqslant \xi \|u-v\|, \forall u,v \in \mathcal{H}_1, s \in \mathcal{H}_2, t \in \mathcal{H}_3.$$

(ii) $N_1(\cdot,\cdot,\cdot)$ is said to be monotone in the first argument if

$$\langle N_1(u,s,t) - N_1(v,s,t), u-v \rangle \geqslant 0, \quad \forall u,v \in \mathcal{H}_1, s \in \mathcal{H}_2, t \in \mathcal{H}_3.$$

(iii) $N_1(\cdot,\cdot,\cdot)$ is said to be strongly monotone in the first argument if there exists a constant $\alpha > 0$ such that

$$\langle N_1(u,s,t) - N_1(v,s,t), u-v \rangle \geqslant \alpha \|u-v\|^2, \forall u,v \in \mathcal{H}_1, s \in \mathcal{H}_2, t \in \mathcal{H}_3.$$

(iv) $N_1(\cdot,\cdot,\cdot)$ is said to be monotone with respect to g_1 in the first argument if

$$\langle N_1(u,s,t) - N_1(v,s,t), g_1(u) - g_1(v) \rangle \geqslant 0, \forall u,v \in \mathcal{H}_1, s \in \mathcal{H}_2, t \in \mathcal{H}_3.$$

(v) $N_1(\cdot,\cdot,\cdot)$ is said to be strongly monotone with respect to g_1 in the first argument if there exists a constant $\beta > 0$ such that

$$\langle N_1(u,s,t) - N_1(v,s,t), g_1(u) - g_1(v) \rangle \geqslant \beta \|u-v\|^2, \forall u,v \in \mathcal{H}_1, s \in \mathcal{H}_2, t \in \mathcal{H}_3.$$

In a similar way, for $i = 2,3$, we can define the Lipschitz continuity and the strong monotonicity (monotonicity) of $N_i : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \longrightarrow \mathcal{H}_i$ with respect to $g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ in the i-th argument.

We also need the following result obtained by Fan, Huang and Thompson [11].

LEMMA 2.1. *Let $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be a single-valued Lipschitz continuous operator with constant τ, $H : \mathcal{H} \longrightarrow \mathcal{H}$ be a strongly η-monotone operator with constant $\gamma > 0$ and $M : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be an (H,η)-monotone operator. Then, the resolvent operator $R_{M,\lambda}^{H,\eta} : \mathcal{H} \longrightarrow \mathcal{H}$ is Lipschitz continuous with constant τ/γ, that is,*

$$\|R_{M,\lambda}^{\mathcal{H},\eta}(x) - R_{M,\lambda}^{\mathcal{H},\eta}(y)\| \leqslant \frac{\tau}{\gamma}\|x-y\|, \forall x,y \in H.$$

3. A New System of Variational Inclusions

In this section, we shall introduce a new system of variational inclusions with (H, η)-monotone operators and construct a new three-step iterative algorithm for solving this system of variational inclusions in Hilbert spaces. In what follows, unless other specified, we always suppose that $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are three Hilbert spaces, $H_i, g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$, $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$, $F_i : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \longrightarrow \mathcal{H}_i$, $(i = 1, 2, 3)$ are single-valued mappings. Let $M_1 : \mathcal{H}_1 \longrightarrow 2^{\mathcal{H}_1}$ be an (H_1, η_1)-monotone operator, $M_2 : \mathcal{H}_2 \longrightarrow 2^{\mathcal{H}_2}$ be an (H_2, η_2)-monotone operator and $M_3 : \mathcal{H}_3 \longrightarrow 2^{\mathcal{H}_3}$ be an (H_3, η_3)-monotone operator. For given $f_1 \in \mathcal{H}_1, f_2 \in \mathcal{H}_2, f_3 \in \mathcal{H}_3, \zeta_1 > 0, \zeta_2 > 0, \zeta_3 > 0$, We consider the following system of variational inclusions with (H, η)-monotone operators, which is to find $(x, y, z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ such that

$$\text{(3.1)} \qquad \begin{cases} f_1 \in F_1(x, y, z) + \zeta_1 M_1\big(g_1(x)\big), \\ f_2 \in F_2(x, y, z) + \zeta_2 M_2\big(g_2(y)\big), \\ f_3 \in F_3(x, y, z) + \zeta_3 M_3\big(g_3(z)\big). \end{cases}$$

Some examples of problem (3.1) include the following.

(i) For $i = 1, 2, 3$, if $f_i = 0, g_i = I_i$ (the identity mapping on $\mathcal{H}_i$) and $\zeta_i = 1$, then Problem (3.1) becomes the following system of variational inclusions with (H, η)-monotone operators, which is to find $(x, y, z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ such that

$$\text{(3.2)} \qquad \begin{cases} 0 \in F_1(x, y, z) + M_1(x), \\ 0 \in F_2(x, y, z) + M_2(y), \\ 0 \in F_3(x, y, z) + M_3(z). \end{cases}$$

Problem (3.2) contains the system of variational inclusions with (H, η)-monotone operators in [**11**] and the system of variational inclusions with H-monotone operators in [**9**] as special cases.

(ii) If $M_1(x) = \Delta_{\eta_1}\varphi_1(x)$, $M_2(y) = \Delta_{\eta_2}\varphi_2(y)$ and $M_3(z) = \Delta_{\eta_3}\varphi_3(z)$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$ and $z \in \mathcal{H}_3$, where $\varphi_i : \mathcal{H}_i \longrightarrow R \cup \{+\infty\}$ is a proper, η_i-subdifferentiable functional and $\Delta_{\eta_i}\varphi_i$ denotes the η_i-subdifferential operator of φ_i (i=1, 2, 3), then problem (3.2) reduces to the following system of variational-like inequalities, which is to find $(x, y, z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ such that

$$\text{(3.3)} \qquad \begin{cases} \big\langle F_1(x, y, z), \eta_1(a, x)\big\rangle + \varphi_1(a) - \varphi_1(x) \geqslant 0, \forall a \in \mathcal{H}_1, \\ \big\langle F_2(x, y, z), \eta_2(b, y)\big\rangle + \varphi_2(b) - \varphi_2(y) \geqslant 0, \forall b \in \mathcal{H}_2, \\ \big\langle F_3(x, y, z), \eta_3(c, z)\big\rangle + \varphi_3(c) - \varphi_3(z) \geqslant 0, \forall c \in \mathcal{H}_3. \end{cases}$$

(iii) If $M_1(x) = \partial\varphi_1(x)$, $M_2(y) = \partial\varphi_2(y)$ and $M_3(z) = \partial\varphi_3(z)$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$ and $z \in \mathcal{H}_3$, where $\varphi_i : \mathcal{H}_i \longrightarrow R \cup \{+\infty\}$ is a proper, convex, lower semicontinuous functional and $\partial\varphi_i$ denotes the subdifferential operator of φ_i (i=1, 2, 3), then problem (3.2) reduces to the following system of variational inequalities, which is to find $(x,y,z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ such that

$$(3.4) \qquad \begin{cases} \langle F_1(x,y,z), a-x\rangle + \varphi_1(a) - \varphi_1(x) \geqslant 0, \forall a \in \mathcal{H}_1, \\ \langle F_2(x,y,z), b-y\rangle + \varphi_2(b) - \varphi_2(y) \geqslant 0, \forall b \in \mathcal{H}_2, \\ \langle F_3(x,y,z), c-z\rangle + \varphi_3(c) - \varphi_3(z) \geqslant 0, \forall c \in \mathcal{H}_3. \end{cases}$$

