

A New Table of Constant Weight Codes

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Abstract—A table of binary constant weight codes of length $n \leq 28$ is presented. Explicit constructions are given for most of the 600 codes in the table; the majority of these codes are new. The known techniques for constructing constant weight codes are surveyed, and also a table is given of (unrestricted) binary codes of length $n \leq 28$.

I. INTRODUCTION

THE MAIN GOAL of this paper is to give an extensive table of lower bounds on $A(n, d, w)$, the maximal possible number of binary vectors of length n , Hamming distance at least d apart, and constant weight w . We also give a table of lower bounds on $A(n, d)$, the maximal possible number of binary vectors of length n and Hamming distance at least d apart (with no restriction on weight).

These functions have been studied by many authors, and were tabulated for $n \leq 24$ in [13], [45], [72], [132]. In the present paper we extend the tables to length $n \leq 28$.

Our main concern is with Table I, the table of constant weight codes. The majority of the 600 codes in this table are new, either because we have discovered nicer versions of existing codes, or (more frequently) because we have found better codes than were known before.

Our goal has been to give either an explicit construction or a reference for every code in the table. With some exceptions a reader should be able to reconstruct any of these codes from the information given here. (This is in contrast to [13], where several codes are simply described as being found by an unstated “miscellaneous construction”.) However, because of space limitations, we have not included explicit listings for the codes constructed in Section XII (indicated by “y” in Table I) when they contain more than 1500 codewords.

Although [13] gives both upper and lower bounds on $A(n, d, w)$ and $A(n, d)$, in the present paper we give only lower bounds, i.e. tables of actual codes. We have not

given upper bounds for several reasons: 1) their calculation is a separate investigation, requiring analytic as opposed to combinatorial methods, 2) all the upper bounds in [13] for $d = 10$ should be rechecked (see the Errata section), 3) it is very difficult to check upper bounds found by others,¹ and 4) the paper is long enough already. However, we do mention the cases where we know that our lower bound is actually the exact value.

$A(n, d, w)$ and $A(n, d)$ are fundamental combinatorial quantities. They are also used in the construction of codes for asymmetric channels [16], [39], [49]–[51], [90], [180], DC-free codes [15], [64], [175], and spherical codes [167].

We would appreciate hearing of any improvements to the tables. Please send them to N. J. A. Sloane, Room 2C-376, AT&T Bell Labs, Murray Hill, NJ 07974, USA; electronic mail address user@mhuxo.att.com.

Notation: The following notation will be used throughout. \mathbb{F}_q denotes the Galois field of order q , while \mathbb{Z}_m denotes the integers modulo m . An $[n, k, d]$ code is a linear code with length n , dimension k and minimal distance d [132]. Bars indicate complements of sets or binary vectors.

To save space we have sometimes written vectors in *hexadecimal*, using $0 = 0000, \dots, 9 = 1001, A = 1010, \dots, F = 1111$, usually omitting leading zeros (so the vectors are right-justified). Superscripts (for example in Table XV) indicate the number of vectors in an orbit. Parentheses inside a vector (for example in Tables XII–XIV) indicate that all simultaneous cyclic shifts of the parenthesized sections are to be used. For example (110)(10) is an abbreviation for the six vectors 11010, 01101, 10110, 11001, 01110, 10101.

A *design* (X, \mathcal{B}) is a set X (of “points”) together with a collection \mathcal{B} of subsets of X (called “blocks”). A $t-(v, k, \lambda)$ design is a design in which $|X| = v$, all blocks contain exactly k points, and any t distinct points of X belong to exactly λ blocks ([14], [27], [31], [77], [96], [163], [176]). A *Steiner system* $S(t, k, v)$ is a $t-(v, k, 1)$ design. A *balanced incomplete block design* is a $2-(v, k, \lambda)$ design. More generally an (r, λ) -design is a design in which each point belongs to exactly r blocks and each pair of points belong to exactly λ blocks (but the blocks need not all contain the same number of points). A *symmetric* design

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¹An extreme case is the recent theorem of Lam *et al.* that there is no projective plane of order 10 [120]. This result, based on *thousands* of hours of computer time and as yet *unchecked* [141], implies $A(111, 20, 11) \leq 110$ (see [132, p. 528]).

TABLE I-A
LOWER BOUNDS ON $A(n, 4, w)$

n, w	3^{t3}	4^{t5}	5	6	7	8	9	10	11	12	13	14
6	4. ^a	3	1	1	0	0	0	0	0	0	0	0
7	7. ^{ts}	7	3	1	1	0	0	0	0	0	0	0
8	8. ^c	14. st	8	4	1	1	0	0	0	0	0	0
9	12. ^s	18. ^s	18	12	4	1	1	0	0	0	0	0
10	13. ^s	30. ^c	36. ^s	30	13	5	1	1	0	0	0	0
11	17. ^s	35. ^{pc}	66. ^s	66	35	17	5	1	1	0	0	0
12	20. ^s	51. ^{p0}	80 ^y	132. st	80	51	20	6	1	1	0	0
13	26. ^c	65. ^s	123 ^y	166 ^y	166	123	65	26	6	1	1	0
14	28. ^s	91. st	169 st	278 st	325 ^{sd}	278	169	91	28	7	1	1
15	35. ^s	105. ^s	237 ^{sd}	389 ^y	585 ^s	585	389	237	105	35	7	1
16	37. ^s	140. st	312 ^{sd}	615 ^y	836 ^{sd}	1170 ^m	836	615	312	140	37	8
17	44. ^s	156. ^{ec}	424 ^{p0}	854 ^{p1}	1416 ^t	1770 ^t	1770	1416	854	424	156	44
18	48. ^s	198. ^c	518 ^y	1260 ^{p1}	2041 ^{p3}	3186 ^t	3540 ^t	3186	2041	1260	518	198
19	57. ^c	228. ^s	684 ^s	1596 ^t	3172 ^{p1}	4667 ^{p0}	6726 ^t	6726	4667	3172	1596	684
20	60. ^s	285. ^{p0}	874 ^y	2280 ^t	4213 ^{p0}	7730 ^{p0}	10039 ^{p0}	13452 ^t	10039	7730	4213	2280
21	70. ^s	315. ^s	1071 ^t	2856 ^t	6120 ^{p0}	10726 ^{p1}	16856 ^{p0}	20188 ^{p2}	20188	16856	10726	6120
22	73. ^s	385. ^s	1386. ^t	3927 ^t	8211 ^{p0}	16354 ^{p1}	25570 ^{p3}	36381 ^{p2}	39688 ^{p3}	36381	25570	16354
23	83. ^s	416 ^{p0}	1771. ^t	5313. ^s	11638 ^t	23276 ^t	40786 ^{p1}	57436 ^{p0}	73794 ^{p1}	73794	57436	40786
24	88. ^s	498. ^{p0}	1895 ^{p0}	7084. st	15554 ^{p0}	34914 ^t	59262 ^{p0}	96496 ^{p0}	116914 ^{p0}	146552 ^{p0}	116914	96496
25	100. ^c	550. ^s	2334 ^{p0}	7772 ^{p1}	21094 ^{p0}	46390 ^{p1}	88411 ^{p0}	140320 ^{p0}	194756 ^{p0}	227168 ^{p0}	227168	194756
26	104. ^c	650. st	2670 ^t	10010 ^{p1}	26920 ^{p0}	65260 ^{p1}	128024 ^{p0}	218853 ^{p1}	315648 ^{p0}	394874 ^{p1}	424868 ^{p0}	394874
27	117. ^s	702. ^s	3276 ^t	12012 ^t	35510 ^{p1}	87709 ^{p0}	184420 ^{p1}	329076 ^{p0}	506444 ^{p1}	672148 ^{p0}	774565 ^{p1}	774565
28	121. ^{t3}	819. st	3718 ^{p0}	15288 ^t	44747 ^{p0}	121403 ^{p0}	259703 ^{p0}	502068 ^{p0}	806303 ^{p0}	1154541 ^{p0}	1399597 ^{p0}	1520224 ^{p0}

TABLE I-B
LOWER BOUNDS ON $A(n, 6, w)$

n, w	4^{t4}	5	6	7	8	9	10	11	12	13	14
8	2.	2	1	1	1	0	0	0	0	0	0
9	3. ^s	3	3	1	1	1	0	0	0	0	0
10	5. ^a	6. ^s	5	3	1	1	1	0	0	0	0
11	6. ^s	11. ^c	11	6	3	1	1	1	0	0	0
12	9. ^s	12. ^c	22. ^{hm}	12	9	4	1	1	1	0	0
13	13. ^c	18. ^s	26. ^c	26	18	13	4	1	1	1	0
14	14. ^c	28. ^s	42. ^c	42. ^{hm}	42	28	14	4	1	1	1
15	15. ^c	42. ^s	70. ^s	69. ^{h1}	69	70	42	15	5	1	1
16	20. ^s	48. ^s	112. ^{t1}	109. ^{h1}	120. ^{t1}	109	112	48	20	5	1
17	20.	68. st	112	166. ^{h1}	184 ^t	184	166	112	68	20	5
18	22. ^{zh}	68	132. ^{sb}	243. ^{h1}	260 ^t	304 ^t	260	243	132	68	22
19	25. ^{t2}	76 ^c	172. ^{sb}	338. ^{sb}	408. ^{sb}	504 ^t	504	408	338	172	76
20	30. ^{z8}	84 ^c	232. ^{t2}	462. ^{t2}	588. ^{t2}	832. ^{t2}	944. ^{t2}	832	588	462	232
21	31. ^s	105 ^c	269. ^{h3}	570. ^{sb}	774 ^y	1184 ^y	1454 ^y	1454	1184	774	570
22	37. ^m	132 ^m	319 ^t	759. ^{h2}	1139 ^y	1792 ^y	2182 ^y	2636. ^{t2}	2182	1792	1139
23	40. ^{t4}	147. ^{t2}	399 ^t	969 ^t	1436 ^t	2271 ^y	2970 ^y	3585 ^y	3585	2970	2271
24	42. ^s	168 ^s	532 ^t	1368 ^t	1882. ^{sd}	3041. ^{ya}	4200 ^y	5267 ^y	5616 ^y	5267	4200
25	50. ^s	210. ^s	700 ^t	1900 ^t	2590. ^{sd}	4127. ^{ya}	6036. ^{ya}	7960 ^y	9031. ^{ya}	9031	7960
26	52. ^c	260. ^s	910. ^s	2600 ^t	3532. ^{sd}	5703 ^y	8695. ^{ya}	12037. ^{ya}	14836. st	15977. st	14836
27	54. ^c	260	1170. ^s	3510. ^s	4786. ^{sd}	7727. ^{ya}	12368. ^{ya}	18096. ^{ya}	23879. st	27553. st	27553
28	63. st	272 ^y	1170	4680. st	6315. ^{sd}	10313. ^{ya}	17447. ^{ya}	28368. st	40188. st	49462. st	52995. st

(or square 2-design) is a 2-design with as many blocks as points.

We have included some codes that are close to the best presently known when they are easy to construct and the best code is not. These nonrecord codes are indicated by ‡.

II. THE TABLES OF $A(n, d, w)$ AND $A(n, d)$

We begin with the main tables, Tables I and II, which give lower bounds on $A(n, d, w)$ and $A(n, d)$. The rest of the paper is devoted to describing the codes in Table I (Table II being largely self-explanatory).

Most of the entries in Table I are new, either because we have improved the lower bound, or because we have found a more symmetric code or more compact definition than was known before. The notes to Table I describe the simplest construction we know for a code with the given parameters. We have usually not attempted to indicate the original discoverer of a code. For as mentioned in Section I our chief concern is to describe these codes explicitly. Those interested in the history of these codes may consult the extensive bibliography (see for example [11], [13], [23], [69], [103], [132], [177], [178]) and the Acknowledgment at the end of the paper. Table II contains one new entry, $A(25, 10) \geq 151$.

TABLE I-C
LOWER BOUNDS ON $A(n, 8, w)$

n, w	5	6	7	8	9	10	11	12	13	14
10	2.	2	1	1	1	1	0	0	0	0
11	2.	2	2	1	1	1	1	0	0	0
12	3. ^j	4. ^d	3	3	1	1	1	1	0	0
13	3.	4.	4	3	3	1	1	1	1	0
14	4. ^s	7. ^d	8. ^s	7	4	3	1	1	1	1
15	6. ^a	10. ^s	15. ^c	15	10	6	3	1	1	1
16	6.	16. ^{q2}	16. ^c	30. ^{hm}	16	16	6	4	1	1
17	7. ^s	17. ^{pc}	24. ^{ec}	34. ^c	34	24	17	7	4	1
18	9. ^s	21. ^{q3}	33. ^s	46. ^f	48. ^s	46	33	21	9	4
19	12. ^s	28. ^s	52. ^f	78. ^s	88. ^s	88	78	52	28	12
20	16. ^s	40. ^s	80. ^s	130. ^f	160. ^f	176. ^f	160	130	80	40
21	21. ^c	56. ^s	120. ^s	210. ^s	280. ^f	336. ^f	336	280	210	120
22	21.	77. ^s	176. ^s	330. ^s	280	616. ^f	672. ^s	616	280	330
23	23. ^c	77	253. ^c	506. ^s	400. ^f	616	1288. ^f	1288	616	400
24	24. ^c	78. ^{x2}	253	759. ^{ss}	640. ⁴	960. ^{4a}	1288	2576. ¹⁴	1288	960
25	30. ^s	100. ^s	254. ^{x2}	759	829. ⁵	1248. ^{qa}	1662. ⁵	2576	2576	1662
26	30.	130. ^{ss}	257. ^f	760. ^{x2}	883. ^f	1519. ⁵	1988. ^{qa}	3070. ^f	3328. ^{qa}	3070
27	30	130	278. ^f	766. ^{3f}	970. ^f	1597. ^f	2295. ^f	3335. ^{qa}	3923. ^f	3923
28	33. ^m	130	296. ^f	833. ^f	1107. ^f	1806. ^m	2756. ^{qa}	4114. ^{qa}	4805. ^{qa}	5280. ⁶

TABLE I-D
LOWER BOUNDS ON $A(n, 10, w)$

n, w	6	7	8	9	10	11	12	13	14
12	2.	2	1	1	1	1	1	0	0
13	2.	2	2	1	1	1	1	1	0
14	2.	2.	2	2	1	1	1	1	1
15	3. ^j	3. ^j	3	3	3	1	1	1	1
16	3.	4. ^j	4. ^j	4	3	3	1	1	1
17	3.	5. ^j	6. ^j	6	5	3	3	1	1
18	4. ^j	6. ^j	9. ^{q2}	10. ^s	9	6	4	3	1
19	4.	8. ^x	12. ^{sb}	19. ^c	19	12	8	4	3
20	5. ^s	10. ^{q2}	17. ^m	20. ^f	38. ^{hm}	20	17	10	5
21	7. ^a	13. ^{sh}	21. ^c	27. ^{pc}	38	38	27	21	13
22	7.	16. ^{pc}	24. ^{sa}	35. ^{pc}	42. ^{ec}	46. ^c	42	35	24
23	8. ^{x2}	20. ^f	33. ^{pc}	45. ^{pc}	54. ^{pc}	63. ^{pc}	63	54	45
24	9. ^{x2}	24. ^c	38. ^{pc}	56. ^c	72. ^c	90. ^{pc}	96. ^c	90	72
25	10. ^s	28. ^{ec}	48. ^{ec}	72. ^{ec}	100. ^c	125. ^c	130. ^{ec}	130	125
26	13. ^{x2}	28	54. ^{pc}	84. ^{pc}	130. ^f	168. ^{pc}	185. ^f	191. ^f	185
27	14. ^{q9}	36. ³	66. ^{pc}	111. ^c	159. ^{pc}	213. ^{sa}	257. ^f	283. ^{qa}	283
28	16. ^m	37. ⁴	78. ^{pc}	132. ^{pc}	195. ^{sd}	280. ^{qa}	356. ^{qa}	414. ^{qa}	435. ^{sd}

Key to Table I

An entry followed by a period is known to be exact.

- a = From a trivial design or its dual (Section III).
- c = Cyclic code (Table XI).
- cm = Conference matrix code ((19) of Section III).
- d = Doubling ((2) of Section III).
- d1 = 2-(25, 9, 3) design (Section III).
- d2 = From (r, λ) -design (Section III).
- ec = Extended cyclic (or "cyclic with fixed point") code (Table XII).
- g = Group code—orbits under a group with more than one generator (Table XV).
- gf = Group code plus extra vectors (Table XV).
- gp = Group code followed by polishing—group code need not be subcode (Table XV).
- gs = From S_2 -sets (Theorem 16 and Table V).
- hm = Hadamard matrix code (Theorem 10).
- hn = From Hadamard matrix of order 12 (Section III).
- h1 = From Theorem 20.
- h2 = From Theorem 21.

- h3 = Hämäläinen (see remark following Theorem 21).
- j = Juxtaposing ((1) of Section III). Details are left to the reader.
- m = Miscellaneous construction (Section XI).
- p0 = Partitioning construction with $n_1 = \lfloor n/2 \rfloor$, $\epsilon = 0$ (Section VI).
- p1 = Partitioning construction with $n_1 = \lfloor n/2 \rfloor$, $\epsilon = 1$ (Section VI).
- p2 = Partitioning construction with $n_1 = \lfloor n/2 \rfloor - 1$, $\epsilon = 0$ (Section VI).
- p3 = Partitioning construction with $n_1 = \lfloor n/2 \rfloor - 1$, $\epsilon = 1$ (Section VI).
- pc = Orbits under a single permutation (Table XIV).
- qi = Quasi-cyclic code, for $2 \leq i \leq 9$ —fixed by a permutation containing i cycles of length n/i (Table XIII).
- s = Section of code below or diagonally down to right, obtained from (5) of Section III.
- sb = Section of code below, obtained by direct examination of the code (Section III).

TABLE I-E
LOWER BOUNDS ON $A(n, 12, w)$

n, w	7	8	9	10	11	12	13	14
14	2.	2	1	1	1	1	1	1
15	2.	2	2	1	1	1	1	1
16	2.	2.	2	2	1	1	1	1
17	2.	2.	2	2	2	1	1	1
18	3. ^{<i>f</i>}	3. ^{<i>d</i>}	4. ^{<i>f</i>}	3	3	3	1	1
19	3.	3.	4.	4	3	3	3	1
20	3.	5. ^{<i>d</i>}	5. ^{<i>f</i>}	6. ^{<i>a</i>}	5	5	3	3
21	3.	5.	7. ^{<i>f</i>}	7. ^{<i>f</i>}	7	7	5	3
22	4. ^{<i>f</i>}	6. ^{<i>d</i>}	8. ^{<i>pc</i>}	11. ^{<i>d</i>}	12. ^{<i>s</i>}	11	8	6
23	4.	6.	10. ^{<i>s</i>}	16. ^{<i>sh</i>}	23. ^{<i>c</i>}	23	16	10
24	4.	9. ^{<i>d</i>}	16. ^{<i>s</i>}	24. ^{<i>c</i>}	24. ^{<i>c</i>}	46. ^{<i>hm</i>}	24	24
25	5. ^{<i>f</i>}	10. ^{<i>q5</i>}	25. ^{<i>d1</i>}	28. ^{<i>pc</i>}	36. ^{<i>cc</i>}	50. ^{<i>cm</i>}	50	36
26	5.	13. ^{<i>d</i>}	26. ^{<i>c</i>}	30. ^{<i>pc</i>}	39. ^{<i>q2</i>}	54. ^{<i>s9</i>}	58. ^{<i>s9</i>}	54
27	6. ^{<i>s</i>}	15. ^{<i>q9</i>}	39. ^{<i>s9</i>}	39. ^{<i>cc</i>}	54. ^{<i>c</i>}	82. ^{<i>t7</i>}	81. ^{<i>t7</i>}	81
28	8. ^{<i>a</i>}	19. ^{<i>pc</i>}	39	48. ^{<i>h</i>}	63. ^{<i>pc</i>}	84. ^{<i>c</i>}	96. ^{<i>pc</i>}	106. ^{<i>m</i>}

TABLE I-F
LOWER BOUNDS ON $A(n, 14, w)$

n, w	8	9	10	11	12	13	14
16	2.	2	1	1	1	1	1
17	2.	2	2	1	1	1	1
18	2.	2.	2	2	1	1	1
19	2.	2.	2	2	2	1	1
20	2.	2.	2.	2	2	2	1
21	3. ^{<i>f</i>}	3. ^{<i>f</i>}	3. ^{<i>f</i>}	3	3	3	3
22	3.	3.	4. ^{<i>f</i>}	4. ^{<i>f</i>}	4	3	3
23	3.	3.	4.	4.	4	4	3
24	3.	4. ^{<i>f</i>}	5. ^{<i>f</i>}	6. ^{<i>f</i>}	6. ^{<i>f</i>}	6	5
25	3.	5. ^{<i>f</i>}	6. ^{<i>f</i>}	7. ^{<i>f</i>}	8. ^{<i>f</i>}	8	7
26	4. ^{<i>f</i>}	6. ^{<i>f</i>}	8. ^{<i>d2</i>}	10. ^{<i>f</i>}	13. ^{<i>q2</i>}	14. ^{<i>s</i>}	13
27	4.	6.	9. ^{<i>f</i>}	13. ^{<i>s</i>}	19. ^{<i>q9</i>}	27. ^{<i>s</i>}	27
28	4.	7. ^{<i>f</i>}	11. ^{<i>s</i>}	21. ^{<i>q4</i>}	28. ^{<i>c</i>}	28. ^{<i>c</i>}	54. ^{<i>hm</i>}

- sd* = Section of code diagonally down to right, obtained by direct examination of the code (Section III).
- sf* = Section of code below or diagonally down to right, followed by addition of extra vectors (Section XI).
- sp* = Section of code below or diagonally down to right, followed by polishing (Section XI).
- ss* = From Steiner systems $S(2, 3, 7)$, $S(3, 4, 8)$, $S(5, 6, 12)$, $S(3, 4, 14)$, $S(3, 4, 16)$, $S(5, 6, 24)$, $S(3, 4, 26)$, $S(3, 4, 28)$ for $d = 4$; $S(3, 5, 17)$, $S(2, 4, 28)$, $S(5, 7, 28)$ for $d = 6$; $S(5, 8, 24)$, $S(3, 6, 26)$ for $d = 8$ (Table IV).
- t1* = From translate of Nordstrom–Robinson code (Section IX).
- t2* = Adding tails to translates of Nordstrom–Robinson code (Section IX).
- t4* = From translate of Golay code (Section IX).
- t5* = Adding tails to translates of Golay code (Section IX).
- t6* = From translate of Karlin’s [27, 14, 7] code (Section IX).
- t7* = From translate of [26, 7, 11] code (Section IX).
- x* = Lexicographic code (Section VIII).
- xh* = Lexicode with seed (Table VIII).
- xy* = Lexicode with seed (Table XVI).
- x2* = Complement of lexicode with sum constraint (Table VII).
- y* = No known structure (Table XVI).

TABLE I-G
VALUES OF $A(n, 16, w)$

n, w	9	10	11	12	13	14
18	2.	2	1	1	1	1
19	2.	2	2	1	1	1
20	2.	2.	2	2	1	1
21	2.	2.	2	2	2	1
22	2.	2.	2.	2	2	2
23	2.	2.	2.	2	2	2
24	3. ^{<i>f</i>}	3. ^{<i>d</i>}	3. ^{<i>f</i>}	4. ^{<i>d</i>}	3	3
25	3.	3.	3.	4.	4	3
26	3.	3.	4. ^{<i>f</i>}	4.	4.	4
27	3.	3.	5. ^{<i>f</i>}	5. ^{<i>f</i>}	6. ^{<i>f</i>}	6
28	3.	4. ^{<i>d</i>}	5.	7. ^{<i>d</i>}	7. ^{<i>f</i>}	8. ^{<i>d</i>}

TABLE I-H
VALUES OF $A(n, 18, w)$

n, w	10	11	12	13	14
20	2.	2	1	1	1
21	2.	2	2	1	1
22	2.	2.	2	2	1
23	2.	2.	2	2	2
24	2.	2.	2.	2	2
25	2.	2.	2.	2	2
26	2.	2.	2.	2.	2
27	3. ^{<i>f</i>}	3. ^{<i>f</i>}	3. ^{<i>f</i>}	3. ^{<i>f</i>}	3
28	3.	3.	3.	4. ^{<i>f</i>}	4. ^{<i>f</i>}

- ya* = Obtained by extending the code above it in the table; no other known structure.
- yd* = Obtained by extending the code diagonally above it to left; no other known structure.
- z2* = [170].
- z3* = Theorem 4.
- z4* = Theorem 6.
- z5* = Theorem 5.
- z8* = [22].
- z9* = [95] (see Table XVI).

Key to Table II

Unmarked entries are either trivial or are obtained by shortening the code below.

An entry followed by a period is known to be exact.

- 1 = Extended Hamming code ([132], p. 23).
- 2 = Conference matrix code ([132], p. 585, [165]).
- 3 = Best ([12], [45], p. 140).
- 4 = From $S(5, 6, 12)$, 6 disjoint words of weight 2 and complements ([45], p. 139, [132], p. 585).
- 5 = Romanov [155]—see Section VI.
- 6 = From Hamming code over GF(5) [79].
- 7 = From the $u|u + v$ construction ([132], p. 76, [166]).
- 8 = Hadamard matrix code ([132], p. 49).
- 8a = “Nadler” code ([13], [129], [132], pp. 75, 79).
- 9 = Nordstrom–Robinson code (Section IX, [5], [132], p. 73).
- 10 = Nonlinear code from Construction X ([132], p. 583, [164], p. 505).
- 11 = From Construction X4 ([132], p. 585, Example 7, [164], p. 507).
- 12 = Wagner [179].

TABLE II
LOWER BOUNDS ON $A(n, d)$

n, d	4	6	8	10	12	14	16	18	20
5	2.	1.	1.	1.	1.	1.	1.	1.	1.
6	4.	2.	1.	1.	1.	1.	1.	1.	1.
7	8.	2.	1.	1.	1.	1.	1.	1.	1.
8	16. ¹	2.	2.	1.	1.	1.	1.	1.	1.
9	20. ²	4.	2.	1.	1.	1.	1.	1.	1.
10	40. ³	6.	2.	2.	1.	1.	1.	1.	1.
11	72	12.	2.	2.	1.	1.	1.	1.	1.
12	144 ⁴	24. ⁸	4.	2.	2.	1.	1.	1.	1.
13	256.	32. ^{8a}	4.	2.	2.	1.	1.	1.	1.
14	512.	64.	8.	2.	2.	2.	1.	1.	1.
15	1024.	128.	16.	4.	2.	2.	1.	1.	1.
16	2048. ¹	256. ⁹	32. ¹⁴	4.	2.	2.	2.	1.	1.
17	2720 ⁵	256	36 ²	6. ¹⁷	2.	2.	2.	1.	1.
18	5248	512	64	10.	4.	2.	2.	2.	1.
19	10496 ⁶	1024	128	20.	4.	2.	2.	2.	1.
20	20480 ⁷	2048 ¹⁰	256	40. ⁸	6. ¹⁷	2.	2.	2.	2.
21	36864	2560 ¹¹	512.	42 ¹⁸	8. ¹⁷	4.	2.	2.	2.
22	73728	4096	1024.	48 ¹⁷	12.	4.	2.	2.	2.
23	147456	8192	2048.	68 ¹⁹	24.	4.	2.	2.	2.
24	294912 ⁷	16384 ¹²	4096. ¹⁵	128 ²⁰	48. ⁸	6. ¹⁷	4.	2.	2.
25	524288	16384	4096	151 ²¹	52 ²	8. ¹⁷	4.	2.	2.
26	1048576	32768	4096	256	64	14.	4.	2.	2.
27	2097152	65536	8192	512	128 ²³	28.	6. ¹⁷	4.	2.
28	4194304 ¹	131072 ¹³	16384 ¹⁶	1024 ²²	128	56. ⁸	8. ¹⁷	4.	2.

- 13 = Shortened nonprimitive BCH code of length 32 ([132], p. 586).
 14 = Reed-Muller code ([132], Chap. 13).
 15 = Golay code ([45], Chaps. 3, 11, [132], Chap. 20).
 16 = Self-dual double circulant code ([45], p. 189, [132], p. 509, [148])—see (53).
 17 = From Hadamard matrices using Levenshtein's construction ([122], [132], p. 50).
 18 = Extended quasi-cyclic code [104].
 19 = Extended cyclic code [105] (see Table XI).
 20 = Hashim-Pozdniakov linear code [88].
 21 = Cyclic code (see Table XI).
 22 = Piret [147].
 23 = Linear code ((51) of Section IX, [87], [89], [132], p. 593).

We see from Table I that the exact value of $A(n, d, w)$ is now known for all lengths $n \leq 11$ (the first undetermined value being $80 \leq A(12, 4, 5) \leq 84$). Similarly, from Table II, $A(n, d)$ (for d even) is known exactly for $n \leq 10$, the first undetermined value being $72 \leq A(11, 4) \leq 79$ (the upper bound is from [12]).

Of course there is no theoretical difficulty in computing $A(n, d)$ or $A(n, d, w)$. One "simply" forms the graph with 2^n or $\binom{n}{w}$ vertices, corresponding to all possible code-words, joins two vertices by an edge if their Hamming distance is at least d , and finds the largest clique. Although several new algorithms have recently been proposed for clique-finding (see [6], [7], [70], [71], [140]), the unsolved problems in Tables I and II appear to be beyond their range. Similarly, finding the largest code invariant under a given permutation group (see Section X) requires finding the largest clique in a graph with weights attached to the vertices.

III. JOHNSON AND RELATED BOUNDS

S. M. Johnson [99]–[102] obtained a number of upper bounds on $A(n, d, w)$, given in Theorems 2, 8 and (7), (16), (18). The cases when equality holds in these bounds (see Conjecture 3 and Theorems 4–7, 9–13) are of particular interest, because of their connection with Steiner systems, block designs and Hadamard matrices. We also mention some related lower bounds obtained from conference matrices.

Theorem 1 (Trivial values):

- a) If d is odd, $A(n, d, w) = A(n, d + 1, w)$.
 b) $A(n, d, w) = A(n, d, n - w)$.
 c) $A(n, d, w) = 1$ if $2w < d$.
 d) If $d = 2w$ then $A(n, d, w) = \lfloor n/w \rfloor$.
 e) $A(n, 2, w) = \binom{n}{w}$.

Remark: Let $N(a, d, w)$ denote the inverse of $A(n, d, w)$, that is, the smallest n for which one can find a words of weight w at mutual distance d , where d is even. Then one can show that

$$N(0, d, w) = 0,$$

$$N(1, d, w) = w,$$

and if $w \geq d/2$ then

$$N(2, d, w) = w + d/2,$$

$$N(3, d, w) = \max\{3d/2, w + d/2\},$$

$$N(4, d, w) = \max\{3d - 2w, \lfloor d + 2w/3 \rfloor, w + d/2\}$$

(if $w < d/2$ these quantities do not exist).

By *juxtaposing* two codes—placing them side by side—we obtain

$$A(n_1 + n_2, d_1 + d_2, w_1 + w_2) \geq \min\{A(n_1, d_1, w_1), A(n_2, d_2, w_2)\}. \quad (1)$$

In particular we have the *doubling construction*:

$$A(2n, 2d, 2w) \geq A(n, d, w). \tag{2}$$

Lower bounds in Table I obtained from (1), (2) are indicated by “j” and “d” respectively.

Theorem 2 (Johnson; [132], p. 527, [176], p. 98):

$$A(n, d, w) \leq \left\lfloor \frac{n}{w} A(n-1, d, w-1) \right\rfloor, \tag{3a}$$

$$A(n, d, w) \leq \left\lfloor \frac{n}{n-w} A(n-1, d, w) \right\rfloor. \tag{3b}$$

In particular

$$A(n, d, w) \leq A(n-1, d, w-1) + A(n-1, d, w). \tag{4}$$

We also use the contrapositives to (3a), (3b): if there exists a code showing that $A(n, d, w) \geq M$, then

$$A(n-1, d, w-1) \geq \left\lceil \frac{wM}{n} \right\rceil, \tag{5}$$

$$A(n-1, d, w) \geq \left\lceil \frac{n-w}{n} M \right\rceil.$$

Lower bounds obtained from (5) are indicated by “s” (for “Section”) in Table I. In some cases these inequalities can be improved by examining the code and finding a column with an exceptionally large number of 0’s or 1’s (and deleting that column). Bounds obtained in this way are indicated by “sb” or “sd” in Table I.

The *first Johnson bound* $J_1(n, d, w)$ is defined to be the smallest upper bound on $A(n, d, w)$ that is obtained by repeatedly applying (3a) and (3b) until Theorem 1 can be used. For example

$$J_1(n, 4, 3) = \begin{cases} \left\lfloor \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor \right\rfloor, & \text{if } n \not\equiv 5 \pmod{6} \\ \left\lfloor \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor \right\rfloor - 1, & \text{if } n \equiv 5 \pmod{6}. \end{cases} \tag{6}$$

Clearly

$$A(n, d, w) \leq J_1(n, d, w), \tag{7}$$

and there are reasons for believing that this bound may be tight when n is sufficiently large.

Conjecture 3: For d, w fixed,

$$A(n, d, w) = J_1(n, d, w) \tag{8}$$

for all sufficiently large n . If true this would be a remarkably strong result (it would imply for example by Theorem 7 that Steiner systems $S(t, k, v)$ exist for all t and k provided v is sufficiently large and satisfies the obvious congruences). Rödl [154] has shown that $A(n, d, w)/J(n, d, w)$ approaches 1 as $n \rightarrow \infty$ for any fixed d and w . An asymptotic result for certain values of w and d growing with n was given in [186]. The conjecture is known to be true in a few cases, as we shall now see.

Theorem 4 (Kirkman [109], Schönheim [157]; see also [77], p. 237, [106], [153], [169]):

$$A(n, 4, 3) = J_1(n, 4, 3) \text{ (see (6)) holds for all } n.$$

Theorem 5 (Brouwer [21], Hanani [81], Kalbfleisch and Stanton [106], Schönheim [157]): For $n \not\equiv 5 \pmod{6}$,

$$A(n, 4, 4) = J_1(n, 4, 4),$$

the values of this function being

$$\frac{n(n-1)(n-2)}{24}, \quad \text{if } n \equiv 2 \text{ or } 4 \pmod{6}, \tag{9}$$

$$\frac{n(n-1)(n-3)}{24}, \quad \text{if } n \equiv 1 \text{ or } 3 \pmod{6}, \tag{10}$$

$$\frac{n(n^2-3n-6)}{24}, \quad \text{if } n \equiv 0 \pmod{6}. \tag{11}$$

For $n \equiv 5 \pmod{6}$ we have $A(5, 4, 4) = 1$, $A(11, 4, 4) = 35$ ([12], [13], [45, p. 141]), but for larger n the values are unknown. See also [86].

Theorem 6 (Brouwer [22], [29]):

$$A(n, 6, 4) = J_1(n, 6, 4) \tag{12}$$

holds for all n except 8, 9, 10, 11, 17, 19 (for these values see Table I-B).

One special case of equality in the first Johnson bound is particularly important.

Theorem 7 (Schönheim [157]; see also [132], p. 528, [160], [176], p. 100):

$$A(n, 2\delta, w) = \frac{n(n-1) \cdots (n-w+\delta)}{w(w-1) \cdots \delta} \tag{13}$$

if and only if a Steiner system $S(w-\delta+1, w, n)$ exists.

In this case equality also holds in (7); the codewords are the blocks of the corresponding Steiner system. The codes obtained in this way are discussed in Section IV.

