A NEW THEORY OF DEPRECIATION OF PHYSICAL ASSETS

By Robert E. Moritz

1. The various methods employed for the computation of the depreciation of a physical asset are as many devices for recovering, by means of a yearly charge to production during the life-time of the asset, its reduction in value. The methods differ according to the answers given to such questions as the following: Should the yearly charge be based on original cost or on replacement value? Should the yearly charge be uniform over the life-time of the asset? If not uniform, should the depreciation charge be proportional to the actual reduction in market value, or, in the case of new plant, should such charges be minimized or wholely deferred during the earlier years when the plant is trying to establish itself? Should interest be disregarded, or should the yearly charges to production be accumulated with interest? If interest is to be considered, should the rate be the effective rate on the capital employed in the business, or the commercial rate? All these questions and others have received careful consideration.[1]

Quite as important as those already mentioned are two other considerations, which have been generally ignored or overlooked. There is first the rather obvious fact that depreciation is inseparably tied up with the question of repairs. Depreciation is greatest when the asset is new, when repair charges are negligible, and it diminishes as repair costs grow. The problem of depreciation, therefore, cannot be adequately treated aside from repairs. Within certain limits depreciation may be compensated by repairs. The yearly depreciation charge to production, therefore, should not be based on original cost or renewal cost alone, but on original cost plus costs of repairs during the life-time of the asset.

In the second place, the life-time of an asset is not a constant

[1]See Saliers, E. A., Depreciation, Principles and Applications, New York (1922).

as is generally assumed, but is a variable which, like depreciation itself, is definitely related to the repair function. Aside from obsolescence, the value of an asset could be kept practically intact indefinitely by sufficiently increasing the outlay for repairs. There is always a threshold period of time when it is a question whether to scrap or to continue to repair, and frequently this threshold extends over a period of years. In short, the life-time of an asset is generally an unknown quantity, the determination of which requires the solution of an equation which expresses the condition that the annual charge to production, necessary to recover the original cost of an asset together with all repair costs, shall be a minimum.[2]

The present paper is an attempt to treat the problem of depreciation from the point of view here suggested. The problem, then, is to determine the life-time of an asset such that the annual charge to production, necessary to cover original cost and all repair charges, shall be a minimum.

2. Let us denote the original cost of the asset by C, the cost of repairs during the first x years by $R(x)$, then the total outlay to be recovered is $C + R(x)$. Furthermore, let $U(x)$ denote the average yearly charge to production necessary to recover the total outlay in x years. Then, disregarding for the present all interest considerations, we have

$$U(x) = \frac{C + R(x)}{x} \tag{1}$$

The notation $U(x)$ suggests that, in general, the unit charge to production will be a function of x.

We shall now define the life-time of an asset as that value of x which will render the value of $U(x)$ in (1) a minimum.

[2] J. S. Taylor, A Statistical Theory of Depreciation, Journal of the American Statistical Association, Vol. 18 (1923), p. 1010, is, I believe, the first writer who recognized in part the principle here set forth. He calls attention to the fact that the useful life of a machine depends both on the manner of distributing depreciation charges and on the assumed interest rate.

The analytical conditions that a function of x may have a minimum are that the first derivative of the function with respect to x be zero and that the second derivative of the function with respect to x be positive. If, as is customary, we denote the first and second derivatives of $U(x)$ by $U'(x)$ and $U''(x)$ respectively, and those of $R(x)$ by $R'(x)$ and $R''(x)$ respectively, we find on differentiating

$$U'(x) = \frac{xR'(x) - C - R(x)}{x^2}$$

$$U''(x) = \frac{x^2R''(x) - 2xR'(x) + 2R(x) + 2C}{x^3} = \frac{R''(x) - 2U'(x)}{x},$$

from which it is evident that the life-time of an asset must satisfy the two conditions

$$(2) \qquad xR'(x) - C - R(x) = 0, \qquad R''(x) > 0$$

In short, the life-time of an asset is given by that root of the equation $xR'(x) - R(x) = C$ which will make $R''(x) > 0$.

For example, let us suppose that the repair function is given by the equation $R(x) = ax^2 + bx + c$. Then $R'(x) = 2ax + b$, $R''(x) = 2a$, and the conditions (2) reduce to

$$ax^2 = C + c, \qquad a > 0$$

The life-time of the asset is therefore equal to $\sqrt{(C+c)/a}$ provided the coefficient a is positive. It is interesting to observe that x is independent of the constant b.