(iv) If $M_1(x) = \partial\delta_{K_1}(x)$, $M_2(y) = \partial\delta_{K_2}(y)$ and $M_3(z) = \partial\delta_{K_3}(z)$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$ and $z \in \mathcal{H}_3$, where $K_i \subset \mathcal{H}_i$ is a nonempty, closed and convex subsets and δ_{K_i} denotes the indicator of K_i for $i = 1,2,3$, then problem (3.4) reduces to the following system of variational inequalities, which is to find $(x,y,z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ such that

$$(3.5) \qquad \begin{cases} \langle F_1(x,y,z), a-x\rangle \geqslant 0, \forall a \in K_1, \\ \langle F_2(x,y,z), b-y\rangle \geqslant 0, \forall b \in K_2, \\ \langle F_3(x,y,z), c-z\rangle \geqslant 0, \forall c \in K_3. \end{cases}$$

Problem (3.5) is the system of inequalities in [4] with the index set $I = \{1,2,3\}$.

(v) If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathcal{H}$ is a Hilbert space, $K_1 = K_2 = K_3 = K$ is a nonempty, closed and convex subset, $F_1(x,y,z) = \rho T(y,x) + x - y$, $F_2(x,y,z) = \lambda T(z,y) + y - z$ and $F_3(x,y,z) = \sigma T(x,z) + z - x$ for all $x,y,z \in K$, where $T : K \times K \longrightarrow \mathcal{H}$ is a mapping on $K \times K$, $\rho, \lambda, \sigma > 0$ are three numbers, then problem (3.5) reduces to the following problem: find $x,y,z \in K$ such that

$$(3.6) \qquad \begin{cases} \langle \rho T(y,x) + x - y, a-x\rangle \geqslant 0, \forall a \in K, \\ \langle \lambda T(z,y) + y - z, a-y\rangle \geqslant 0, \forall a \in K, \\ \langle \sigma T(x,z) + z - x, a-z\rangle \geqslant 0, \forall a \in K. \end{cases}$$

Moreover, if $\sigma = 0$, Problem (3.6) becomes the problem introduced and studied by Verma [**23**].

(viii) If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathcal{H}$ is a Hilbert space, $K_1 = K_2 = K_3 = K$ is a nonempty, closed and convex subset, $F_1(x,y,z) = \rho T_1(y) + x - y$, $F_2(x,y,z) = \lambda T_2(z) + y - z$ and $F_3(x,y,z) = \sigma T_3(x) + z - x$ for all $x,y,z \in K$, where $T_1, T_2, T_3 : K \longrightarrow \mathcal{H}$ are three mappings on K, $\rho, \lambda, \sigma > 0$ are three numbers, then problem (3.5) reduces to the following problem: find $x,y,z \in K$ such that

$$(3.7) \qquad \begin{cases} \langle \rho T_1(y) + x - y, a-x\rangle \geqslant 0, \forall a \in K, \\ \langle \lambda T_2(z) + y - z, a-y\rangle \geqslant 0, \forall a \in K, \\ \langle \sigma T_3(x) + z - x, a-z\rangle \geqslant 0, \forall a \in K, \end{cases}$$

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Moreover, if $T_1 = T_2 = T_3 = T$ and $\sigma = 0$, Problem (3.7) becomes the problem introduced and studied by Verma [**21, 22, 24, 20**].

It is worthy noting that problem (3.1)–(3.4), problem (3.6) and (3.7) are all new mathematical models.

In brief, for a suitable choice of mappings $\eta_i, H_i, g_i, M_i, F_i$, the element f_i, and the space $\mathcal{H}_i$, it is easy to see that the problem (3.1) includes a number of mathematical models studied in [**21, 22, 24, 20, 23, 16, 7, 9, 11**] and the reference therein.

4. Existence of the solution

In this section, we shall prove existence and uniqueness for solutions of problem (3.1) and its special cases. For our main results, we give a characterisation of the solution of problem (3.1) as follows.

Lemma 4.1. *For $i = 1,2,3$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be a single-valued operator, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be a strictly η-monotone operator, $M_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be an (H_i, η_i)-monotone operator, then, $(x,y,z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ is a solution of problem (3.1), if and only if*

$$\begin{cases} g_1(x) = R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x) - \rho F_1(x,y,z)\big), \\ g_2(y) = R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}\big(\lambda f_2 + H_2 g_2(y) - \lambda F_2(x,y,z)\big), \\ g_3(z) = R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3 + H_3 g_3(z) - \delta F_3(x,y,z)\big), \end{cases}$$

where

$$\begin{aligned} &R^{H_1,\eta_1}_{M_1,\rho\zeta_1} = (H_1 + \rho\zeta_1 M_1)^{-1},\ R^{H_2\eta_2}_{M_2,\lambda\zeta_2} = (H_2 + \lambda\zeta_2 M_2)^{-1}, \\ &R^{H_3,\eta_3}_{M_3,\delta\zeta_3} = (H_3 + \delta\zeta_3 M_3)^{-1}, \rho > 0, \lambda > 0, \delta > 0, \zeta_1 > 0, \zeta_2 > 0 \text{ and } \zeta_3 > 0 \end{aligned}$$

are constants.

Proof: $(x,y,z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ is a solution of problem (3.1)

$$\iff \begin{cases} f_1 \in F_1(x,y,z) + \zeta_1 M_1\big(g_1(x)\big), \\ f_2 \in F_2(x,y,z) + \zeta_2 M_2\big(g_2(y)\big), \\ f_3 \in F_3(x,y,z) + \zeta_3 M_3\big(g_3(z)\big). \end{cases}$$

$$\iff \begin{cases} \rho f_1 \in \rho F_1(x,y,z) + \rho\zeta_1 M_1\big(g_1(x)\big), \\ \lambda f_2 \in \lambda F_2(x,y,z) + \lambda\zeta_2 M_2\big(g_2(y)\big), \\ \delta f_3 \in \delta F_3(x,y,z) + \delta\zeta_3 M_3\big(g_3(z)\big), \end{cases}$$

$$\iff \begin{cases} \rho f_1 + H_1\big(g_1(x)\big) - \rho F_1(x,y,z) \in (H_1 + \rho\zeta_1 M_1)\big(g_1(x)\big), \\ \lambda f_2 + H_2\big(g_2(y)\big) - \lambda F_2(x,y,z) \in (H_2 + \lambda\zeta_2 M_2)\big(g_2(y)\big), \\ \delta f_3 + H_3\big(g_3(z)\big) - \delta F_3(x,y,z) \in (H_3 + \delta\zeta_3 M_3)\big(g_3(z)\big) \end{cases}$$

$$\iff \begin{cases} g_1(x) = R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x) - \rho F_1(x,y,z)\big), \\ g_2(y) = R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}\big(\lambda f_2 + H_2 g_2(y) - \lambda F_2(x,y,z)\big), \\ g_3(z) = R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3 + H_3 g_3(z) - \delta F_3(x,y,z)\big) \end{cases}$$