Theorem 7 enables us to write down immediately the parameters of the codes corresponding to the blocks of a Steiner system. The blocks of an arbitrary $2-(v, k, \lambda)$ design form a code showing that $A(v, d, k) \geq b$, but in general d is not determined by the other parameters. However, the blocks of the *dual* or *transpose* design (obtained by interchanging points and blocks) form a code showing that

$$A(b, 2(r-\lambda), r) \geq v. \tag{14}$$

If the design is symmetric these two codes have the same parameters. Theorem 9 describes a case where equality holds both in (14) and in another of Johnson’s bounds.

Theorem 8 (Johnson [99]; [132, p. 526]): Let $A(n, 2\delta, w) = M$, and write $wM = an + b$, $0 \leq b < n$. Then (by considering the sum of the inner products of all pairs of codewords)

$$(n-b)a(a-1) + ba(a+1) \leq (w-\delta)M(M-1). \tag{15}$$

Theorem 8 rules out certain combinations of n, d, w, M (see [132, p. 526] for an example). The *second Johnson bound* $J_2(n, d, w)$ is defined to be the largest value of M permitted by Theorem 8 (possibly infinity). Clearly

$$A(n, d, w) \leq J_2(n, d, w). \tag{16}$$

The proof of Theorem 8 shows that if equality holds in (15) then every pair of distinct codewords has inner prod-

uct $w - \delta$, and also that the $M \times n$ array formed by the codewords contains $n - b$ columns of weight a and b columns of weight $a + 1$. This is the dual of an (r, λ) -design with $r = w$, $\lambda = w - \delta$ in the notation of Section I. We mention three examples of such designs.

1) A trivial example occurs when

$$n = \binom{k}{a}, \quad M = k, \quad b = 0, \quad w = \binom{k-1}{a-1}, \quad (17)$$

and the columns of the array consist of all possible binary vectors of weight a . Such codes are denoted by "a" in Table I.

2) In the special case $a = 2$ (when $wM < 3n$), equality holds in (15) if and only if there is a 1-design consisting of b triples from M points, such that no pair is covered more than $w - \delta$ times. For example $A(26, 14, 9) = 6$ can be obtained in this way; we omit the details.

3) To obtain $A(26, 14, 10) = 8$, by the previous remarks we must find 24 triples and 2 quadruples from an 8-set. If the 8-set is $\{a, b, c, d, A, B, C, D\}$ we may use the triples $\{i, j, K\}$ and $\{i, J, K\}$ and the quadruples $\{a, b, c, d\}$, $\{A, B, C, D\}$.

We shall discuss equality in (16) at the end of this section. We next discuss a weaker version of Theorem 8, also due to Johnson (obtained by ignoring the condition that certain variables must be integers), which states that

$$A(n, 2\delta, w) \leq \frac{\delta n}{w^2 - wn + \delta n}, \quad (18)$$

provided the denominator is positive ([99]; [132], p. 525, [176], p. 97).

Theorem 9 (Semakov and Zinoviev [160]; [176], p. 99): Equality holds in (18) if and only if there exists a 2-design with parameters $b = n$, $r = w$, $\lambda = w - \delta$, $v = \delta n / (w^2 - wn + \delta n)$, $k = \delta w / (w^2 - wn + \delta n)$.

In Theorem 9 the codewords are the blocks of the dual design; equality then holds in (14). There are lists of small block designs in [77], [133]. The entry $A(25, 12, 9) = 25$ in Table I for example is obtained from Theorem 9 using a symmetric $2-(25, 9, 3)$ design. There are in fact 78 inequivalent designs with these parameters [58], [133], and so exactly 78 inequivalent codes showing $A(25, 12, 9) = 25$. One example is given in Table XIV.

Hadamard matrices also determine some values of $A(n, d, w)$.

Theorem 10 ([160]; see also [13], Theorem 15, [132], p. 528): A Hadamard matrix H_n of order $n \geq 1$ exists if and only if

$$A\left(n, \frac{n}{2}, \frac{n}{2}\right) = 2n - 2.$$

The code is constructed from the rows of H_n and $-H_n$ by making the entries of the first row of H_n equal to $+1$, then changing $+1$ to 0 and -1 to 1 in every row. The optimality of these codes follows by applying (3) once and then (18).

A modification of this construction shows that $A(14, 6, 7) \geq 42$. Take a Hadamard matrix of order 12, with first row and column containing $+1$'s, replace $+1$'s by 0 's and -1 's by 1 's, omit the first (zero) row, and label the remaining rows ρ_0, \dots, ρ_{10} . Then the code

$$\begin{array}{lll} \rho_0 & 0 & 1 \\ \overline{\rho_0} & 0 & 1 \\ \rho_i & 1 & 0 \quad (1 \leq i \leq 10) \\ \overline{\rho_i} & 1 & 0 \quad (1 \leq i \leq 10) \\ \rho_0 + \rho_i & 0 & 1 \quad (1 \leq i \leq 10) \\ \overline{\rho_0 + \rho_i} & 0 & 1 \quad (1 \leq i \leq 10) \end{array}$$

shows that $A(14, 6, 7) \geq 42$. This idea appears to succeed only for a Hadamard matrix of order 12 .²

Similarly if a conference matrix ([132], p. 55, [165]) of order $n \equiv 2 \pmod{4}$ exists then

$$A\left(n-1, \frac{n-2}{2}, \frac{n-2}{2}\right) \geq 2n-2. \quad (19)$$

Equality holds in (19) for $n = 6, 10, 14, 18$; we would like to know if it holds in general.

Honkala *et al.* [95] show that if a conference matrix of order n exists then

$$A(2n, n, n-1) \geq 2n, \quad (20)$$

if n is a multiple of 4 then

$$A(2n+1, n, n-1) \geq 2n + A\left(n, \frac{n}{2}, \frac{n-2}{2}\right), \quad (21)$$

and if n is a multiple of 4 and a conference matrix of order either $n/2$ or $n+2$ exists then

$$A(2n+1, n, n-1) \geq 3n. \quad (22)$$

For example (22) gives an alternative proof that $A(25, 12, 11) \geq 36$ (cf. Table IV).

Finally we return to (16). In 1985 Honkala [92] obtained a considerable generalization of Theorem 10 by showing that equality holds in (16) over a wide range of values. The proof is constructive, by juxtaposing (see (1)) appropriate combinations of Hadamard matrices of orders up to n .

Theorem 11 (Honkala [92]): Provided Hadamard matrices of all orders $4k \leq n$ exist,

$$A(n, d, w) = J_2(n, d, w)$$

holds whenever

$$n - d \leq w \leq d. \quad (23)$$

In particular, if (23) holds and $n < 2d$, then

$$A(n, d, w) = \begin{cases} 2u, & \text{if } i - v \leq 2j \leq i + v, \\ 2u - 1, & \text{otherwise,} \end{cases}$$

where $i = 2d - n$, $j = d - w$ and $d = ui + v$ with $0 \leq v < i$.

²A fact which ultimately depends on the existence of the outer automorphism of the symmetric group S_6 .

TABLE III
SOURCES FOR PROOF THAT CERTAIN ENTRIES IN TABLE I MARKED
WITH PERIOD ARE EXACT*

Value	Reference
$A(12, 6, 5) = 12$	[132], p. 530.
$A(13, 6, 5) = 18$	[132], p. 531.
$A(14, 6, 7) = 42$	Exhaustive search (see Section XII).
$A(17, 6, 4) = 20$	[20].
$A(19, 6, 4) = 25$	[170].
$A(16, 8, 7) = 16$	Exhaustive search (see Section XII).
$A(17, 8, 7) = 24$	Exhaustive search (see Section XII).
$A(17, 8, 8) = 34$	[149].
$A(22, 8, 5) = 21$	By the Bose-Connor theorem ([18], [96]) a square divisible design $GD(5, 1, 2; 11 \times 2)$ does not exist.
$A(24, 8, 12) = 2576$	Linear programming [13].
$A(26, 8, 5) = 30$	[25].
$A(20, 10, 8) = 17$	Exhaustive search (see Section XII).
$A(21, 10, 7) = 13$	From Theorem 8, if $A(21, 10, 7) \geq 14$ then in fact $A(21, 10, 7) = 15$; but (see [77]) no $2-(15, 5, 2)$ design exists.
$A(22, 10, 7) = 16$	Exhaustive search (see Section XII).
$A(23, 10, 7) = 21$	Exhaustive search (see Section XII).
$A(28, 12, 8) = 19$	From Theorem 8, if $A(28, 12, 8) \geq 20$ then in fact $A(28, 12, 8) = 21$; but (see [77]) no $2-(21, 6, 2)$ design exists.

*Exactness of entries not listed here follows from the Johnson bounds (3), (7), (16).

The example $A(26, 14, 10) = 8$ previously mentioned shows that equality may also hold in (16) outside the "Honkala region" (23).

Most proofs that the entries in Table I followed by a period are exact can be obtained from the Johnson bounds (3), (7), (16)). Sources for proofs that the other entries are exact are given in Table III.

IV. STEINER SYSTEMS

In this section we give a highly compressed survey of Steiner systems (defined in Section I). Because some constructions are available only in obscure sources, and because in Section VI we shall require not just a single Steiner system but as many disjoint ones as possible, Table IV contains a number of explicit constructions for small Steiner systems.

As we saw in the previous section (Theorem 7), if a Steiner system $S(t, k, v)$ exists, it contains $b = \binom{v}{t} / \binom{k}{t}$ blocks, and then $A(v, 2k - 2t + 2, k) = b$. If $S(t, k, v)$ exists then so does the *contracted* (or *derived*) design $S(t - 1, k - 1, v - 1)$, formed from all the blocks containing (say) the last point.

Steiner triple systems $S(2, 3, v)$ exist if and only if $v \equiv 1$ or $3 \pmod{6}$ (Kirkman [109]; see Theorem 4). *Steiner quadruple systems* $S(3, 4, v)$ exist if and only if $v \equiv 2$ or $4 \pmod{6}$ (Hanani [81]; see Theorem 5).

Theorem 12 (Hanani [82], [83]): $S(2, 4, v)$ exists if and only if $v \geq 4$ and $v \equiv 1$ or $4 \pmod{12}$; in these cases

$$A(v, 6, 4) = v(v - 1)/12. \tag{24}$$

$S(2, 5, v)$ exists if and only if $v \geq 5$ and $v \equiv 1$ or $5 \pmod{20}$;

in these cases

$$A(v, 8, 5) = v(v - 1)/20. \tag{25}$$

Theorem 13 (See [53], [52], Chap. 6; [84]): For any prime power q and any nonnegative integer d there exist Steiner systems $S(2, q, q^d)$ (from the lines in the affine geometry $AG(d, q)$), $S(2, q + 1, q^d + q^{d-1} + \dots + q + 1)$ (from the lines in the projective geometry $PG(d, q)$), $S(2, q + 1, q^3 + 1)$ (from a unital in $PG(2, q^2)$), $S(2, 2^a, 2^{a+b} + 2^a - 2^b)$ for $a \leq b$ (from a complete 2^a -arc in $PG(2, 2^b)$), $S(3, q + 1, q^2 + 1)$ (from an elliptic quadric in $PG(3, q)$), or more generally $S(3, q + 1, q^d + 1)$ (from subfield sublines of a projective line). Thus

$$A(q^d, 2q - 2, q) = \frac{q^{d-1}(q^d - 1)}{q - 1}, \tag{26}$$

$$A(q^d + q^{d-1} + \dots + q + 1, 2q, q + 1) = \frac{(q^{d+1} - 1)(q^d - 1)}{(q^2 - 1)(q - 1)}, \tag{27}$$

$$A(q^3 + 1, 2q, q + 1) = q^2(q^2 - q + 1), \tag{28}$$

$$A(2^{a+b} + 2^a - 2^b, 2^{a+1} - 2, 2^a) = (2^b + 1)(2^b - 2^{b-a} + 1), \tag{29}$$

$$A(q^d + 1, 2q - 2, q + 1) = \frac{q^{d-1}(q^{2d} - 1)}{q^2 - 1}. \tag{30}$$

The only Steiner systems presently known with $t \geq 4$ are $S(5, 6, 12)$, $S(5, 8, 24)$ (Mathieu, Witt [182], [183], [45]), $S(5, 6, 24)$, $S(5, 7, 28)$, $S(5, 6, 48)$, $S(5, 6, 84)$ (Denniston [56]), and $S(5, 6, 72)$ (Mills [138]); and their contractions with $t = 4$. All these 5-designs are invariant under the group $PSL(2, v - 1)$ for the appropriate value of v .

Table IV lists the known Steiner systems (or $t - (v, k, 1)$ designs) with $v \leq 28$ and $2 \leq t < k$; b denotes the number of blocks. For each design we give either a construction or a reference, as well as information about the number of inequivalent designs (up to permutation of coordinates) and the number of disjoint designs. Some designs which are contractions of others have been omitted.

The Steiner triple systems on at most 21 points are treated in great detail in [156]. For further information about Steiner systems see [1], [3], [45], [52], [61], [62], [84], [96], [117], [118], [121], [127], [128], [139], [152], [168], [182], [183].

V. GRAHAM-SLOANE TYPE BOUNDS

We next discuss lower bounds on $A(n, d, w)$, roughly in order of increasing complexity. The bounds described in this section require very little computation (and correspondingly produce the weakest results). For $d \geq 6$ we give several different bounds, all roughly comparable (although only codes from Theorem 16 are needed for the present version of Table I).

TABLE IV
STEINER SYSTEMS

$S(2,3,7) = PG(2,2)$; $b = 7$. Cyclic: $\{(1011000)\}$; unique. Exactly two disjoint designs exist, the other being $\{(1101000)\}$.
 $S(3,4,8) = AG(3,2)$; $b = 14$. Extended cyclic: $\{(1011000)1, (010011100)\}$; unique. Exactly two disjoint designs exist, the other being $\{(1101000)1, (00101110)\}$ [32].
 $S(2,3,9) = AG(2,3)$; $b = 12$. It is also the unital in $PG(2,4)$; unique. In each of the following matrices, the 12 triples formed by the rows, columns and six generalized diagonals form an $S(2,3,9)$. The 7 matrices together yield 7 disjoint designs, partitioning the set of all $\binom{9}{3}$ triples [109].

124	128	125	129	123	126	127
378	943	983	743	469	357	346
956	765	476	586	785	489	598

There are exactly two inequivalent partitions [9].
 $S(3,4,10)$; $b = 30$. Cyclic: unique. Exactly five disjoint designs exist, in a unique way [115]. For example, we may take the five designs defined by the columns of the following array. The six hexadecimal vectors in a column together with their images under the permutation $(1,2,3,4,5)(6,7,8,9,10)$ form an $S(3,4,10)$; the five columns give five disjoint designs (cf. Section X).

27	2B	2D	2E	36
3A	3C	33	35	39
69	65	74	72	6A
B1	A6	B8	A3	AC
E4	F0	E2	E8	E1
170	162	168	161	164

$S(4,5,11)$; $b = 66$. Unique. Exactly two disjoint designs exist, in a unique way [115]. For example, the designs $\{7A0^{55}, 712^{11}\}$ and $\{782^{55}, 748^{11}\}$ defined by the group of order 55 generated by $(1,2,3,\dots,11)$ and $(2,5,6,10,4)(3,9,11,8,7)$ (cf. Table XV).
 $S(5,6,12)$; $b = 132$. Unique. Exactly two disjoint designs exist, in a unique way [115]. An $S(5,6,12)$ may be obtained from an $S(4,5,11)$ by appending 1 and adjoining the complements of all blocks. Two disjoint $S(5,6,12)$'s arise from the two previous disjoint $S(5,6,11)$'s. Alternatively, two disjoint designs arise from the supports of the codewords of weight 6 in the two extended ternary quadratic residue codes of length 12 whose zeros are the residues and nonresidues modulo 11 respectively [2], [3].

$S(2,3,13)$; $b = 26$. There are exactly two inequivalent designs ([77], p. 237), one of which is cyclic (Table XI). Denniston [54] (see also [116]) partitioned the set of all $\binom{13}{3}$ triples into 11 disjoint designs.
 $S(2,4,13) = PG(2,3)$; $b = 13$. Cyclic; unique. Chouinard [36] partitioned the set of all $\binom{13}{4}$ 4-sets into 55 disjoint designs.
 $S(3,4,14)$; $b = 91$. There are exactly four inequivalent designs, one of which is given in Table XV [137]. A set of four disjoint designs (not hitherto known to exist) is obtained by applying the permutations $(1,2,4,7,10,13,12,14,3,6,8,5,9,11)$, $(1,4,8,10,2,11,3,5,7,6,12,13,9)$ and $(1,6,9,14,8,11,12,10,5,4,3)(2,7,13)$ to the design in Table XV. See also [116].
 $S(2,3,15)$; $b = 35$. There are exactly 80 inequivalent designs [38], [78], one of which is $PG(3,2)$. Denniston [54], [55] partitioned the set of all $\binom{15}{3}$ triples into 13 disjoint designs.
 $S(2,4,16) = AG(2,4)$; $b = 20$. The weight 4 words in a translate of the Nordstrom–Robinson code (see Table X); unique. There exists an $S(3,4,16)$ that is tiled by 7 copies of $S(2,4,16)$.
 $S(3,4,16)$; $b = 140$. For example, the planes in $AG(4,2)$, the weight 4 words in the $\{16,11,4\}$ Hamming code. There are at least 31301 inequivalent designs [126], [127]. At least 8 pairwise disjoint designs exist (see [112], [124], [125] and the Appendix).
 $S(3,5,17)$; $b = 68$. Unique (see Theorem 13 and Table XV). At least two disjoint designs exist [136], [185].
 $S(3,4,20)$; $b = 285$. From the partitioning construction (Section VI). There are at least 10^{17} inequivalent designs [126], and exactly 29 cyclic designs [142]. At least 15 disjoint designs exist [67].
 $S(3,4,22)$; $b = 385$. There are exactly 21 inequivalent cyclic designs, and exactly 5 disjoint cyclic designs [59]. At least 11 disjoint designs exist [144], [145].
 $S(5,6,24)$; $b = 7084$. See Table XV. At least three inequivalent designs exist [56], [74].
 $S(5,8,24)$; $b = 759$. The weight 8 words in the $[24,12,8]$ Golay code. Unique [182], [183], [132], [45]. At least 9 disjoint designs exist [114].
 $S(2,5,25) = AG(2,5)$; $b = 30$. Unique.
 $S(3,4,26)$; $b = 210$. See Theorem 5 or [81]. At least 13 disjoint designs exist [146].
 $S(3,6,26)$; $b = 130$. Unique (see Theorem 13, Table XI, [33]).
 $S(2,3,27)$; $b = 117$. Many examples, one of which is $AG(3,3)$.
 $S(2,4,28)$; $b = 63$. See [52], [84], Theorem 13. At least 154 inequivalent designs exist [24], [26].
 $S(3,4,28)$; $b = 819$ [81]. At least 18 disjoint designs exist [66].
 $S(5,7,28)$; $b = 4680$. See [56] or Table XV.
 $S(2,6,31) = PG(2,5)$; $b = 31$. Cyclic: $\{0,1,3,8,12,18\} \pmod{31}$; unique.

The prototype of these bounds is the following.

Theorem 14 ([72]):

$$A(n,4,w) \geq \frac{1}{n} \binom{n}{w}. \tag{31}$$

This is established by defining a map f from vectors $a = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{F}_2^n$ to \mathbb{Z}_n by

$$f(a) = \sum_{i=0}^{n-1} ia_i \pmod{n}. \tag{32}$$

Then it is easy to see that the sets $C_0^{(w)}, C_1^{(w)}, \dots, C_{n-1}^{(w)}$, where

$$C_k^{(w)} = \{a \in \mathbb{F}_2^n : f(a) = k, \quad wt(a) = w\}, \tag{33}$$

form a partition of the set of all $\binom{n}{w}$ binary vectors of weight w into n disjoint codes each with Hamming distance 4. Since one of the $C_k^{(w)}$ must contain at least as many words as the average, (31) follows.

Furthermore (31) can be replaced by

$$A(n,4,w) \geq \max_{k=0,1,\dots,n-1} |C_k^{(w)}|, \tag{34}$$

although in practice this gives little improvement on (31).

Kløve [110] generalized Theorem 1 by replacing \mathbb{Z}_n with an arbitrary abelian group G of order n , defining f by

$$f(a) = \sum_{g \in G} a_g g. \tag{35}$$

He found an explicit formula for the best lower bound for $A(n,d,w)$ (for given n and w) obtained from (34) using the optimal choice of G . Again these results do not greatly differ from those obtained from (31). See also Delsarte and Piret [50].

For $d \geq 6$ there are several competing analogues of Theorem 14.

Theorem 15 ([72]): If $\delta \geq 3$ and n is a prime power then

$$A(n, 2\delta, w) \geq \frac{1}{n^{\delta-1}} \binom{n}{w}. \tag{36}$$

Proof: Let $n = q$ be a prime power, and write $\mathbb{F}_q = \{\alpha_0, \dots, \alpha_{q-1}\}$. The proof replaces (32) by the map f from \mathbb{F}_2^n to $\mathbb{F}_q^{\delta-1}$ given by

$$f(a) = (T_1(a), T_2(a), \dots, T_{\delta-1}(a)), \tag{37}$$

where

$$\begin{aligned} T_1(a) &= \sum_{a_i=1} \alpha_i, \\ T_2(a) &= \sum_{\substack{i < j \\ a_i = a_j = 1}} \alpha_i \alpha_j, \\ T_3(a) &= \sum_{\substack{i < j < k \\ a_i = a_j = a_k = 1}} \alpha_i \alpha_j \alpha_k, \\ &\dots \end{aligned}$$

For any $\mathbf{k} \in \mathbb{F}_q^{\delta-1}$ it can be shown that the code

$$C_{\mathbf{k}}^{(w)} = \{a \in \mathbb{F}_2^n : f(a) = \mathbf{k}, \quad wt(a) = w\}$$

has minimal distance 2δ , and (36) follows.

A subset $S = \{s_1, \dots, s_n\}$ of an abelian group G is called an S_t -set of size n if all the sums

$$s_{i_1} + s_{i_2} + \dots + s_{i_t}$$

for $1 \leq i_1 < i_2 < \dots < i_t \leq n$ are distinct in G (cf. [72], [73], [75], [76]). S_t -sets are relevant here because of the following result.

Theorem 16 (Compare [72]): If there exists an $S_{\delta-1}$ -set of size n in an abelian group of order m then

$$A(n, 2\delta, w) \geq \frac{1}{m} \binom{n}{w}.$$

Proof: Replace (35) by $f(a) = \sum_{i=1}^n s_i a_i$.

In 1962 Bose and Chowla ([17], [76]; see also [72]) constructed an S_t -set of size $q+1$ in \mathbb{Z}_m for $m = (q^{t+1} - 1)/(q - 1)$, for any prime power q . Another construction is the following.

Theorem 17: For any prime power q there is an S_t -set of size q in \mathbb{Z}_m for $m = q^t - 1$.

Proof: Let ξ be a primitive element of the field of order q^t , and let F be the subfield of order q . The set $\{s \in \mathbb{Z}_m \mid \xi^s - \xi \in F\}$ is the desired S_t -set.

Furthermore the columns of a parity-check matrix for an $[n, k, d = 2t + 1]$ binary linear code, together with the zero vector, form an S_t -set of size $n + 1$ in the abelian group \mathbb{Z}_2^{n-k} . (The corresponding constant weight code is the collection of words of weight w in some coset of the code extended with an (anti-) parity check bit.)

Let $v_2(n)$ denote the smallest m such that an S_2 -set of size n exists in \mathbb{Z}_m , and more generally let $v(n)$ be the smallest m such that an S_2 -set of size n exists in some

TABLE V
UPPER BOUNDS FOR S_2 -SETS
 m IS THE SMALLEST NUMBER KNOWN SUCH THAT $v(n) \leq m$

n	m	n	m	n	m
2	2^a	11	99^c	20	381^h
3	3^a	12	123^f	21	?
4	6^a	13	152^c	22	?
5	11^a	14	183^h	23	?
6	16^b	15	222^f	24	512^f
7	24^c	16	255^f	25	624^f
8	40^a	17	?	26	651^h
9	52^d	18	256^k	27	728^f
10	72^a	19	360^f	28	757^h

abelian group of order m . In [73] it is shown that $v_2(n) \sim n^2$ as $n \rightarrow \infty$, but for $v(n)$ it is known only that

$$\binom{n}{2} \leq v(n) < n^2 + O(n^{36/23}).$$

Table V gives the best upper bounds presently known on $v(n)$ for $2 \leq n \leq 28$.

Key to Table V

Elements of the elementary abelian groups \mathbb{Z}_2^k are written in decimal; for example $0101 \in \mathbb{Z}_2^4$ is written as 5.

- a = Optimal S_2 -set in \mathbb{Z}_m (see Table IV in [73]).
- b = In \mathbb{Z}_2^4 use $\mathbf{0}$ and columns of parity-check matrix for $[5, 1, 5]$ code.
- c = $\{(0,0), (1,0), (2,0), (4,0), (0,1), (7,1), (0,2)\}$ in $\mathbb{Z}_2^3 \times \mathbb{Z}_3$.
- d = $\{(0,0), (0,1), (0,2), (1,0), (1,4), (1,8), (2,2), (2,5), (2,8)\}$ in $\mathbb{Z}_2^2 \times \mathbb{Z}_{13}$.
- e = $\{0, 1, 2, 4, 7, 15, 26, 45, 54, 66, 83\}$ in \mathbb{Z}_{99} .
- f = $\{0, 1, 2, 4, 7, 14, 23, 31, 48, 59, 74, 92\}$ in \mathbb{Z}_{123} .
- g = $\{0, 1, 2, 4, 7, 12, 20, 35, 63, 77, 106, 115, 132\}$ in \mathbb{Z}_{152} .
- h = Perfect difference set [8].
- i = $\{0, 1, 2, 4, 7, 12, 20, 29, 46, 69, 92, 116, 140, 170, 191\}$ in \mathbb{Z}_{222} .
- j = Theorem 17.
- k = In \mathbb{Z}_2^8 use $\mathbf{0}$ and columns of parity-check matrix for $[17, 9, 5]$ cyclic code.
- l = In \mathbb{Z}_2^9 use $\mathbf{0}$ and columns of parity-check matrix for Wagner's $[23, 14, 5]$ code [179].

For example, when $n = 25$, Theorem 16 and Table V imply that

$$A(25, 6, 12) \geq \left\lceil \frac{1}{624} \binom{25}{12} \right\rceil = 8334\ddagger.$$

(\ddagger indicates a code which does not yield the best lower bound known for this value of $A(n, d, w)$, but is easy to construct.) The entries labeled "gs" in Table I are obtained in this way. The entries in the third column of Table V appear quite weak, and the construction of better S_2 -sets would probably improve several entries of Table I-B.

We also mention an unpublished general lower bound of Zaptcioglu [184], although in the range of our tables it

does not lead to any new bounds. Suppose C attains $A(n, d, w) = M$, where $d = 2\delta$. A d -neighbor of $c \in C$ is any $v \in \mathbb{F}_2^n$ such that $wt(v) = w$ and $\text{dist}(v, c) < d$. Let $GN(n, d, w, i)$ denote the number of d -neighbors of the i th word of C that are also d -neighbors of the j th word of C for some $j < i$. Then Zaptcioglu begins with the equality

$$A(n, d, w) = \frac{\binom{n}{w} + \sum_{i=2}^M GN(n, d, w, i)}{\sum_{i=0}^{\delta-1} \binom{w}{i} \binom{n-w}{i}}. \quad (38)$$

This is proved by multiplying both sides by the denominator, and using a straightforward counting argument. Since $GN \geq 0$ we immediately obtain the "Gilbert bound" of [72]:

$$A(n, d, w) \geq G(n, d, w),$$

where

$$G(n, d, w) = \frac{\binom{n}{w}}{\sum_{i=0}^{\delta-1} \binom{w}{i} \binom{n-w}{i}}.$$

Zaptcioglu shows that (38) has the following stronger consequence.

Theorem 18 (Zaptcioglu [184]): For $d \geq 4$ and $w \geq 3$ we have

$$A(n, d, w) \geq G(n, d, w) + \frac{\sum_{i=0}^{\delta-1} \binom{w}{i} \binom{n-w-1}{i}}{\sum_{i=0}^{\delta-1} \binom{w}{i} \binom{n-w}{i}} \cdot [A(n-1, d, w) - G(n-1, d, w)].$$

VI. THE PARTITIONING CONSTRUCTION

The partitioning construction, used by several authors ([13], [49], [68], [149], [150]), produces good lower bounds for codes with minimal distance 4. It is related to a generalization of Theorem 14 (see subsection 4).

A partition $\Pi(n, w) = (X_1, \dots, X_m)$ is a collection of disjoint sets or classes X_1, \dots, X_m , each of which is a code of length n , distance 4 and constant weight w , and whose union contains all $\binom{n}{w}$ vectors of weight w . The vector $\pi(n, w) = (|X_1|, \dots, |X_m|)$ with integer components is the index vector of the partition $\Pi(n, w)$, and

$$\pi(n, w) \cdot \pi(n, w) = \sum_{i=1}^m |X_i|^2$$

is its norm. We always assume $|X_1| \geq \dots \geq |X_m|$. When there are several different partitions available for a given n and w we often denote them by $\Pi_1(n, w), \Pi_2(n, w), \dots$, and their index vectors by $\pi_1(n, w), \pi_2(n, w), \dots$.

The direct product $\Pi(n_1, w_1) \times \Pi(n_2, w_2)$ of two partitions $(X_1, \dots, X_{m_1}), (Y_1, \dots, Y_{m_2})$ consists of the vectors

$$\{(u, v) : u \in X_i, v \in Y_i, 1 \leq i \leq m\},$$

where $m = \min\{m_1, m_2\}$. This set (which is only part of the final code) clearly has length $n_1 + n_2$, distance 4, weight $w_1 + w_2$, and contains

$$\pi(n_1, w_1) \cdot \pi(n_2, w_2) = \sum_{i=1}^m |X_i| |Y_i| \quad (39)$$

words.

The construction: To obtain a code of length n , distance 4 and weight w by the partitioning construction we write $n = n_1 + n_2$, choose $\epsilon = 0$ or 1, and take the union of the direct products

$$\begin{aligned} &\Pi(n_1, \epsilon) \times \Pi(n_2, w - \epsilon), \\ &\Pi(n_1, \epsilon + 2) \times \Pi(n_2, w - \epsilon - 2), \\ &\Pi(n_1, \epsilon + 4) \times \Pi(n_2, w - \epsilon - 4), \\ &\dots \end{aligned} \quad (40)$$

It is apparent that this code does have the required properties, and contains

$$\begin{aligned} &\pi(n_1, \epsilon) \cdot \pi(n_2, w - \epsilon) + \pi(n_1, \epsilon + 2) \cdot \pi(n_2, w - \epsilon - 2) \\ &+ \pi(n_1, \epsilon + 4) \cdot \pi(n_2, w - \epsilon - 4) + \dots \end{aligned} \quad (41)$$

words.

As an illustration we construct a code showing that $A(18, 4, 7) \geq 2041$. We take $n_1 = 8, n_2 = 10, \epsilon = 1$ and form the union of the direct products

$$\begin{aligned} &\Pi(8, 1) \times \Pi_2(10, 6), \\ &\Pi(8, 3) \times \Pi_1(10, 4), \\ &\Pi(8, 5) \times \Pi(10, 2), \\ &\Pi(8, 7) \times \Pi(10, 0), \end{aligned} \quad (42)$$

(see below and Table VI). The corresponding index vectors are

$$\begin{aligned} \pi(8, 1) &= (1, 1, 1, 1, 1, 1, 1, 1), \\ \pi(10, 6) &= (30, 30, 30, 30, 26, 25, 22, 15, 2), \end{aligned}$$

so the first direct product contains

$$1 \cdot 30 + 1 \cdot 30 + 1 \cdot 30 + 1 \cdot 30 + 1 \cdot 26 + 1 \cdot 25 + 1 \cdot 22 + 1 \cdot 15 = 208$$

words;

$$\begin{aligned} \pi(8, 3) &= (8, 8, 8, 8, 8, 8, 8, 8), \\ \pi_1(10, 4) &= (30, 30, 30, 30, 30, 22, 22, 12, 2, 2), \end{aligned}$$

so the second direct product contains

$$8 \cdot 30 + \dots + 8 \cdot 30 + 8 \cdot 22 + 8 \cdot 22 = 1552$$

words;

$$\begin{aligned} \pi(8, 5) &= (8, 8, 8, 8, 8, 8, 8, 8), \\ \pi(10, 2) &= (5, 5, 5, 5, 5, 5, 5, 5, 5, 5), \end{aligned}$$

so the third direct product contains

$$8 \cdot 5 + \dots + 8 \cdot 5 = 280$$

words; and

$$\begin{aligned} \pi(8, 7) &= (1, 1, 1, 1, 1, 1, 1, 1), \\ \pi(10, 0) &= (1), \end{aligned}$$

so the last direct product contains $1 \cdot 1 = 1$ word. The total number of codewords is

$$208 + 1552 + 280 + 1 = 2041.$$

Codes obtained from the partitioning construction are indicated by “ $p0$ ”, \dots , “ $p3$ ” in Table I-A. The values of n_1, n_2, ϵ are as follows:

$$\begin{aligned} \text{for type } p0, \quad n_1 &= \left\lfloor \frac{n}{2} \right\rfloor, \quad n_2 = n - n_1, \quad \epsilon = 0, \\ \text{for type } p1, \quad n_1 &= \left\lfloor \frac{n}{2} \right\rfloor, \quad n_2 = n - n_1, \quad \epsilon = 1, \\ \text{for type } p2, \quad n_1 &= \left\lfloor \frac{n}{2} \right\rfloor - 1, \quad n_2 = n - n_1, \quad \epsilon = 0, \\ \text{for type } p3, \quad n_1 &= \left\lfloor \frac{n}{2} \right\rfloor - 1, \quad n_2 = n - n_1, \quad \epsilon = 1. \end{aligned}$$

The partitions needed are listed in Table VI. (We do not take the space to indicate the particular partitions used in each construction.) Partitioning also gives $A(16, 4, 5) \geq 305\ddagger$.

We next discuss the choice of a good partition. When applying the partitioning construction in a situation where several different partitions are available, we see from (39), (41) that we should choose pairs $\Pi(n_1, w_1), \Pi(n_2, w_2)$ so as to maximize the inner product $\pi(n_1, w_1) \cdot \pi(n_2, w_2)$. (For example we use $\Pi_2(10, 6)$ rather than $\Pi_1(10, 6)$ in the first line of (42), so as to maximize the inner product with $\pi(8, 1) = (1, 1, 1, 1, 1, 1, 1, 1)$.)

We say that one partition $\Pi(n_1, w_1)$ dominates another $\Pi'(n_1, w_1)$ if

$$\pi(n_1, w_1) \cdot \pi(n_2, w_2) \geq \pi'(n_1, w_1) \cdot \pi(n_2, w_2) \quad (43)$$

holds for all choices of n_2, w_2 and all possible index vectors $\pi(n_2, w_2)$. If a partition is dominated it need never be used in the construction.

There is a simple test for dominance.

Theorem 19: $\pi(n_1, w_1) = (a_1, \dots, a_m)$ dominates $\pi'(n_1, w_1) = (b_1, \dots, b_{m'})$ if and only if

$$\sum_{i=1}^j a_i \geq \sum_{i=1}^j b_i, \quad \text{for all } j = 1, \dots, \max\{m, m'\}.$$

Proof: The components of the index vector $\pi(n_2, w_2)$ in (43) are nonincreasing positive integers, and any such vector is a positive combination of vectors of the form $(1, 1, \dots, 1, 0, \dots, 0)$.