∎

THEOREM 4.1. *For $i = 1,2,3$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be Lipschitz continuous with constant $\sigma_i > 0$, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be strongly η_i-monotone and Lipschitz continuous with constants $\gamma_i > 0$ and $\tau_i > 0$, respectively, $g_i : \mathcal{H}_i \to \mathcal{H}_i$ be strongly monotone and Lipschitz continuous with constants $\beta_i > 0$ and $\theta_i > 0$, respectively, $M_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be an (H_i, η_i)-monotone operator. Let $F_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_1$ be a single-valued mapping such that F_1 is strongly monotone with respect to $\widehat{g_1}$ and Lipschitz continuous in the first argument with constants $r_1 > 0$ and $s_1 > 0$, respectively, where $\widehat{g_1} : \mathcal{H}_1 \to \mathcal{H}_1$ is defined by $\widehat{g_1}(x) = H_1 \circ g_1(x) = H_1\big(g_1(x)\big), \forall x \in \mathcal{H}_1$, F_1 is Lipschitz continuous in the second and third arguments with constants $t_1 > 0$ and $a_1 > 0$, respectively; $F_2 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_2$ be a single-valued mapping such that F_2 is strongly monotone with respect to $\widehat{g_2}$ and Lipschitz continuous in the second argument with constants $r_2 > 0$ and $s_2 > 0$, respectively, where $\widehat{g_2} : \mathcal{H}_2 \to \mathcal{H}_2$ is defined by $\widehat{g_2}(y) = H_2 \circ g_2(y) = H_2\big(g_2(y)\big), \forall y \in \mathcal{H}_2$, F_2 is Lipschitz continuous in the third and first arguments with constants $t_2 > 0$ and $a_2 > 0$, respectively; $F_3 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_3$ be a single-valued mapping such that F_3 is strongly monotone with respect to $\widehat{g_3}$ and Lipschitz continuous in the third argument with constants $r_3 > 0$ and $s_3 > 0$, respectively, where $\widehat{g_3} : \mathcal{H}_3 \to \mathcal{H}_3$ is defined by $\widehat{g_3}(z) = H_3 \circ g_3(z) = H_3\big(g_3(z)\big), \forall z \in \mathcal{H}_3$, F_3 is Lipschitz continuous in the first and second arguments with constants $t_3 > 0$ and $a_3 > 0$, respectively. If there exist constants $\rho > 0, \lambda > 0$, and $\delta > 0$ such that,*

(4.1)

$$\begin{cases} \gamma_1\gamma_2\gamma_3\sqrt{1 - 2\beta_1 + \theta_1^2} + \sigma_1\gamma_2\gamma_3\sqrt{\tau_1^2\theta_1^2 - 2\rho r_1 + \rho^2 s_1^2} + \sigma_3\delta\gamma_1\gamma_2 t_3 + \sigma_2\lambda\gamma_1\gamma_3 a_2 < \gamma_1\gamma_2\gamma_3, \\ \gamma_1\gamma_2\gamma_3\sqrt{1 - 2\beta_2 + \theta_2^2} + \sigma_2\gamma_1\gamma_3\sqrt{\tau_2^2\theta_2^2 - 2\lambda r_2 + \lambda^2 s_2^2} + \sigma_1\rho\gamma_2\gamma_3 t_1 + \sigma_3\delta\gamma_1\gamma_2 a_3 < \gamma_1\gamma_2\gamma_3, \\ \gamma_1\gamma_2\gamma_3\sqrt{1 - 2\beta_3 + \theta_3^2} + \sigma_3\gamma_1\gamma_2\sqrt{\tau_3^2\theta_3^2 - 2\delta r_3 + \delta^2 s_3^2} + \sigma_2\lambda\gamma_1\gamma_3 t_2 + \sigma_1\rho\gamma_2\gamma_3 a_1 < \gamma_1\gamma_2\gamma_3 \end{cases}$$

Then, problem (3.1) admits a unique solution.

PROOF: For any given $\rho, \lambda, \delta > 0$, define $T_\rho : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_1, S_\lambda : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_2, P_\delta : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_3$ by

$$\begin{aligned} T_\rho(x,y,z) &= x - g_1(x) + R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x) - \rho F_1(x,y,z)\big), \\ S_\lambda(x,y,z) &= y - g_2(y) + R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}\big(\lambda f_2 + H_2 g_2(y) - \lambda F_2(x,y,z)\big), \\ P_\delta(x,y,z) &= z - g_3(z) + R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3 + H_3 g_3(z) - \delta F_3(x,y,z)\big), \end{aligned} \tag{4.2}$$

for all $(x,y,z) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$.

For any $(x_1,y_1,z_1), (x_2,y_2,z_2) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$, it follows from (4.2) that

$$\big\|T_\rho(x_1,y_1,z_1) - T_\rho(x_2,y_2,z_2)\big\|$$

$$= \Big\| x_1 - g_1(x_1) + R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x_1) - \rho F_1(x_1,y_1,z_1)\big) - \Big(x_2 - g_1(x_2) + R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x_2) - \rho F_1(x_2,y_2,z_2)\big)\Big)\Big\|$$

$$\leqslant \Big\| x_1 - x_2 - \big(g_1(x_1) - g_1(x_2)\big)\Big\| + \Big\| R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x_1) - \rho F_1(x_1,y_1,z_1)\big) - R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x_2) - \rho F_1(x_2,y_2,z_2)\big)\Big\|. \tag{4.3}$$

Since g_1 is strongly monotone and Lipschitz continuous with constants β_1 and θ_1, we have

$$\Big\| x_1 - x_2 - \big(g_1(x_1) - g_1(x_2)\big)\Big\|^2 = \|x_1 - x_2\|^2 - 2\big\langle g_1(x_1) - g_1(x_2), x_1 - x_2\big\rangle + \big\|g_1(x_1) - g_1(x_2)\big\|^2 \leqslant (1 - 2\beta_1 + \theta_1^2)\|x_1 - x_2\|^2, \tag{4.4}$$

It follows from Lemma 2.1 that

$$\Big\| R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x_1) - \rho F_1(x_1,y_1,z_1)\big) - R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x_2) - \rho F_1(x_2,y_2,z_2)\big)\Big\|$$
$$\leqslant \frac{\sigma_1}{\gamma_1}\Big\| H_1\big(g_1(x_1)\big) - H_1\big(g_1(x_2)\big) - \rho\big(F_1(x_1,y_1,z_1) - F_1(x_2,y_2,z_2)\big)\Big\|$$
$$\leqslant \frac{\sigma_1}{\gamma_1}\Big\| H_1\big(g_1(x_1)\big) - H_1\big(g_1(x_2)\big) - \rho\big(F_1(x_1,y_1,z_1) - F_1(x_2,y_1,z_1)\big)\Big\| + \frac{\sigma_1\rho}{\gamma_1}\|F_1(x_2,y_1,z_1) - F_1(x_2,y_2,z_1)\| + \frac{\sigma\rho}{\gamma_1}\|F_1(x_2,y_2,z_1) - F_1(x_2,y_2,z_2)\|. \tag{4.5}$$

Since H_1 and g_1 are Lipschitz continuous with constants τ_1 and θ_1, respectively, F_1 is strongly monotone with respect to $\widehat{g_1}$ and Lipschitz continuous in the first argument with constants r_1 and s_1, respectively, we have

$$\Big\| H_1\big(g_1(x_1)\big) - H_1\big(g_1(x_2)\big) - \rho\big(F_1(x_1,y_1,z_1) - F_1(x_2,y_1,z_1)\big)\Big\|^2$$
$$\leqslant \Big\| H_1\big(g_1(x_1)\big) - H_1\big(g_1(x_2)\big)\Big\|^2 - 2\rho\Big\langle F_1(x_1,y_1,z_1) - F_1(x_2,y_1,z_1), H_1\big(g_1(x_1)\big) - H_1\big(g_1(x_2)\big)\Big\rangle + \rho^2\big\|F_1(x_1,y_1,z_1) - F_1(x_2,y_1,z_1)\big\|^2$$
$$\leqslant \tau_1^2\big\|g_1(x_1) - g_1(x_2)\big\|^2 - 2\rho r_1\|x_1 - x_2\|^2 + \rho^2 s_1^2\|x_1 - x_2\|^2$$
$$\leqslant (\tau_1^2\theta_1^2 - 2\rho r_1 + \rho^2 s_1^2)\|x_1 - x_2\|^2. \tag{4.6}$$

Since F_1 is Lipschitz continuous in the second and third arguments with constants t_1 and a_1, respectively, we have

$$\big\|F_1(x_2,y_1,z_1) - F_1(x_2,y_2,z_1)\big\| \leqslant t_1\|y_1 - y_2\|, \tag{4.7}$$
$$\big\|F_1(x_2,y_2,z_1) - F_1(x_2,y_2,z_2)\big\| \leqslant a_1\|z_1 - z_2\|. \tag{4.8}$$

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It follows from (4.3)-(4.8) that

(4.9)
$$\|T_\rho(x_1,y_1,z_1)-T_\rho(x_2,y_2,z_2)\| \leqslant \left(\sqrt{1-2\beta_1+\theta_1^2}+\frac{\sigma_1}{\gamma_1}\sqrt{\tau_1^2\theta_1^2-2\rho r_1+\rho^2 s_1^2}\right)\|x_1-x_2\| + \frac{t_1\sigma_1\rho}{\gamma_1}\|y_1-y_2\|+\frac{a_1\sigma_1\rho}{\gamma_1}\|z_1-z_2\|.$$

Similarily, we have

$$\begin{aligned}
&\|S_\lambda(x_1,y_1,z_1)-S_\lambda(x_2,y_2,z_2)\|\\
&\leqslant \left\|y_1-y_2-\big(g_2(y_1)-g_2(y_2)\big)\right\| + \Big\|R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}\big(\lambda f_2+H_2g_2(y_1)-\lambda F_2(x_1,y_1,z_1)\big)\\
&\qquad -R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}\big(\lambda f_2+H_2g_2(y_2)-\lambda F_2(x_2,y_2,z_2)\big)\Big\|\\
&\leqslant \sqrt{1-2\beta_2+\theta_2^2}\|y_1-y_2\|\\
&\qquad +\frac{\sigma_2}{\gamma_2}\Big\|H_2\big(g_2(y_1)\big)-H_2\big(g_2(y_2)\big)-\lambda\big(F_2(x_1,y_1,z_1)-F_2(x_1,y_2,z_1)\big)\Big\|\\
&\qquad +\frac{\sigma_2\lambda}{\gamma_2}\|F_2(x_1,y_2,z_1)-F_2(x_2,y_2,z_1)\|+\frac{\sigma_2\lambda}{\gamma_2}\|F_2(x_2,y_2,z_1)-F_2(x_2,y_2,z_2)\|\\
&\leqslant \left(\sqrt{1-2\beta_2+\theta_2^2}+\frac{\sigma_2}{\gamma_2}\sqrt{\tau_2^2\theta_2^2-2\lambda r_2+\lambda^2 s_2^2}\right)\|y_1-y_2\|\\
&\qquad +\frac{a_2\sigma_2\lambda}{\gamma_2}\|x_1-x_2\|+\frac{t_2\sigma_2\lambda}{\gamma_2}\|z_1-z_2\|. \qquad (4.10)
\end{aligned}$$

And

$$\begin{aligned}
&\|P_\delta(x_1,y_1,z_1)-P_\delta(x_2,y_2,z_2)\| \leqslant \left\|z_1-z_2-\big(g_3(z_1)-g_3(z_2)\big)\right\|\\
&\qquad +\Big\|R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3+H_3g_3(z_1)-\delta F_3(x_1,y_1,z_1)\big)\\
&\qquad -R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3+H_3g_3(z_2)-\delta F_3(x_2,y_2,z_2)\big)\Big\|\\
&\leqslant \sqrt{1-2\beta_3+\theta_3^2}\|z_1-z_2\|\\
&\qquad +\frac{\sigma_3}{\gamma_3}\Big\|H_3\big(g_3(z_1)\big)-H_3\big(g_3(z_2)\big)-\delta\big(F_3(x_1,y_1,z_1)-F_3(x_1,y_1,z_2)\big)\Big\|\\
&\qquad +\frac{\sigma_3\delta}{\gamma_3}\|F_3(x_1,y_1,z_2)-F_3(x_2,y_1,z_2)\|+\frac{\sigma_3\delta}{\gamma_3}\|F_3(x_2,y_1,z_2)-F_3(x_2,y_2,z_2)\|\\
&\leqslant \left(\sqrt{1-2\beta_3+\theta_3^2}+\frac{\sigma_3}{\gamma_3}\sqrt{\tau_3^2\theta_3^2-2\delta r_3+\delta^2 s_3^2}\right)\|z_1-z_2\|+\frac{t_3\sigma_3\delta}{\gamma_3}\|x_1-x_2\|\\
&\qquad +\frac{a_3\sigma_3\delta}{\gamma_3}\|y_1-y_2\|. \qquad (4.11)
\end{aligned}$$

It follows from (4.9)-(4.11) that

$$\|T_\rho(x_1,y_1,z_1)-T_\rho(x_2,y_2,z_2)\|$$

$$
\begin{aligned}
&+\big\|S_\lambda(x_1,y_1,z_1)-S_\lambda(x_2,y_2,z_2)\big\|+\big\|P_\delta(x_1,y_1,z_1)-P_\delta(x_2,y_2,z_2)\big\|\\
\leqslant&\left(\sqrt{1-2\beta_1+\theta_1^2}+\frac{\sigma_1}{\gamma_1}\sqrt{\tau_1^2\theta_1^2-2\rho r_1+\rho^2 s_1^2}+\frac{a_2\sigma_2\lambda}{\gamma_2}+\frac{t_3\sigma_3\delta}{\gamma_3}\right)\|x_1-x_2\|\\
&+\left(\sqrt{1-2\beta_2+\theta_2^2}+\frac{\sigma_2}{\gamma_2}\sqrt{\tau_2^2\theta_2^2-2\lambda r_2+\lambda^2 s_2^2}+\frac{a_3\sigma_3\delta}{\gamma_3}+\frac{t_1\sigma_1\rho}{\gamma_1}\right)\|y_1-y_2\|\\
&+\left(\sqrt{1-2\beta_3+\theta_3^2}+\frac{\sigma_3}{\gamma_3}\sqrt{\tau_3^2\theta_3^2-2\delta r_3+\delta^2 s_3^2}+\frac{a_1\sigma_1\rho}{\gamma_1}+\frac{t_2\sigma_2\lambda}{\gamma_2}\right)\|z_1-z_2\|\\
\leqslant& k\big(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|\big), \qquad (4.12)
\end{aligned}
$$