A partition $\Pi(n, w)$ is *optimal* if it dominates all other $\Pi'(n, w)$ with the same n and w , or just *maximal* if it is not itself dominated by any other $\Pi'(n, w)$.

In the remainder of this section we describe techniques for finding good partitions.

1) By taking complements the existence of a $\Pi(n, w)$ implies the existence of a $\Pi(n, n - w)$ with the same index vector.

2) For $w = 0$ and 1 there are trivial partitions with index vectors

$$\begin{aligned} \pi(n, 0) &= (1), \\ \pi(n, 1) &= (1, 1, \dots, (n \text{ times})), \end{aligned}$$

and for $w = 2$ there are well-known partitions

$$\begin{aligned} \pi(n, 2) &= \left(\frac{n}{2}, \frac{n}{2}, \dots, (n-1 \text{ times}) \right), \quad n \text{ even}, \\ \pi(n, 2) &= \left(\frac{n-1}{2}, \frac{n-1}{2}, \dots, (n \text{ times}) \right), \quad n \text{ odd} \end{aligned}$$

see [136]. All these partitions are optimal.

3) If we have a lower bound $A(n, 4, w) \geq M$ there is always the partition

$$\pi(n, w) = \left(M, 1, 1, 1, \dots, \left(\binom{n}{w} - M \text{ times} \right) \right).$$

(This is useful when the inner product with $\pi(n, \bullet) = (1)$ is to be maximized.)

4) The results of [72]—see Theorem 14—show that a partition $\Pi(n, w)$ always exists with $m \leq n$ classes. In many cases—for example if n is prime—the index vector for this partition is

$$\pi(n, w) = \left(\frac{1}{n} \binom{n}{w}, \frac{1}{n} \binom{n}{w}, \dots, (n \text{ times}) \right).$$

5) A number of optimal partitions with $w = 3$ are available in the literature. It is known that, if $n \equiv 1$ or $3 \pmod{6}$, $n \neq 7$, then the set of all $\binom{n}{3}$ triples can be partitioned into $n - 2$ mutually disjoint Steiner triple systems—implying that there is an optimal partition

$$\pi(n, 3) = \left(\frac{n(n-1)}{6} (n-2 \text{ times}) \right).$$

This result is due to Lu [130], [131]. (The manuscript of [131] was incomplete at the time of the author’s death, but the six unfinished values of n have since been dealt with by Teirlinck [174].) Earlier results on this problem were given by Cayley [32], Denniston [54], [55], [57], Kirkman [109], Phelps [143], Schreiber [158], Teirlinck [171], Wilson [181]. When (as in this case) the set of all $\binom{n}{w}$ vectors of weight w can be partitioned into disjoint designs all having the same parameters, the designs are said to form a “large set.” Further results on partitions may be found in [4], [19], [60], [65]–[68], [85], [113], [119], [144], [146], [159]–[162], [172], [173].

6) Van Pul [149], [150] and Etzion–Van Pul [68] found the partitions $\Pi(6, 3)$, $\Pi(7, 3)$, $\Pi(8, 4)$, $\Pi(10, 3)$, $\Pi_1(10, 4)$ mentioned in Table VI. (However only $\Pi_1(10, 4)$ is given explicitly in [68].)

7) In situations not covered by the preceding comments we use the computer to find good partitions. Our methods are based on the following considerations. a) Finding a good partition is a graph coloring problem. For if we construct the graph whose vertices represent $\binom{n}{w}$ binary vectors of weight w , and join two vertices by an edge if

and only if the vertices are Hamming distance 2 apart, then a partition $\Pi(n, w) = (X_1, \dots, X_m)$ describes a coloring of the vertices using m colors, the classes X_1, \dots, X_m being the color classes. b) A useful heuristic for finding a good partition is to maximize the norm of the index vector. This is only a heuristic, for we already saw in the previous example that there are situations where partitions with less than the maximal norm are preferable. However, a partition with the greatest possible norm is always maximal. A second heuristic is to minimize the number of color classes. c) Good methods of choosing the initial classes X_1, X_2, \dots of $\Pi(n, w)$ are to use a maximal independent set algorithm, to use as many disjoint Steiner systems as possible (see Tables IV), or more generally to look for as many disjoint (or almost disjoint) copies as possible of the largest known code of length n and weight w . (Some partition obtained by repeatedly removing maximal independent sets is maximal.) We then look for a coloring of the remaining vertices with the greatest norm.

David Johnson [98] has developed a simulated annealing program for graph coloring, which attempts to maximize the sum of the squares of the color class sizes. Some of the partitions given in Table VI below were found (in part) using this program. Others were found by various iterative procedures. Johnson's program uses Kempe-chain interchanges for graph coloring; a recent alternative suggestion by Berge [10] may lead to better colorings and hence better partitions.

8) The methods of 7) are only successful for n up to about 14. For $n \geq 12$ we also made use of Etzion and Van Pul's "Construction B" for combining partitions ([68]). This construction works as follows. Given partitions $\Pi(l, u)$ and $\Pi(m, v)$ (for all $u, 0 \leq u \leq \min(l, w)$ and $v, 0 \leq v \leq \min(m, w)$) we construct a partition $\Pi(n, w)$, where $n = l + m$, by repeatedly using the partitioning construction (cf. (40)). More precisely, start with two sufficiently long rows of empty buckets—the "odd" row and the "even" row. For each pair (u, v) with $u + v = w$ we consider the given partitions $\Pi(l, u) = (X_1, \dots, X_r)$ and $\Pi(m, v) = (Y_1, \dots, Y_s)$, and distribute the $r_1 r_2$ direct products $X_i \times Y_j$ into the first $r = \max(r_1, r_2)$ buckets in the row with the same parity as u , where each bucket gets $\min(r_1, r_2)$ codes $X_i \times Y_j$, such that no bucket contains two codes $X_i \times Y_j$ and $X_{i'} \times Y_{j'}$ where $i = i'$ or $j = j'$. The result will be that the buckets form a partition $\Pi(n, w)$. (At least r buckets are required because of the conditions $i \neq i'$ and $j \neq j'$. A distribution into r buckets is possible because the complete bipartite graph K_{r_1, r_2} has an edge coloring with r colors.) For example, take $(n, w) = (12, 4)$. Using partitions with index vectors $\pi(6, 0) = \pi(6, 6) = (1)$, $\pi(6, 1) = \pi(6, 5) = (1, 1, 1, 1, 1, 1)$, $\pi(6, 2) = \pi(6, 4) = (3, 3, 3, 3, 3)$, $\pi(6, 3) = (4, 4, 4, 4, 2, 2)$ we fill five buckets in the even row with $1 \cdot 3 + 5 \cdot 3 \cdot 3 + 3 \cdot 1 = 51$ words each, and six buckets in the odd row with $\pi(6, 1) \cdot \pi(6, 3) + \pi(6, 3) \cdot \pi(6, 1) = 40$ words each, to obtain a partition $\Pi(12, 4)$ with index vector $(51, 51, 51, 51, 51, 40, 40, 40, 40, 40, 40)$.

Table VI gives the index vectors of the nontrivial partitions used in constructing the codes marked "p0," "p1,"

"p2," "p3" in Table I. In view of 1) we only give partitions $\Pi(n, w)$ with $w \leq n/2$. The partitions marked "A" are given explicitly in the Appendix. Partitions known to be optimal are marked with an asterisk. For some values of n and w several different partitions $\Pi_i(n, w)$ ($i = 1, 2, \dots$) are given (i is given in column 3), no one of which dominates any of the others. In many cases we have found other partitions besides those in Table VI, which although not needed for the partitioning construction, nevertheless are not dominated by the partitions in the table. It would be nice to replace these with smaller sets of maximal partitions. For $\Pi(10, 4)$ there are at least two maximal partitions, and, as will be shown below, certainly no optimal $\Pi(10, 4)$ exists.

Key to Table VI

- * = Optimal.
- Λ = See Appendix.
- B = Construction B.
- GS = From Theorem 14.
- (1) = Shortened $\Pi(9, 3)$
- (2) = Shortened $\Pi(10, 4)$.
- (3) = Shortened $\Pi(13, 3)$.
- (4) = Shortened $\Pi(15, 3) = (35, 35, \dots, (13 \text{ times}))$ of [55].

Optimal and maximal partitions: We first show that the partitions marked with an asterisk in Table VI are optimal. Obviously $\pi(n, w)$ is optimal if all parts (except perhaps the smallest) have size $A(n, 4, w)$, which proves the optimality of $\pi(8, 3)$, $\pi(9, 3)$, etc.

$\pi(6, 3) = (4, 4, 4, 4, 2, 2)$ is optimal.

Proof: Any code attaining $A(6, 4, 3) = 4$ is equivalent to

$$C_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & \bullet \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

We must show that $(4, 4, 4, 4, 4)$ and $(4, 4, 4, 4, 3, 1)$ are impossible. Since C_1 has column sums 2, the last four vectors of the partition also have column sums 2, and are therefore equivalent either to C_1 or to

$$C_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The vectors of C_2 cannot be partitioned $3+1$, so the last four vectors of the partition are also equivalent to C_1 , and the partition is $(4, 4, 4, 4, 4)$. Let the first class be C_1 . Then the four vectors 100011, 100101, 110001, 101001 have mutual distances 2 and each lies in a different one of the other four classes. It is now easy to check by hand that these classes cannot be completed.

$\pi(7, 3) = (7, 7, 6, 6, 5, 4)$ is optimal.

Proof: We know from Table IV that there do not exist three classes of size 7. Also $(7, 7, 6, 6, 6, 1, 1, 1)$ is

TABLE VI
NONTRIVIAL PARTITIONS USED TO CONSTRUCT CODES IN TABLE I

n	w	i	m	Norm	Source	Index Vector of $\Pi_i(n, w)$
6	3	1	6	72	A*	4.4.4.4.2.2
7	3	1	6	211	A*	7.7.6.6.5.4
8	3	1	7	448	(1)*	8.8.8.8.8.8.8
8	4	1	6	844	A*	14.14.12.12.10.8
9	3	1	7	1008	Table IV*	12.12.12.12.12.12.12
9	4	1	8	2066	A	18.18.18.18.16.15.15.8
9	4	2	10	2036	(2)	18.18.18.18.18.14.13.7.1.1
10	3	1	10	1530	A*	13.13.13.13.13.13.13.13.3
10	4	1	10	5620	[68]	30.30.30.30.30.22.22.12.2.2
10	4	2	9	5614	A	30.30.30.30.26.25.22.15.2
10	5	1	8	8044	A	36.36.34.34.29.29.27.27
11	3	1	10	2731	A	17.17.17.17.17.17.17.16.16.14
11	3	2	11	2713	A	17.17.17.17.17.17.17.17.16.12.1
11	3	3	11	2705	A	17.17.17.17.17.17.17.17.17.10.2
11	4	1	11	10724	A	35.35.35.34.33.33.33.32.31.25.4
11	4	2	11	10616	A	35.35.35.35.35.33.33.32.28.21.8
11	5	1	10	25066	A	66.66.60.60.54.45.44.40.26.1
11	5	2	10	25046	A	66.66.60.60.54.45.44.42.22.3
12	3	1	11	4400	(3)*	20.20.20.20.20.20.20.20.20.20.20
12	4	1	11	22903	A	51.51.51.51.51.46.45.44.44.34.27
12	4	2	12	22843	A	51.51.51.51.49.48.48.42.42.37.23.2
12	4	3	12	22815	A	51.51.51.51.49.48.48.42.42.40.15.7
12	4	4	12	22795	A	51.51.51.51.49.48.46.44.43.37.20.4
12	4	5	12	22755	A	51.51.51.51.49.48.48.45.39.36.22.4
12	4	6	12	22663	A	51.51.51.51.49.48.48.45.41.32.22.6
12	5	1	12	55860	A	80.80.80.80.72.70.69.67.67.62.48.17
12	5	2	13	55350	A	80.80.80.80.75.72.71.69.63.55.40.23.4
12	6	1	10	99952	A	132.132.120.120.110.94.90.76.36.14
12	6	2	10	99776	A	132.132.120.120.110.94.90.72.42.12
12	6	3	10	99072	A	132.132.120.110.110.97.91.75.47.10
13	3	1	11	7436	Table IV*	26.26.26.26.26.26.26.26.26.26.26
13	4	1	13	42165	A	65.65.65.65.62.61.60.57.57.53.52.45.8
13	4	2	13	42163	A	65.65.65.65.65.62.61.60.58.55.54.52.45.8
13	4	3	13	42147	A	65.65.65.65.62.60.60.58.57.54.49.47.8
13	4	4	13	42015	A	65.65.65.65.62.60.60.58.57.56.53.37.12
13	4	5	13	41975	A	65.65.65.65.62.62.60.59.56.55.49.40.12
13	4	6	13	41795	A	65.65.65.65.62.61.61.59.58.54.50.32.18
13	5	1	13	135679	A	123.123.121.115.110.109.109.102.99.92.84.72.28
13	5	2	13	135557	A	123.123.121.115.110.109.109.101.99.93.86.68.30
13	5	3	14	135437	A	123.122.121.114.110.109.109.102.97.91.85.77.26.1
13	5	4	13	134757	A	123.123.123.116.110.109.106.100.98.92.81.68.38
13	5	5	14	134753	A	123.123.123.116.110.109.107.104.97.89.83.62.40.1
13	6	1	14	239106	A	166.166.160.156.143.143.139.135.131.122.107.100.46.2
13	6	2	14	239082	A	166.166.160.156.144.142.138.137.131.120.106.102.46.2
13	6	3	13	238832	A	166.166.160.156.143.142.138.136.130.120.111.97.51
13	6	4	14	238698	A	166.166.160.156.145.142.139.136.131.118.113.91.50.3
13	6	5	14	238384	A	166.166.160.156.145.142.139.136.131.119.112.88.52.4
13	6	6	13	238116	A	166.166.160.156.145.144.137.132.127.118.111.98.56
13	6	7	14	237556	A	166.166.160.156.145.142.140.136.131.118.106.86.59.5
14	3	1	13	10192	(4)*	28.28.28.28.28.28.28.28.28.28.28.28
14	4	1	13	79393	A	91.91.91.91.81.79.78.77.74.73.71.62.42
14	4	2	14	79357	A	91.91.91.91.80.79.78.78.75.74.71.60.41.1
14	4	3	14	79339	A	91.91.91.91.81.79.79.77.76.71.67.67.38.2
14	4	4	13	79269	A	91.91.91.91.82.79.78.77.75.72.67.62.45
14	5	1	15	291280	A	169.169.165.156.156.152.149.144.143.137.134.121.118.80.9
14	5	2	15	290646	A	169.169.165.156.155.153.151.147.143.137.134.120.112.76.15
14	5	3	16	290288	A	169.169.165.156.155.153.151.147.142.137.133.124.109.75.16.1
14	5	4	15	289872	A	169.169.163.156.155.152.149.148.142.139.132.131.102.76.19
14	6	1	14	680081	B	253.252.243.243.243.243.212.212.212.212.212.212.42
14	7	1	14	913176	B	282.282.280.280.271.271.271.271.271.271.271.70.70
14	7	2	15	887552	B	292.292.280.280.272.272.242.242.242.242.242.242.48.2

impossible, for it would shorten to either $\pi(6,3) = (4,4,4,4,4)$ or $(4,4,4,4,3,1)$. (Any class of size 6 must shorten in three ways to a class of size 4 and in four ways to a class of size 3.) Therefore $(7,7,6,6,5,4)$ is optimal. This also implies the optimality of

$$\pi(8,4) = (14,14,12,12,10,8).$$

Second, we point out that there is no optimal partition $\pi(10,4)$. For $\pi_1(10,4) = (30,30,30,30,22,22,12,2,2)$ is maximal, as we now show, while $\pi_2(10,4) = (30,30,30,30,26,25,22,15,2)$ has fewer classes. Since $A(10,4,4) = 30$, by Theorem 7, no class has size greater than 30. From [115]—see Table IV—the maximal number of classes of size 30 is five, this can occur in an essentially unique way. We use the particular set of five given in Table IV. When these are removed the remaining $\binom{10}{4} - 5 \times 30 = 60$ vectors consist of

$$\begin{array}{lll} (01111) & (00000) & (5) \\ (00000) & (01111) & (5) \\ (00011) \times (00011) & & (25) \\ (00101) \times (00101) & & (25). \end{array}$$

Each of the first five vectors (and each of the second five) must be a different color. Among the final 50 vectors there cannot be a color class of size 21, because if so then that class would contain at least three words of form $***** (00011)$ or three of form $***** (00101)$. Thus $\pi_1(10,4)$ is maximal.

Concluding remarks:

1) If $n = 2t$, $w = t$, t odd, we may take $\epsilon = 0$ and obtain

$$A(2t,4,t) \geq \sum_{w=0}^{t-1} \text{norm } \Pi(t,w). \quad (44)$$

Similarly if t is even we get

$$A(2t,4,t) \geq \max \left\{ \sum_{w=0,2,\dots,t} \text{norm } \Pi(t,w), \sum_{w=1,3,\dots,t-1} \text{norm } \Pi(t,w) \right\}. \quad (45)$$

2) Using their Construction B, Etzion and Van Pul [68] show that if n is of the form 2^k ($k \geq 2$) or $3 \cdot 2^k$ ($k \geq 2$) and w is even then Theorem 14 can be replaced by

$$A(n,4,w) \geq \frac{1}{n-1} \binom{n}{w}. \quad (46)$$

From the partitions in Table VI (especially $\Pi_2(10,4)$) this now also holds for $n = 5 \cdot 2^k$ ($k \geq 2$).

3) Romanov's construction [155] showing that $A(16,3) \geq 2720$ (see Table II) also uses partitioning. We write the codewords in the form (a,b) , where length $(a) = 9$, length $(b) = 7$. On the left side a has weight 0, 3, 6, or 9, and we make use of Kirkman's partition

$$\pi(9,3) = (12,12,12,12,12,12,12)$$

of the set of triples on 9 points into 7 disjoint copies $\mathcal{S}_1, \dots, \mathcal{S}_7$ of $S(2,3,9)$ (see Tables IV, VI). On the right

we partition the set of all 128 7-bit words into eight disjoint translates $\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_7$ of the $[7,4,3]$ Hamming code. Romanov's code then consists of the vectors $(0, \mathcal{H}_0), \mathcal{S}_i \times \mathcal{H}_i$ ($1 \leq i \leq 7$), and their complements.

VII. LOWER BOUNDS OBTAINED BY MODIFYING CODES WITH A LARGER MINIMAL DISTANCE

The following inequalities, due to Zinovicv [187], Van Pul [149], [150], and Honkala *et al.* [94], resemble those of Section V in requiring very little computation. They produce good lower bounds for codes with $d = 6$. We follow the treatment of Honkala *et al.* [94].

Theorem 20 ([187], [149], [150], [94]):

a) For $0 \leq g < \min\{w, \delta\}$ and $0 \leq k < n$ we have

$$A(n-k, 2\delta-2g, w-g) \geq \frac{1}{\binom{n}{k}} A(n, 2\delta, w) \sum_{i=0}^g \binom{w}{i} \binom{n-w}{k-i}.$$

b) For $0 \leq g \leq w$, $0 \leq k < n$ and $k-g < \delta$, we have

$$A(n-k, 2\delta-2k+2g, w-g) \geq \frac{1}{\binom{n}{k}} A(n, 2\delta, w) \sum_{i=g}^k \binom{w}{i} \binom{n-w}{k-i}.$$

Proof: Suppose \mathcal{C} attains the bound $A(n, 2\delta, w)$. For any k -subset S of the coordinates let c_S denote the projection of $c \in \mathcal{C}$ into S , and let $c_{\bar{S}}$ denote the projection onto the other coordinates. A new code \mathcal{C}_S with length $n' = n - k$, $d' = 2\delta - 2g$ and $w' = w - g$ is obtained by taking all words $c_{\bar{S}}$ for which $c \in \mathcal{C}$ and $wt(c_S) = i$ for some $0 \leq i \leq g$, and complementing any $g - i$ 1's. A counting argument shows that

$$\sum_{|S|=k} |\mathcal{C}_S| = A(n, 2\delta, w) \sum_{i=0}^g \binom{w}{i} \binom{n-w}{k-i},$$

and a) follows. To prove b) we take all words $c_{\bar{S}}$ for which $c \in \mathcal{C}$ and $wt(c_S) = i$ for some $g \leq i \leq k$, and complement any $i - g$ 0's.

The lower bounds on $A(17,6,7)$ and $A(18,6,7)$ in Table I-B are obtained from Theorem 20a by taking $g = 1$ and \mathcal{C} to be the Steiner system $S(5,8,24)$. The lower bounds on $A(15,6,7)$, $A(16,6,7)$ follow similarly using a particular choice of S .

*Theorem 21 (Honkala *et al.* [94]):*

$$A(n-2, d-2, w-1) \geq A(n, d, w).$$

Proof: We modify the codewords c for a code \mathcal{C} attaining $A(n, d, w)$ as follows. If c ends with 00, complement the final 1, while if c ends with 11, complement the final 0. Now omit the last two coordinates of all words.

The lower bounds $A(22,6,7) \geq 759$, $A(22,6,11) \geq 2576\ddagger$ follow by taking \mathcal{C} to be $S(5,8,24)$ or the code attaining $A(24,8,12) = 2576$.

H. Hämmäläinen [80] used a modification of this argument to show $A(21,6,6) \geq 269$. Start with the $S(5,8,24)$

TABLE VII
SUM-CONSTRAINED LEXICODES

Bound	s
$A(12, 4, 6) = 132$	21
$A(24, 8, 18) \geq 78$	175
$A(25, 8, 18) \geq 254$	175
$A(26, 8, 18) \geq 760$	175
$A(26, 8, 19) \geq 256\ddagger$	188
$A(19, 10, 11) = 12$	84
$A(23, 10, 17) = 8$	181
$A(24, 10, 18) = 9$	203
$A(28, 10, 14) \geq 415\ddagger$	130

lexicode (see the following section), and take the $759 - 330 = 429$ words that do not end with 00. If such a word ends with 11 complement the final 0. By omitting the last two coordinates we obtain 429 words of length 22, distance 6 and weight 7. Using this set as the seed for a lexicographic code (see the following section) we get another code \mathcal{C} showing $A(22, 6, 7) \geq 759$, in which (labeling the coordinates 1 to 22 from right to left) 269 words have a 1 in coordinate 2. Hence $A(21, 6, 6) \geq 269$.

VIII. LEXICOGRAPHIC CODES

Lexicographic codes are studied in detail in [44], and we refer to that paper for the general theory. Here we just consider constant weight lexicode, which give easily computed lower bounds on $A(n, d, w)$ that are often reasonably good and in some cases give the best bounds known.

The *constant weight lexicographic code* (or *lexicode*, for short) with length n , Hamming distance d and weight w is obtained by starting with the empty code, considering all binary vectors of the given length and weight in lexicographic order (beginning with $00 \dots 011 \dots 1$), and adding them to the code if they have the desired Hamming distance from it.

This is a “no-input” construction. The most remarkable example of a lexicode is the Steiner system $S(5, 8, 24)$ (see [44], [45]); other examples are indicated by “ x ” in Table I.

Several variations are possible. The vectors of complementary weight $n - w$ may be used instead, or the vectors may be considered in Gray code order, or both. For example $A(25, 12, 9) = 25$ and $A(27, 12, 18) = 39$ also arise as Gray lexicode.

Another modification, a *sum-constrained lexicode*, only considers binary vectors $(a_0, a_1, \dots, a_{n-1})$ that satisfy the constraint

$$\sum_{i=0}^{n-1} ia_i \geq s, \tag{47}$$

where s is specified in advance. For example the choice $s = 21$ produces the Steiner system $S(5, 6, 12)$ ([44], [45]). Other examples are given in Table VII. Although a considerable amount of computing is needed to discover the best value of s , once found this gives a succinct definition of the code.

A more powerful modification is to start with an initial

TABLE VIII
LEXICODES WITH A SEED

Bound	Seed
$A(18, 6, 4) = 22$	1422, 24410
$A(26, 6, 10) \geq 8189\ddagger$	$A(25, 6, 9) \geq 4100\ddagger$ from Table XV
$A(26, 6, 17) \geq 5407\ddagger$	$A(25, 6, 16) \geq 4100\ddagger$ from Table XV
$A(27, 6, 9) \geq 7198\ddagger$	$A(25, 6, 9) \geq 4100\ddagger$ from Table XV
$A(27, 6, 10) \geq 11656\ddagger$	$A(25, 6, 10) \geq 5700\ddagger$ from Table XV
$A(28, 6, 9) \geq 9577\ddagger$	$A(25, 6, 9) \geq 4100\ddagger$ from Table XV
$A(26, 8, 13) \geq 3004\ddagger$	The lexicode $A(24, 8, 12) = 2576$
$A(27, 8, 13) \geq 3601\ddagger$	The lexicode $A(24, 8, 12) = 2576$
$A(21, 10, 7) = 13$	310CC, 54524
$A(23, 12, 10) = 16$	58AA1C, 60E4A6

set of vectors (the “seed”) instead of the empty set. Some codes (labeled “ xy ” in Table I) found this way are best described in the condensed notation introduced in Section XII, and are listed in Table XVI. Others (indicated by “ xh ” in Table I) are given in Table VIII. In the latter table, if the seed has shorter word length than the final code, we pad the seed by adding prefixes $0 \dots 01 \dots 1$ of the appropriate weight on the left.

IX. CONSTANT WEIGHT CODES FROM TRANSLATES OF LINEAR CODES

A number of good constant weight codes may be obtained from translates of linear codes (and from translates of the Nordstrom–Robinson code, which behaves in many ways like a linear code). If \mathcal{C} is an $[n, k, d]$ binary linear code, let $B^{(w)}(u)$ be the set of vectors of weight w in the translate $u + \mathcal{C}$, $u \in \mathbb{F}_2^n$. Then

$$A(n, d, w) \geq \max_{u \in \mathbb{F}_2^n} |B^{(w)}(u)|. \tag{48}$$

Furthermore (cf. [28]) the code

$$(B^{(w-1)}(u), 1) \cup (B^{(w)}(u), 0)$$

shows that

$$A(n + 1, d, w) \geq \max_{u \in \mathbb{F}_2^n} \{|B^{(w-1)}(u)| + |B^{(w)}(u)|\}. \tag{49}$$

A not very systematic search through known linear codes has yielded the following examples. The 14 vectors

$$(1000000000000)(11101011100000) \tag{50}$$

span Karlin’s [28, 10, 8] self-dual code ([132], p. 509, column 2, first code). We apply (49) to the [27, 10, 7] code \mathcal{C} obtained by deleting the last coordinate. With $u = 0$ we obtain $A(28, 8, 13) \geq 4668\ddagger$, $A(28, 8, 14) \geq 5280$, with $u = 1F$ we obtain $A(28, 8, 10) \geq 1652\ddagger$, with $u = A$ we obtain $A(28, 8, 11) \geq 2666\ddagger$, and with $u = 15$ we obtain $A(28, 8, 12) \geq 3780\ddagger$.

The generator matrix

011	011	011	011	011	001111111111
011	101	101	101	101	110011111111
101	011	110	101	110	111100111111
101	110	011	110	101	111111001111
110	110	101	011	110	111111110011
110	101	110	110	011	111111111100
110	110	110	101	101	010101010101

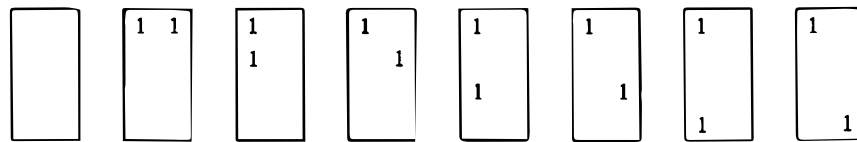
(51)

TABLE IX
WEIGHT DISTRIBUTIONS OF TRANSLATES OF [23, 12, 7] GOLAY CODE

#\i	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	0	0	0	0	0	253	506	0	0	1288
23	0	1	0	0	0	0	77	176	176	330	616	672
253	0	0	1	0	0	21	56	112	240	400	546	672
1771	0	0	0	1	5	16	48	120	240	400	560	658

defines a [27, 7, 12] code (a less symmetrical code with these parameters is given in [87]). Again we apply (49) to the code \mathcal{C} formed by omitting the last coordinate. With $u = 0$ we obtain $A(27, 12, 12) \geq 82$ and with $u = 92120$ we get $A(27, 12, 13) \geq 81$.

Table IX gives the weight distributions of the cosets of the [23, 12, 7] perfect Golay code ([132], Chap. 2). The first column gives the number of cosets with the given



weight distribution. (In both Tables IX and X the weight distributions are symmetric about $n/2$.) Using (49) we obtain $A(24, 8, 9) \geq 640$, as well as the other entries labeled "t4" in Table I.

The Nordstrom–Robinson code: The nonlinear Nordstrom–Robinson code of length 16, distance 6 and 256 codewords [132], [5], [107] produces a number of good constant weight codes, as was first observed by Semakov and Zinoviev [159]. We work inside the [24, 12, 8] extended Golay code \mathcal{S} and represent codewords of \mathcal{S} by 4×6 arrays called MOG's (or miracle octad generators). These have been described in several references (see [40], [42], [43], [46]–[48] and especially [45], pp. 303–304) and we do not repeat the definition here. We label the first 8 coordinates as follows (cf. [45], p. 316):

∞	0		
3	2		
5	1		
6	4		

By deleting these 8 coordinates from the codewords of \mathcal{S} we obtain (two copies of) the [16, 11, 4] Hamming code \mathcal{H} , while the codewords of \mathcal{S} that vanish on these 8 coordinates yield the [16, 5, 8] first-order Reed–Muller code \mathcal{R} . We order the coordinates by reading down the columns, from left to right. When the Golay code defined by the MOG coordinates is read in this way it coincides with the lexicographic version of this code ([44], [45], p. 327).

Let \mathcal{R}_i ($0 \leq i \leq 6$) denote the words of \mathcal{R} that have 1's in coordinates ∞ and i , and 0's elsewhere in the first 8

TABLE X
WEIGHT DISTRIBUTIONS OF TRANSLATES OF NORDSTROM–ROBINSON CODE THAT PARTITION THE SPACE

#\i	0	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	112	0	30
16	0	1	0	0	0	42	0	85	0
120	0	0	1	0	14	0	63	0	100
112	0	0	0	5	0	33	0	90	0
7	0	0	0	0	20	0	48	0	120

coordinates, with the first 8 coordinates deleted. Each \mathcal{R}_i is a translate of \mathcal{R} containing 16 words of weight 6 and 16 of weight 10, and

$$\mathcal{N} = \mathcal{R} \cup \mathcal{R}_0 \cup \mathcal{R}_1 \cup \dots \cup \mathcal{R}_6$$

is the Nordstrom–Robinson code. Thus \mathcal{N} consists of the words of \mathcal{S} that begin with one of

with these first 8 coordinates deleted.

Let α_0 denote any of the four octads (weight 8 words in \mathcal{S}) that have 1's at $\infty, 3, 5, 6$ and 0's at $0, 1, 2, 4$. Similarly α_i ($0 \leq i \leq 6$) is any octad that meets the first 8 coordinates just in ∞ and $3+i, 5+i, 6+i \pmod{7}$. The seven translates $\alpha_i + \mathcal{N}$ (with the first 8 coordinates deleted) together with \mathcal{N} itself form a partition of the Hamming code \mathcal{H} . We remark that all seven translates $\alpha_i + \mathcal{N}$ are equivalent, all pairs of such translates are equivalent, and there are two inequivalent ways to choose three translates.

The Nordstrom–Robinson code also has the property that certain of its translates partition the whole space of vectors of length 16. The weight distributions of these translates are given in Table X; the last row describes the translates $\alpha_i + \mathcal{N}$. From Table X and (48) we obtain $A(16, 6, 6) \geq 112$, $A(16, 6, 8) \geq 120$.

The decomposition of \mathcal{H} into 8 translates of \mathcal{N} shows in particular that the 448 weight-6 words in \mathcal{H} can be partitioned as

$$448 = 112 + 7 \times 48,$$

where each part has minimal distance 6 (see the $i=6$ column of Table X). There is however a better partition of these 448 words. Let \mathcal{S}_1 denote the words of \mathcal{S} that have exactly two 1's in the first 8 coordinates, in coordinates $(\infty, 0)$, $(\infty, 1)$, $(\infty, 2)$, \dots , or $(\infty, 6)$, with the first 8 coordinates deleted. Similarly \mathcal{S}_2 is obtained from the words of \mathcal{S} that have 1's in coordinates $(0, 1)$, $(0, 2)$, \dots , $(0, 6)$; \mathcal{S}_3 from $(1, 2)$, $(1, 3)$, \dots , $(1, 6)$; \mathcal{S}_4 from $(2, 3)$, $(2, 4)$, $(2, 5)$, $(2, 6)$; \mathcal{S}_5 from $(3, 4)$, $(3, 5)$, $(3, 6)$; and \mathcal{S}_6 from $(4, 5)$, $(4, 6)$, $(5, 6)$. Each \mathcal{S}_i is a union of translates of \mathcal{R} . Let $\mathcal{S}_i^{(6)}$ denote the weight 6 words in \mathcal{S}_i . Then $\mathcal{S}_1^{(6)}, \dots, \mathcal{S}_6^{(6)}$ contain 112, 96, 80, 64, 48, 48 words, re-

spectively. Since any two pairs defining an \mathcal{S}_i have a point in common, and \mathcal{S} has minimal distance 8, it follows that each $\mathcal{S}_i^{(6)}$ is a constant weight code of minimal distance 6. All the $\mathcal{S}_i^{(6)}$ are contained in \mathcal{H} , and so they partition the weight 6 words of \mathcal{H} as

$$448 = 112 + 96 + 80 + 64 + 48 + 48.$$

There is a similar result for weight 5 words. Let \mathcal{H}' denote the Hamming code translated by 00...01. Then \mathcal{H}' is partitioned into 8 translates of \mathcal{N} (one from the second row of Table X, seven from the penultimate row), which partitions the weight 5 words in \mathcal{H}' as

$$273 = 42 + 7 \times 33$$

(see the $i = 5$ column of Table X). An alternative partition

$$273 = 42 + 36 + 4 \times 33 + 3 \times 21$$

where each part has minimal distance 6 may be obtained as follows. We denote the nine parts by $\mathcal{D}_1^{(5)}, \dots, \mathcal{D}_9^{(5)}$, where $\mathcal{D}_1^{(5)}$ consists of the weight 5 words in the translate of Θ_1 by 00...01 with the first 8 coordinates deleted, and $\Theta_1 = \mathcal{S}_1$, $\Theta_2 = \mathcal{S}_2$. For $j = 0, 1, 2, 3$, Θ_{j+3} consists of the words of \mathcal{S} having either two or four 1's in the first eight coordinates, such that these 1's are a subset or superset of $\{1, 3, 4\} + j \pmod{7}$. Θ_7 consists of the words of \mathcal{S} with two or four 1's in the first eight coordinates, such that these 1's are either the set $\{2, 3\}$ or are a superset of $\{2, 3, \bullet\}$. Θ_8 and Θ_9 are defined in the same way as Θ_7 , replacing 2, 3 by 4, 6 and 1, 5 respectively.

Adding tails to translates: The remaining codes in this section are found by adding tails to translates of the Nordstrom–Robinson and Golay codes. We denote by B_i^w the set of vectors of weight w in a translate of either of these codes by a vector of weight t . Thus $|B_i^w|$ is given by the entry in Table IX or X in the column headed w and in the row in which the first nonzero entry occurs in column t .

The following codes in Table I are obtained from the Nordstrom–Robinson code \mathcal{N} .

$A(20, 6, 6) \geq 232 = 112 + 6 \times 20$ from $B_0^6\{0000\}$, $(B_4^4)^i\{1100\}$, $0 \leq i \leq 5$, where $(B_4^4)^i$ ($0 \leq i \leq 5$) represents the vectors of weight 4 in six translates of the type $\alpha_i + \mathcal{N}$ described in the last row of Table X, and $\{1100\}$ denotes all six binary 4-tuples of weight 2.

$A(20, 6, 7) \geq 462 = 30 + (112 + 96 + 80 + 64) + 4 \times 20$ from $\mathcal{R}^{(7)}\{0000\}$, $\mathcal{S}_i^{(6)}\{1000\}$ ($1 \leq i \leq 4$) and $(B_4^4)^j\{0111\}$ ($0 \leq j \leq 3$), where $\mathcal{R}^{(7)}$ is obtained by complementing the final 1 in each of the 30 weight-8 words in \mathcal{R} , and $\mathcal{S}_i^{(6)}$ is defined above.

$A(20, 6, 8) \geq 588 = 120 + (112 + 96 + 80 + 64 + 48 + 48) + 20$ from $(B_4^8)^0\{0000\}$, $\mathcal{S}_j^{(6)}\{1100\}$, $(B_4^4)^0\{1111\}$, $1 \leq j \leq 6$.

$A(20, 6, 9) \geq 832 = 4 \times 120 + (112 + 96 + 80 + 64)$ from $(B_4^8)^i\{1000\}$, $\mathcal{S}_j^{(6)}\{0111\}$, $0 \leq i \leq 3$, $1 \leq j \leq 4$.

$A(20, 6, 10) \geq 944 = 112 + 6 \times 120 + 112$ from $B_0^{10}\{0000\}$, $(B_4^8)^i\{1100\}$, $B_0^6\{1111\}$, $0 \leq i \leq 5$.

$A(21, 6, 10) \geq 1382\ddagger = 112 + 7 \times 120 + 30 + (112 + 96 + 80 + 64 + 48)$ from $B_0^{10}\{0000\}$, $(B_4^8)^i\{1100\}$, $0 \leq i \leq 6$, $B_0^8\{00011\}$, $\mathcal{S}_j^{(6)}\{01111\}$, $1 \leq j \leq 5$.

$A(22, 6, 8) \geq 1116\ddagger = 6 \times 90 + 42 + 42 + 7 \times 33 + 7 \times 33 + 6 \times 5$ from $(B_3^7)^i\{100000\}$, $B_1^5\{111000\}$, $B_1^5\{000111\}$, $(B_3^5)^j\{t_i\}$, $(B_3^5)^j\{t_j\}$, $(B_3^3)^i\{011111\}$, $0 \leq i \leq 5$, $0 \leq j \leq 6$, where $\{t_0, \dots, t_6\} = \{110001, 110010, 110100, 101001, 101010, 101100, 011001\}$, and $(B_3^7)^i$ ($0 \leq i \leq 5$) represents the vectors of weight 7 in six cosets of the type described in the penultimate row of Table X, the cosets being chosen to have Hamming distance 4 apart. This code contains exactly 21 holes, all of which may be adjoined, yielding $A(22, 6, 8) \geq 1137\ddagger$. Further optimization by the methods described in Section XII gives $A(22, 6, 8) \geq 1139$ (see Table XVI).

$A(22, 6, 9) \geq 1736\ddagger = 6 \times 120 + 112 + 112 + 7 \times 48 + 7 \times 48 + 6 \times 20$ from $(B_4^8)^i\{100000\}$, $B_0^6\{111000\}$, $B_0^6\{000111\}$, $(B_4^6)^j\{t_i\}$, $(B_4^4)^j\{t_j\}$, $(B_4^4)^i\{011111\}$, $0 \leq i \leq 5$, $0 \leq j \leq 6$. This can be improved to $A(22, 6, 9) \geq 1768\ddagger$ by first adding 28 holes in lexicographic order, then replacing the eight words FE3, 1DDA, 2A815A, 26419A, 30C523, 3300DA, 3304A3, 3C01A3 by the twelve words 208F23, 210CE3, 2403E3, 24199A, 26605A, 28155A, 2AA09A, 300CDA, 30E063, 33211A, 332223, 3C2823. By shortening this code we obtain $A(21, 6, 9) \geq 1092\ddagger$.

$A(22, 6, 10) \geq 2180\ddagger = 6 \times 90 + 85 + 85 + 7 \times 90 + 7 \times 90 + (42 + 36 + 4 \times 33)$ from $(B_3^9)^i\{100000\}$, $B_1^7\{111000\}$, $B_1^7\{000111\}$, $(B_3^7)^j\{t_i\}$, $(B_3^7)^j\{t_j\}$, $\mathcal{D}_k^{(5)}\{111110\}$, $0 \leq i \leq 5$, $0 \leq j \leq 6$, $1 \leq k \leq 6$, where $\mathcal{D}_k^{(5)}$ is defined above.

$A(22, 6, 11) \geq 2636 = 2 \times (448 + 30 + 7 \times 120)$ from $\mathcal{S}_i^{(6)}\{011111\}$ ($1 \leq i \leq 6$), $B_0^8\{111000\}$, $(B_4^8)^j\{t_i\}$ ($0 \leq j \leq 7$), and their complements.

$A(23, 6, 5) \geq 147 = 7 \times 20 + 7$ from $(B_4^4)^i\{1000000\}$, $00000000\{1000000\}\{1110100\}$, $0 \leq i \leq 6$.

The following codes are similarly obtained from the $[23, 12, 7]$ Golay code:

$$A(26, 8, 11) \geq 1858\ddagger \quad (B_3^8\{111\}, B_3^9\{110\}, B_3^{10}\{100\}, B_3^{11}\{000\}),$$

$$A(27, 8, 11) \geq 2047\ddagger \quad (B_0^7\{1111\}, B_0^8\{1110\}, B_0^{11}\{0000\}),$$

$$A(27, 8, 12) \geq 3082\ddagger \quad (B_0^8\{1111\}, B_0^{11}\{1000\}, B_0^{12}\{0000\}).$$

The final set of codes in this section come from the $[24, 12, 8]$ Golay code \mathcal{G} . Now B_i^w denotes the vectors of weight w in a translate of \mathcal{G} by a vector of weight t (see [46], [132], p. 69).

$A(25, 8, 9) \geq 829$ is obtained from the vectors $(B_0^8)^i\{1\}$, $(B_0^{12})^j\{0\}$, where $(B_0^8)^i$ consists of the $759 - 210 = 549$ words of weight 8 in \mathcal{G} not ending 000, and $(B_0^{12})^j$ consists of the 280 words of weight 12 ending 111 with these three 1's complemented (Kaikkonen [105]).

$A(25, 8, 10) \geq 1232\ddagger = 960 + 272$ is obtained from a translate of \mathcal{G} containing 360 words of weight 8 (denoted by B_4^8) and 960 words of weight 10 (denoted by B_4^{10}). We first take the 960 words $B_4^{10}\{0\}$. Any vector $u1$, where u is obtained by complementing any 0 in a vector of B_4^8 , is at distance 8 from the initial 960 words; there are $360 \times 16 = 5760$ such vectors, and we must find a subset of them at Hamming distance 8 apart. We could take the 240 out of the 360 that have a 0 in a particular coordinate and complement that coordinate, obtaining $A(25, 8, 10) \geq$

960 + 240 = 1200‡. However we can do better. Consider the subset of the 360 words with at most one 1 in a particular set of three coordinates, and in these coordinates replace 000 by 100, 100 by 110, 010 by 011 and 001 by 101. Consider first a random set of 3 out of the 24 coordinates. The probability that a vector containing 8 1's and 16 0's has at most a single 1 in three coordinates is

$$\frac{\binom{16}{3} + \binom{8}{1}\binom{16}{2}}{\binom{24}{3}} = 0.7510 \dots$$

so at least $360 \times 0.7510 \dots = 270.36 \dots$ (hence 271) words can be added in this way. A particular choice of three coordinates gives 272. Thus $A(25, 8, 10) \geq 960 + 272 = 1232‡$. By computer search it was found that 288 words can be added, yielding $A(25, 8, 10) \geq 960 + 288 = 1248$.

$A(25, 8, 11) \geq 1662 = 1218 + 444$ is similarly obtained from a translate of \mathcal{S} containing 640 words (B_3^9) of weight 9 and 1218 words (B_3^{11}) of weight 11. We first take the 1218 words $B_3^{11}0$, and look for a subset of the $640 \times 15 = 9600$ vectors $u1$ that can be adjoined, where u is obtained by complementing any 0 in a vector of B_3^9 . If we take the 400 out of the 640 with a 0 in a particular coordinate we get $A(25, 8, 11) \geq 1218 + 400 = 1618‡$. Again we can do better by using a subset of the 640 that have at most a single 1 in a particular set of three coordinates. By averaging we find that at least $640 \times 0.6917 \dots = 442.68 \dots$ (hence 443) words can be added in this way. A particular choice of three coordinates gives 444, so $A(25, 8, 11) \geq 1218 + 444 = 1662$.

$A(26, 8, 10) \geq 1519 = 759 + 760$ is found by starting with the 759 words $B_0^8 11$, and looking for a subset of the vectors $u00$ to adjoin, where u is obtained by complementing any two 1's in a word of B_0^{12} . We take the $120 + 4 \times 160 = 760$ words of B_0^{12} that have at most a single 0 in a set of four coordinates (see Fig. 2.15 of [132]), and replace 1111 by 0011, 0111 by 0001, 1011 by 1000, 1101 by 0100 and 1110 by 0010.

$A(26, 8, 12) \geq 3026‡ = 2576 + 450$ is obtained in a similar way from $B_0^{12}00$ and $B_0^8 11$. There are $130 + 4 \times 80 = 450$ words in B_0^8 with at most a single 1 in a set of four coordinates (see Fig. 2.14 of [132]), and we now apply the complementary transformation to the previous one (replacing 0000 by 1100, etc.). This construction can be improved as follows. We use the lexicographic version of \mathcal{S} , so that the octad $11 \dots 100 \dots 0 \in \mathcal{S}$. There are 256 octads in \mathcal{S} of the form $\{1000\}\{1000\}y$ (with each 1 in any of four positions). For each of these we form the vector $(1100)(1100)y11$, and adjoin these vectors to $B_0^{12}00$. This extends by "minimal degree lexicography" (see Section XII) to give $A(26, 8, 12) \geq 3070$.

$A(26, 8, 13) \geq 3328 = 2576 + 752$ is found by starting with the 2576 words $B_0^{12}01$. There are 35420 vectors of weight 13 at distance 8 from this set; they have the form $u10$, where u is the union of three words of weight 8 in \mathcal{S} all at mutual distance 8 (see [45], Fig. 10.1, [46]). By com-

puter it was found that 752 of these vectors can be adjoined to the 2576.

$A(27, 8, 12) \geq 3146‡ = 2576 + 210 + 3 \times 120$ is obtained from $B_0^{12}000$ and $B_0^8 111$, modifying the vectors in B_0^8 that have at most a single 1 in a set of three coordinates.

X. CODES FROM PERMUTATION GROUPS

The codes in this section are unions of orbits under a nontrivial permutation group. Let G be a permutation group permuting the symbols $\{1, \dots, n\}$. The orbit of a vector $x = (x_1, \dots, x_n)$ under G is the set of all vectors $x^g = (x_{g(1)}, \dots, x_{g(n)})$, $g \in G$.

We first discuss groups generated by a single permutation π .

If π is a cycle of length n (equal to the length of the code), the code is a *cyclic code*, indicated by "c" in Table I. Orbit representatives are listed in Table XI.

If π is a cycle of length $n - 1$, the code is an *extended cyclic code*, or "cyclic with a fixed point," indicated by "ec" in Table I. Orbit representatives are listed in Table XII.

If the permutation consists of a number of cycles of equal length the code is *quasi-cyclic* (see Table XIII). If there are i cycles of length n/i the code is indicated by " qi " in Table I.

The remaining codes defined by a single permutation ("polycyclic" codes) are listed in Table XIV, and indicated by "pc" in Table I.

The final table in this section (Table XV) lists codes that are defined by a group G (of order g) having more than one generator. These *group codes* are indicated by "g" in Table I.

The first column in Table XV gives the parameters of the code and [in brackets] the abstract type of G . The notation $q:r$ indicates that G is isomorphic to a group of permutations of \mathbb{F}_q of the form $x \rightarrow ax + b$, where a belongs to the multiplicative subgroup of \mathbb{F}_q^* of order r , and $b \in \mathbb{F}_q$. (The colon indicates a semidirect product as in [41].) $Q(16)$ is a generalized quaternion group ([97], p. 91).

The final column of Table XV gives orbit representatives for the code, written in hexadecimal and right justified, with the orbit size as superscript. For example the first orbit representative for $A(12, 4, 6) = 132$ is BE^{55} , indicating that the vector

11	10	9	8	7	6	5	4	3	2	1
0	0	0	1	0	1	1	1	1	1	0

defines an orbit of size 55. The coordinates are numbered from right to left.

Of course the full automorphism group of a code constructed in this section may be much larger than the group we use to construct it. For example, the first code in Table XV has as automorphism group the Mathieu group M_{12} of order 95040.

TABLE XI
CYCLIC CODES

Bound	Other Representatives (in Hexadecimal)
$A(22,9) \geq 68$	0, C984F, 11BDB5, 284347, 3FFFFFF (Ref. [105])
$A(25,10) \geq 151$	0, 33947, BC5D3, 1492D5, 23EEBF, 2D3ED3, 358D99
$A(8,4,3) = 8$	B
$A(13,4,3) = 26$	13, 85
$A(18,4,4) = 198$	17, 63, D1, 129, 303, 419, 445, 885, A09, 1089, 1421
$A(19,4,3) = 57$	43, 89, 405
$A(25,4,3) = 100$	D, 841, 2201, 8101
$A(26,4,3) = 104$	D, 441, 4201, 8101
$A(11,6,5) = 11$	97
$A(12,6,5) = 12$	97
$A(13,6,4) = 13$	B1
$A(13,6,6) = 26$	1AB, 279
$A(14,6,4) = 14$	53
$A(14,6,6) = 42$	BB, 4C7, 52D
$A(15,6,4) = 15$	813
$A(19,6,5) \geq 76$	A7, 1503, 420B, 8449 (Ref. [108])
$A(20,6,5) \geq 84$	3043, 11111, 14025, 20017, 40883 (Ref. [108])
$A(21,6,5) \geq 105$	343, 1017, 21049, 28083, 40423 (Ref. [108])
$A(26,6,4) = 52$	20B, 10811
$A(27,6,4) = 54$	883, 4025
$A(15,8,7) = 15$	537
$A(16,8,7) = 16$	112F
$A(17,8,8) = 34$	B9D, 2DA3
$A(21,8,5) = 21$	985
$A(23,8,5) = 23$	410B
$A(24,8,5) = 24$	20B1
$A(26,8,6) = 130$	68B, 20139, 49015, 81843, 110A11
$A(19,10,9) = 19$	5793
$A(20,10,9) \geq 20$	1129F
$A(21,10,8) \geq 21$	1112F
$A(22,10,11) \geq 46$	12E6F, 3ED19, 155555
$A(24,10,7) \geq 24$	12E11
$A(24,10,9) \geq 56$	13A35, 84537, B0B0B
$A(24,10,10) \geq 72$	5348F, 85DC9, 88CB7
$A(24,10,12) \geq 96$	4BE2F, 519F7, 1A4EE5, 1DAC99
$A(25,10,10) \geq 100$	7B621, 8591F, 9AA4D, 151867
$A(25,10,11) \geq 125$	1D4B7, 42F37, B63A5, D954D, 1CF223
$A(26,10,10) \geq 130$	7B20B, 8165F, 1C7131, 235499, 654A49
$A(27,10,9) \geq 111$	19535, 85A2D, 1518E1, 923245, 1249249
$A(27,10,12) \geq 252$	5AE4F, 1322BF, 2A0EEB, 3A55C5, 43F195, 4668DB, 5AB943, 62DE51, 9493CD, 9C4E27
$A(23,12,11) = 23$	299AF
$A(24,12,10) = 24$	DE245
$A(24,12,11) \geq 24$	A65F1
$A(26,12,9) \geq 26$	289CB
$A(27,12,11) \geq 54$	2CC789, 42FA23
$A(28,12,12) \geq 84$	11D5E3, 532679, A17A1B
$A(28,14,12) \geq 28$	8C97C5
$A(28,14,13) \geq 28$	A2993F

TABLE XII
EXTENDED CYCLIC (OR "CYCLIC WITH A FIXED POINT") CODES

Bound	Orbit Representatives
$A(17,4,4) \geq 156$	(000000001011)0, (00000000110001)0, (00000001100001)0, (000000001100)1, (00000010001010)0, (00000000101000)0, (00000100010100)0, (00000000101000)1, (00001001001000)0, (00010010000100)0, (00010001000100)0
$A(17,8,7) = 24$	(111000010001010)0, (1011000010110000)1
$A(22,10,10) \geq 42$	(10010010001110101000)1, (101000111001101110000)0
$A(25,10,7) \geq 28$	(0000000010010001)0, (0001000100010001000)1
$A(25,10,8) \geq 48$	(100010001010100000110100)0, (0000000001100100101000)1
$A(25,10,9) \geq 72$	(000001010111010100)0, (00000100011001000011011)0, (00000001101001100100)0
$A(25,10,12) \geq 130$	(000001010101111000)1, (0000010011001110101101)0, (00001000011110101000)1, (000100101100111010100)1, (0000011101010001000111)1, (00100111001001110001)0, (01010101010101010101)0
$A(27,10,11) \geq 208\ddagger$	(000010001101010100100)1, (0000001111001100001001101)0, (000100110010101010001010)0, (000001001110010001110000)1, (0000100011010100101101)0, (000000100011101001110011)0, (0000001000000001011011)1, (00000001001000111101010)0
$A(28,10,10) \geq 192\ddagger$	(000000100100001011100100)1, (000000010100111010000110)0, (00000101100101001000100)0, (0000000010000111101011000)0, (000000010110000100111000)1, (000000011001001001101)0, (0000011010110001000010101)0, (00100100100100100100100)1, (00010001010000111000101)0, (00000000010110100011100)1, (0000010001010001010011101)0, (00010010100101001000101)1, (00001001000110001011010)0, (00000101001110010011000)0, (000001010100001000110101)1, (00000100011100100001111000)0, (000000110000101110100100)1, (00000010010100001011101)0
$A(25,12,11) \geq 36$	(001101000000110101011)0, (0001001011100001001011)0
$A(27,12,10) \geq 39$	(00100000101110010000101)0, (00000000101000100111001)1

A number of different computer programs were used to find the group-invariant codes described in this section. The following seems to be the most efficient method. Given a permutation group G , we first find its orbits on the 0-element subset of the coordinates (there is only one!). Given representatives for the i -element subsets, we extend these in all possible ways by a singleton and find among the vectors thus obtained the (lexicographically minimal) representatives for the orbits on $(i+1)$ -subsets. At the same time this tells us how often each type of i -subset occurs in an $(i+1)$ -subset of given type. This process is continued until representatives for the i -element subsets with $i \leq w$ have been found.

For $t = w - d/2 + 1$, we form a matrix B indexed by orbits of w -sets and t -sets, specifying how often a t -set is covered by the vectors in a given orbit of w -sets. Orbits of w -sets for which the corresponding row of B contains an entry greater than 1 can be discarded.

We now define a graph on the remaining w -set orbits, joining two of them when they do not cover the same t -set, i.e., when the corresponding rows are orthogonal. The largest codes invariant under G are obtained as the largest weighted cliques in this graph, where the weights are the orbit sizes. This method requires enough space to store the $T_w \times T_t$ (0,1)-matrix B , where T_i is the number of orbits on i -sets.

TABLE XIII
CASE-CYCLIC CODES

Bound	Orbit Representatives
$A(16,8,6) = 16$	(00110101)(00011000), (00011000)(01010011)
$A(18,8,6) = 21$	(110100)(100000)(110000), (000010)(110100)(100001), (000011)(100001)(010100), (010101)(010101)(000000), (000000)(000000)(111111)
$A(18,10,8) = 9$	(00010011)(00010110)
$A(20,10,7) = 10$	(0000101011)(00001001)
$A(26,10,6) = 13$	(000000011001)(0000010000)
$A(27,10,6) = 14$	(010)(000)(000)(001)(010)(101)(000)(000), (010)(000)(000)(100)(000)(100)(000)(101)(010), (000)(010)(100)(000)(000)(001)(001)(000)(011), (000)(001)(100)(000)(110)(010)(000)(001)(000), (111)(111)(000)(000)(000)(000)(000)(000)(000), (000)(000)(111)(111)(000)(000)(000)(000)(000) (Ref. [105])
$A(27,10,7) \geq 36$	(00001011)(00000100)(01000100), (00010001)(001010000)(000110000), (00000001)(001000001)(100001110), (000001001)(01100100)(00000000)
$A(28,10,7) \geq 37$	(000001)(000001)(100001)(100000), (000001)(010000)(0011000)(0100100), (000000)(000001)(0010101)(0001101), (0001001)(1001000)(000001)(0011000), (0001001)(0100101)(0000110)(000000), (111111)(000000)(000000)(000000), (000000)(000000)(111111)(000000)
$A(25,12,8) = 10$	(1100)(1100)(10100)(10100)(00000), (00100)(00001)(11000)(00100)(11010)
$A(26,12,11) \geq 39$	(000100111011)(001101010000), (000110001100)(001100010101), (0011011001000)(00001110010)
$A(27,12,8) = 15$	(001)(000)(010)(000)(000)(001)(010)(101)(101), (100)(010)(010)(001)(000)(110)(110)(000)(000), (011)(000)(010)(001)(101)(000)(001)(000)(010), (001)(011)(100)(101)(000)(000)(000)(010)(100), (000)(011)(000)(000)(011)(001)(100)(100)(010) (Ref. [105])
$A(26,14,12) = 13$	(000100100111)(000100101011)
$A(27,14,12) \geq 19$	(000)(000)(111)(111)(000)(000)(000)111, (110)(110)(100)(110)(100)(000)(100)(110)100, (011)(011)(100)(001)(110)(110)(000)(100)010, (011)(110)(011)(100)(100)(100)(011)(000)001, (100)(100)(000)(011)(100)(010)(011)(011)011, (010)(100)(001)(000)(011)(110)(100)(011)011, (010)(001)(101)(100)(000)(011)(011)(010)110
$A(28,14,11) = 21$	(100000)(0011101)(011000)1, (101000), (101000)(100000)(100110)(110100), (010110)(010011)(100000)(1101000)

XI. MISCELLANEOUS CONSTRUCTIONS

In this section we give some isolated constructions that do not fit into any other category.

The group code \mathcal{C} showing that $A(16,4,8) \geq 1164\ddagger$ (see Table XV) leads to three other good codes.

$A(16,4,8) \geq 1170$. In \mathcal{C} , replace the ten words FF, AF5, 11EE, 24DB, 7D82, BE41, C03F, CF30, F30C, FF00 by the sixteen words 1EF, 2F7, 4DF, 8FD, 10FE, 20FB, 07F, 7F02, 80BF, BF01, DF10, EF20, F704, FB08, FD80, E40.

$A(14,4,6) \geq 276\ddagger$. Shorten \mathcal{C} by taking the 275 words with a 1 in the first (i.e., left-most) and tenth coordinates, and adjoin 58B.

$A(14,4,7) \geq 317\ddagger$. Shorten \mathcal{C} by taking the 314 words beginning 10, and adjoin BF, B4B, 3F80.

$A(22,6,4) = 37$. Take a Kirkman triple system of order 15 ([14], [30]), i.e., a Steiner system $S(2,3,15)$ in which the 35 blocks are partitioned into seven "parallel classes"

TABLE XIV
OTHER CODES GENERATED BY A SINGLE PERMUTATION,
AND RELATED CODES

$A(11,4,4) = 35$	(001000)(011)(10), (110001)(100)(00), (101000)(100)(01), (000011)(100)(01), (00100)(011)(00), (011011)(000)(00), (100100)(000)(11), (101010)(000)(10)	$A(26,10,9) \geq 84$	(0000000100001001111011)(01), (0000000100010111000011)(10), (00010010110001101000101)(00), (00001000110100001000110)(01)
$A(17,8,6) = 17$	(11101000000)(100)(10), (10010010010)(011)(00), (101010101010)(000)(00)	$A(26,10,11) \geq 168$	(00000110100001011101001)(10), (000100110100100110010111)(00), (00000110001010111100101)(00), (0000011000011111)(01), (000010010101110001)(01), (00000011001010100101011)(11), (00000101010000110011101)(10)
$A(21,10,9) \geq 27$	(1001)(0011)(1010)(0100)00101, (0001)(0101)(0011)(0011)00011, (1010)(0010)(0011)(0011)01100, (0110)(0011)(0001)(0101)10001, (0011)(0110)(1000)(0101)01010, (0010)(0101)(0110)(1001)11000, (1111)(0000)(111)(0000)00010, (00)(00)(00)(1)(00)(00)(1)(1)10110, (00)(1)(00)(00)(11)(1)(00)(00)10110 (Ref. [105])	$A(27,10,8) \geq 66$	(00101010111)(1000000001)(000), (00000001111)(000101000100)(010), (00000100101)(10001001010)(100), (000010001001)(100010000001)(110), (000000010001)(10001110100)(000), (000011000011)(00100001000)(011)
$A(22,10,7) = 16$	(11100)(10010)(10000)(10000)00, (10000)(10000)(00110)(01001)10, (10000)(10000)(01001)(00110)01, (00000)(11111)(00000)(00000)11	$A(27,10,10) \geq 159$	(00000010000100111011101)(000), (00000101001101010010101)(100), (00001000100100000011111)(001), (0000010101000100010010111)(001), (000001101000101100100011)(001), (00000000000000010011)(011), (00100100100100100101)(110)
$A(22,10,9) \geq 35$	(000011)(0011000)(1001010), (0001001)(0100001)(1111100)0, (000001)(0110011)(010001)1, (0001101)(010001)1, (0001101)(001011)(100000)1, (0011101)(1100000)(001001)0	$A(28,10,8) \geq 78$	(0000000001010001101100001)01, (000000001000100100111)10, (0000001001000011010101)00
$A(23,10,8) \geq 33$	(111000000)(0010100010)1, (10100001000)(1010010001)1, (11100010010)(0000011001)0 (Ref. [105])	$A(28,10,9) \geq 132$	(00000000100001100100111)(0101), (000010010100011001001)(0100), (00001001010101000001)(0001), (000000001011000010111)(1100), (000001111010000110001)(0000), (00000101100100001011001)(0010)
$A(23,10,9) \geq 45$	(111101001000100010000)10, (111010001100001000010)01, (100100100100100100)11 (Ref. [80])	$A(28,10,9) \geq 132$	(00000000100001100100111)(0101), (000010010100011001001)(0100), (00001001010101000001)(0001), (000000001011000010111)(1100), (000001111010000110001)(0000), (00000101100100001011001)(0010)
$A(23,10,10) \geq 54$	(0000001011)(0001010011)111, (0011011101)(0101100001)000, (0001101001)(0000111101)00, (0001001011)(111000100)010, (0000111001)(0110010110)001, (0101010101)(1010101010)000, (0000000000)(1111111111)000, then adjoin 3DE003	$A(22,12,9) = 8$	(11010000)(11010000)(1100)(10)
$A(23,10,11) \geq 63$	(0000010101001101111)00, (00001111010110001001)01, (00010100111100100011)10	$A(25,12,9) = 25$	(00000101101)(001100)(1100)(10)0, (000011000011)(100100)(010)(00)1, (001100110011)(010101)(0000)(00)0, (010101010101)(000000)(0000)(11)1, (00000000000)(111111)(0000)(11)1
$A(24,10,8) \geq 38$	00000000(11111111)(00000000), 00000000(00000000)(11111111), 111(1000)0(10001000)(10001000), 100(1000)00(110100)0100011, plus first 4 cyclic shifts right of each 010(00110)(10001000)(01000011), 010(01001)(10001000)(00110100), 001(00110)(0100011)(10001000), 001(01001)(00110100)(10001000), 000(00110)(00110100)(00110100), 000(01001)(01000011)(01000011)	$A(25,12,9) = 25$	(000000010101)(100001)(001)1, (000111000111)(100100)(1010)(00)0, (001100110011)(101010)(0000)(00)1, (001001001001)(110110)(0000)(11)0, (010101010101)(000000)(0101)(11)0, (00000000000)(111111)(1111)(00)0
$A(24,10,11) \geq 90$	(0010101100110100110101)(000), (0000100110001010011101)(11), (000001110101100001111)(10), (000000101011101100001)(10), (01010101010101010101)(00)	$A(26,12,10) \geq 30$	(11110001001000101000110)00, every 4th shift of (01100000000010000111010)11 1100000000(100)(100)(010)(001)(001)(010), 001100000(100)(010)(100)(001)(001), 0000110000(100)(001)(010)(100)(010), 0000110000(100)(001)(010)(100)(010), 0000001100(100)(001)(010)(100)(100), 000000011(100)(010)(001)(001)(010)(100), 1010101010(111)(000)(000)(000)(000), 1001010101(000)(111)(000)(000)(000)(000), 0110100101(000)(000)(111)(000)(000)(000), 0101101010(000)(000)(000)(111)(000)(000) (Ref. [80])
$A(26,10,8) \geq 54$	(000011101001)(000100000001)10, (000000000011)(100100101011)00, (000010000101)(000100110100)01, (000001001101)(100011000010)01, (000101000101)011000011000)00	$A(28,12,11) \geq 63$	(00100100100100100100101)(11)0, (000010111001000010111001)(100)0, (00001010010011101010001)(100)0, (00000001000100011010011)(011)1, (000101100101001110011001)(01)0, (00000011001101010101111)(000)1, (000011000111101000100101)(110)1, (00000101111100011101001)(001)0

each containing five disjoint blocks. Add one further point “at infinity” to each parallel class, yielding 35 words of length 22 and weight 4, and adjoin $A(7,6,4) = 2$ words on the 7 extra points.

$A(22,6,5) \geq 132$. In the weight 5 words of the [11,6,5] ternary Golay code, replace 0 by 00, 1 by 01 and 2 by 10.

$A(28,8,5) = 33$. Start with the affine plane $AG(2,5)$, containing 30 5-sets (the lines) on 25 points, and adjoin three points X_1, X_2, X_3 . Choose three noncollinear points P_1, P_2, P_3 , and in the line P_1P_2 replace P_1 by X_3 , in the line P_2P_3 replace P_2 by X_3 , in the line P_3P_1 replace P_3 by X_3 , and adjoin the 5-set $P_1P_2P_3X_1X_2$. Repeat this with three further noncollinear points Q_1, Q_2, Q_3 (replacing Q_1, Q_2, Q_3 in the three lines by X_2 and adjoining $Q_1Q_2Q_3X_1X_3$), and again with three noncollinear points R_1, R_2, R_3 , making sure that $\{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$, $\{P_1, P_2, P_3, R_1, R_2, R_3\}$ and $\{Q_1, Q_2, Q_3, R_1, R_2, R_3\}$ are conics in the affine plane. The final code is shown in Fig. 1.

$A(20,10,8) = 17$ is constructed in Fig. 2.

$A(28,10,6) = 16$ follows by shortening the Steiner system $S(2,6,31) = PG(2,5)$.

Finally Kaikkonen [105] observed that if n is even and $d' = \min\{n, 2d\}$ then

$$A(2n, d', n) \geq A(n, d) + A\left(n, d, \frac{n}{2}\right).$$

This is obtained by replacing 0 by 01 and 1 by 10 in the code attaining $A(n, d)$, and 0 by 00 and 1 by 11 in the code attaining $A(n, d, n/2)$. For example $A(28, 12, 14) \geq A(14, 6) + A(14, 6, 7) = 106$. Many generalizations are possible, for example using ternary codes, but do not seem to lead to new records in the range of our tables.

XII. SEARCHING FOR CODES WITH A COMPUTER; CODES WITH NO KNOWN STRUCTURE

In preparing Table I we made use of several computer programs that searched for codes. Two kinds of programs were used, exhaustive search methods and heuristic (non-exhaustive) methods. In discussing running times, besides the usual variables n, d, w and M (the number of

codewords), we use U to denote $\binom{n}{w}$, the size of the universe of possible codewords.

Exhaustive search methods: We explain our exhaustive search technique by describing the proof that $A(14, 6, 7) = 42$. A code with 42 words was constructed in Section III, so it suffices to show that no code exists with 43 words. In principle we must consider all possible subsets of 43 words from a universe of size $U = \binom{14}{7} = 3432$.

However, the size of this search space may be greatly reduced. First, from (5), any code attaining $A(14, 6, 7) = 43$ must contain a subcode C' with $n = 13$, $d = 6$, $w = 7$, $M = 22$, and C' must contain a subcode C'' with $n = 12$, $d = 6$, $w = 7$, $M = 11$. A previous exhaustive search has determined that there are precisely 95 inequivalent choices for C'' . We may now restrict our search to codes that contain one of these codes as the first 11 words.

Next, we need not consider every subset of the weight-7 14-bit vectors as a possible code. Only sets of vectors with all pairwise distances ≥ 6 need be considered. All such sets may be generated by a standard backtracking algorithm, indeed in lexicographic order.

The search space may be further reduced by noticing that any M -word code C has $M!n!/|\text{Aut}(C)|$ isomorphic versions (obtained by permuting the n coordinates and the M codewords). We wish our search to find exactly one (or at any rate very few) of the codes in each such equivalence class. The $M!$ factor is avoided by requiring that any code generated must be in lexicographic order—a condition readily incorporated into the backtracking algorithm. A large part of the $n!$ factor is automatically removed by the fact that the first 11 words form one of the subcodes C'' previously mentioned, and the first 22 words form a subcode C' .

More generally we may require all generated codes to be in “canonical form”: namely lexicographically least under any permutation of coordinates and corresponding resorting of codewords, while preserving the property that the Johnson subcode $A(n-1, d, w)$ lies in the first $n-1$ coordinates and constitutes the first $\lfloor (n-w)M/n \rfloor$ words (and so on recursively for the subcodes of this code, ...).

Proving that a code is canonical is difficult, but some simple tests can readily show that a code is not canonical. If any departure from canonical form does occur during the backtracking, we may immediately prune that branch of the search.

The combination of all of these ideas made this initially intractable search problem solvable (in less than 18 minutes on an IBM 3090 model S computer, including the time to generate the 95 inequivalent codes C''), using a program written in a combination of Fortran and IBM-370 assembly language. The search tree contained about 90 million nodes, 309704 of them being codes C' with $n = 13$, $d = 6$, $w = 7$, $M = 22$ (we did not attempt to sort these into equivalence classes). Other exact values of $A(n, d, w)$ found in this way are given in Table III.

Heuristic search methods: For problems too large to be attacked by exhaustive search, we must be content with heuristic algorithms. Perhaps the most straightforward

heuristic is simply to run the exhaustive search described above, or a variant of it, performing an incomplete search, and keep the best code found.

Often we are given a good partial code (obtained for example by shortening another code), and wish to complete it. Several heuristic methods are available. The lexicographically least code containing some given “seed” subcode is readily found in $O(UM)$ steps. A similar, but more powerful code-extension heuristic is “minimal-degree lexicography.” Consider the H -vertex graph formed by the H holes in a partial code (two vertices being joined by an edge if the corresponding vectors differ in less than d places). Remove the lexicographically least vertex of minimal degree from this graph (as well as all its neighbors), and place it in the code; continue doing this (using the sequence of successively smaller graphs that arise) until no further augmentation is possible. This procedure takes $O(UM + H^2)$ time.