Where

$$
\begin{aligned}
k=\max\Bigg\{&\sqrt{1-2\beta_1+\theta_1^2}+\frac{\sigma_1}{\gamma_1}\sqrt{\tau_1^2\theta_1^2-2\rho r_1+\rho^2 s_1^2}+\frac{a_2\sigma_2\lambda}{\gamma_2}+\frac{t_3\sigma_3\delta}{\gamma_3}\\
&\sqrt{1-2\beta_2+\theta_2^2}+\frac{\sigma_2}{\gamma_2}\sqrt{\tau_2^2\theta_2^2-2\lambda r_2+\lambda^2 s_2^2}+\frac{a_3\sigma_3\delta}{\gamma_3}+\frac{t_1\sigma_1\rho}{\gamma_1}\\
&\sqrt{1-2\beta_3+\theta_3^2}+\frac{\sigma_3}{\gamma_3}\sqrt{\tau_3^2\theta_3^2-2\delta r_3+\delta^2 s_3^2}+\frac{a_1\sigma_1\rho}{\gamma_1}+\frac{t_2\sigma_2\lambda}{\gamma_2}\Bigg\}.
\end{aligned}
$$

Define $\|\cdot\|_1$ on $\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_3$ by $\big\|(x,y,z)\big\|_1=\|x\|_1+\|y\|_1+\|z\|_1, \forall(x,y,z)\in\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_3$. It is easy to see that $(\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_3,\|\cdot\|_1)$ is a Banach space. For any given $\rho,\lambda,\delta>0$, define $W_{\rho,\lambda,\delta}:\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_3\to\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_3$ by

$$W_{\rho,\lambda,\delta}(x,y,z)=\big(T_\rho(x,y,z),S_\lambda(x,y,z),P_\delta(x,y,z)\big),\forall(x,y,z)\in\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_3.$$

By (4.1), we know that $0<k<1$, it follows from (4.12) that

$$\big\|W_{\rho,\lambda,\delta}(x_1,y_1,z_1)-W_{\rho,\lambda,\delta}(x_2,y_2,z_2)\big\|_1\leqslant k\big\|(x_1,y_1,z_1)-(x_2,y_2,z_2)\big\|_1.$$

This shows that $W_{\rho,\lambda,\delta}$ is a contraction operator. Hence, there exists a unique $(x,y,z)\in\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_3$, such that

$$W_{\rho,\lambda,\delta}(x,y,z)=(x,y,z),$$

that is

$$
\begin{aligned}
g_1(x)&=R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1+H_1g_1(x)-\rho F_1(x,y,z)\big),\\
g_2(y)&=R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}\big(\lambda f_2+H_2g_2(y)-\lambda F_2(x,y,z)\big),\\
g_3(z)&=R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3+H_3g_3(z)-\delta F_3(x,y,z)\big).
\end{aligned}
$$

By lemma 4.1, (x,y,z) is the unique solution of problem (3.1).This completes this proof. ∎

REMARK 4.1. By Theorem 4.1, it is easy to get the existence and uniqueness of solutions for the special cases of problem (3.1), now we give three examples as follows.

For $i = 1,2,3$, let $g_i = I_i$ (the identity map on $\mathcal{H}_i$) and $f_i = 0$, then $\theta_i = \beta_i = 1$, by Theorem 4.1, we have

COROLLARY 4.2. *For $i = 1,2,3$, let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be Lipschitz continuous with constant $\sigma_i > 0$, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be strongly η-monotone and Lipschitz continuous with constants $\gamma_i > 0$ and $\tau_i > 0$, respectively, $M_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be an (H_i, η_i)-monotone operator. Let $F_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_1$ be a single-valued mapping such that F_1 is strongly monotone with respect to H_1 and Lipschitz continuous in the first argument with constants $r_1 > 0$ and $s_1 > 0$, respectively, F_1 is Lipschitz continuous in the second and third arguments with constants $t_1 > 0$ and $a_1 > 0$, respectively; $F_2 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_2$ be a single-valued mapping such that F_2 is strongly monotone with respect to H_2 and Lipschitz continuous in the second argument with constants $r_2 > 0$ and $s_2 > 0$, respectively, F_2 is Lipschitz continuous in the third and first arguments with constants $t_2 > 0$ and $a_2 > 0$, respectively; $F_3 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_3$ be a single-valued mapping such that F_3 is strongly monotone with respect to H_3 and Lipschitz continuous in the third argument with constants $r_3 > 0$ and $s_3 > 0$, respectively, F_3 is Lipschitz continuous in the first and second arguments with constants $t_3 > 0$ and $a_3 > 0$, respectively. If there exist constants $\rho > 0, \lambda > 0$, and $\delta > 0$ such that,*

$$(4.13)\qquad \begin{cases} \sigma_1\gamma_2\gamma_3\sqrt{\tau_1^2 - 2\rho r_1 + \rho^2 s_1^2} + \sigma_3\delta\gamma_1\gamma_2 t_3 + \sigma_2\lambda\gamma_1\gamma_3 a_2 < \gamma_1\gamma_2\gamma_3, \\ \sigma_2\gamma_1\gamma_3\sqrt{\tau_2^2 - 2\lambda r_2 + \lambda^2 s_2^2} + \sigma_1\rho\gamma_2\gamma_3 t_1 + \sigma_3\delta\gamma_1\gamma_2 a_3 < \gamma_1\gamma_2\gamma_3, \\ \sigma_3\gamma_1\gamma_2\sqrt{\tau_3^2 - 2\delta r_3 + \delta^2 s_3^2} + \sigma_2\lambda\gamma_1\gamma_3 t_2 + \sigma_1\rho\gamma_2\gamma_3 a_1 < \gamma_1\gamma_2\gamma_3 \end{cases}$$

then problem (3.2) admits a unique solution.