A simple probabilistic variant of these two methods uses “coin tossing.” Namely, each time the extension heuristic could add a new vector to the current code, we toss a (possibly biased) coin, and add the vector only if the coin toss comes up “heads.” Otherwise (“tails”) we discard the vector and continue. Alternatively, deterministic skip-selection methods (such as a systematic backtrack search, or a greedy procedure) may be used.

Another kind of approximate optimization is based on “local search.” We say that a code is “ k -optimal” if its cardinality cannot be increased (while maintaining the minimal distance) by deleting k words and adding holes. Thus lexico-codes (and other codes found by a greedy algorithm) are 0-optimal, while codes attaining $A(n, d, w)$ are k -optimal for all k . If k is small then an M -word code C may be tested for k -optimality (and if not k -optimal an improvement found) in $O(UM + \binom{M-1}{k-1}kT)$ time, where T is the size of the following bipartite graph.

This graph, which is constructed at the beginning of the search, has two sets of vertices, a blue vertex for every codeword of C and a red vertex for every vector not in C that “bites” (lies at distance less than d from) k or fewer words of C ; an edge joins each such vector to the codewords that it “bites.” Once this graph is constructed, no further hole-finding need be done.

We now consider all possible k -subsets S of C (the blue vertices), and see if their deletion will increase the size of C , using a trivial exhaustive search among the red vertices that are connected only to S .

The following “polishing” procedure was often able to improve even these k -optimal codes. We begin by constructing the bipartite graph described above, maintaining the red vertices (those not in the code) in a queue. If there is a red vertex h of degree 1 we exchange h with the codeword c to which it is connected, adding h to the code in place of c , which is placed at the end of the queue. When there are several choices for h we choose the one closest to the front of the queue, i.e., which has been out of the code longest. This procedure is then repeated. If after some fixed number (e.g., 1000) of itera-

TABLE XV
CODES DEFINED BY GROUPS WITH MORE THAN ONE GENERATOR, AND RELATED CODES

Bound [Group]	g	Generating Permutations	Orbit Representatives
$A(12,4,6) = 132$ [11:5]	55	(1,2,3,...,11), (2,5,6,10,4)(3,9,11,8,7)	$5F^{55}, 83D^{55}, ED^{11}, 897^{11}$
$A(14,4,4) = 91$ [7:6]	42	(1,2,3,4,5,6,7)(8,9,10,11,12,13,14), (1,14)(2,12,5,13,3,10)(4,8,6,11,7,9)	$186^{21}, 198^{21}, 1E0^{21},$ $17^{14}, D1^{14}$
$A(14,4,5) \geq 169$ [13:12]	156	(1,2,3,...,13), (1,2,4,8,3,6,12,11,9,5,10,7)	$5D^{78}, 2053^{52}, 6B^{39}$
$A(14,4,7) \geq 316\ddagger$ [7:6]	42	(1,2,3,4,5,6,7)(8,9,10,11,12,13,14), (1,3,2,6,4,5)(8,10,9,13,11,12)	$19F^{42}, 3B9^{42}, 5D5^{42}, 793^{42},$ $7A5^{42}, BAC^{42}, FB0^{42}, 5CB^{14},$ $FE^7, 3F80^1$
$A(15,4,5) \geq 234\ddagger$	42	same group as above	$19A^{42}, 1A6^{42}, 418C^{42}, 1D4^{21},$ $385^{21}, 7A0^{21}, 4099^{21}, 4382^{21},$
$A(15,4,6) \geq 382\ddagger$	42	same group as above then adjoin vectors 1F, 67, 79	$18F^{42}, 399^{42}, 3AA^{42}, 5C6^{42}, 783^{42},$ $4394^{42}, 4585^{42}, 40BC^{21}, 4193^{21},$ $43C8^{21}, 5B4^{14},$
$A(16,4,6) \geq 592\ddagger$ [2 ⁴ :5]	80	same group as above then adjoin 3F, FA0, 1790, 1B88, 1D84, 1E82, 1F40, 2F01, 6780, 7980, 7E00	$BB^{80}, 1F1^{80}, 2BC^{80}, 297^{80},$ $30F^{80}, 94E^{80}, 6E^{80}, 378^{80},$ $2CE^{16}, 365^{16}$
$A(16,4,8) \geq 1164\ddagger$ [PSL(2,7)]	168	(1,2,3,4,5,6,7)(9,10,11,12,13,14,15), (2,3,5)(4,7,6)(10,11,13)(12,15,14), (1,8)(2,7)(3,4)(5,6)(9,16)(10,15)(11,12)(13,14)	$75D^{168}, 76B^{168}, F78^{168}, 1F23^{168},$ $1F2C^{168}, 173A^{84}, 737^{56}, 1F15^{56},$ $F0F^{42}, 3FC^{28}, 3FCO^{28}, 17E8^{14},$ $1DE2^{14}, FF00^1, FF^1$
$A(18,4,5) \geq 516\ddagger$ [3 ² :8]	72	(1,5,9)(2,3,8)(4,7,6)(10,14,18)(11,12,17)(13,16,15), (1,2,3,4,5,6,7,8)(10,11,12,13,14,15,16,17) then adjoin 643, C86, 1A0C, 3418, 6814, C068, 14281, 1A090, 22111, 24422, 28844, 31088	$21B^{72}, 64C^{72}, 6A4^{72}, E14^{72},$ $E81^{72}, F02^{72}, 1E40^{72},$
$A(20,4,6) \geq 2280$ [PSL(2,19)]	3420	(1,2,3,...,19), (2,5,17,8,10,18,12,7,6)(3,9,14,15,19,16,4,13,11), (1,20)(2,19)(3,10)(4,7)(5,15)(6,16)(8,9)(11,18)(12,13)(14,17)	$5F^{710}, F3^{570}$
$A(20,4,10) \geq 13452$	3420	same group as above	$BDF^{3420}, FBD^{3420}, DFB^{1710},$ $EF7^{1710}, 12FF^{1710}, 157F^{1140}, F6I^{342}$

tions the code has not been improved, the vector at the front of the queue is added to the code and its neighbors are removed and placed at the end of the queue (temporarily decreasing the size of the code).

In the range of our tables we are able to achieve k -optimality for values of k ranging 2-5. A less conservative attack would allow k -alterations for much larger values of k , considering only a small fraction of possible k -sets, without trying to achieve k -optimality. One such heuristic code-improver is the following.

- 1) Perform a permutation on the coordinates of the code.

- 2) Perturb every codeword by "pushing" it until it is lexicographically as small as possible.
- 3) Sort the (permuted and pushed) codewords into lexicographic order.
- 4) Remove the (lexicographically) last k words from the code, and attempt to replace them by more than k words, using some exhaustive or heuristic search method.
- 5) Go back to Step 1) (and repeat as many times as desired).

In this procedure one can use very large values of k , e.g., 20% of the codewords. This is one of a class of possible

TABLE XV (Continued)

Bound [Group]	g	Generating Permutations	Orbit Representatives
$A(24,4,6) = 7084$ [PSL(2,23)]	6072	(1,2,3,...,23), (2,3,5,9,17,10,19,14,4,7,13)(6,11,21,18,12,23,22,20,16,8,15), (1,24)(2,23)(3,12)(4,16)(5,18)(6,10)(7,20)(8,14). (9,21)(11,17)(13,22)(15,19)	$6F^{3036}$, $1CD^{3036}$, 197^{1012}
$A(24,4,8) \geq 34914$	6072	same group as above	$5BD^{6072}$, $67D^{3036}$, $72F^{3036}$, $9DB^{3036}$, ADD^{3036} , $B67^{3036}$, $EE5^{3036}$, $F39^{3036}$, $FA3^{3036}$, $12BD^{3036}$, $149F^{759}$, $351B^{759}$
$A(28,4,6) \geq 15288$ [PSL(2,13)]	1092	(1,2,5,3,10,6,12,4,9,11,8,7,13). (15,16,19,17,24,20,26,18,23,25,22,21,27), (1,3,5,7,9,11)(2,4,6,8,10,12)(15,17,19,21,23,25). (16,18,20,22,24,26), (1,7)(2,6)(3,5)(8,12)(9,11)(13,14)(15,21)(16,20). (17,19)(22,26)(23,25)(27,28)	$C01C^{1092}$, $C09A^{1092}$, $C0A6^{1092}$, $1C0B0^{1092}$, $2C112^{1092}$, $2C700^{1092}$, $3C041^{1092}$, $3C208^{1092}$, 4076^{546} , $412E^{546}$, $1C128^{546}$, $1C888^{546}$, $2C034^{546}$, $2C260^{546}$, $BC100^{546}$, $DC040^{546}$, $C033^{273}$, $C303^{273}$, $C330^{273}$, $AC00A^{273}$, $AC021^{273}$, $AD080^{273}$, $1C1C0^{182}$, $2E420^{182}$, DB^{91} , $6C000^{91}$
$A(17,6,5) = 68$ [PSL(2,16)]	4080	(1,6,13,5,4,2,15,10,14,12,3,9,7,11,8), (1,16)(2,3)(4,5)(6,7)(8,9)(10,11)(12,13)(14,15), (2,3)(4,9)(5,7)(6,8)(10,14)(11,13)(12,15)(16,17)	CE^{68}
$A(17,6,8) \geq 184$ [Q(16)]	16	(1,2,...,8)(9,10,...,16), (1,9,5,13)(2,16,6,12)(3,15,7,11)(4,14,8,10)	$195B^{16}$, $1C37^{16}$, $2D0F^{16}$, 5267^{16} , 6857^{16} , $8A4F^{16}$, $C41F^{16}$, 10337^{16} , $1492B^{16}$, 18547^{16} , $1A02F^{16}$, 6633^4 , $1i^2$, 5555^2
$A(18,6,8) \geq 248\ddagger$ [2^4.2]	32	(1,2)(3,4)(5,6)...(15,16), (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13,15)(14,16), (1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16), (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16), (3,4)(7,8)(9,13)(10,14)(11,16)(12,15)(17,18)	11593^{32} , $11A6C^{32}$, 31472^{32} , $147D^{16}$, $172B^{16}$, $17D4^{16}$, $1D47^{16}$, $1E1E^{16}$, $555A^{16}$, $3111E^{16}$, 31247^{16} , $11EE^8$, $12B7^8$, $333C^8$
$A(18,6,9) \geq 304$	384	(2,3)(6,7)(9,13)(10,15)(11,14)(12,16), (2,11,12)(3,16,14)(4,6,7)(5,13,9)(8,10,15), (2,5,3,8)(6,7)(9,16,15,11)(10,12,13,14)(17,18), (1,2)(3,4)...(17,18)	$103F3^{96}$, $11B1B^{96}$, $10F69^{48}$, $1FCA^{32}$, $307AC^{32}$
$A(19,6,9) \geq 504$ [2^4.S ₃]	96	the generators for $A(18,6,8) \geq 248$ together with (2,3,4)(6,7,8)(10,11,12)(14,15,16)(17,18,19)	$3162E^{96}$, $103CF^{48}$, $11B27^{48}$, $11E4B^{48}$, 13369^{48} , 13535^{48} , $135CA^{48}$, 70356^{48} , $10F55^{24}$, $111BB^{24}$, $114D7^{24}$

optimizers that work on the principle of “cut a hole in the code, then refill it.” The additional step of “pushing” codewords in order to artificially “expand” the hole can further increase the power of the heuristic.

The codes labeled “y,” “ya,” or “yd” in Table I were obtained by one or more of the previous methods. Those with fewer than 1500 words are listed in full in Table

XVI. Most of these codes are at least 2-optimal. They are described in compressed notation, obtained as follows. The words are first sorted into lexicographic order, yielding a sequence of words c_1, c_2, \dots, c_M (say). The compressed notation for this code is $(\alpha_1, \alpha_2, \dots, \alpha_M)$. To find α_i ($1 \leq i \leq M$), let u_1, u_2, \dots be the list of all vectors, arranged in lexicographic order and following c_{i-1} in the

TABLE XV (Continued)

Bound [Group]	g	Generating Permutations	Orbit Representatives
$A(22,6,6) \geq 319$ [11:5]	55	(1,2,3,...,11)(12,13,14,...,22), (2,5,6,10,4)(3,9,11,8,7)(13,16,17,21,15)(14,20,22,19,18)	1925 ⁵⁵ , 1B50 ⁵⁵ , 5868 ⁵⁵ , 7881 ⁵⁵ , 15D00 ⁵⁵ , A3A ¹¹ , DC4 ¹¹ , 4B820 ¹¹ , 74804 ¹¹
$A(23,6,8) \geq 1265\ddagger$ [23:22]	506	(1,2,3,...,23), (2,3,5,9,17,10,19,14,4,7,13)(12,23,22,20,16,8,15,6,11,21,18)	A6F ²³ , 2FC1 ²³ , 4B99 ²³ , 61E9 ¹¹ , B4A3 ²³
$A(25,6,9) \geq 4100\ddagger$ [5 ² : 2 A ₄]	600	(1,2,...,5)(6,...,10)(11,...,15)(16,...,20)(21,...,25), (2,6,25)(3,11,19)(4,16,13)(5,21,7)(8,10,20)(9,15,14)(12,24,22)(17,18,23), (2,11,20)(3,21,9)(4,6,23)(5,16,12)(7,8,18)(10,13,15)(14,25,24)(17,22,19)	8CAF ⁶⁰⁰ , A47B ⁶⁰⁰ , EBC2 ⁶⁰⁰ , 2CF6 ³⁰⁰ , 9637 ²⁰⁰ , 9A9B ²⁰⁰ , 9D4B ²⁰⁰ , C531 ²⁰⁰ , C99E ²⁰⁰ , CE4D ²⁰⁰ , D4C7 ²⁰⁰ , D86E ²⁰⁰ , 1A8B3 ²⁰⁰ , 2B869 ²⁰⁰ A73E ⁶⁰⁰ , B375 ⁶⁰⁰ , D2E7 ⁶⁰⁰ , 9E21 ⁶⁰⁰ , D733 ⁶⁰⁰ , 19B96 ⁶⁰⁰ , D7F ⁶⁰⁰ , 11153D ⁶⁰⁰ , 1D5A6 ³⁰⁰ , 1EAB2 ²⁰⁰ , 11A176 ²⁰⁰ , 2B969 ²⁰⁰ B1E1 ⁶⁰⁰ , BE7A ⁶⁰⁰ , CF1F ⁶⁰⁰ , 19D97 ⁶⁰⁰ , 1A5BE ⁶⁰⁰ , 1C6AF ⁶⁰⁰ , 1D772 ⁶⁰⁰ , 10B45F ⁶⁰⁰ , 119F2C ⁶⁰⁰ , 12274F ⁶⁰⁰ , 12CCB9 ⁶⁰⁰ , 1086FB ³⁰⁰ , 113979 ³⁰⁰
$A(25,6,10) \geq 5700\ddagger$	600	same group as above	122F ³²⁷⁶ , 4F9 ¹⁴⁰⁴
$A(25,6,11) \geq 7200\ddagger$	600	same group as above	
$A(28,6,7) = 4680$ [PSL(2,27)]	9828	(1,14,27)(2,10,4)(3,22,13)(5,19,8)(6,18,21)(7,12,11). (9,16,26)(15,17,23)(20,24,25), (1,3,5,...,25)(2,4,6,...,26), (1,14)(2,13)(3,12)(4,11)(5,10)(6,9)(7,8)(15,26)(16,25). (17,24)(18,23)(19,22)(20,21)(27,28)	
$A(28,12,10) \geq 48$ [3 ²]	9	(1,2,3)(4,5,6)...(16,17,18), (4,5,6)(10,11,12)(13,15,14)(16,18,17)(19,21,20)(22,24,23)	90C264C ⁹ , 911291A ⁹ , 9128A25 ⁹ , 9249143 ⁹ , E001C0F ¹ , 1F8000F ¹ , 007007F ¹ , 000E38F ¹ , FC00388 ¹ , 038FC08 ¹ , E07E004 ¹ , 0001FF4 ¹ , 03F0382 ¹ , 1C0E072 ¹ , E380071 ¹ , 1C71C01 ¹

lexicographic order, that have distance $\geq d$ from the subcode $\{c_1, \dots, c_{i-1}\}$. If $c_i = u_r$ we set $\alpha_i = r - 1$. Informally, given c_1, \dots, c_{i-1} , we must skip α_i lexicographic words to get c_i . Koschnick [111], who independently discovered this compressed notation, refers to $(\alpha_1, \dots, \alpha_M)$ as a *skip-vector* for the code.

For example the skip-vector for a lexicographic code itself is simply $0, 0, \dots, 0$ (M times), which we abbreviate to 0^M .

The algorithm for decompressing this notation is equally simple. To recover a code $\{c_1, \dots, c_M\}$, showing that $A(n, d, w) \geq M$, first form the sequence $\alpha_1, \dots, \alpha_M$ by expanding each symbol a^k to a, a, \dots, a (k times). Then c_i ($1 \leq i \leq M$) is the $(\alpha_i + 1)$ st vector c in lexicographic order starting at c_{i-1} such that $wt(c) = w$ and the distance from c to $\{c_1, \dots, c_{i-1}\}$ is at least d .

In a sense the codes in Table XVI are our failures. At least one of the authors (NJAS) believes that every value

of $A(n, d, w)$ in the range of our tables should be attained by a code with some mathematical structure. Experience has shown that sooner or later most random codes in this range are superseded. We hope this will happen to the codes in Table XVI.

ERRATA IN EARLIER WORKS

In [13], Table II-D, $A(16, 10, 7) = A(16, 10, 9) = 4$ (not 3). In Table III-A, $T(1, 2, 7, 16, 10) = 8$ (not ≤ 6), $T(1, 3, 7, 16, 10) = 12$ (not ≤ 9), $T(1, 4, 7, 16, 10) = 16$ (not ≤ 12), $T(1, 5, 7, 16, 10) = 16$ (not ≤ 15). In Table III-D, $T(2, 4, 7, 16, 10) \geq 19$ (not ≤ 18). We do not at present know how these errors affect the upper bounds in [13] (nor papers such as [180] that make use of these bounds). Until further checks are made, all the upper bounds in [13] for codes with $d = 10$ obtained by linear programming should be regarded with suspicion. We have also been

01111	00000	00000	00000	00000	00100
00000	11111	00000	00000	00000	00000
00000	00000	11011	00000	00000	10000
00000	00000	00000	11111	00000	00000
00000	00000	00000	00000	10111	01000
10000	00000	10000	10000	10000	00100
01000	01000	01000	01000	01000	00000
00100	00100	00100	00100	00100	00000
00010	00010	00000	00010	00010	10000
00001	00001	00001	00001	00001	00000
10000	01000	00100	00010	00001	00000
01000	00100	00010	00001	10000	00000
00100	00010	00001	10000	01000	00000
00010	00001	10000	01000	00100	00000
00001	10000	01000	00100	00010	00000
10000	00100	00001	01000	00010	00000
01000	00010	10000	00100	00001	00000
00100	00001	01000	00010	10000	00000
00010	10000	00100	00001	01000	00000
00001	01000	00010	10000	00100	00000
10000	00010	01000	00001	00100	00000
01000	00001	00100	10000	00010	00000
00100	10000	00010	01000	00001	00000
00010	01000	00001	00100	10000	00000
00001	00000	10000	00010	01000	01000
10000	00001	00010	00100	01000	00000
00000	10000	00001	00010	00100	00001
00100	01000	10000	00001	00010	00000
00010	00100	01000	10000	00000	01000
00001	00000	00100	01000	10000	10000
11000	10000	00000	00000	00000	11000
00000	00100	00000	00000	01001	10100
00000	00010	00110	00000	00000	01100

Fig. 1. $A(28, 8, 5) = 33$.

unable to recover the “miscellaneous construction” of [13] that produced $A(18, 6, 6) \geq 144$, and in the present paper have replaced it by $A(18, 6, 6) \geq 132$. In [13], p. 89, col. 2, the words “equivalent to determining $D(t, k, v)$, where” are illegible in some copies. In (5), line 5, change 197 to 297.

In [44, (5)], the range is $a' < a, b' < b$. On page 348 the last line of Table XIII should read $A(24, 10, 12) \geq 80$.

In [45], p. 141, caption to Fig. 5.1, the length of code is 11 (not 10).

In [50], p. 12, line 21, $|\Gamma_{12}| = 112952$ (not 11_{10}).

In [72], p. 40, $T(1, 6, 6, 15, 10) = 8$ (not 7). On p. 40, column 1, line 3, change 554 to 553. In Table IV, $A(16, 10, 7) = A(16, 10, 9) = 4$ (not 3).

In [147], $A_3^2 = 56$ (not 99).

In [150], Table 2, $A(19, 4, 5) \geq 644$ (not 664).

ACKNOWLEDGMENT

We thank several correspondents who kindly sent us codes they had discovered. Since some codes were found independently by several people, and others have been transformed or “beautified” (and so do not appear in the final table in their original form) it is appropriate to record these constructions here. To keep this list to manageable size we mention only codes that are as good as the current record, were unpublished at the time of writing, and were found since the publication of [13].

1000	1000	1100	0101	1001
1000	0100	0011	1010	1001
1000	0010	0011	0101	0110
1000	0001	1100	1010	0110
0100	0110	1000	1100	0101
0100	0110	0100	0011	1010
0100	1001	0010	0011	0101
0100	1001	0001	1100	1010
0010	1010	0110	1000	1100
0010	0101	0110	0100	0011
0010	1010	1001	0010	0011
0010	0101	1001	0001	1100
0001	0011	1010	0110	1000
0001	1100	0101	0110	0100
0001	1100	1010	1001	0010
0001	0011	0101	1001	0001
1111	1111	0000	0000	0000

Fig. 2. $A(20, 10, 8) = 17$.

T. Etzion and C. L. M. van Pul [68] showed $A(17, 4, 6) \geq 854$,

H. Hämäläinen [80] showed $A(18, 6, 9) \geq 304$, $A(19, 6, 9) \geq 504$, $A(23, 10, 9) \geq 45$, $A(25, 10, 8) \geq 48$, $A(26, 14, 11) = 10$, $A(27, 14, 11) = 13$, $A(28, 12, 8) \geq 19$, $A(28, 12, 10) \geq 48$.

I. Honkala [91] showed $A(22, 12, 9) = 8$, $A(25, 12, 8) = 10$. I. Honkala [93] showed $A(26, 14, 9) = 6$, $A(26, 14, 10) = 8$, $A(28, 14, 9) = 7$.

M. K. Kaikkonen [105] showed $A(22, 9) \geq 68$, $A(25, 8, 9) \geq 829$, $A(21, 10, 9) \geq 27$, $A(23, 10, 8) \geq 33$, $A(24, 10, 9) \geq 56$, $A(25, 10, 6) \geq 10$, $A(25, 10, 8) \geq 48$, $A(26, 10, 6) \geq 13$, $A(27, 10, 6) \geq 14$, $A(25, 12, 8) \geq 10$, $A(28, 12, 14) \geq 106$, $A(26, 14, 10) = 8$, $A(26, 14, 11) = 10$, $A(27, 14, 11) = 13$, $A(28, 14, 9) = 7$, $A(28, 14, 11) \geq 21$,

C. L. M. van Pul [149] showed $A(17, 4, 5) \geq 424$, $A(17, 4, 6) \geq 854$, $A(19, 6, 7) \geq 338$, $A(19, 6, 8) \geq 408$, $A(21, 6, 7) \geq 570$, $A(17, 8, 8) = 34$, $A(18, 8, 6) = 21$, $A(22, 10, 7) \geq 16$, $A(22, 10, 8) \geq 24$. C. L. M. van Pul [150] showed $A(17, 6, 7) \geq 166$, $A(18, 6, 7) \geq 243$, and

S. Rankinen [151] showed $A(20, 6, 8) \geq 588$.

After this paper was submitted we received a preprint by K.-U. Koschnick [111], which independently showed $A(12, 4, 5) \geq 80$, $A(18, 8, 6) = 21$, $A(22, 10, 7) \geq 16$, and improved one of our entries by showing $A(23, 10, 7) \geq 20$. A code with the latter parameters and equivalent to Koschnick’s is given in Table XVI.

Other codes were found by Chen, Jin, and Fan [34], Cheswick [35], Chung and Kumar [37], Darwish and Bose [49], Dueck and Scheuer [63], Lin [123], Zaptcioglu [184] and Zinoviev and Litsyn [188]. We thank all of these correspondents.

David Johnson kindly allowed us to use his simulated annealing graph coloring programs [98]. We also thank John Conway for many helpful discussions. Iiro Honkala, Heikki Hämäläinen and Markku Kaikkonen for informing us of a large number of codes that they had found, and Tuvi Etzion and Kevin Phelps for several helpful comments. Tuvi Etzion also helped us find some of the partitions in Table VI (cf. [65]). Aaron Grosky and Ralph Knag provided valuable assistance in running our programs at Bell Labs.

The automorphism groups of certain codes in this paper were computed using B. D. McKay’s graph-automor-

TABLE XVI
CODES WITH NO KNOWN STRUCTURE, DESCRIBED BY SKIP-VECTORS

A(12,4,5) ≥ 80
 $0^3, 1^7, 3, 1, 0, 3, 2, 0, 2^3, 4, 0, 2, 0^2, 1, 0^9, 1, 0^6, 1, 0, 2, 0, 1, 0^{10}, 1, 0^5, 1^2, 0^2, 1, 0, 1, 0^{16}$

A(13,4,5) ≥ 123
 $0^2, 2, 1, 1, 0, 1^2, 4, 2, 0, 1, 0, 6, 1, 4, 3, 0^2, 4, 0, 1, 0^2, 2, 4, 0, 4, 0, 1, 0^7, 1, 0^7, 2, 0^2, 2, 3, 0, 1, 0^3, 4, 0, 2, 0^5, 1, 0^6, 1, 0^5, 1^2, 0^9, 1, 0^3, 1, 0^6, 1, 0^{25}$

A(13,4,6) ≥ 166
 $0^3, 1^2, 3^2, 1, 0^2, 1, 0, 1, 2, 1, 2, 1, 0, 1, 0^4, 2, 3, 0^3, 1, 3, 0^2, 1, 0^2, 3, 1, 0^4, 1, 0^5, 1^2, 0, 1^2, 0, 2, 0, 1, 0^{12}, 1, 0^{12}, 1, 0^{13}, 1, 0, 1, 0^2, 3, 1, 0^2, 1, 0^3, 1, 0^{16}, 1, 0^8, 1, 0^2, 1, 3, 2, 0^9, 1, 0^{16}$

A(14,4,6) ≥ 278
 $0^6, 1, 3, 0, 5, 0^3, 1, 2, 1^2, 0^6, 2, 3, 2, 0, 1, 0, 3, 0^5, 1, 0^{12}, 4, 0^4, 1, 0^{10}, 2, 0^7, 4, 1, 0^{20}, 1^2, 2, 0^3, 1^2, 0, 1^2, 0^{13}, 2, 0, 1^2, 0^3, 2, 0^{10}, 1, 0^6, 1, 0^{23}, 1, 0^5, 1, 0^6, 1, 0^3, 2, 0^2, 2, 0, 1, 0^7, 1, 0^2, 2, 0^7, 1, 0, 1, 0^7, 1, 0^{20}, 1, 0^8, 1, 0^2, 1, 0, 3, 0, 1, 0^{22}$

A(14,4,7) ≥ 325
 $0^5, 6, 7, 0, 1, 0^2, 4, 7, 2, 3, 0, 4, 2, 1, 0^5, 3, 0^2, 1, 0^2, 2, 0, 3, 1, 0, 1^2, 2, 3, 0^4, 1^2, 0^6, 1, 0^2, 1, 0^4, 1^2, 0, 1, 0^2, 1, 0^8, 1^2, 5, 0, 4, 0, 1, 4, 3, 0, 1, 0, 1, 0, 2, 0^3, 4, 0^2, 1, 0, 2, 1, 0, 1, 0^{20}, 1^2, 0^5, 1, 0^{31}, 3, 0, 6, 1, 0^5, 1, 0^5, 1, 0^5, 2, 0^3, 1^2, 0^4, 1, 0^{11}, 1^2, 0^4, 1, 0^{40}, 1, 0^3, 1^2, 0^{69}$

A(15,4,10) ≥ 237 (has a shortening to **A(14,4,10) = 91**)
 $0^7, 1, 0^3, 3, 2^3, 0, 1, 0^2, 1^2, 0, 1, 0^3, 2, 0^4, 5, 4, 0, 2, 0^4, 1, 0^6, 1, 2^2, 1, 0, 2, 0^5, 1, 0^{34}, 3^2, 1, 0^2, 1^2, 2, 0, 1^2, 0, 3, 2, 0, 5, 1^2, 0^2, 1, 0^4, 1, 0^4, 2, 0^5, 1, 0, 3, 0, 2^2, 0^4, 1, 0^7, 1^4, 2, 0, 1, 0^3, 1^2, 0^2, 1, 0, 2, 0^2, 2, 0, 1, 0^6, 1, 0^5, 2, 0, 1, 0^2, 1, 0, 1, 0^2, 1^2, 0^5, 1, 0^4, 4, 1, 0^7, 1, 0^{25}$

A(15,4,6) ≥ 389
 $0^2, 11, 3, 4^2, 9, 5, 1, 0, 5, 0, 2, 1^2, 2^2, 0, 1, 4, 0, 3, 2, 0^4, 2, 3, 1, 4, 0, 1, 0, 2, 0, 2, 11, 0, 3, 5, 2, 0^3, 2, 1^2, 3, 0^3, 2, 1, 0^2, 1, 0^3, 1, 0, 1, 2, 0^2, 2, 1, 0^3, 1, 4, 0, 4, 1^2, 5, 2, 0^3, 9, 0, 2, 0, 1, 0^3, 6, 0, 3, 0^6, 4, 0, 1, 0^3, 1^2, 2, 0, 1, 0^3, 1, 0^5, 1^2, 0^5, 1, 0, 1, 0, 1, 0^3, 4, 5, 1, 4, 3, 0^4, 2, 0^4, 2, 0^2, 1, 0, 1^2, 0, 1, 3, 0^2, 1, 0, 1, 0^2, 1^3, 0^3, 1, 4, 0, 1, 0^2, 1^2, 0^{11}, 1, 0^3, 3, 0, 1^2, 0, 1, 0, 1, 0^{16}, 1, 0^{16}, 2, 0, 1, 2, 0^2, 1, 0, 2, 1, 0^2, 1^2, 2, 1, 0^2, 1, 0^4, 1, 0, 1, 0^5, 1, 0, 1^2, 0, 1, 0^5, 7, 0^2, 1, 0, 1, 0^{12}, 1, 0^{24}, 1, 0^{17}, 1, 0^4, 1, 0^{14}, 1, 0^{30}$

A(16,4,11) ≥ 312
 $0^{105}, 2, 3, 4, 1, 0, 1, 4, 3, 2, 0, 4, 1, 0, 3, 0^4, 1, 0^3, 2^2, 0, 5, 1, 3, 0, 1, 0, 3, 0^2, 1, 0^3, 2, 1, 4, 0^9, 1, 0^{15}, 1, 0^{10}, 1, 3, 0, 2^2, 0, 1, 0, 1^2, 0^2, 1, 0, 5, 0, 1^2, 0^2, 1, 0^3, 1^2, 0, 1, 0^2, 1, 0^8, 1, 0, 1, 0^2, 1, 2, 1, 0, 3, 0^2, 2, 0^5, 1, 2, 0^5, 1, 0^2, 1, 0^{10}, 1^2, 3, 0^3, 1^2, 0^{15}, 1, 0^{15}, 1, 0^5, 1, 0^6$

A(16,4,6) ≥ 615
 $0^3, 2^2, 4, 9, 2, 14, 0, 2, 0, 2, 0, 5, 2, 7, 1^2, 0, 1, 5, 0, 2, 1, 4, 1, 4^2, 2^2, 0^4, 1, 5, 1^2, 0^4, 2, 0, 2, 0, 5, 0, 1, 2, 0, 2, 5, 3, 1^2, 0^4, 1, 0, 1, 0^3, 1^3, 3, 4, 0, 1, 0, 6, 3, 0^3, 4, 2, 1, 0^5, 1, 0^2, 3, 1^2, 0^2, 1^3, 0, 2, 0^6, 1, 0, 1^3, 0^4, 1, 0, 1^2, 0^3, 1^2, 0^6, 1, 0, 4, 0^2, 1, 0, 2, 1, 0^2, 3, 0, 5^2, 1, 7, 0^3, 1, 0^3, 1, 0^2, 2, 0, 1, 0^2, 1, 0^3, 1, 0, 1, 2^2, 0, 2, 1, 0, 2, 0^6, 1, 0^4, 1, 0, 1, 0^7, 1, 2, 0^{20}, 1, 0^4, 3, 0^5, 1, 3, 0, 1^2, 0, 5, 0, 1^2, 0^2, 1, 0, 4, 2, 0^2, 1^2, 0^3, 4, 0^2, 1^2, 0^7, 1, 0^4, 1, 0^2, 1^2, 0, 1, 0^7, 2, 4, 1, 0, 1^2, 0, 1, 0, 2, 0^6, 1, 0, 2, 0^6, 1, 0, 2, 0^6, 1, 0^{17}, 1, 0^3, 1, 0^{19}, 1, 0^{18}, 1, 0^4, 1^2, 0^3, 1, 0^6, 1, 0^{20}, 1, 0^7, 1, 0^2, 1, 0^4, 1, 0, 1, 0^6, 1, 0^6, 2, 0, 1, 0^{11}, 1, 0^2, 1, 0^5, 1, 0^3, 1, 0^4, 1, 0^7, 1, 0^{28}, 1^2, 0^5, 1, 0^6, 1, 0^{10}, 1, 0^{17}$

A(18,4,13) ≥ 518
 $0^7, 10, 18, 7, 3, 6, 4, 1, 4, 2^2, 3, 4, 0, 2, 3, 4, 1, 2, 1^2, 0, 2, 1, 0^3, 1, 0^2, 4, 15, 4, 2^2, 0, 2, 4^2, 3, 0^2, 3, 2, 1, 3, 6, 1^2, 2, 1, 4, 1, 0^2, 1^2, 21, 0, 5, 3, 2, 1, 2, 3^3, 2, 0^6, 4, 3, 0^5, 6, 0^2, 2, 1^2, 3^2, 0, 1, 3, 0^5, 1, 5, 1, 3, 0, 1, 0^3, 2, 0^4, 1^3, 0^5, 1, 0^2, 1, 2, 0^2, 1, 0^3, 1, 0^{10}, 32, 9, 0, 2, 5, 2, 0, 7, 1, 0, 4, 1^3, 0^2, 3, 0^3, 5, 0^4, 1^2, 6, 3, 0, 1, 0, 1, 2^2, 1^2, 3, 1, 3, 0, 2, 1^3, 4, 0, 1^2, 0^2, 1^3, 3, 1, 0, 4, 1, 0, 2, 0^4, 1, 0, 3, 7, 3, 0^2, 3, 0, 1, 0, 2, 0^3, 4, 1, 0^2, 1^2, 0^2, 3, 5, 0^5, 1, 0^4, 1, 0^8, 3, 6, 2, 4, 0, 1, 4, 1, 2, 0^2, 2, 0, 1^3, 0, 1, 0^2, 1, 0, 4, 0^4, 1, 2, 0^3, 4, 0^5, 3^3, 0, 2, 0^6, 1, 0^2, 1, 0^8, 1^2, 0^7, 4, 0^4, 2, 1, 0^5, 2, 0^2, 3, 0, 1^2, 0, 4, 0^3, 2, 1^2, 0^5, 1, 2, 0, 4, 0, 3, 0^5, 2, 0, 3, 1^2, 2, 0^3, 3, 0^7, 1^2, 0^{11}, 1, 6, 0, 4, 1^2, 0^2, 1, 0^2, 2, 0, 1, 0, 1, 0, 1, 0^5, 5, 0^{20}, 1, 2, 0^{16}, 1, 0^5, 1, 0^6, 1, 0^2, 1, 0^6, 1, 0^{10}, 1, 2, 0^{20}$

A(20,4,5) ≥ 874
 $0^2, 13, 5, 8^2, 5, 18, 21, 1, 3, 0, 1, 5, 0, 15, 2, 7, 0, 8, 0, 2, 1, 2, 1, 5, 2, 11, 15, 3, 5, 0, 9, 1, 7, 3, 6, 7, 4, 2, 0^2, 2, 1, 2, 1, 9, 5, 3, 6, 0, 3, 1, 3, 0, 11, 1, 4^2, 3, 0, 1, 0, 1, 4, 1^2, 0, 1^2, 0^4, 2, 1, 6, 0, 1, 15, 2, 1, 3, 0, 3, 1, 0^2, 7, 0, 1, 3, 1^2, 3, 1, 8, 2, 0, 2, 0, 2^2, 4, 3, 1^2, 0^2, 1, 0^2, 2, 0, 9, 2, 4, 1, 0, 3^2, 2, 0^3, 4, 0^2, 1, 3, 1, 0^2, 1, 2, 0, 1, 0, 4, 0^3, 2^3, 3^2, 0^2, 2, 0, 1, 6, 1, 0, 1, 0^2, 1, 0^2, 2, 0^3, 1, 0, 3, 2^2, 1, 0^2, 2^2, 0^2, 4, 1, 0^3, 2, 0, 4, 0, 4, 0, 2, 0^2, 1, 0, 1, 0^4, 4, 2, 0^3, 1, 0, 1, 0^5, 2, 0^3, 2, 0^{13}, 2, 0, 1, 0^5, 1, 0^5, 1, 0^4, 3, 0, 3, 0, 2, 0^4, 2, 3, 1, 0, 1, 2, 0, 3, 0^3, 1, 0, 1, 0^3, 1, 0, 1^2, 0^2, 1, 0^6, 1, 2, 0, 1, 0^2, 1, 0^{14}, 2, 0^3, 1, 0^4, 1, 0^4, 1, 0^7, 1, 0^2, 3, 0, 1, 0^{20}, 6, 2, 0^2, 2, 3, 0, 1, 2, 0^3, 4, 1, 0, 2, 1, 0^2, 3, 0^2, 1, 2^2, 0^{12}, 1, 0, 1, 0^8, 1, 0^5, 1, 0^4, 1, 0, 1, 0^2, 2, 0^7, 1, 0^9, 1^2, 0^8, 1,$

TABLE XVI (Continued)

$0^{17}, 2, 0^3, 1, 0^{27}, 2, 1, 0, 4, 0^3, 3, 0^3, 2, 1, 0^5, 1, 0, 1, 0, 1, 0^2, 1, 0^7, 1, 0^9, 1^2, 0, 1, 0^{11}, 1, 0, 2, 0^{19}, 1, 0, 1, 0^3, 1, 0^{26}, 1, 0^{34}, 1, 0^{29}, 2, 0^3, 2, 1, 0^{10}, 1, 0^7, 1, 2, 0^3, 1, 0^3, 1^2, 0^{10}, 1, 0^{17}, 1, 0^{21}, 1, 0^5, 1, 0^3, 1^2, 0^5, 1, 0^8, 1, 0^{16}, 1, 0^{21}, 1, 0^{39}$

A(18,6,8) ≥ 260

$0^2, 10^2, 13, 8, 9, 30, 8, 1^2, 0, 9, 10, 9, 6, 0, 2^2, 5, 4, 2, 0^4, 1, 0, 1, 0, 2, 1, 3, 2, 0^{26}, 2, 3, 1, 0, 1, 0^3, 1, 0^{31}, 31, 16, 7^2, 17, 24, 4, 1, 0, 4, 1, 10, 9, 2, 7, 0^2, 2, 0, 21, 0, 7, 2, 7, 3^2, 0, 3, 2, 3^2, 1, 32, 7, 14, 16, 0, 10, 2, 1, 8, 4^2, 10, 1^3, 4, 3, 4, 12, 1, 5, 2, 10, 0, 3^2, 1^2, 0^2, 7, 2, 0, 1, 0^4, 1, 1, 0^9, 1, 0^{59}$

A(21,6,8) ≥ 774

139,694,443,214,933,307,286,778,117,622,712,220,438,242,328,471,22,4,55,10,0,62,23,2,1,2,10,0,16,12,1,4,12,0,2^2,6,3,0^3,22,10,7,2,4,2,1,7,0,2,0,1,0^3,1,0^8,17,0,1^2,11,0^2,1,0^3,4,0,1,0^{24},86,7,54,12,1,0,1,30,0,19,2,0,2,22,0,1,10,8,7,5^2,2,3,0,2,1,0,1,9,5,0^2,2,0^2,1,0^{14},4,2,0^2,1,0^2,1,0^{16},1,0^{15},11,79,0,8,23,1,15,3,5,2,4^2,0^2,5,1,0^2,2,0^2,2,1,0^4,3,0^4,3,0^8,1,0^9,5,0,2,0^5,2,1^2,0^{29},6,2,16,3,1,10,6,1,5,3,0,4,0,4,1,0,3,1,0^2,2,1,0,1,0^3,1,0^3,3,0^6,3,0^4,1,0,13,5,0,1,0,9,1,0,5,3,0^4,3^2,1,4,0^2,1,0^7,7,0^{21},2,0^{15},1,0^2,1,0^{21},3,1,2,0,3,1,2,6,0^3,7,2,9,3,1,3,1,3,6,0,15,4,0,1,0,7,1,4,1^2,10,1,10,12,13,9,0,1,0,2,0,2,0^3,17,0^2,6,0^2,5,0,2,0^5,1,0^{17},1,3,5,1,0,4,2^2,0,1,0^9,1,0^2,1,0^2,1^2,0^5,1^2,0^4,1,0^{24},1,0^{25},10,7,20,3,11,2,1,5,0^4,1^2,0^3,2,1^2,0,1,0^8,14,15,0,1,8,1,0^2,1,2,0^4,2,0^6,1,0^2,1,0^3,1,0,4,0^2,1,0^9,1,0^{15},1^4,2,0^3,2,0^{25},2,0,1,0^{61}

A(21,6,9) ≥ 1184

11591,0,66,15,8,78,9,8,19,0^2,3,16,10,2,11,0,5,0,3,2,0^7,17,5,1^2,2,0,1,5,0,2,1,0^{21},4,1,0^2,1^3,0^9,1,0^{43},46,1,49,1,29,8,28,2,0^3,1,6,0,1,6,0^6,1,0^4,1,4,0^2,1,0^4,1,0^{27},1,0^{56},1,4,0,1,2,0^2,1,0,2,0^2,1,0,3,14,0,11,4,18,0,2^2,1,0^{111},16,1^2,0^3,6,0,1,2,0,3,0,1,0^2,5,6,12,4,6,10,2,0^2,8,2,0,1,4,3,0^2,1,0,4,0,2,0,1^2,0,19,0^5,1,2,0^2,1,0^{106},61,0,7,0,1^2,0,2^2,0^4,1^2,2,0^3,3,1,0,1,0^2,1,0^2,1^2,0,1,0,1,0,1,0^4,1,2,0^2,1,0^3,2^2,0^3,2^2,1,33,3,0,2,5,0^2,2,1^2,0^2,1,0^8,1,0^6,1,0^4,1,0,1,0,1,0^5,1^2,0^4,2,0^4,1^2,0,2,1,0^6,2,0^2,1,0^{12},1,0^3,2,0,1,0^{15},1,0^{31},1,0^{28},1^2,0,1,0^{19},1,0^2,1,0^2,2,0^7,1,0^{12},1,0,6,3,2^3,7,1,2,0,5,1,0^2,3,2,6,0,2,0,1,0^{10},1,0^3,1,6,1,0^8,1,0^7,1,0^4,3,0^2,1,0,1,0^2,1,0,1,0,2,6,3,0,1^2,0,1,2,0^5,2,4,1,0^7,2,0^6,1,0^{33},1,0^5,1,0,1,0^{11},1,0^{13},2,0,3,0,1,0^3,1,0^{37},1,0^{50}

A(21,6,10) ≥ 1454

$0^6, 6, 7, 2, 0, 4, 19, 10, 10, 5, 28, 8, 5, 3, 59, 3, 0, 1, 6^2, 0, 3, 1, 8, 0, 4, 2, 3, 0^2, 1, 0^4, 2, 0, 2^2, 0, 74, 0^2, 24, 3, 0, 3, 2, 7, 3, 5, 0^2, 6, 1, 0^5, 9, 0, 1, 0, 1, 0, 13, 5, 4, 7, 12, 2, 3, 4, 6, 0^{11}, 1, 0^7, 4, 2, 1^2, 0^4, 47, 2, 4^2, 0^2, 2, 0^5, 1, 0^2, 1, 0, 1, 0^2, 5, 0, 2, 0, 1, 0, 14, 4, 8, 7, 0, 3, 0^4, 3, 1, 0^6, 1, 0^9, 1, 2, 0^5, 5, 3, 1, 2, 0, 1^3, 0^{13}, 1, 0^5, 2, 0^7, 4, 1, 0^{17}, 1, 0^4, 1, 0^{12}, 50, 1, 2, 1, 0, 1, 0^7, 1, 0^7, 3, 0, 2, 0, 2, 1, 6, 0^3, 3, 0^9, 2, 0^4, 1, 0^5, 1, 2, 0^2, 1, 0^8, 1, 0^3, 1^2, 0^{20}, 1, 0^2, 1, 0, 1, 2, 0^{35}, 1^2, 0^{17}, 1, 0^{18}, 3, 0^{10}, 1, 0^5, 1, 0^{15}, 1, 0^{60}, 10, 35, 2, 5^2, 0, 8, 4, 0, 2, 4, 2, 0, 1, 4, 18, 3, 9, 2, 5, 1, 0^3, 1, 3, 2, 0, 2, 1, 0, 10, 13, 1, 4, 0^2, 2, 4, 0, 1, 0, 1, 0, 1, 6, 0, 2, 1, 3^2, 1, 3, 0, 1, 0, 2, 0^2, 4, 0^7, 10, 6, 0, 3^2, 5, 1, 0^2, 2, 1, 0^2, 1^2, 2, 0^2, 4, 0^4, 2, 3, 2, 0^4, 2, 1, 0^3, 1, 0^3, 1, 0, 1, 0^5, 1, 0, 2, 1, 0^{18}, 1^2, 5, 4, 7, 0, 1, 0^3, 3, 0, 1, 0^3, 1^2, 0, 1, 0, 1, 0, 3, 1, 0, 1, 0, 1, 0^2, 1, 0^{11}, 2, 0^3, 1, 3, 0^5, 1, 0, 1, 0^2, 1, 0^6, 1, 0^2, 1, 0^{15}, 1, 0^{10}, 1, 0^4, 2, 0^{10}, 1, 0, 1, 0^2, 1, 0^8, 8, 2, 0^2, 4, 0^2, 1^2, 3, 0^6, 2, 0, 1, 0, 3, 1^2, 3, 0, 4, 5, 0, 1^2, 0^2, 1, 0^{10}, 2, 0, 1^2, 2, 0^2, 3, 1, 0, 1^2, 2, 0^{13}, 1^2, 0^2, 1, 2, 0^7, 1, 0, 1, 0^2, 1^2, 0, 1, 0, 1, 2, 1, 0, 2^2, 0^{15}, 1, 0^2, 1, 2, 0^{45}, 1^2, 0^{19}, 1, 3, 0, 1, 0, 2^2, 0^{33}, 1, 0^{55}, 2, 1, 0^{14}, 1, 0^{49}, 4, 0, 1, 0, 1, 0^{24}, 1, 0^{29}, 1^2, 0^{39}, 1, 0^{42}, 1, 0^{53}, 1, 0^{137}$

A(22,6,8) ≥ 1139

139,694,443,214,933,307,286,778,117,622,712,220,438,242,328,471,22,4,55,10,0,62,23,2,1,2,10,0,16,12,1,4,12,0,2^2,6,3,0^3,22,10,7,2,4,2,1,7,0,2,0,1,0^3,1,0^8,17,0,1^2,11,0^2,1,0^3,4,0,1,0^5,1,0^{17},86,7,54,12,1,0,1,30,0,19,2,0,2,22,0,1,10,8,7,5^2,2,3,0,2,1,0,1,9,5,0^2,2,0^2,1,0^{14},4,2,0^2,1,0^2,1,0^{16},1,0^{15},99,0,8,23,1,15,3,5,2,4^2,0^2,5,1,0^2,2,0^2,2,1,0^4,3,0^4,3,0^8,1,0^9,5,0,2,0^5,2,1^2,0^{29},39,5,3,18,8,1,6,5,1,4,0,4,1,0,4,1,2,0,2,1,0,1,0^3,1,0^2,5,0^6,3,0,1,0^2,1,0,13,5,0,1,0,9,1,0,5,3,0^4,3^2,1,4,0^2,1,0^7,7,0^{21},2,0^{15},1,0^2,1,0^{20},57,0,11,1,3,1,0,1,0,1,3,0,1,0^{11},2,0^8,19,2,5,0^3,1,0^6,1,0^9,1,0^{10},1,2,1,0,1,0^2,1,0^5,1,0^3,1,2,0^2,1,0^6,1,3,5,1,0,4,2^2,0,1,0^9,1,0^2,1,0^2,1^2,0,1,0^3,1^2,0^4,1,0^{24},2,0^{25},49,2,0^2,1,0^{20},1,0^6,28,0^2,3,0^4,1,0^4,1^2,0,1,0^4,2,0,1,0^9,3,0^2,2,1,0^{13},1,0^{12},2,3,0^{27},1,0^3,9,1^2,7,3,0,1,4,0^4,1,0^2,1,0^3,3,0,3,1^3,0,1,0^3,1,0^2,1,0^5,1,0^5,1,0^2,1,0^2,1,0^{57},7,21,5^2,3,2,0,4,24,31,17,7,5,15,1,0,3,1,0^2,1,0^{28},23,0^2,2,3,1^2,0^3,1^2,0^4,1,0^{11},1,0^2,1,0^2,1,0^{31},2,0^{26},3,0^4,4,0^{98},1,0^{55}

A(23,6,8) ≥ 1436

139,694,443,214,933,307,286,778,117,622,712,220,438,242,328,471,22,4,55,10,0,62,23,2,1,2,10,0,16,12,1,4,12,0,2^2,6,3,0^3,22,10,7,2,4,2,1,7,0,2,0,1,0^3,1,0^8,17,0,1^2,11,0^2,1,0^3,4,0,1,0^5,1,0^{17},86,7,54,12,1,0,1,30,0,19,2,0,2,22,0,1,10,8,7,5^2,2,3,0,2,1,0,1,9,5,0^2,2,0^2,1,0^{14},4,2,0^2,1,0^2,1,0^{16},1,0^{15},99,0,8,23,1,15,3,5,2,4^2,0^2,5,1,0^2,2,0^2,2,1,0^4,8,0^3,5,2,0^2,3,0^4,1,0^9,5,0,3,0^5,2,1^2,0^4,1,0,1^2,0^{15},1,0^6,65,5,3,19,13,6^2,1,6,0,6,1,5,6,3,0,7,1^2,3,0^3,2,1,10,0^2,1^2,0^2,5,0,1,0^2,1,14,5,0,1,0,9,1,0,5,3,0^4,4,3,1,4,0^2,1,0^7,8,0^{21},3,0^{12},1,0^2,1,0^2,1,0^8,1,0^{10},15,3,43,20,11^2,5,3,7,3,0,4,0^3,4,1,2,1^3,0^2,1,0^4,10,18,5,6,2,0^2,2,0^2,1^6,0^5,1,0,1,0^7,1,2,1,0,1,0^2,1,0,1,0,1,0^2,1,0,2,0^3,1,0^7,1,3,5,1,0,4,2^2,0,1,0^9,1,0^2,1,0^2,1,3,0,1^2,

TABLE XVI (Continued)

$0^2, 1^2, 0^4, 1, 0^{24}, 2, 0^{25}, 52, 2, 1, 0, 1, 0^2, 2, 0, 1, 0^{12}, 3, 1, 0^2, 1, 35, 0^2, 3, 0^4, 1, 3, 0, 2, 1, 2, 0, 1, 0, 1, 2, 1, 0^5, 1^2, 0^2, 3, 2, 0, 3, 1, 2, 2, 0^4, 1, 0^2, 1^2, 0^{12}, 1, 0, 1, 0, 1, 3, 0^2, 1, 0, 1, 2, 1, 0, 1, 0^6, 2, 0^4, 6, 11, 4, 0^2, 6, 0^4, 1, 0^2, 1, 0^2, 3, 0^2, 2, 0^4, 1, 0, 1, 0^3, 1, 3, 0, 3, 0, 3, 1^3, 0, 1, 0^3, 1, 0^2, 1, 0^5, 1, 0^5, 2, 0^2, 1, 0^2, 1, 0^5, 1, 0^{10}, 2, 0, 1, 0^2, 2, 0^{15}, 1, 0^9, 2, 4, 2, 11, 0^2, 3, 2, 17, 1^2, 6, 0^2, 7, 1, 0^3, 14, 4, 2, 6, 7, 2, 7, 20, 0, 5, 1, 0, 1, 11, 2, 5, 0, 1^2, 0^3, 1, 0^{12}, 1, 0^4, 31, 0^3, 1^2, 2, 0^3, 1, 0^2, 2, 0^2, 1, 0^3, 3^2, 0^{15}, 2, 0^4, 1, 0^2, 1, 0^2, 1, 0^3, 2, 0^2, 1, 0^2, 1, 0, 2, 0^2, 2, 1, 2, 0^2, 2, 1, 0, 2^2, 0, 6, 0^4, 1, 0, 1, 0^6, 1, 0, 1, 0^4, 2, 1, 0^5, 1, 0^2, 1, 0^{12}, 1, 0^8, 1, 0^7, 1, 0^{12}, 2, 1, 0^{22}, 1, 0, 1^2, 0^{20}, 1, 0^2, 1, 0, 1, 2, 1, 5, 1, 0^2, 7, 2, 0, 1, 0^2, 1, 2, 0^2, 1^2, 0, 3, 0^4, 1, 0, 1, 0^2, 2, 0^5, 1^2, 0^2, 2^2, 1, 0, 1, 0, 1, 0^3, 1, 0^2, 2, 0^2, 1, 0^4, 1, 0^3, 1, 2, 1, 2, 3, 0, 2, 4, 3, 2, 4, 0, 2, 0, 3, 2, 1^3, 0, 1, 3, 1, 0^3, 1, 0, 2, 0^3, 1, 0, 2, 0, 1, 2, 0, 3, 0^3, 1, 0, 4, 0^2, 1, 0^7, 1^2, 2, 1^4, 0^7, 1, 3, 0^4, 3, 1, 0^2, 1, 0^2, 1, 2, 0^3, 1, 3, 2, 0, 4, 0, 1, 0^2, 1^2, 2, 0^5, 3, 0^3, 1, 0, 1, 0, 1, 0^3, 1, 0^4, 1^2, 0^4, 1, 0^2, 7, 2, 0^3, 1^4, 0^5, 1, 0^2, 1, 0^{10}, 1, 3, 2, 0^{10}, 2, 0^{21}, 2, 0^4, 1, 0^3, 5, 0, 1, 0, 2, 0^4, 1^2, 0^4, 1^2, 0^2, 1^2, 2, 0^{21}, 1, 0^{14}, 1, 0^5, 1, 0^{32}$

A(28,6,5) ≥ 272

$0^2, 148, 162, 22, 123, 80, 17, 71, 291, 299, 220, 5, 37, 183, 19, 28, 123, 9, 90, 55, 9, 260, 121, 70, 8, 101, 1, 11, 46, 48, 75, 2, 4, 31, 21, 6, 24, 2, 14, 0^2, 48, 7, 58, 4, 1, 41, 35, 1, 17, 2, 12, 0, 9, 19, 7, 1, 2, 9, 66, 35, 3^2, 5, 9, 0^2, 7, 2, 8, 3, 2, 15, 6, 0, 3, 1, 15, 3, 13, 0, 3, 1, 0, 4, 0, 9, 2, 0, 1, 0^3, 15, 16, 3, 11, 0, 15, 7, 0, 18, 0, 7, 2^2, 4, 1, 0, 1, 0^3, 1, 0^3, 11^2, 5, 2, 4, 11, 2^2, 3, 1, 0, 3, 5, 2, 0, 2, 0^2, 1, 0^3, 1, 0, 1, 0^6, 7, 0, 5, 0, 5, 3, 10, 12, 4, 1, 0, 3, 0, 4, 0, 2, 0, 1^2, 0, 1, 0^4, 1, 0^2, 1^2, 0^4, 1, 4, 0, 1^3, 7, 2, 0, 3, 1, 0^5, 1, 0^2, 1, 0^4, 1, 0^2, 1, 0^6, 1, 0^4, 3, 0^3, 1, 2, 1, 0^2, 1, 0, 2, 1, 0^{12}, 3, 0^{24}$

A(25,8,10) ≥ 1248

$196702, 124, 0^2, 22, 0, 5, 3, 0, 8, 0^5, 6, 0^2, 3, 0^5, 18, 7, 0^2, 2, 0, 1, 0^2, 2, 0^5, 2, 0^8, 6, 0^{23}, 12, 0^{87}, 850, 0^{23}, 10, 0^{23}, 4, 0^{23}, 8, 0^{87}, 478, 0^{23}, 4, 0^{23}, 2, 0^{23}, 4, 0^{87}, 444, 0^{15}, 8, 0^{15}, 4, 0^{47}, 192, 0^{159}, 240, 0^{79}, 120, 0^{159}, 1138, 0^{15}, 162, 0^{135}, 17, 15, 7^2, 0, 2, 3, 0^9, 12, 0^{39}, 16, 6, 0, 14, 0^3, 1, 0^7, 14, 0^{24}, 23, 1, 4, 5, 0^{12}, 12, 0^{23}$

A(26,8,19) ≥ 257

$30027, 1557, 7514, 6608, 3164, 1865, 3262, 695, 400, 12, 243, 460, 241, 614, 94, 0, 391, 195, 111, 212, 252, 1065, 316, 234, 17, 32, 140, 1833, 132, 8, 0, 483, 34, 5, 0, 3, 9, 4, 0^2, 10, 0^2, 13, 188, 0, 7, 8, 21, 1, 0, 3, 0, 1, 2^2, 0^3, 1, 4, 0, 1, 4, 1, 2, 1, 0, 1, 0, 1, 0^3, 1, 0, 2, 0^5, 1, 0, 11, 0, 1^2, 0^5, 1, 0^{10}, 1, 0, 1, 0^5, 3, 0^{17}, 1, 0^{18}, 1, 0^{23}, 1, 0^{47}, 1, 0^3, 1, 0^{10}, 1, 0, 2, 0^8, 1^2, 0^8$

A(26,8,9) ≥ 883

$0^4, 1, 9, 2^2, 0, 6, 2, 1, 7, 0, 113, 65, 138, 32, 409, 5, 0, 53, 0, 5, 11, 0, 3, 9, 51, 43, 0, 1, 33, 0, 25, 4, 0^2, 39, 0, 4, 6, 0^9, 5, 52, 30, 0^2, 5, 0^2, 1, 0^6, 2, 4, 2, 4, 0^2, 6, 0^{15}, 30, 3, 0^5, 1, 0^3, 2, 0^7, 1^2, 0^7, 1, 0^{13}, 1, 0^{18}, 21, 14, 0^3, 1, 0, 1, 0^3, 1, 0, 1, 0^{17}, 1, 0^7, 1, 0^8, 1, 0^3, 1, 0^{10}, 1, 0^{24}, 1, 0^5, 1, 0^5, 1, 0^4, 2, 0^6, 1, 0^6, 1, 0^8, 1, 0^8, 1, 0, 2, 0^{16}, 1, 0^7, 1^2, 0^2, 1, 0^5, 1, 0^{12}, 1, 0^{24}, 1, 0^5, 1, 0^{16}, 1^2, 4, 0^2, 1, 0^2, 1, 0^3, 1, 0^3, 1, 0, 2, 0^9, 1, 0, 1, 0^{32}, 1, 0^5, 1, 0^4, 1, 0^6, 1, 0^7, 1, 0^{11}, 1, 0^3, 1^2, 0^{11}, 2, 0^5, 1, 0^{20}, 1, 0^{17}, 1, 0^{11}, 1^2, 0^{12}, 1, 0^2, 1, 0^{10}, 1, 0^{15}, 5, 10, 0^2, 1^2, 0^2, 4, 2, 1, 0, 1, 5, 0^2, 2, 1, 0, 1, 2, 4, 1, 0, 2, 1^2, 3, 1, 4, 0, 2, 0^2, 1, 5, 1^3, 0, 1^2, 0, 3, 1, 0^3, 1, 2, 0^2, 2, 0^2, 1, 2^2, 3, 0, 1, 0, 1, 0, 1, 0, 3, 1, 0, 2^2, 0, 2, 0^3, 1, 3, 4, 1, 0^8, 2, 0, 1, 0, 1, 0, 1, 0^2, 4, 3, 0, 1^3, 0^2, 1, 0, 1, 2^2, 1^4, 0, 1^2, 0^3, 1^2, 6, 1, 0^2, 1, 0^4, 1, 0^3, 2, 1, 0^2, 1, 0, 1^2, 0, 3, 0, 2, 1, 0, 1, 3, 2, 0, 1, 3, 0, 1^2, 0, 2^2, 0^3, 3, 0^2, 3, 0, 2, 0, 3, 0^3, 3, 0^3, 2, 0^6, 3, 0^2, 1, 3, 0^4, 3^2, 0^4, 1, 0^6, 1, 0, 1, 8, 1, 6, 0, 6, 0, 1, 0^2, 1, 0, 1, 0^{33}, 1^2, 0^{22}$

A(27,8,7) ≥ 278

$2273, 164, 5, 224, 161, 154, 40, 2, 44, 135, 0^2, 45, 4, 9, 15, 2^2, 8, 4, 7, 12, 100, 13, 0^2, 8, 0^2, 2, 4, 3, 2, 0, 4, 0, 9, 0^3, 2, 0^3, 1, 3, 2, 0^8, 9, 22, 0, 4, 0^4, 1, 0, 1, 0, 1^2, 0, 1, 0^7, 1, 0^3, 1, 0^3, 12, 0^{10}, 1, 0^{11}, 1, 0, 2, 0^8, 1^2, 0^4, 1^2, 0^5, 11, 5, 0, 1^3, 0^3, 1^4, 0, 1, 0^3, 1, 0^4, 1, 0^4, 1, 0, 1, 0^3, 1^2, 0^2, 2, 1, 0^3, 1, 0^3, 1, 0^2, 1, 0^2, 1, 0, 15, 48, 13, 4, 6, 8, 12, 17, 26, 1, 53, 43, 16, 8, 14, 13, 0, 4, 0, 6, 7^2, 4, 5, 0, 1, 0, 15, 3, 8, 25, 6^3, 0^2, 12, 8, 1, 3, 0^5, 2, 1, 0^6, 4, 0^3, 4, 0^3, 1, 0, 2, 0^7, 1, 0^{18}$

A(27,8,19) ≥ 766

$1, 0, 1, 0^7, 1^2, 0^{59}, 1, 0^{12}, 1, 0^{681}$

A(27,8,9) ≥ 970

$0^4, 1, 9, 2^2, 0, 6, 2, 1, 7, 0, 3, 0, 1, 9, 1^2, 0, 3, 1, 0^2, 2, 0, 49, 17, 84, 12, 44, 64, 3, 50, 163, 1, 0, 43, 0, 12, 8, 0, 5, 0^3, 7, 2, 0, 159, 6, 10, 21, 0, 23, 1, 7, 0^3, 1, 0^6, 30, 0, 3, 0^3, 4, 0^{10}, 1, 8, 5, 21, 30, 0^2, 1, 2, 0^4, 3, 0^4, 1, 0^3, 1, 0^{11}, 1, 0^7, 1, 0^{15}, 2, 16, 1, 22, 0^2, 1, 0^3, 1, 0, 1, 0^7, 1, 4, 0, 1, 0^{10}, 5, 0^{23}, 1, 0, 1, 0^{10}, 1, 0^6, 1, 0^{13}, 11, 1^2, 4, 0^{10}, 1, 0, 1, 2, 0^2, 1, 0^{10}, 2, 0^{21}, 1^2, 0^{14}, 1, 0^3, 1, 0^2, 1, 2, 1, 0^8, 1, 0^3, 1, 0^8, 2, 0^5, 1, 0^5, 1, 0^{13}, 1, 0^{10}, 1, 0^2, 6, 0^{10}, 1, 0^8, 1, 0, 1, 0^6, 1, 0^{20}, 1, 0^6, 1, 0^9, 1, 0^7, 1, 0^{16}, 1^2, 0^7, 1, 3, 0^5, 1, 0^6, 1, 0^4, 1, 0^2, 1^2, 0^5, 1, 0^2, 1^2, 0^9, 1, 0^4, 2, 0^4, 1, 0, 1, 0^8, 1, 0^4, 1, 0^4, 1, 0^2, 1, 0^6, 1, 0^8, 2, 0^{12}, 5, 0^2, 1^4, 0, 1, 0, 2, 0^3, 1, 0^2, 3, 0, 1, 0, 1, 5, 0^3, 1, 0^4, 1^3, 2, 0, 2, 0, 3, 15, 0, 1^6, 0^4, 2, 1, 2, 0, 1, 0^2, 4, 0^6, 2^2, 3, 1, 0^2, 3, 0, 1^3, 5, 0^3, 2, 0^2, 1, 0, 1, 0, 3, 2, 0^5, 3, 0^2, 2^2, 0, 1, 0, 2, 1, 2, 4, 0, 1, 3, 0^3, 3, 1^2, 0, 1, 0^2, 1, 3, 0, 2, 1, 2, 1^4, 2, 3, 2, 0^2, 1, 0^2, 1, 0^2, 3, 1^2, 0^3, 3, 9, 0^2, 7, 1, 0^3, 1, 0, 13, 8, 15, 3, 0, 1^2, 11, 0, 15, 3, 4, 3, 2, 8, 4, 8, 3, 0, 1, 0^2, 1, 6, 0^3, 1, 0, 1, 3, 4^2, 2, 1^2, 4, 0^2, 3, 2, 5, 0, 4, 2^3, 1^2, 0^4, 2, 0^3, 3, 0^4, 2^2, 0, 1, 0, 2, 0^2, 3, 0, 1, 6, 0, 2^2, 0, 1^2, 0^6, 1, 0^8, 2, 0^5, 1, 5, 1, 7, 4, 1, 7, 2, 0, 4, 2, 0^3, 20, 17, 0, 1, 3, 2, 1, 0, 4, 3, 0^5, 4, 0^3, 13, 0, 8, 2, 1, 0^2, 12, 0^2, 3, 1^3, 2, 0^2, 2^2, 0^5, 1, 0^5, 1, 0, 1, 0, 1, 0^6, 9, 1, 3, 0, 1, 2, 1, 5, 0^5, 3, 0^4, 1, 0, 1, 0^4, 1, 0, 1, 0^5, 1, 0^3, 1, 0, 1, 0^3, 1, 0^{13}$

A(28,8,21) ≥ 296

TABLE XVI (continued)

2616.169.1019.39.293².484.166.831.33.369.134.377.17.34.69.1.215.201.406.7.220.1.91.52.39.5.193.
219.0.89.2.61.1.40.26.13.5.3.10.50.7.60.4.38.57.1.0.2.1.7.20.1.6.0.11.5.6².3.21.6.22.0.16.6.5.10.3.
16.10.2².6.1.2.3.1.4.1.50.434.82.4.6.79.17.20.38.29.166.1.7.3.8.6.7.40.24.12.9.27.25.5.12.6.24.6.
11.2.11.0.4.1.4.5.51.0.2².11.0.1.2.0.2².1.7.13.0.6.1³.0.2³.4.14.18.85.28.1.35.31.14.64.0.6.7.1.0.3.1.
6.19.0.2.3.11.1.8.0.4.17.1.10.11.7.0.1.2.8.1.13.8.12.2.0.3.0³.1.2.0².1.9.2.0.7.1².2.14.0.2².4.8.0³.2.
0.1.3.4.1.0³.1.3.0⁴.2.0².1.0.8.6.0.2.0².2.0.4.0.2.1.0.2.0².2.0.2.1.0⁵.3.1.0.1.0⁵.1².0⁹.1.0⁸.1.0¹⁴.1.0²
A(28,8,20) ≥ 833
0³.5.21.15.2.23.30.8.2.19.8.2.8².2².8.1.2.51.6.0.4.17.7.2.9.24.7.9.2².7.1.0.20.0.14.13.48.12.2².0.
14.4.2.0².17².6.12.24.10.0².1.9.1.5.33.22.9.17.0.2².3.2.1.2.1.3.0.1.0.1.2.23.10.2.4.1².7.1.21.2.4.5.
1³.0.2.0.59.2.9.12.3.1.4.2.6.12.9.0.5.2.1.7.0.16.0.25.10.0.5.3.24.0.3.11.1.2.12.0.3.19.3.2³.7.0.1.22.
0.1.0².2².1.0².20.0.2.0².2².8.0.1.0.1.11.3.0.3².8.0.1².0⁴.1.3.2.0².1.0⁴.1².19.2².0³.1.0.4.0.2.0⁴.3.
0².2².0.4.1.4.1².0³.3.0³.4.0.1.0².1.0².9.0².1.0.11.3.1².2.1.0.1.2.1².0⁴.1.0.1³.7.6.0.1.0².2.1.0².1.0².
2².0⁸.6.0.1.2.1.14.2.3.0.1².2.1².2.7.12.1.8.7.1.0.1.0².8.0.2.1.0².5.2.12.0.8.0.14.11.3.0.2.3.0.1².0.3.
0.4.2².0.5.0².2.9.0.3.0.3.0.1².2.0.2.0².2.0².2.6.0².5.10.1.3.0.5.1.0².9.0.1.2.0.1.0.4.0.1².0.1.2.0.2.
7.0².3.1.4.0³.2.11.0².1.0⁴.8.0².12.1.0.10.2.0.1.0².1.0⁴.4.0⁴.3.0.1.0⁶.4.2.1.0.1.0¹¹.1.0⁸.1.0.1.0.3.
0³.6.0.2.0³.2.7.0.18.0².5.18.2³.0.1.0.3.1.2.3.0.2.1.2.1.4.3.2.1.0².1².0.12.7.1³.2.1.2.3³.1.0⁷.2.0.4.
0².2.0.1.0.1.2.0.2.1.0².5.0³.8.0².4.1³.0³.3.0³.2.0².6.0.3.1.2.0⁷.6.0³.3.0⁵.1.0².2.1.0⁴.1.0⁴.2.0.1³.
0².2.0.5.4.0.4.1.0⁷.1³.0.1.0³.1.0³.3.2.3².1².0².1.0³.6.0.4.2.0.1.0.1.0⁷.1.0.1.0³.2.0.4.1².0².1.0².1.
0².1².0⁵.2.0.3.0³.1.4.0².1.0².2.0⁷.1.4.1.0⁷.1.0³.2.0.1.3.0²².1.0.2³.0³.1.0.1.0⁵.1.0⁸.1.0¹⁰.1.0.1³.0².
2.0².4.0¹³.1.0².1.0¹⁰.1.0³.2².0¹⁸