For $i = 1,2,3$, if $\eta_i(x_i, y_i) = x_i - y_i$, $H_i = I_i$ and $M_i = \partial\varphi_i$, then $\sigma_i = \gamma_i = \tau_i = 1$, by Corollary 4.2, we have

COROLLARY 4.3. *For $i = 1,2,3$, let $\varphi_i : \mathcal{H}_i \longrightarrow R \cup \{+\infty\}$ be a proper, convex, lower semicontinuous functional, $F_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_1$ be a single-valued mapping such that F_1 is strongly monotone and Lipschitz continuous in the first argument with constants $r_1 > 0$ and $s_1 > 0$, respectively, F_1 is Lipschitz continuous in the second and third arguments with constants $t_1 > 0$ and $a_1 > 0$, respectively; $F_2 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_2$ be a single-valued mapping such that F_2 is strongly monotone and Lipschitz continuous in the second argument with constants $r_2 > 0$ and $s_2 > 0$, respectively, F_2 is Lipschitz continuous in the third and first arguments with constants $t_2 > 0$ and $a_2 > 0$, respectively; $F_3 : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_3$ be a single-valued mapping such that F_3 is strongly monotone and Lipschitz continuous in the third argument with constants $r_3 > 0$ and $s_3 > 0$, respectively, F_3 is Lipschitz continuous in the first and second arguments with constants $t_3 > 0$ and $a_3 > 0$, respectively. If there exist constants $\rho > 0, \lambda > 0$, and $\delta > 0$ such* *that,*

$$(4.14)\qquad \begin{cases} \sqrt{1-2\rho r_1+\rho^2 s_1^2}+\delta t_3+\lambda a_2<1,\\ \sqrt{1-2\lambda r_2+\lambda^2 s_2^2}+\rho t_1+\delta a_3<1,\\ \sqrt{1-2\delta r_3+\delta^2 s_3^2}+\lambda t_2+\rho a_1<1. \end{cases}$$

Then, problem (3.4) admits a unique solution.

Let $\mathcal{H}_1=\mathcal{H}_2=\mathcal{H}_3=\mathcal{H}$ be a Hilbert space, $K_1=K_2=K_3=K$ be a nonempty, closed and convex subset, $M_1(x)=M_2(x)=M_3(x)=\partial\delta_K(x)$, $F_1(x,y,z)=\rho T_1(y)+x-y$, $F_2(x,y,z)=\lambda T_2(z)+y-z$ and $F_3(x,y,z)=\sigma T_3(x)+z-x$ for all $x,y,z\in K$, by Corollary 4.3, we have

Corollary 4.4. *For $i=1,2,3$, let $T_i:K\longrightarrow\mathcal{H}$ be strongly monotone and Lipschitz continuous in the first argument with constants $r_i>0$ and $s_i>0$, respectively. If there exist constants $\rho>0,\lambda>0$, and $\delta>0$ such that,*

$$(4.15)\qquad \begin{cases} 0<\rho<\dfrac{r_1}{s_1^2}<1,\\ 0<\lambda<\dfrac{r_2}{s_2^2}<1,\\ 0<\delta<\dfrac{r_3}{s_3^2}<1 \end{cases}$$

Then, problem (3.7) admits a unique solution.

5. Iterative algorithm and convergence

In this section, we shall construct some three-step iterative algorithm for approximating the unique solution of problem (3.1) and its special cases and discuss the convergence analysis of these Algorithms.

Lemma 5.1. ([11]) *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers that satisfy the following conditions.*

(1) $0\leqslant k_n<1, n=0,1,2\dots$ and $\limsup_n k_n<1$,

(2) $c_{n+1}\leqslant k_n c_n, n=0,1,2\dots,$

then c_n converges to 0 as $n\to\infty$.

Algorithm 5.1. For $i=1,2,3$, let H_i,M_i,F_i,g_i,η_i be the same as in Theorem 4.1. For any given $(x_0,y_0,z_0)\in\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_3$, define the three-step iterative sequence $\{(x_n,y_n,z_n)\}$ by

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)\Big[x_n-g_1(x_n)+R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1+H_1g_1(x_n)-\rho F_1(x_n,y_n,z_n)\big)\Big],$$

$$(5.1)\qquad y_{n+1}=\alpha_n y_n+(1-\alpha_n)\Big[y_n-g_2(y_n)+R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}\big(\lambda f_2+H_2g_2(y_n)-\lambda F_2(x_n,y_n,z_n)\big)\Big],$$

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$$z_{n+1} = \alpha_n z_n + (1-\alpha_n)\Big[z_n - g_3(z_n) + R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3 + H_3 g_3(z_n) - \delta F_3(x_n, y_n, z_n)\big)\Big],$$

where

$$0 \leqslant \alpha_n < 1, \quad \text{and} \quad \limsup_n \alpha_n < 1. \tag{5.2}$$

THEOREM 5.1. *For $i = 1,2,3$, let $H_i, M_i, F_i, g_i, \eta_i$ be the same as in Theorem 4.1. Assume that all the conditions of theorem 4.1 hold. Then (x_n, y_n, z_n) generated by Algorithm 5.1 converges strongly to the unique solution (x, y, z) of problem (3.1).*

PROOF: By Theorem 4.1, problem (3.1) admits a unique solution (x, y, z), it follows from Lemma 4.1 that

$$\begin{aligned}
x &= \alpha_n x + (1-\alpha_n)\Big[x - g_1(x) + R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x) - \rho F_1(x,y,z)\big)\Big],\\
y &= \alpha_n y + (1-\alpha_n)\Big[y - g_2(y) + R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}\big(\lambda f_2 + H_2 g_2(y) - \lambda F_2(x,y,z)\big)\Big],\\
z &= \alpha_n z + (1-\alpha_n)\Big[z - g_3(z) + R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3 + H_3 g_3(z) - \delta F_3(x,y,z)\big)\Big].
\end{aligned} \tag{5.3}$$

It follows from (5.1) and (5.3) that

$$\begin{aligned}
\|x_{n+1} - x\| = \Big\|&\alpha_n(x_n - x) + (1-\alpha_n)\Big[\Big(x_n - g_1(x_n) - \big(x - g_1(x)\big)\Big)\\
&+ R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x_n) - \rho F_1(x_n, y_n, z_n)\big)\\
&- R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x) - \rho F_1(x,y,z)\big)\Big]\Big\|\\
\leqslant\ & \alpha_n\|x_n - x\| + (1-\alpha_n)\Big\|x_n - g_1(x_n) - \big(x - g_1(x)\big)\Big\|\\
&+(1-\alpha_n)\Big\|R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x_n) - \rho F_1(x_n, y_n, z_n)\big)\\
&- R^{H_1,\eta_1}_{M_1,\rho\zeta_1}\big(\rho f_1 + H_1 g_1(x) - \rho F_1(x,y,z)\big)\Big\|.
\end{aligned} \tag{5.4}$$

Since g_1 is strongly monotone and Lipschitz continuous with constants β_1 and θ_1, we have

$$\Big\|x_n - x - \big(g_1(x_n) - g_1(x)\big)\Big\|^2 \leqslant (1 - 2\beta_1 + \theta_1^2)\|x_n - x\|^2. \tag{5.5}$$