A(28,8,19) ≥ 1107
0⁵.28.0³.33.12.13.0.9.5.0⁴.13.11.17.9.2.19.0⁴.4.3.0.7.0².10.0².7.0³.8.1.2.0.1.0.19.0⁵.9.0.12.0¹².5.
0.4.0³.103.1.0.4.0⁴.132.0.2.0³.14.392.13.0⁴.56.0⁴.28.0.2.0.4.46.0².4.0.52.0².8.1.0³.335.0³.42.0.16.
11.0.22.0.6.0.2.0.14.0.1.0.291.0³.69.0.21.0².5.3².0.9.0².5.0².104.586.14.12.8.0².3.4.0⁶.1.0.17.0².
13.0³.3.0².1.3.0.4.0².37.5.90.2.0³.273.13.0.49.150.0⁷.407.98.22.46.57.55.0³.82.0⁷.7.312.46.281.9.
47.36.0⁷.173.73.459.0.2.0⁷.2.0⁹.214.43.45.133.14.65.30.15.50.67.64.56.24.6.1.253.0³.3.0.3.0.237.
161.176.0.94.38.0².1.0³.1.82.61.42.51.95.11.0³.2.0¹¹.28.82.41.120.0.27.21.25.0².32.0².18.0.1.0².
28.107.81.62.2.14.1.0².9.0⁴.1.0³.4.0.94.2.59.40.28.10.12.10.0.9.7.9.2.0⁶.3.0¹².2.0².1.21.47.15.9.26.
3³.8.9.1.3.1².0¹⁰.1.0⁵.2.1.29.9.23.9.18.20.1.0.13.1.0.11.6.1³.25.0.15.91.1.0².30.1.0².2.3.0³.6.16.9.
12.0.22.0³.11.2.0.23.286.0⁵.84.44.5.8.16.0².6.0⁴.1.3.0².1.0.4.0.15.75.2.35.1.0³.61.36.0².12.2.0².
15.76.4.12.6.9.3.0³.10.2.1.8.47.15.25.0³.9.0².2.0.1.0.5.1.0⁵.31.71.82.26.19.5.0³.4.0⁴.56.4.70.0².10.
9.3.1.0².3.8.3.9.8.0.9.97.3.6.0.11.24.2.8.9.6.5.6².0².1.0.5.0⁶.3.0³.1.0³.1.83.5.20.8².2.11.2.0.1.5.
0⁸.1.0².2.1.0².6.0².1².0⁹.2.1.6.40.3.0.4.0.1.0.5.15.0.4.2.12.2.3.308.7.2.1.23.10.3.124.1.12.35.0².
17.0.6.0.4.0².11.9.12.42.0.2.1².0³.2.0.1.0².10.4.0.3.0.9.3.0.5.0.4.0⁴.1.4.1.0.1².5.2².0.5.49.10.0³.4.
1.0.2.0.3.15.7.2.0.8.2.0⁵.2.0.10.5.0³.4.7.9.8.1.2.0³.12.5.9.0.65.12.7.19.0.3.0².6.0.5.0⁷.4.1.2.0.1.5.
3.0⁹.1.0².5.1.0.1.0².2.0¹².1.0³.8.39.1.0³.4.0.11.1.7.1.6.2.0⁶.1.3.0⁴.3.9.3.0.3.2².0.3.0⁸.3.1.0¹¹.1.5.
1.0.3.1.2.1.0¹².1.0³.1.0.1.0².1.0⁴.4.0³.1.0³.2.0.13.2.1.4.0.1.0³.3.0².2.1.0².1.0.1.0³.2².0.1.0⁵.1.0¹³.
1.0¹³.1².0².1.0⁴.1.0¹¹.2.0³.1.0⁴.1.0³.1.0³³.1.0¹⁰.1.0²⁵

A(23,10,7) ≥ 20
31030.8098.707.3722.291.84.959.43.9.2.39.0³.3.0⁵

A(26,10,12) ≥ 185
0⁴.9.804.1634.31.0.653.287.18.333.12.7.19.1108.2.108.41.8.10.177.2.0.9.11.13.3.1.6.23.45.6.16.1.
0².21.0.18.4.25.0.5.21.1.0².1.3.1.2.1².0².1².0.52.18.3.6.7.0.12.4².2.0.1⁴.0.7.0².2.0.2.0⁶.15.9.2.1².
3.1².6.12.2.0.10.0.10.58.1.2.9.1.9.6.7.0.7.0.1.5.2².0⁹.4.3.0.28.9.10.9.24.7.0².8.4.3.0.6.5.0².1.0⁷.
54.36.0.13.5.1.18.1².2.4.1.14.1.0³.1.0¹³

A(26,10,13) ≥ 191
0².56.420.4213.616.3042.732.98.10921.2166.577.1260.0.299.606.249.4566.1544.579.478.0.143.145.
150.224.361.38.431.267.241.0.34.2033.404.108.1062.0.1172.89.242.160.371.0.74.22².17.129.101.
105.31.14.64.14.31.27.12.0.67.3.33.0².11.3.94.119.63.15.1².6.40.75.69.30.0.45.31.7.93.71.24.41.14.
98.40.7.6.2.3.2.5.2.18.14.2.39.22.1.3.0.11.3.0.1.2.0.2.3.1.6.0.2.0².2.0⁵.3.62.0.23.150.5.6.79.29.8.6.
72.8.13.2.12.62.29.42.0.4.0⁶.4.0.2.0.5.1².0.9.7.1.2.0.2.4².1.0².2.7.0².1.0¹⁷

A(27,10,11) ≥ 213
52263.46674.39014.12594.7191.49805.257.1176.8505.2624.2904.1024.411.5583.9562.3356.214.1045.
3611.2612.291.935.641.227.406.430.2926.3213.1039.2077.44.486.200.22.156.171.481.183.51.412.27.

TABLE XVI (continued)

93,42,38,91,9,17,3,1410,278,244,96,100,17,33,132,24,10,6,31,106,11,7,4,15,74,11,55,6,3,22,4,8,0,1,0,29,0,10,6,11,0,1³,0¹¹,190,10,52,15,5,37,3,0,5,2,8,21,1,3,1,0²,15,0²,9,5,0⁴,2,0,5,0,1,0⁴,1,0³⁶,47,3,1³,2,5,1,0,1²,3,2,1,0,1,0²,2,0¹³,1,0⁵,1,0³,1²,0

A(27,10,12) ≥ 257

0¹,1212,865,887,143,95,906,134,674,11,56,2,0,44,58,634,6,0,39,157,8,47,14,8²,23,9²,50,31,11,0,3²,5,17,0³,5,0,7,0,4,6,0,3,0,5,0⁶,1,0⁴,38,20,9,5,13,3,11,17,2,17,4,0,2,13,4,0²,3,0⁵,1²,0,2,11,2⁴,1,0³,2,0²,23,42,20,16,10,0,6,3,7,0,2,0²,13,3,0,1,0,6,0⁶,1²,3,0³,36,35,4,0,5,0,3²,4,0⁴,7,0,1,0⁶,8,11,6,7,1,4,3,1²,4,1,0,8,2²,0,1,0²,3,0,1,0⁶,23,8,2,1,4,2,3,0³,18,8,12,5,12,1,2,1,4,0,4,1²,0³,1²,0,1,2,0³,17,6,13,2,26,4,18,3,6,2²,0⁴,1²,2,0,2,0⁵,2,0⁴,2,0⁹

A(27,10,13) ≥ 283

0²,56,420,4213,616,3042,732,98,10921,2166,577,1260,0,299,606,249,4566,1544,579,478,0,143,145,150,224,361,38,431,267,241,0,34,2033,404,108,1062,0,1172,89,242,160,371,0,74,22²,17,129,101,105,31,14,64,14,31,27,12,0,67,3,33,0²,11,3,94,119,63,15,1²,6,40,75,69,30,0,45,31,7,93,71,24,41,14,98,40,7,6,2,3,2,5,2,18,14,2,39,22,1,3,0,11,3,0,1,2,0,2,3,1,6,0,2,0²,2,0⁵,3,62,0,23,150,5,6,79,29,8,6,72,8,13,2,12,62,29,42,0,4,0⁶,4,0,2,0,5,1²,0,9,7,1,2,0,2,4²,1,0²,2,7,0²,1,0¹⁷,43,14,13,20,7,0,1,9,14,19,10,4,0,10,0,3,1,0⁵,1,16,4,3,0⁵,6,0,4,1,0⁴,83,23,1,76,8,0³,46,0,10,1,0³,21,2,7,0³,3,2,0⁵,2,0,4,1,0²,2,0⁵,1,0¹²

A(28,10,18) ≥ 195

0³,56,286,1296,3684,5186,1441,138,0,833,5,115,4737,1382,956,86,373,96,51,2263,1799,21,135,242,103,342,104,49,48,113,75,17,8,2,1,667,1,269,55,58,19,35,365,86,2,92,15,37,4,21,0,11,0²,12,15,12,0,10,16,1,3,16,0,31,8,2,1²,9,0,1,0,2,5,1,2,0²,10,0,4,0³,2,4,0³,6,0⁴,1,0,1,0⁵,1,0⁵,649,116,343,515,44,27,21,20,33,258,58,39,14,49,30,3,7,6,21,31,49,29,30,2,9,1,8,0,22,23,1,6,13,7,93,5,55,10,6,1,0,10,3,0,2,1,2,0,1,0,16,0¹⁴,1²,0,9,0²,1,0¹²

A(28,10,11) ≥ 280

52263,46674,39014,12594,7191,49805,257,1176,8505,2624,2904,1024,411,5583,9562,3356,214,1045,3611,2612,291,935,641,227,406,430,2926,3213,1039,2077,44,486,200,22,156,171,481,183,51,412,27,93,42,38,91,9,17,3,1410,278,244,96,100,17,33,132,24,10,6,31,106,11,7,4,15,74,11,55,6,3,22,4,8,0,1,0,29,0,10,6,11,0,1³,0¹¹,190,10,52,15,5,37,3,0,5,2,8,21,1,3,1,0²,15,0²,9,5,0⁴,2,0,5,0,1,0⁴,1,0³⁶,47,3,1³,2,5,1,0,1²,3,2,1,0,1,0²,2,0¹³,1,0⁵,1,0³,1²,0,11,4,5,0,3,1,0,3,0,1³,0²,1,0²,1,0⁵,1,0,1,65,17,13,16,6,4,2,0⁴,3,5,0³,1,2,4,12,2,1,2,1,2,0,2,0⁵,1²,0⁸

A(28,10,12) ≥ 356

0¹,1212,865,887,143,95,906,134,674,11,56,2,0,44,58,634,6,0,39,157,8,47,14,8²,23,9²,50,31,11,0,3²,5,17,0³,5,0,7,0,4,6,0,3,0,5,0⁶,1,0⁴,38,20,9,5,13,3,11,17,2,17,4,0,2,13,4,0²,3,0⁵,1²,0,2,11,2⁴,1,0³,2,0²,23,42,20,16,10,0,6,3,7,0,2,0²,13,3,0,1,0,6,0⁶,1²,3,0³,36,35,4,0,5,0,3²,4,0⁴,7,0,1,0⁶,8,11,6,7,1,4,3,1²,4,1,0,8,2²,0,1,0²,3,0,1,0⁶,23,8,2,1,4,2,3,0³,18,8,12,5,12,1,2,1,4,0,4,1²,0³,1²,0,1,2,0³,17,6,13,2,26,4,18,3,6,2²,0⁴,1²,2,0,2,0⁵,2,0⁴,2,0⁹,6,1,0,10,7,4,3,1,2²,5,1,7,1,17,7,6,13,19,0,16,8,3²,0³,1,3,1,0²,1²,3,1,0³,1,2,0,87,16,64,31,6,28,23,3,2,18,9,16,7,8,1,0⁵,1,0²,2,1,4,0,6,1,3,0²,2,1²,0²,3,0,1,0¹⁷

A(28,10,13) ≥ 414

0²,56,420,4213,616,3042,732,98,10921,2166,577,1260,0,299,606,249,4566,1544,579,478,0,143,145,150,224,361,38,431,267,241,0,34,2033,404,108,1062,0,1172,89,242,160,371,0,74,22²,17,129,101,105,31,14,64,14,31,27,12,0,67,3,33,0²,11,3,94,119,63,15,1²,6,40,75,69,30,0,45,31,7,93,71,24,41,14,98,40,7,6,2,3,2,5,2,18,14,2,39,22,1,3,0,11,3,0,1,2,0,2,3,1,6,0,2,0²,2,0⁵,3,62,0,23,150,5,6,79,29,8,6,72,8,13,2,12,62,29,42,0,4,0⁶,4,0,2,0,5,1²,0,9,7,1,2,0,2,4²,1,0²,2,7,0²,1,0¹⁷,43,14,13,20,7,0,1,9,14,19,10,4,0,10,0,3,1,0⁵,1,16,4,3,0⁵,6,0,4,1,0⁴,83,23,1,76,8,0³,46,0,10,1,0³,21,2,7,0³,3,2,0⁵,2,0,4,1,0²,2,0⁵,1,0¹²,1,7,6,19,3,2,8,4,6,0,6,5,1,5,0,7,3,4,1,0,14,30,15,10,20,3,0,2,4²,7,0²,2,3,0,3,0²,1,4²,0,2,1,0,5,0,1,0,1,0³,26,67,37,6,28,22,19,4,7,2,9,2,19,0,5,18,3,7,24,0²,8,1,0,1,0,1,0,9,0,2³,0²,6,1,3,0,2,1,0²,1,0,2,1,0²,2,1,0³,1,2,0⁵,1,0,1,0¹³

A(28,10,14) ≥ 435

0³,1150,96,5557,4719,2069,4941,1017,10572,1545,2879,3746,675,0,842,229,295,389,1949,163,2268,2130,9212,0,47,1563,161,174,210,429,0,218,1161,3037,254,700,65,4877,681,5297,706,0,330,820,60,34,411,436,1620,124,873,885,1639,503,513,0,408,643,160,1301,1860,396,5²,28,444,558,161,286,344,544,305,183,0,65,2416,95,3964,1032,1720,38,105,851,58,1061,312,32,980,1003,187,198,0,60,508,221,249,117,53,187,28,107,1,424,210,12,460,305,176,6,4,0,214,19,0,8²,84,0,100,165,42,26,32,5,13,0,34,2,16,24,0,25,0,22,8,11,2,6,0³,2,4,1,0,130,398,240,208,257,40,52,232,106,6,2²,106,3,17,523,4,215,115,17,20,6,142,3,1,12,93,1,0,8,4,71,43,15,2,24,5,13,16,26,0,3,1,0,4,2,13,1,0²,2,3,2,4,3,7,8,2,1,0,2,0,1,3,0³,42,1,35,0,12,0,1²,2,0²,6,1²,2,0,2,1,2,1,0,1,0,10,0,1³,0,1³,0⁵,1,0,3,1,0³,1,0,1,0²²,69,270,157,82,24,3,97,54,9,184,7,32,28,21,47,48,5,6,52,21,7,13,2,50,8,3,54,0²,31,2,11,67,2,21,42,3,2,11,1,9,1²,5,21,0,34,0,6,0²,2,0,2,1,0,11,6,2²,3,1,3²,0¹²,7,4,2,6,1³,3,0,1,3,4,8,0³,3,2,0²,2,7,2,0,1,2,0²,2,1³,0,2,0²,1,0⁵,2²,0,1,0⁷,1²,0,1,0¹⁹

APPENDIX TABLE

$\pi(7,3) = (7,7,6,6,5,4)$, norm=211: 32165426131543522314621534146242315
 $\pi(8,4) = (14,14,12,12,10,8)$, norm=844
 5214334215143251362435261246153612442163516421625342631523415124334125
 $\pi_1(9,4) = (18,18,18,18,16,15,15,8)$, norm=2066
 17623428635817463454125177316422563873426148517253184325231283746465712458534721
 6133574126187263635452417345612167327645741523
 $\pi(10,3) = (13,13,13,13,13,13,13,3)$, norm=1530
 594279368731456885696A815924134132727981346526724153678986723131249257A794658148
 5932136745528129734872A96981635445986137
 $\pi_2(10,4) = (30,30,30,30,26,25,22,15,2)$, norm=5614
 54712763112843623648312585431276459317626584345225436176184725341716326423357614
 21487575321755814263362415413281825637142436751247573365365542824145671317242816
 39246314182173853275647321562417418366342578654321
 $\pi(10,5) = (36,36,34,34,29,29,27,27)$, norm=8044
 28657343817173624561257134426131247583724512686231547863743812415158264512867245
 38645273718164382552658738264312368411347415623164551244873216281724536372485163
 72416563782738537214614567332718412585622361476418737325413364815225814354218471
 312648782563
 $\pi_1(11,3) = (17,17,17,17,17,17,16,16,14)$, norm=2731
 7A453288175962311934912386465964A5768914257463279AA1372816496A7593475825382A12A1
 586423157964A27A8658571324358916A82413974536989A1581276279483786A34532714A691643
 79A25
 $\pi_2(11,3) = (17,17,17,17,17,17,17,16,12,1)$, norm=2713
 5A13869754634872517937298A2483491561863279716289436A4157927415163A7286485627353A
 8219748246325B93A18569712A9254686A13954729156843A11A632A75498743985238475916857A
 64312
 $\pi_3(11,3) = (17,17,17,17,17,17,17,17,10,2)$, norm=2705
 25816741797459633968436A349852128571936255A2328741891746369782851454A12217639478
 9635817496A385612747A9163364A952925B78419583A74236831256729A4A5618931645978224A8
 1B573
 $\pi_1(11,4) = (35,35,35,34,33,33,33,32,31,25,4)$, norm=10724
 89546341268B217762426598A84675BA7131A59832B319A46B37A545933761472218A9A291324175
 169568453A452893718693A42645742596832791A815365163718A4227A375534938152949371426
 842798A6195856127654A393645782A3A81266A41571386742293745172979583619251A84A58341
 5239637615497864812369285751368AA6742A47837629181932548492613A686184795732A52374
 917429138A
 $\pi_2(11,4) = (35,35,35,35,35,33,33,32,28,21,8)$, norm=10616
 A951842513754A3B3251A287473914656828946261735593672A9486145874B3921AB7517464329A
 962389871528196338614A57973A8219576148A2B324A638B978431261775828A995642163758B43
 1175243569A426133A8625428A361257513467598426945871763285694718AA3B62915437182913
 9875442B159378276145825A6331947571282546364A25158634972615A445363452717298693A71
 8486719A32
 $\pi_1(11,5) = (66,66,60,60,54,45,44,40,26,1)$, norm=25066
 93521728164179693243516743257448136274353299216532181547423817125643546365741675
 25683212178596213428531647372912411395784842638345269411247539513641279275842586
 14327215816738125637418268474354913212365681944526373712724418825616354873541462
 32148535621771625945236881453412935878271537412329657214814563127359148724153326
 48651385742923196815231822463941435764293146317587247635212164789931513289425643
 67251524614A36514533842842712371653458722135796185813721286934
 $\pi_2(11,5) = (66,66,60,60,54,45,44,42,22,3)$, norm=25046
 63521827184179693243519843258446137264353287218532171548423718125743546365941695
 25673212187576213426531648382712411385794742637345268411249539513641289285A42576
 1432821561793712573841728749435461321236569174452638361282441772561A354683541482
 32147535621881725745239661453412635676281538412329758214614573128357146824153326
 47651385742823196715231722463941435864293146317578248635212164978831513279425A43

APPENDIX TABLE (Continued)

69251524714638514533742642812381653457822135896175713821276934
 $\pi_1(12,4) = (51,51,51,51,46,45,44,44,34,27)$, norm=22903
 864134B319A5564215638972A89AA4241791542B782763566978421634839A51B2B856396B5978A2
 678235123BB413872456596182A57683145471A91A24737694241A575231863B824152A89634B743
 586A31513B97294A59762115247179A38AB94639512B432658B275754318A82479236A5A36941825
 3618B497A81623415A377152948291434617B1465978B3A53B9826938532BA96645B182617926475
 83A5782341532713269526B9A8781397845157481356278314B469AA1425B2673941A519A3843527
 686329469732B5AB97238461467593129A895187342846517A6418B69572593142493673A2516872
 5AB81417A4325B9
 $\pi_2(12,4) = (51,51,51,51,49,48,48,42,42,37,23,2)$, norm=22843
 A234B42B3B37A42659717886A9851B65A19185698567959B71A186B7392484321624375B898A1156
 17A68596873427136452279436274313248A43255869A189156BA518A6187979562A43829534437A
 2473A242B1336254815769736247432952431BA15861596782B6717199858A659865B1453A242B63
 312743421A3A4722754395168A2643725A34431A219A87B718596412743632479432876891569A85
 13A85167CB7168596B916599A781A58214353247743262A43527A3443162167A9541352427B33B21
 495861A895B673342763147225843596BA197B18AC61A8597493524A432612437679158586719BB6
 79852A8A1596423
 $\pi_3(12,4) = (51,51,51,51,49,48,48,42,42,40,15,7)$, norm=22815
 BC41343A1136B94758626897A8952C75A28295789576858962B2B7A63A191342741365A989A2257
 26A7B587963146237154468137461323419A13455978A2982578A529A7296868574A1394B531136A
 4163A41492337451C25678637416134A541329A25872587694A7626298B59B75897582153A414C73
 324613142A3A1644651385279A4713645A31132A428A96B629587C24613734168134967982578A95
 2BA952763162795B7A827588A692A59421353416613474A13546A3113274276A8512354146833942
 185972A985B761314673216445913587BA286C2BA972A9586383541A1347241367682595976283C7
 68954A9A258714B
 $\pi_4(12,4) = (51,51,51,51,49,48,46,44,43,37,20,4)$, norm=22795
 31AC2821CB233186579498B6A9A54865A4747568B569757A849476AA372181324621395B977A4456
 4B96A58679312843615227813628134321891325596A74A845687549764B9798562B1372A531139A
 2193A212A43362518459679362181327521347A458645A698B768494B785A9657B65A4153A212763
 342913124A3A1922851375468B2613925A311348248B79A8475B61429136321A813279678456B875
 4AA754692394685768746577BAA4957241353219813262813529B31134624697A514352129833824
 1A576497A596B831296341A225813586C74B9748A3649A5712835217132642138698457576849296
 AB753C7A4596B18
 $\pi_5(12,4) = (51,51,51,51,49,48,48,45,39,36,22,4)$, norm=22755
 3C24B4C92327B43658717996ABA51965A181856BA567858971B186A72843932416432759A8891156
 17A6B5A6872347126354479326473212439932455B6A81AB156B8519861A787956493284A523327B
 43729434A1226453B157687264373248543218A159615967A4867171A895AB658A6591352A434862
 214732341A29374475328516B94632745B233219419A87B7185A6C14732624379324876891569A85
 12A851673B7169586A816588A791B58413252437732464932547923321641678A531254347A22941
 395861A8A5B672234762137445A32596A81A781B9C61B9587392543832461432767915858671A3B6
 7A85498A1596B42
 $\pi_6(12,4) = (51,51,51,51,49,48,48,45,41,32,22,6)$, norm=22663
 A24C34392B352946785159B6A9A71B67C181876AB765878951B186A52843932416432579B8891176
 15A6C7A685234512637445932645321243993247796A819B176A8719861A585A764B328495233259
 4352943491226473A175685264353248743218B17A617965AC865151B8A7AB678A67913729434862
 214532341929354457328716A94632547923321941AB85C5187B62145326243593248568B176BA87
 13A87165945169786A816788A591B78413272435532464932745B233216416589731274345A22941
 397861A8A7B654234562135447A32796A81B581A9261B9785392743832461432565917878651ABB6
 5A87398A17964C2
 $\pi_1(12,5) = (80,80,80,80,72,70,69,67,67,62,48,17)$, norm=55860
 965317A143671252437CB371B58274431A6998C15246383524931865165783A927244BA4953895B2
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APPENDIX TABLE (Continued)

A793164A527498312B972B1934A5A5863BA1362438294598A815B73547361242783A8311B9245839
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 $\pi_2(12.5) = (80,80,80,80,75,72,71,69,63,55,40,23,4)$, norm=55350
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 615A327B1795649B628427345831A2318456C93B2671A31A8C4A4B597895A23137C1562B
 $\pi_1(12.6) = (132,132,120,120,110,94,90,76,36,14)$, norm=99952
 31542766754122374813646275134216576238594A15381925782131482843564917156629872265
 431786214546523A7123816493125445218367389185472361253487154215732837446328132588
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 51A49583267561243157264631847322145766724513
 $\pi_2(12.6) = (132,132,120,120,110,94,90,72,42,12)$, norm=99776
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 144171279567253249634819A71562351245617322735348162485618563431277124381438A2635
 17327474532611525467213456361A8791632572315424832712543899165826451246549153269A
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 51849593267561243157264631947322145766724513
 $\pi_3(12.6) = (132,132,120,110,110,97,91,75,47,10)$, norm=99072
 32541677654211364923747165234127567139596825382415681232481843574926257718861175
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 244262169576153149734829A6257135214572631163534827148572857343216621438243981735
 2631646453172251547612345737289692731561325414A31621543898265816452147549253179A
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APPENDIX TABLE (Continued)

$\pi_1(13,4) = (65,65,65,65,62,61,60,57,57,53,52,45,8)$, norm=42165
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$\pi_2(13,4) = (65,65,65,65,62,61,60,58,55,54,52,45,8)$, norm=42163
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B957A286D2926A51B54378B913624A67C57B223B15A18C4B3989261573544A9D1CCD175631B9B6CA
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B59AC8C51AB8426789CA3116768B475542B45869AC7B51398AB2A35178949142682D7351A5236824
ABC193362B1767385A4516B142C53A23C89B726443B6A71748596C23C45236BA8257A97C1426A68B
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$\pi_3(13,4) = (65,65,65,65,62,60,60,60,58,57,54,49,47,8)$, norm=42147
B326A54137963D47341825C43932867B3A16124572AC8C8A786245941852931C7A74B62C599C481B
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$\pi_4(13,4) = (65,65,65,65,62,60,60,58,57,56,53,37,12)$, norm=42015
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$\pi_5(13,4) = (65,65,65,65,62,60,59,56,55,49,40,12)$, norm=41975
B235861429C52C1A214C3671272395A827454316A387B87CAB53169148639245A591CD3765A71D49
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41C38947A6B8B27341652917B236B918547C431A263C82A5A9735684578A4B6914A6513B72C

$\pi_6(13,4) = (65,65,65,65,62,61,61,59,58,54,50,32,18)$, norm=41795
C24765312ABC293A23194583282497B6271714359468B689CB7435A31654821A97A3DB495ABA3C1D
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A958826BD2727B51A543868913724C78758B223A15D16D493B682715935449D716CB185731ACD78B
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APPENDIX TABLE (Continued)

97593A146817643D892AB4A5C88196D97A955617C564383526C1B65D2137BC21485768B129536827
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 $\pi_1(13,5) = (123,123,121,115,110,109,109,102,99,92,84,72,28)$, norm=135679
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 $\pi_2(13,5) = (123,123,121,115,110,109,109,101,99,93,86,68,30)$, norm=135557
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 $\pi_3(13,5) = (123,122,121,114,110,109,109,102,97,91,85,77,26,1)$, norm=135437
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APPENDIX TABLE (*Continued*)

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 5A463C2
 $\pi_4(13,5) = (123,123,123,116,110,109,106,100,98,92,81,68,38)$, norm=134757
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 $\pi_5(13,5) = (123,123,123,116,110,109,107,104,97,89,83,62,40,1)$, norm=134753
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 $\pi_1(13,6) = (166,166,160,156,143,143,139,135,131,122,107,100,46,2)$, norm=239106
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APPENDIX TABLE (Continued)

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$\pi_2(13,6) = (166,166,160,156,144,142,138,137,131,120,106,102,46,2)$, norm=239082

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$\pi_3(13,6) = (166,166,160,156,143,142,138,136,130,120,111,97,51)$, norm=238832

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$\pi_4(13,6) = (166,166,160,156,145,142,139,136,131,118,113,91,50,3)$, norm=238698

APPENDIX TABLE (Continued)

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$\pi_5(13,6) = (166,166,160,156,145,142,139,136,131,119,112,88,52,4)$, norm=238384

1472356C36AD71927AA82D1CB493B2896551827C614DA3BA89473154B63921B4B3C258664524751A
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$\pi_6(13,6) = (166,166,160,156,145,144,137,132,127,118,111,98,56)$, norm=238116

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APPENDIX TABLE (Continued)

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 $\pi_7(13,6) = (166,166,160,156,145,142,140,136,131,118,106,86,59,5), \text{norm}=237556$
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 879C81742A6642B36742C3B81A517B454A698746A1538421B3C25DA93149576A123B6278E6B89824
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 2135B875165AC4D35C732B7D41BA64214B9D1B624D832523D591428C3DA752324811639B5657B3C4
 39C21368B7162C9D486E3C2581B75B1A4729
 $\pi_1(14,4) = (91,91,91,91,81,79,78,77,74,73,71,62,42), \text{norm}=79393$
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 55D82168217143D63D48CB3659B1542389CA11ABD74A674925A8134672592347C6D3B47C95649C23
 8A5B762BA45718CB81354561A2592DA8A8164BA1231D74C1732D9B18D4273A515629C6B3A8193CB4
 A77C45B328597C24B612823574A463392456DA5BB6CD24391A8B675C1A29A71348ACDB83179A6219
 B7B4135D56784261483595343CB1828791A7B2CD56418267C3CB13BD89A7957A1148263524936C23
 54C89AA7962451D9428AB27365ADBA3861CC643D928747325DB16C5764D83241B918A5673B16491C
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 7A6249C318195B761DA9498B62353C254A8152C7B3649A43152B7682C189547A1D46721AB63C7D32
 954638AB1DC3B2D5A764A67483512CB543682A719
 $\pi_2(14,4) = (91,91,91,91,80,79,78,78,75,74,71,60,41,1), \text{norm}=79357$
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 95D83168317142C62945DB265AB154328AC911DB8749674A3C981246735A3247C6D2B47CA564AC32
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APPENDIX TABLE (Continued)

977C45B2385A7C34B61383257494622A3456C95BB6C9342A198B675C193A9712489CDE8217A9631A
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 53C78445139B583B2978725C91A43A154296C1B847B29D3B3A78621245A1B5B291783AB39C1489A6
 3C2CA62758434186BA591D79312C8A4BA655A28D1325417B849C326AC96B734856835497D122CDB8
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 A7164618A721348C9B35CA59281382C9B6DA58673134BC1A593218D62C9D64A84BA2795B5627D431
 79634AD2181A5B761DCA4A8B63252C35498153C7B624A492153B7683C18A54791D467319B62A7D23
 A546289B1DC2B3D59764967482513CB542683971A
 $\pi_3(14.4) = (91.91, 91.91, 81.79, 79.77, 76.71, 67.67, 38.2)$, norm=79399
 71C9462818C913591BA52CDE141B3867149C64325E8B2A7597B463A36279C18B35C842435696248B
 1D7C43258673DA28B39C8927A8B12CA3A416295DC45B713D286B7A9CB4A7593D6B8516434287C372
 55A9317931D142672A456B2758B1C43298CA116A9D4D764538A9124763583246C7D2B46C85748D332
 9A5B673BA45C196B91254571A3583DA9AC174BA1321B64C1623B8D19C4362A515738C7B2A9182DB4
 AC6D458239586C34A713932564A4722834579B56B7DC342D1A9B765C1A38A61249AC8B9216DA7318
 B6C41259576C4371492585242CB1939681A6B3CA57419376B2CB12BA68CD85691149372534827B32
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 3C2C872659434197B85A196A312CB84BC755829A1325416C84AC327BCA7B6349579354A6D1225ABD
 382A67B349C5C27B4583264752691B194B21A5C43B681D7369AC1A6178753AD42958641343896724
 861747198621349CDB38C85A291392CA57B859763134BC185A3219C72DAC74894C826A5B5726D431
 6D7348C219185B671CA8489B73252B354C9153A6B72484A2153D6793C198546A1847631A872D6C23
 854729AB1CD2B3D5A674A76492513CB542793A618
 $\pi_4(14.4) = (91.91, 91.91, 82.79, 78.77, 75.72, 67.62, 45)$, norm=79269
 638C49283B7C315B3A55279A343617CC348794125BA62865BCA49171927DB3CD15B742415B7924C7
 3C764125AD618B2D8165AD29BCB32891A4372856D4A6731B2C9DA67864BA5D1869A53841429B7182
 55C913771353426D284A69285D635412BA9C339B7A467D4815DA3246815D12496782947C9584AD12
 6A5B7B1C64593A8A73254583B15A17D8D93B48631237A4C3621CB93B741A2C535918BD62C8382694
 C79B45621A5C8D149531A12DB464822814677B5C98BA142835C8769C391A6A324DB6A9723DB65138
 CC54327A97B541D34725A5242CA3717D9369618C5B43C19D82893296B67A75DC334A192514726C12
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 192CA9275D41436A79583AB6132ACB467955C2D6312543B8748C12A6B96A81475D71548BA32257AA
 152B97A14865D2C84C612D4A725D36384D2368C416B93B7189CD39A3CC6818B4297A54314167BA24
 6839483BAC23149B751AAD5B2731C2875699568713146B385A123DC92A9C74664B9278575C296413
 6C7148923D3957BA36B847CA612528154693519C6824A4B235178CD1B39D5496364CB1369A27CB21
 A5452A6D3C82A1759864AC7482531B85427A19B36
 $\pi_1(14.5) = (169.169, 165.156, 156.152, 149.144, 143.137, 134.121, 118, 80.9)$, norm=291280
 864C73E574DB519391AB23D415CE4B1A7B989A7CC34E2B68AD132D6CAC65219B8A358D1219448B35
 ADE62AC62747B29568714453327DB9A3AD571BE2998B659126BAB828C695A4C276AC189978345B73
 656A8A9E3727245AEE2F11E2B3D8B75451793841BD9AC7297736142B5CE6939352A14DCD1C95B463
 D6EDC87A98121EBA7382528A64B72B616289EC5493DD8C5275D873DE9A6816C95425C16844A81372
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 43C3A48771C6531BE2B5E6D937281A4C19834D7A16C34E1D3B1697AB9CB659E442838C2A54F3DBC2
 9B4185275DA16395DB286E194531D24B5A6372A5BCA4E86159D47C65A9BC3139C153C4A221B76B39
 83251DAD47B87E6CB5587DB139271D56A62495C48C562A1422C587E69EA32E67D94A517368C394B7
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APPENDIX TABLE (Continued)

69B3DEC769E4346A5A57B3DC8171B29318B4DA3ADBC5E12E24C67615A3E8891435C646C72819D565
 42A2ADE12CD743D54C91BDA378FE6C723792164BD18946A1B5A85273D5D434C9E6219BBE6371A68A
 5719EB3A97C491C35EAB5F42D86243B7183492B65712AC78D1942673264957A1687123DD325BCB42
 68E69187A598AE43C17B9865CDA9D4452A3CB52B1A4EC32C7136CE3A6172B43A8914175D269A2181
 B2952BA73DC648B96354AA4792CBE9618A75735B846B283C9D435A62137284551476C312B8D9A5DE
 A29B6471C3DE82563C914B967D14C258C1397CDABE7E43C98B746851235A12BBDE9AB235C61D3EB2
 1895529A3E8D19562D4D43786BEDB4329657526B8A138C98C751421A463C792BEC1A736B854294D2
 61367842C9A6D3CB8134C9A15B248658337D121E64D2A75B954CA78E3746912C91A236E81B9D8AC3
 57

$\pi_3(14.5) = (169, 169, 165, 156, 155, 153, 151, 147, 143, 137, 134, 120, 112, 76, 15)$, norm=290646

964D72E574CB518281AB32C415DC4B1A7B898A76D24C3B69EE123A6DAE65318B9A259E1318449B25
 ACE63AD63747B38569714452237CBDA2AC571BE3889B658136BAB939D685A4D376AD1988C9245B72
 656A9A8E2737345E8E2B11E3B2C9B75451782941BC8AD7387726143B5DE6828253A14CDC1D85B462
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 63A97B6EB325419A54E197A62D2963C586EA7294515B17391C68B452D3328696A4B496812BE39C28
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 42D2A9771D6521B83B5E6DD2739184E1892CA7A16C24B1D2B1687AB8EB658E443929D3A54D2CBE3
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 68B2ECD7A864246E5A57B2CC9171B38219B4EA2A8BD5C13E34D67615D2E9981425A646D72918C565
 43A3ABE13BC742C54D819BA27DFE6E732783164BC1984691B5A95372C5C424C8D6318BB86271A69A
 5718DB2AC7D481D25EAB5C43A96342B7192483B65713A579F1843672364E57A1697132EC235BDB43
 69E68197A58DAC42B17B8965CEA8F4453A2DB53B1A4BF23D7126EC2A6173942CE814175C368A3191
 B3853BA72CD649B86254AA4783CDC8619A75725D946B392D8C425A63127394551476D213E9C8A5DA
 F38B6471D2CB93562F814E867A14D359C1287BD8A67B42B89A746951325913CBCE8AB325D61A2EB3
 198553CA2E9D18563E4C42796B5CB4238657736C9A129C89A751431D462A783ECF1D726B9543C483
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 57

$\pi_3(14.5) = (169, 169, 165, 156, 155, 153, 151, 147, 142, 137, 133, 124, 109, 75, 16, 1)$, norm=290288

964E73E574DB518381AB23D415CD4B1A7B898A76C34D2B69EE132A6CAC65218B9A359C1218449B35
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 656A9A8E3727245A8E2B11D2B3C9B75451783941BD8AD7287736142B5CE6838352A14CDC1C85B463
 E6EAC97A89121EBA6392529A64B72DB1629A7C5483ED8C5275D973EC8A6916C85425C16844D91372
 AEB37CD41293ED241A538E819ED5A6DB7539168AA4912BDBC1879B7234953482DD3967CAE14157AE
 371A3514126FA3B7C2664AB9834C8C5D2B46516D7AC539B26B72C982143856DA18C7A5E62B37468A
 3D17C491326974C51B5623482E7215C4A59D67599B886C3A84D7543727DB3634C16D8192ED172B58
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 5A12B75C123584945AAB73CCA617235DBE68D1BB7EA672C4D636895D759CA12382A9B1475B3C7213
 45694251D928235C14A7C6D1DE96C817926938DA756361D94B245A587B2C84476398614B67BAD227
 43C3A49771C6531B82B5E6DC3729184E1893DA7A16D34E1D3B1687AB8CB658E442939C2A54E3DBF2
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 93251DAC47E97A6CB5597DA13B271856A62485C49C562E1422C597B68DA32D67B84A517369C384E7
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 68B3DEC7A864346D5A57B3DC9171B28319B4DA3A8BC5E12E24C67615C3E9981435C646C72918D565

APPENDIX TABLE (Continued)

42A2ABE12CA743B54D81DBA379FC6C723782164BD19846A1B5A95273D5A434C8GD218BB86371A69A
 5718CB3AD7C481C35EAE5F42F96243B7193482B65712AC79D1842673264D57A1697123CE325BCB42
 69D68197A589AD43B17B8965EBA8D4452A3EC52B1A4BE32C7136CE3A6172B43A9814175D268A2191
 C2852BA73DC649B86354AA4782DCD8619A75735B946B293C8D435A62137294551476C312E9D8A5FD
 A28B6471D3EB92563E814C867A14C259D1387BC8A67B43B89A746951235912CEDF8AB235C61A3EB2
 198552EA3C9E18562F4D43796B5DB4328657726D9A139D89C751421E463A782FEC1D736B9542D482
 61367B428DE6A38B9134C8A15E24965C337D121C94D2A75B854DA79A3746812B81A236F91B8E9AC3
 57
 $\pi_4(14,5) = (169,169,163,156,155,152,149,148,142,139,132,131,102,76,19)$, nonn=289872
 B4763834A858B2AC52D415EC2DB86C294F956DB47DA9134A772E1EB69D5A12AB346A8B2127B6B743
 679416531898F1797C32835D419C637A496C25B1EF568A721949C51386E57A91D8C326B5EBA7469D
 3C4366B57F17139B8D1A2281C8AAC74362583B42AE9C751A57962E1DCD65A89D41423C552BAEC436
 B8A439B457212976E391D14CC3BB146231587A7985C9DAE1379A53B4A58A2646871892A784962CB1
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 C629752A216B4B4E51D37CEAC57BEAB31976425BD8A393C17391468127956FD82C6537BD1357AC65
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 55A9B1625C1B16672DBC3982A7C7A42481DBCC3857B7324E5B1373598914AF578D89326E64BC411C
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 8432961CD7C26E4C831A6524579231B7E869C187B34D5B6259C37C64A7DB8236724A8E41125639AD
 681C2B5B493EFC3BADA79B32E4182835641BC9A57843162911E4B36735A91AF85D7762C9848D56C4
 2A91A3A628514D76D23849784D299C5EA531D2B634859378BAC14A152632159B47839A58BE6A71CC
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 C2561B31AE53C5829B12A4786F1E872A6D3C1745C8D69831247EA7514AEBDC93E5A1264D8D9B3432
 67C5D8B83967C537EB5CE598E2C2A1462E3B9DA4959C821C173FEB2A546C392D4AB39A8E13294BCB
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 3926394574AB72CCEB9D834168A145672DAB5144C32145B87268196517B3A532786215B8B15C96A1
 83D3924ECBAEC9D5827BE5399C7BED4616483E16243DFB13925A46BD527156C3492B2A781FD81252
 B14A1BD9934A785AC83769C3518864A276C56A59CDAE13B844597E312A5163AC26875C21487B5689
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 24B73169B5A326781AD7EBDC45A566717E9C91C46823A454B892312345B8961E382958C669B14D51
 7283A4D17996DA7C32C473A2B51675CA4B3A21297C51C86497DB453843E5621382D1B7CA2E85D3A6
 B9

phism program NAUTY ([134], [135]). We thank him for letting us use this program.

APPENDIX
 TABLES OF PARTITIONS

This Appendix gives a number of partitions of the set of all $\binom{n}{w}$ binary vectors of length n and weight w into disjoint codes with distance 4 (see Section VI and especially Table VI). To explain the notation,

$\pi(6,3) = (4,4,4,4,2,2)$, norm = 72: 23541364214136122543

indicates that there is a partition of the 20 6-bit, weight-3 binary words into 6 disjoint distance-4 codes, of respective sizes 4, 4, 4, 4, 2, and 2. The norm is $4^2 + 4^2 + 4^2 + 2^2 + 2^2 = 72$. The lexicographically first word of $\Pi(6,3)$ (namely 000111) is in code 2, the second word (namely 001011) is in code 3, the third is in code 5, ... and the 20th word (namely 111000) is in code 3.

The digit-string giving the code-numbers (or "colors") is in a constant-width font with 80 characters per line, and the colors are represented by 1, 2, 3, ..., 9, A, B, C,

REFERENCES

- [1] E. F. Assmus, Jr. and J. D. Key, "On an infinite class of Steiner systems with $t = 3$ and $k = 6$," *J. Combin. Theory*, vol. 42A, pp. 55-60, 1986.
- [2] E. F. Assmus, Jr. and H. F. Mattson, Jr., "Disjoint Steiner systems associated with the Mathieu groups," *Bull. Amer. Math. Soc.*, vol. 72, pp. 843-845, 1966.
- [3] ———, "New 5-designs," *J. Combin. Theory*, vol. 6, pp. 122-151, 1969.
- [4] R. D. Baker, "Partitioning the planes of $AG_{2m}(2)$ into 2-designs," *Discrete Math.*, vol. 15, pp. 205-211, 1976.
- [5] R. D. Baker, J. H. van Lint, and R. M. Wilson, "On the Preparata and Goethals codes," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 342-345, 1983.
- [6] E. Balas, V. Chvátal, and J. Nešetřil, "On the maximum weighted clique problem," *Math. Oper. Res.*, vol. 12, pp. 522-535, 1987.
- [7] E. Balas and C. S. Yu, "Finding a maximal clique in an arbitrary graph," *SIAM J. Comput.*, vol. 15, pp. 1054-1068, 1986.
- [8] L. D. Baumert, "Cyclic difference sets," *Lecture Notes Math.*, vol. 182. New York: Springer-Verlag, 1971.
- [9] S. Bays, "Une question de Cayley relative au problème des triades de Steiner," *Enseignement Math.*, vol. 19, pp. 57-67, 1917.
- [10] C. Berge, "A new color change to improve the coloring of a graph," *Discrete Appl. Math.*, vol. 24, pp. 25-28, 1989.
- [11] M. R. Best, " $A(11,4,4) = 35$ or some new optimal constant weight codes," Math. Centr. Report ZN 71/77, Amsterdam, Feb. 1977.

- [12] —, "Binary codes with a minimum distance of four," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 738–742, 1980.
- [13] —, A. E. Brouwer, F. J. MacWilliams, A. M. Odlyzko, and N. J. A. Sloane, "Bounds for binary codes of length less than 25," *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 81–93, 1978.
- [14] T. Beth, D. Jungnickel, and H. Lenz, *Design Theory*. New York: Cambridge University Press, 1986.
- [15] M. Blåum, "A (16, 9, 6, 5, 4) error-correcting dc free block code," *IEEE Trans. Inform. Theory*, vol. 34, no. 1, pp. 138–141, Jan. 1988.
- [16] B. Bose and T. R. N. Rao, "Theory of unidirectional error correcting/detecting codes," *IEEE Trans. Comput.*, vol. 31, pp. 521–530, 1982.
- [17] R. C. Bose and S. Chowla, "Theorems in the additive theory of numbers," *Comment. Math. Helvet.*, vol. 37, pp. 141–147, 1962–63.
- [18] R. C. Bose and W. S. Connor, "Combinatorial properties of group divisible incomplete block designs," *Ann. Math. Stat.*, vol. 23, pp. 367–383, 1952.
- [19] D. R. Breach and A. P. Street, "Partitioning sets of quadruples into designs II," *J. Comb. Math. Comb. Computing*, vol. 3, pp. 41–48, 1988.
- [20] A. E. Brouwer, " $A(17, 6, 4) = 20$ or the nonexistence of the scarce design $SD(4, 1; 17, 21)$," Math. Cntr. Report ZW62, Amsterdam, Dec. 1975.
- [21] —, "On the packing of quadruples without common triples," *Ars. Comb.*, vol. 5, pp. 3–6, 1978.
- [22] —, "Optimal packings of K_4 's into a K_n ," *J. Comb. Theory*, vol. 26A, pp. 278–297, 1979.
- [23] —, "A few new constant weight codes," *IEEE Trans. Inform. Theory*, vol. 26, p. 366, 1980.
- [24] —, "Some unitals on 28 points and their embeddings in projective planes of order 9," in *Geometries and Groups*, M. Aigner and D. Jungnickel, Eds., Lect. Notes Math., vol. 893. New York: Springer-Verlag, 1981, pp. 183–188.
- [25] —, "Partial linear spaces with one point and one line more than affine planes—Report on results of K. Metsch as communicated by A. Blokhuis." unpublished manuscript, 1989.
- [26] —, "A unital in the Hughes plane of order nine," preprint.
- [27] —, "Block designs," *Discrete Math.*, vol. 77, pp. 55–56, 1989.
- [28] A. E. Brouwer, P. Delsarte and P. Piret, "On the (23, 14, 5) Wagner code," *IEEE Trans. Inform. Theory*, vol. 26, pp. 742–743, 1980.
- [29] A. E. Brouwer, A. Schrijver, and H. Hanani, "Group divisible designs with block-size four," *Discrete Math.*, vol. 20, pp. 1–10, 1977.
- [30] P. J. Cameron, *Parallelisms of Complete Designs*. New York: Camb. Univ. Press, 1976.
- [31] P. J. Cameron and J. H. van Lint, *Graphs, Codes and Designs*. New York: Camb. Univ. Press, 1980.
- [32] A. Cayley, "On the triadic arrangements of seven and fifteen things," *Phil. Mag.*, vol. 37, pp. 50–53, 1850; *Collected Mathematical Papers*, vol. I, pp. 481–484.
- [33] Y. Chen, "The Steiner system $S(3, 6, 26)$," *J. Geometry*, vol. 2, pp. 7–22, 1972.
- [34] Z. Chen, F. Jin, and P. Fan, "On a new binary (22, 13, 5) code," *IEEE Trans. Inform. Theory*, vol. 36, no. 1, pp. 228–229, Jan. 1990.
- [35] W. R. Cheswick, personal communication.
- [36] L. G. Chouinard II, "Partitions of the 4-subsets of a 13-set into disjoint projective planes," *Discrete Math.*, vol. 45, pp. 297–300, 1983.
- [37] H. Chung and P. V. Kumar, "Optical orthogonal codes—New bounds and an optimal construction," *IEEE Trans. Inform. Theory*, vol. 36, no. 4, July 1990.
- [38] F. N. Cole, L. D. Cummings, and H. S. White, "The complete enumeration of triad systems in 15 elements," *Proc. Nat. Acad. Sci. USA*, vol. 3, 1917, pp. 197–199.
- [39] S. D. Constantin and T. R. N. Rao, "On the theory of binary asymmetric error correcting codes," *Inform. Contr.*, vol. 40, pp. 20–36, 1979.
- [40] J. H. Conway, "The miracle octad generator," in *Topics in Group Theory and Computation*, M. P. J. Curran, Ed. New York: Academic Press, 1977, pp. 62–68.
- [41] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *ATLAS of Finite Groups*. New York: Oxford Univ. Press, 1985.
- [42] J. H. Conway, R. A. Parker, and N. J. A. Sloane, "The covering radius of the Leech lattice," *Proc. Roy. Soc. Lon.*, vol. 380A, pp. 261–291, 1982.
- [43] J. H. Conway and N. J. A. Sloane, "Laminated lattices," *Ann. Math.*, vol. 116, pp. 593–620, 1982.
- [44] —, "Lexicographic codes: Error-correcting codes from game theory," *IEEE Trans. Inform. Theory*, vol. 32, pp. 337–348, 1986.
- [45] —, *Sphere Packings, Lattices and Groups*. New York: Springer-Verlag, 1988.
- [46] —, "Orbit and coset analysis of the Golay and related codes," *IEEE Trans. Inform. Theory*, vol. 36, no. 5, pp. 1038–1050, Sept. 1990.
- [47] R. T. Curtis, "On subgroups of $\cdot 0. 1$. Lattice stabilizers," *J. Alg.*, vol. 27, pp. 549–573, 1973.
- [48] —, "A new combinatorial approach to M_{24} ," *Math. Proc. Camb. Phil. Soc.*, vol. 79, pp. 25–42, 1976.
- [49] N. Darwish and B. Bose, "New lower bounds for constant weight and asymmetric error correcting codes," preprint.
- [50] P. Delsarte and P. Piret, "Spectral enumerators for certain additive-error-correcting codes over integer alphabets," *Inform. Contr.*, vol. 48, pp. 193–210, 1981.
- [51] —, "Bounds and constructions for binary asymmetric error-correcting codes," *IEEE Trans. Inform. Theory*, vol. 27, pp. 125–128, 1981; correction, vol. 36, no. 4, p. 954, July 1990.
- [52] P. Dembowski, *Finite Geometries*. Berlin: Springer-Verlag, 1968.
- [53] R. H. F. Denniston, "Some maximal arcs in finite projective planes," *J. Combin. Theory*, vol. 6, pp. 317–319, 1969.
- [54] —, "Some packings with Steiner triple systems," *Discrete Math.*, vol. 9, pp. 213–227, 1974.
- [55] —, "Sylvester's problem of the 15 schoolgirls," *Discrete Math.*, vol. 9, pp. 229–233, 1974.
- [56] —, "Some new 5-designs," *Bull. Lond. Math. Soc.*, vol. 8, pp. 263–267, 1976.
- [57] —, "Four doubly resolvable complete 3-designs," *Ars. Comb.*, vol. 7, pp. 265–272, 1979.
- [58] —, "Enumeration of symmetric designs (25, 9, 3)," *Ann. Discrete Math.*, vol. 15, pp. 111–127, 1982.
- [59] I. Diener, "On cyclic Steiner systems $S(3, 4, 22)$," pp. 301–313 of [128].
- [60] J. Doyen, "Constructions of disjoint Steiner triple systems," *Proc. Amer. Math. Soc.*, vol. 32, pp. 409–416, 1972.
- [61] J. Doyen and A. Rosa, "An updated bibliography and survey of Steiner systems," *Ann. Discrete Math.*, vol. 7, pp. 317–349, 1980.
- [62] —, "A bibliography and survey of Steiner systems," preprint, Feb. 1989.
- [63] G. Dueck and T. Scheuer, "Threshold accepting: A general purpose optimization algorithm appearing superior to simulated annealing," preprint.
- [64] T. Etzion, "Constructions of error-correcting DC-free block codes," *IEEE Trans. Inform. Theory*, vol. 36, no. 4, pp. 889–905, July 1990.
- [65] —, "Optimal partitions for triples," *J. Comb. Theory*, to appear.
- [66] —, personal communication.
- [67] T. Etzion and A. Hartman, "Towards a large set of Steiner quadruple systems," *SIAM J. Discrete Math.*, to appear.
- [68] T. Etzion and C. L. M. van Pul, "New lower bounds for constant weight codes," *IEEE Trans. Inform. Theory*, vol. 35, no. 6, pp. 1324–1329, Nov. 1989.
- [69] A. A. El Gamal, L. A. Hemachandra, I. Shperling, and V. K. Wei, "Using simulated annealing to design good codes," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 116–123, 1987.
- [70] L. Gerhards and W. Lindenberg, "Cliques detection for nondirected graphs: Two new algorithms," *Computing*, vol. 21, pp. 295–322, 1979.
- [71] M. Goldberg and T. Spencer, "A new parallel algorithm for the maximal independent set problem," *SIAM J. Comput.*, vol. 18, pp. 419–427, 1989.
- [72] R. L. Graham and N. J. A. Sloane, "Lower bounds for constant weight codes," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 37–43, 1980.
- [73] —, "On additive bases and harmonious graphs," *SIAM J. Alg. Discrete Methods*, vol. 1, pp. 382–404, 1980.
- [74] M. J. Grannell and T. J. Griggs, "A note on the Steiner systems $S(5, 6, 24)$," *Ars. Comb.*, vol. 8, pp. 45–48, 1979.
- [75] R. K. Guy, *Unsolved Problems in Number Theory*. New York: Springer-Verlag, 1981.
- [76] H. Halberstam and K. F. Roth, *Sequences*. New York:

- Springer-Verlag, 1966.
- [77] M. Hall, Jr., *Combinatorial Theory*. Watham, MA: Blaisdell, 1967.
- [78] M. Hall, Jr. and J. D. Swift, "Determination of Steiner triple systems of order 15," *Math. Tables Aids Comput.*, vol. 9, pp. 146–152, 1955.
- [79] H. Härmäläinen, "Two new binary codes with minimum distance three," *IEEE Trans. Inform. Theory*, vol. 34, p. 875, 1988.
- [80] —, personal communication.
- [81] H. Hanani, "On quadruple systems," *Canad. J. Math.*, vol. 12, pp. 145–157, 1960.
- [82] —, "The existence and construction of balanced incomplete block designs," *Ann. Math. Stat.*, vol. 32, pp. 361–386, 1961.
- [83] —, "A balanced incomplete block design," *Ann. Math. Stat.*, vol. 36, p. 711, 1965.
- [84] —, "Balanced incomplete block designs and related designs," *Discrete Math.*, vol. 11, pp. 255–369, 1975.
- [85] A. Hartman, "The existence of resolvable Steiner quadruple systems," *J. Comb. Theory*, vol. 44A, pp. 182–206, 1987.
- [86] A. Hartman, W. H. Mills, and R. C. Mullin, "Covering triples by quadruples: An asymptotic solution," *J. Combin. Theory*, vol. 41A, pp. 117–138, 1986.
- [87] A. A. Hashim and A. G. Constantinides, "Some new results on binary linear block codes," *Electron. Lett.*, vol. 10, pp. 31–33, 1974.
- [88] A. A. Hashim and V. S. Pozdniakov, "Computerized search for binary linear codes," *Electron. Lett.*, vol. 12, pp. 350–351, 1976.
- [89] H. J. Helgert and R. D. Stinaff, "Minimum-distance bounds for binary linear codes," *IEEE Trans. Inform. Theory*, vol. 19, pp. 344–356, 1973.
- [90] T. Helleseth and T. Kløve, "On group theoretic codes for asymmetric channels," *Inform. Contr.*, vol. 49, pp. 1–9, 1981.
- [91] I. Honkala, "Lower bounds for constant weight codes," (in Finnish), Master's dissertation, Mathematics Dept., Univ. Turku, Turku, Finland, Jan. 1985.
- [92] —, "Some optimal constant weight codes," *Ars. Comb.*, vol. 20B, pp. 43–47, 1985.
- [93] —, personal communication.
- [94] I. Honkala, H. Härmäläinen, and M. K. Kaikkonen, "A modification of the Zinoviev lower bound for constant weight codes," *Discrete Appl. Math.*, vol. 11, pp. 307–310, 1985.
- [95] —, "Some lower bounds for constant weight codes," *Discrete Appl. Math.*, vol. 18, pp. 95–98, 1987.
- [96] D. R. Hughes and F. C. Piper, *Design Theory*. Cambridge: Cambridge Univ. Press, 1985.
- [97] B. Hupert, *Endliche Gruppen I*. New York: Springer-Verlag, 1967.
- [98] D. S. Johnson, C. R. Aragon, L. A. McGeoch, and C. Schevon, "Optimization by simulated annealing: An experimental evaluation. Part 1," *Oper. Res.*, vol. 37, pp. 865–892, 1989.
- [99] S. M. Johnson, "A new upper bound for error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-8, pp. 203–207, 1962.
- [100] —, "Improved asymptotic bounds for error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-9, pp. 198–205, 1963.
- [101] —, "On upper bounds for unrestricted binary error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-17, pp. 466–478, 1971.
- [102] —, "Upper bounds for constant weight error correcting codes," *Discrete Math.*, vol. 3, pp. 109–124, 1972.
- [103] G. A. Kabatiansky and V. I. Panchenko, "Packings and coverings of Hamming space with unit spheres," (in Russian), *Probl. Peredach. Inform.*, vol. 24, no. 4, pp. 3–16, 1988.
- [104] M. K. Kaikkonen, "A new four-error correcting code of length 20," *IEEE Trans. Inform. Theory*, vol. 35, no. 6, p. 1344, Nov. 1989.
- [105] —, personal communication.
- [106] J. G. Kalbfleisch and R. G. Stanton, "Maximal and minimal coverings of $(k-1)$ -tuples by k -tuples," *Pac. J. Math.*, vol. 26, pp. 131–140, 1968.
- [107] W. M. Kantor, "On the inequivalence of generalized Preparata codes," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 345–348, 1983.
- [108] R. E. Kibler, "Some new constant weight codes," *IEEE Trans. Inform. Theory*, vol. 26, pp. 364–365, 1980.
- [109] T. P. Kirkman, "On a problem in combinations," *Cambridge and Dublin Math. J.*, vol. 2, pp. 191–204, 1847.
- [110] T. Kløve, "A lower bound for $A(n, 4, w)$," *IEEE Trans. Inform. Theory*, vol. 27, pp. 257–258, 1981.
- [111] K.-U. Koschnick, "On the construction of some new constant weight codes," preprint.
- [112] A. Kotzig and J. Turgeon, "On certain constructions for Latin squares with no subsquares of order two," *Discrete Math.*, vol. 16, pp. 263–270, 1976.
- [113] E. S. Kramer, "Some triple system partitions for prime powers," *Utilitas Math.*, vol. 12, pp. 113–116, 1977.
- [114] E. S. Kramer and S. S. Magliveras, "Some mutually disjoint Steiner systems," *J. Combin. Theory*, vol. 17A, pp. 39–43, 1974.
- [115] E. S. Kramer and D. M. Mesner, "Intersections among Steiner systems," *J. Combin. Theory*, vol. 16A, pp. 273–285, 1974.
- [116] —, "The possible (impossible) systems of 11 disjoint $S(2, 3, 13)$'s $(S(3, 4, 14)$'s)," *Utilitas Math.*, vol. 7, pp. 55–58, 1975.
- [117] —, "Admissible parameters for Steiner systems $S(t, k, v)$ with a table for all $(v-t) \leq 498$," *Utilitas Math.*, vol. 7, pp. 211–222, 1975.
- [118] —, " t -Designs on hypergraphs," *Discrete Math.*, vol. 15, pp. 263–296, 1976.
- [119] D. L. Kreher and S. P. Radziszowski, "The existence of simple $6-(14, 7, 4)$ designs," *J. Combin. Theory*, vol. 43A, pp. 237–243, 1986.
- [120] C. W. H. Lam, L. Thiel, and S. Swiercz, "The non-existence of finite projective planes of order 10," *Canad. J. Math.*, vol. 41, pp. 1117–1123, 1989.
- [121] H. Lenz, "Tripling Steiner quadruple systems," *Ars. Combin.*, vol. 20, pp. 193–202, 1985.
- [122] V. I. Levenshtein, "The application of Hadamard matrices to a problem in coding," *Probl. Kibernetiki*, vol. 5, pp. 123–136, 1961. English translation in *Probl. Cybern.*, vol. 5, pp. 166–184, 1961.
- [123] Shen Lin, personal communication.
- [124] C. C. Lindner, "A note on disjoint Steiner quadruple systems," *Ars. Combin.*, vol. 3, pp. 271–276, 1977.
- [125] —, "On the construction of pairwise disjoint Steiner quadruple systems," *Ars. Combin.*, vol. 19, pp. 153–156, 1985.
- [126] C. C. Lindner and A. Rosa, "There are at least 31,021 non-isomorphic Steiner quadruple systems of order 16," *Utilitas Math.*, vol. 10, pp. 61–64, 1976.
- [127] —, "Steiner quadruple systems—A survey," *Discrete Math.*, vol. 22, pp. 147–181, 1978.
- [128] C. C. Lindner and A. Rosa, Eds., "Topics on Steiner systems," *Annals of Discrete Mathematics*, vol. 7. Amsterdam: North-Holland, 1980.
- [129] J. H. van Lint, "A new description of the Nadler code," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 825–826, 1972.
- [130] J. X. Lu, "On large sets of disjoint Steiner triple systems I–VI," *J. Combin. Theory*, vol. 34A, pp. 140–183, 1983; vol. 37A, pp. 136–192, 1984.
- [131] —, "On large sets of disjoint Steiner triple systems VII," unfinished manuscript.
- [132] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam: North-Holland, 1979.
- [133] R. Mathon and A. Rosa, "Tables of parameters of BIBDs with $r \leq 41$ including existence, enumeration, and resolvability results," *Ann. Discrete Math.*, vol. 26, pp. 275–308, 1985.
- [134] B. D. McKay, "NAUTY user's guide (version 1.2)," Dept. Comput. Sci. Tech. Rep. TR-CS-87-03, Australian Nat. Univ., Canberra, Australia, 1987.
- [135] —, "NAUTY notes," preprint, Feb. 2, 1989.
- [136] E. Mendelsohn and A. Rosa, "One-factorizations of the complete graph—A survey," *J. Graph Theory*, vol. 9, pp. 43–65, 1985.
- [137] N. S. Mendelsohn and S. H. Y. Hung, "On the Steiner systems $S(3, 4, 14)$ and $S(4, 5, 15)$," *Utilitas Math.*, vol. 1, pp. 5–95, 1972.
- [138] W. H. Mills, "A new 5-design," *Ars. Combin.*, vol. 6, pp. 193–195, 1978.
- [139] R. C. Mullin and A. Rosa, "Orthogonal Steiner systems and generalized Room squares," in *Proc. 6th Manitoba Conf. Num. Math.*, Winnipeg, 1976. Congr. Numer. XVIII, Util. Math. Publ. Inc., Winnipeg, 1977, pp. 319–323.
- [140] P. M. Pardalos and G. P. Rodgers, "A branch and bound algorithm for the maximum clique problem," preprint.
- [141] I. Peterson, Search yields math proof no one can check," *Science*, vol. 134, p. 406 Dec. 1988.
- [142] K. T. Phelps, "On cyclic Steiner quadruple systems $S(3, 4, 20)$," pp. 277–300 of [128].
- [143] —, "A class of large sets of Steiner triple systems of order 15," preprint.
- [144] —, "A class of 2-chromatic SQS(22)," preprint.
- [145] —, personal communication.

- [146] K. T. Phelps and A. Rosa, "2-chromatic Steiner quadruple systems," *Europ. J. Combin.*, vol. 1, pp. 253-258, 1980.
- [147] P. Piret, "Good linear codes of length 27 and 28," *IEEE Trans. Inform. Theory*, vol. IT-26, p. 227, 1980.
- [148] V. Pless, "The children of the (32, 16) doubly even codes," *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 738-746, 1982.
- [149] C. L. M. van Pul, "On bounds on codes," M.Sc. dissertation, Dept. of Math. and Comput. Sci., Eindhoven Univ. of Technol., Aug. 1982.
- [150] —, "Some distance problems in coding theory," Ph.D. dissertation, Dept. of Math. and Comput. Sci., Eindhoven Univ. of Technol., 1987.
- [151] S. Rankinen, personal communication.
- [152] D. K. Ray-Chaudhuri and R. M. Wilson, "Solution of Kirkman's schoolgirl problem," in *Combinatorics*, T. S. Motzkin, Ed., *Proc. Symp. Pure Math.*, vol. 19, pp. 187-203, 1971.
- [153] M. Reiss, "Über eine Steinersche Combinatorische Aufgabe welche im 45sten Bande dieses Journals. Seite 181, gestellt worden ist," *J. Reine Angew. Math.*, vol. 56, pp. 326-344, 1859.
- [154] V. Rödl, "On a packing and covering problem," *Europ. J. Combin.*, vol. 6, pp. 69-78, 1985.
- [155] A. M. Romanov, "New binary codes with minimal distance three," (in Russian). *Probl. Peredach. Inform.*, vol. 19, no. 3, pp. 101-102, 1983.
- [156] A. Rosa, R. A. MATHON, and K. T. Phelps, "Small Steiner triple systems and their properties," *Ars. Combin.*, vol. 15, pp. 3-110, 1983.
- [157] J. Schönheim, "On maximal systems of k -tuples," *Studia Scient. Math. Hungar.*, vol. 1, pp. 363-368, 1966.
- [158] S. Schreiber, "Covering all triples on n marks by disjoint Steiner systems," *J. Combin. Theory*, vol. 15A, pp. 347-350, 1973.
- [159] N. V. Semakov and V. A. Zinoviev, "Complete and quasi-complete balanced codes," (in Russian). *Probl. Peredach. Inform.*, vol. 5, no. 2, pp. 14-18, 1969. English translation in *Probl. Inform. Transm.*, vol. 5, no. 2, pp. 11-13, 1969.
- [160] —, "Balanced codes and tactical configurations," (in Russian). *Probl. Peredach. Inform.*, vol. 5, no. 3, pp. 28-37, 1969. English translation in *Probl. Inform. Transm.*, vol. 5, no. 3, pp. 22-28, 1969.
- [161] M. J. Sharry and A. P. Street, "Partitioning sets of triples into designs I," *Ars. Combin.*, vol. 26B, pp. 51-66, 1988.
- [162] —, "Partitioning sets of triples into designs II," *J. Combin. Math. Combin. Computing*, vol. 4, pp. 53-68, 1988.
- [163] S. S. Shrikhande, "Affine resolvable balanced incomplete block designs: A survey," *Aequat. Math.*, vol. 14, pp. 251-269, 1976.
- [164] N. J. A. Sloane, S. M. Reddy, and C. L. Chen, "New binary codes," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 503-510, 1972.
- [165] N. J. A. Sloane and J. J. Seidel, "A new family of codes obtained from conference matrices," *Ann. NY Acad. Sci.*, vol. 175 Article 1, pp. 363-365, 1970.
- [166] N. J. A. Sloane and D. S. Whitehead, "A new family of single-error correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 717-719, 1970.
- [167] Warren D. Smith and N. J. A. Sloane, "Bounds for spherical codes," preprint.
- [168] R. G. Stanton and I. P. Goulden, "Graph factorization, general triple systems, and cyclic triple systems," *Aequat. Math.*, vol. 22, pp. 1-28, 1981.
- [169] R. G. Stanton and M. J. Rogers, "Packings and coverings by triples," *Ars. Combin.*, vol. 13, pp. 61-69, 1982.
- [170] D. Stinson, "Determination of a packing number," *Ars. Combin.*, vol. 3, pp. 89-114, 1977.
- [171] L. Teirlinck, "On the maximum number of disjoint Steiner triple systems," *Discrete Math.*, vol. 6, pp. 299-300, 1973.
- [172] —, "On making two Steiner triple systems disjoint," *J. Combin. Theory*, vol. 23A, pp. 349-350, 1977.
- [173] —, "On large sets of disjoint ordered designs," *Ars. Combin.*, vol. 25, pp. 31-37, 1988.
- [174] —, "A completion of Lu's determination of the spectrum for large sets of disjoint Steiner triple systems," preprint.
- [175] H. C. A. van Tilborg and M. Blaum, "On error-correcting balanced codes," *IEEE Trans. Inform. Theory*, vol. 35, no. 5, pp. 1091-1095, Sept. 1989.
- [176] V. D. Tonchev, *Combinatorial Configurations, Designs, Codes, Graphs*. New York: Longman, Harlow, 1988.
- [177] T. Verhoeff, "An updated table of minimum-distance bounds for binary linear codes," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 665-680, 1987.
- [178] —, "An updated table of minimum-distance bounds for binary linear codes," (revised), preprint, Jan. 1989.
- [179] T. J. Wagner, "A search technique for quasi-perfect codes," *Inform. Contr.*, vol. 9, pp. 94-99, 1966.
- [180] J. H. Weber, C. de Vroedt, and D. E. Boekke, "Bounds and constructions for binary codes of length less than 24 and asymmetric distance less than 6," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1321-1331, 1988.
- [181] R. M. Wilson, "Some partitions of all triples into Steiner triple systems," in *Hypergraph Seminar*, Ohio State Univ., 1972; *Lecture Notes Math.*, vol. 411. New York: Springer-Verlag, 1974, pp. 267-277.
- [182] E. Witt, "Die 5-fach transitiven Gruppen von Mathieu," *Abh. Math. Sem. Univ. Hamburg*, vol. 12, pp. 256-264, 1938.
- [183] —, "Ueber Steinersche Systeme," loc. cit., vol. 12, pp. 265-275, 1938.
- [184] A. Zaptcioglu, "Entwicklung störsicherer Binärcodes," Diplomarbeit, Lehrstuhl für Informatik, Technische Universität Braunschweig, Germany, Aug. 6, 1981.
- [185] L. Zhu, "A pair of orthogonal Steiner systems $S(3,5,17)$," *Discrete Math.*, vol. 31, pp. 111-113, 1980.
- [186] V. A. Zinoviev, "Cascade equal-weight codes and maximal packings," *Probl. Contr. and Inform. Theory*, vol. 12, no. 1, pp. 3-10, 1983.
- [187] —, "On a generalization of the Johnson bound for constant-weight codes," (in Russian). *Probl. Peredach. Inform.*, vol. 20, no. 3, pp. 105-108, 1984.
- [188] V. A. Zinoviev and S. N. Litsyn, "On recurrent relations for the cardinality of equal-weight codes," in *Proc. Internat. Workshop Algebraic and Combin. Coding Theory*, Varna, Bulgaria, 1988, pp. 181-183.