By Lemma 2.1, we have

$$\begin{aligned}
\Big\|R^{H_1,\eta_1}_{M_1,\lambda\zeta_1}&\big(\rho f_1 + H_1 g_1(x_n) - \rho F_1(x_n, y_n, z_n)\big) - R^{H_1,\eta_1}_{M_1,\lambda\zeta_1}\big(\rho f_1 + H_1 g_1(x) - \rho F_1(x,y,z)\big)\Big\|\\
&\leqslant \frac{\sigma_1}{\gamma_1}\Big\|H_1\big(g_1(x_n)\big) - H_1\big(g_1(x)\big) - \rho\big(F_1(x_n, y_n, z_n) - F_1(x, y_n, z_n)\big)\Big\|\\
&\quad + \frac{\sigma_1\rho}{\gamma_1}\|F_1(x, y_n, z_n) - F_1(x, y, z_n)\| + \frac{\sigma_1\rho}{\gamma_1}\|F_1(x, y, z_n) - F_1(x,y,z)\|.
\end{aligned} \tag{5.6}$$

Since H_1 is Lipschitz continuous with constant τ_1, F_1 is strongly monotone with respect to $\widehat{g_1}$ and Lipschitz continuous in the first argument with constants r_1 and s_1, we have

$$\left\|H_1(g_1(x_n)) - H_1(g_1(x)) - \rho(F_1(x_n,y_n,z_n) - F_1(x,y_n,z_n))\right\|^2 \leqslant (\tau_1^2\theta_1^2 - 2\rho r_1 + \rho^2 s_1^2)\|x_n - x\|^2. \tag{5.7}$$

Since F_1 is Lipschitz continuous in the second and third arguments with constants $t_1 > 0$ and $a_1 > 0$, respectively, we have

$$\|F_1(x,y_n,z_n) - F_2(x,y,z_n)\| \leqslant t_1\|y_n - y\|, \tag{5.8}$$

and

$$\|F_1(x,y,z_n) - F_2(x,y,z)\| \leqslant a_1\|z_n - z\|. \tag{5.9}$$

It follows from (5.4)–(5.9) that

$$\begin{aligned}\|x_{n+1} - x\| &\leqslant \alpha_n\|x_n - x\| + (1-\alpha_n)\sqrt{1-2\beta_1+\theta_1^2}\|x_n - x\| \\ &\quad +(1-\alpha_n)\left(\frac{\sigma_1}{\gamma_1}\sqrt{\tau_1^2\theta^2 - 2\rho r_1 + \rho^2 s_1^2}\|x_n - x\| + \frac{t_1\sigma_1\rho}{\gamma_1}\|y_n - y\| \right.\\ &\quad \left. +\frac{a_1\sigma_1\rho}{\gamma_1}\|z_n - z\|\right) \\ &\leqslant \alpha_n\|x_n - x\| + (1-\alpha_n)\left(\sqrt{1-2\beta_1+\theta_1^2} + \frac{\sigma_1}{\gamma_1}\sqrt{\tau_1^2\theta_1^2 - 2\rho r_1 + \rho^2 s_1^2}\right)\|x_n - x\| \\ &\quad +(1-\alpha_n)\frac{t_1\sigma_1\rho}{\gamma_1}\|y_n - y\| + (1-\alpha_n)\frac{a_1\sigma_1\rho}{\gamma_1}\|z_n - z\|,\end{aligned} \tag{5.10}$$

Similarly, we have

$$\begin{aligned}\|y_{n+1} - y\| &= \left\|\alpha_n(y_n - y) + (1-\alpha_n)\Big[\big(y_n - g_2(y_n) - (y - g_2(y))\big) \right.\\ &\qquad +R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}(\lambda f_2 + H_2 g_2(y_n) - \lambda F_2(x_n,y_n,z_n)) \\ &\qquad \left. -R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}(\lambda f_2 + H_2 g_2(y) - \lambda F_2(x,y,z))\Big]\right\| \\ &\leqslant \alpha_n\|y_n - y\| + (1-\alpha_n)\left\|y_n - g_2(y_n) - (y - g_2(y))\right\| \\ &\qquad +(1-\alpha_n)\left\|R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}(\lambda f_2 + H_2 g_2(y_n) - \lambda F_2(x_n,y_n,z_n)) \right.\\ &\qquad \left. -R^{H_2,\eta_2}_{M_2,\lambda\zeta_2}(\lambda f_2 + H_2 g_2(y) - \lambda F_2(x,y,z))\right\| \\ &\leqslant \alpha_n\|y_n - y\| + (1-\alpha_n)\sqrt{1-2\beta_2+\theta_2^2}\|y_n - y\| \\ &\qquad +(1-\alpha_n)\left(\frac{\sigma_2}{\gamma_2}\sqrt{\tau_2^2\theta_2^2 - 2\lambda r_2 + \lambda^2 s_2^2}\|y_n - y\| + \frac{t_2\sigma_2\lambda}{\gamma_2}\|z_n - z\|\right.\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{a_2\sigma_2\lambda}{\gamma_2}\|x_n - x\|\Big) \\
&\leqslant \alpha_n\|y_n - y\| + (1-\alpha_n)\Big(\sqrt{1-2\beta_2+\theta_2^2} + \frac{\sigma_2}{\gamma_2}\sqrt{\tau_2^2\theta_2^2 - 2\lambda r_2 + \lambda^2 s_2^2}\Big)\|y_n - y\| \\
&\quad + (1-\alpha_n)\frac{t_2\sigma_2\lambda}{\gamma_2}\|z_n - z\| + (1-\alpha_n)\frac{a_2\sigma_2\lambda}{\gamma_2}\|x_n - x\|, \qquad (5.11)
\end{aligned}
$$

and

$$
\begin{aligned}
\|z_{n+1} - z\| &= \Big\|\alpha_n(z_n - z) + (1-\alpha_n)\Big[\Big(z_n - g_3(z_n) - \big(z - g_3(z)\big)\Big) \\
&\quad + R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3 + H_3 g_3(z_n) - \delta F_3(x_n, y_n, z_n)\big) \\
&\quad - R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3 + H_3 g_3(z) - \delta F_3(x, y, z)\big)\Big]\Big\| \\
&\leqslant \alpha_n\|z_n - z\| + (1-\alpha_n)\Big\|z_n - g_3(z_n) - \big(z - g_3(z)\big)\Big\| \\
&\quad + (1-\alpha_n)\Big\|R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3 + H_3 g_3(z_n) - \delta F_3(x_n, y_n, z_n)\big) \\
&\quad - R^{H_3,\eta_3}_{M_3,\delta\zeta_3}\big(\delta f_3 + H_3 g_3(z) - \delta F_3(x, y, z)\big)\Big\| \\
&\leqslant \alpha_n\|z_n - z\| + (1-\alpha_n)\sqrt{1-2\beta_3+\theta_3^2}\|z_n - z\| \\
&\quad + (1-\alpha_n)\Big(\frac{\sigma_3}{\gamma_3}\sqrt{\tau_3^2\theta_3^2 - 2\delta r_3 + \delta^2 s_3^2}\|z_n - z\| \\
&\quad + \frac{t_3\sigma_3\delta}{\gamma_3}\|x_n - x\| + \frac{a_3\sigma_3\delta}{\gamma_3}\|y_n - y\|\Big) \\
&\leqslant \alpha_n\|z_n - z\| + (1-\alpha_n)\Big(\sqrt{1-2\beta_3+\theta_3^2} + \frac{\sigma_3}{\gamma_3}\sqrt{\tau_3^2\theta_3^2 - 2\delta r_3 + \delta^2 s_3^2}\Big)\|z_n - z\| \\
&\quad + (1-\alpha_n)\frac{t_3\sigma_3\delta}{\gamma_3}\|x_n - x\| + (1-\alpha_n)\frac{a_3\sigma_3\delta}{\gamma_3}\|y_n - y\|, \qquad (5.12)
\end{aligned}
$$

It follows from (5.10)-(5.12) that

$$
\begin{aligned}
&\|x_{n+1} - x\| + \|y_{n+1} - y\| + \|z_{n+1} - z\| \\
&\quad \leqslant \alpha_n\big(\|x_n - x\| + \|y_n - y\| + \|z_n - z\|\big) \\
&\qquad + (1-\alpha_n)k\big(\|x_n - x\| + \|y_n - y\| + \|z_n - z\|\big) \\
&\quad = \big(k + (1-k)\alpha_n\big)\big(\|x_n - x\| + \|y_n - y\| + \|z_n - z\|\big). \qquad (5.13)
\end{aligned}
$$

Where k is defined by

$$
\begin{aligned}
k = \max\Big\{&\sqrt{1-2\beta_1+\theta_1^2} + \frac{\sigma_1}{\gamma_1}\sqrt{\tau_1^2\theta_1^2 - 2\rho r_1 + \rho^2 s_1^2} + \frac{t_3\sigma_3\delta}{\gamma_3} + \frac{a_2\sigma_2\lambda}{\gamma_2}, \\
&\sqrt{1-2\beta_2+\theta_2^2} + \frac{\sigma_2}{\gamma_2}\sqrt{\tau_2^2\theta_2^2 - 2\lambda r_2 + \lambda^2 s_2^2} + \frac{t_1\sigma_1\rho}{\gamma_1} + \frac{a_3\sigma_3\delta}{\gamma_3}, \\
&\sqrt{1-2\beta_3+\theta_3^2} + \frac{\sigma_3}{\gamma_3}\sqrt{\tau_3^2\theta_3^2 - 2\delta r_3 + \delta^2 s_3^2} + \frac{t_2\sigma_2\lambda}{\gamma_2} + \frac{a_1\sigma_1\rho}{\gamma_1}\Big\}.
\end{aligned}
$$

It follows from hypothesis (4.1) that $0 < k < 1$.

Let $a_n = \|x_n - x\| + \|y_n - y\| + \|z_n - z\|, k_n = k + (1-k)\alpha_n$. Then, (5.13) can be rewritten as $a_{n+1} \leqslant k_n a_n, n = 0, 1, 2 \ldots$. By (5.2), we know that $\limsup_n k_n < 1$, it follows from Lemma 5.1 that

$$a_n = \|x_n - x\| + \|y_n - y\| + \|z_n - z\| \text{ converges to } 0 \text{ as } n \longrightarrow \infty.$$

Therefore, (x_n, y_n, z_n) converges to the unique solution (x, y, z) of problem (3.1). This completes the proof. ∎

We can also construct some new three-step iterative algorithms for the special cases of problem (3.1). For examples, we give the following iterative algorithms for problem (3.2), (3.4) and (3.7), respectively.

ALGORITHM 5.2. For $i = 1, 2, 3$, let H_i, M_i, F_i, η_i be the same as in Corollary 4.2. For any given $(x_0, y_0, z_0) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$, define the three-step iterative sequence $\{(x_n, y_n, z_n)\}$ by

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)\Big[R^{H_1,\eta_1}_{M_1,\rho}\big(H_1(x_n) - \rho F_1(x_n, y_n, z_n)\big)\Big], \\ y_{n+1} &= \alpha_n y_n + (1-\alpha_n)\Big[R^{H_2,\eta_2}_{M_2,\lambda}\big(H_2(y_n) - \lambda F_2(x_n, y_n, z_n)\big)\Big], \\ z_{n+1} &= \alpha_n z_n + (1-\alpha_n)\Big[R^{H_3,\eta_3}_{M_3,\delta}\big(H_3(z_n) - \delta F_3(x_n, y_n, z_n)\big)\Big], \end{aligned} \tag{5.14}$$

where α_n satisfies the hypothesis (5.2).

ALGORITHM 5.3. For $i = 1, 2, 3$, let φ_i, F_i be the same as in Corollary 4.3. For any given $(x_0, y_0, z_0) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$, define the three-step iterative sequence $\{(x_n, y_n, z_n)\}$ by

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)\Big[J^{\rho}_{\varphi_1}\big(x_n - \rho F_1(x_n, y_n, z_n)\big)\Big], \\ y_{n+1} &= \alpha_n y_n + (1-\alpha_n)\Big[J^{\lambda}_{\varphi_2}\big(y_n - \lambda F_2(x_n, y_n, z_n)\big)\Big], \\ z_{n+1} &= \alpha_n z_n + (1-\alpha_n)\Big[J^{\delta}_{\varphi_3}\big(z_n - \delta F_3(x_n, y_n, z_n)\big)\Big], \end{aligned} \tag{5.15}$$

where $J^{\rho}_{\varphi_i} = (I + \partial\phi_i)^{-1}, i = 1, 2, 3$ and α_n satisfies the hypothesis (5.2).

ALGORITHM 5.4. For $i = 1, 2, 3$, let T_i be the same as in Corollary 4.4. For any given $(x_0, y_0, z_0) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$, define the three-step iterative sequence $\{(x_n, y_n, z_n)\}$ by

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)\big[P_K(x_n - \rho T_1(y_n)\big], \\ y_{n+1} &= \alpha_n y_n + (1-\alpha_n)\Big[P_K\big(y_n - \lambda T_2(z_n)\big)\Big], \\ z_{n+1} &= \alpha_n z_n + (1-\alpha_n)\big[P_K(z_n - \delta T_3(x_n)\big], \end{aligned} \tag{5.16}$$

where P_K is the projection of $\mathcal{H}$ onto K and α_n satisfies the hypothesis (5.2).

By using similar argument with the proof of Theorem 5.1, we have the following Corollaries.

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Corollary 5.2. *For $i = 1,2,3$, let H_i, M_i, F_i, η_i be the same as in Corollary 4.2. Assume that all the conditions of Corollary 4.2 hold. Then (x_n, y_n, z_n) generated by Algorithm 5.2 converges strongly to the unique solution (x,y,z) of problem* (3.2).

Corollary 5.3. *For $i = 1,2,3$, let φ_i, F_i be the same as in Corollary 4.3. Assume that all the conditions of Corollary 4.3 hold. Then (x_n, y_n, z_n) generated by Algorithm 5.3 converges strongly to the unique solution (x,y,z) of problem* (3.4).

Corollary 5.4. *For $i = 1,2,3$, let T_i be the same as in Corollary 4.4. Assume that all the conditions of Corollary 4.4 hold. Then (x_n, y_n, z_n) generated by Algorithm 5.4 converges strongly to the unique solution (x,y,z) of problem* (3.7).

Remark 5.1. It is easy to see that the results in this paper extend and generalise those results in [**21, 22, 24, 20, 23, 16, 7, 9, 11**] and the references therein.

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