3. In the preceding discussion no allowance was made for the salvage value of the asset. Let us denote the scrap-value of the asset after x years by $S(x)$, then the average yearly de-

preciation, interest again not considered, is

$$(3) \qquad U(x) = \frac{C + R(x) - S(x)}{x}.$$

and the conditions which will make $U(x)$ a minimum are

$$(4) \quad x\left[R'(x) - S'(x)\right] - \left[C + R(x) - S(x)\right] = 0, \quad R''(x) > S''(x).$$

If the scrap-value is a constant, both $S'(x)$ and $S''(x)$ vanish, and the life-time of the asset is determined by

$$(5) \qquad xR'(x) - C - R(x) + S(x) = 0, \quad R''(x) > 0.$$

The conditions (2) include (4) if we replace $R(x)$ by $R(x) - S(x)$, that is, if in the outset we diminish the repair function by the salvage value at time x ; to include (5) it is sufficient to replace C, the original cost of the asset, by $C - S$, the difference between original cost and scrap value. With these modifications we may treat (2) as representing the general case.

4. To avoid any possible confusion, let us denote by $T(x)$ the total outlay to be recovered by uniform annual charges to production during the life-time of the asset. Taking account of the residual value $S(x)$ we see that

$$(6) \quad T(x) = C + R(x) - S(x), \qquad T'(x) = R'(x) - S'(x)$$

and (3) and (4) take the simpler forms

$$(7) \quad U(x) = T(x)/x, \qquad xT'(x) - T(x) = 0, \quad R''(x) > S''(x).$$

From the first and second of the equations (7) follows:

$$(8) \qquad T'(x) = U(x)$$

which may be appropriately called the life-equation of an asset since its solution yields the life-time of the asset as defined in 1.

5. When the repair function and the salvage function are known, the real roots of the life-equation may be found either by direct methods or by methods of approximation. However, in the great majority of cases which occur in practice the value of $T(x)$ is given only empirically, from the recorded experience relating to the asset in question, and the data available may not lend itself to analytical treatment. In all such cases the life-time of the asset may be determined approximately by means of the following simple graphic method.

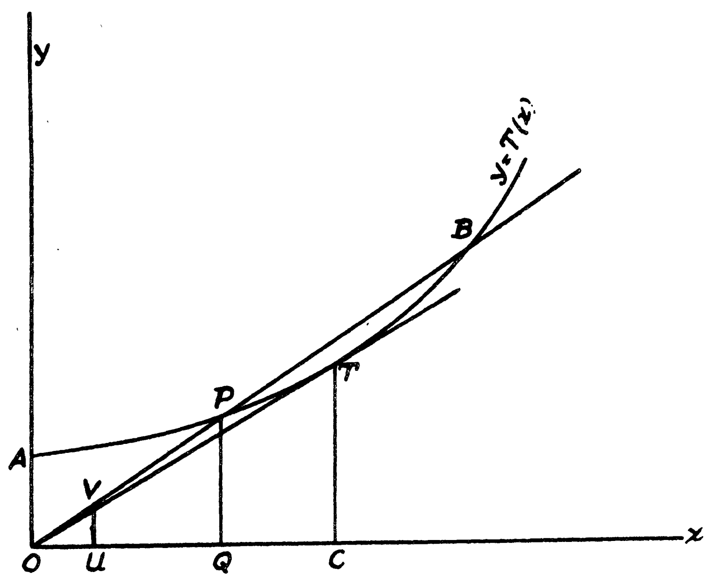


FIGURE 1.

Let AB (Fig. 1) represent the graph of the equation $y = T(x)$, constructed in Cartesian coordinates. We shall call it the total outlay graph, because the ordinate y of any point (x, y) on this graph represents the total outlay during the period of x years if the asset were scrapped at the end of this period. The straight line, OP, joining the origin O to any point P on AB, we shall call the uniform charge to production graph. It enables us to determine at sight the aggregate amount that must be charged to production during any given period of time in order to recover the total outlay OP for the time x on the basis of uniform distribution over the entire period x. If x is expressed in years, the ordinate UV, of the point on OP whose abscissa is unity, will represent the uniform charge to production per year, which is required to recover the total outlay for x years.

Now it is obvious that this unit charge UV will vary with the slope of the line OP. It will be least when the slope is least, that is to say, when the point P is such that the line OP is tangent to the total outlay graph. The abscissa, OC, of the point of contact, T, is then the life-time of the asset under consideration.

To determine the life-time of an asset, interest considerations being disregarded, we need therefore only construct the total outlay graph AB, then draw the tangent OT, and finally measure the abscissa of the point of contact T

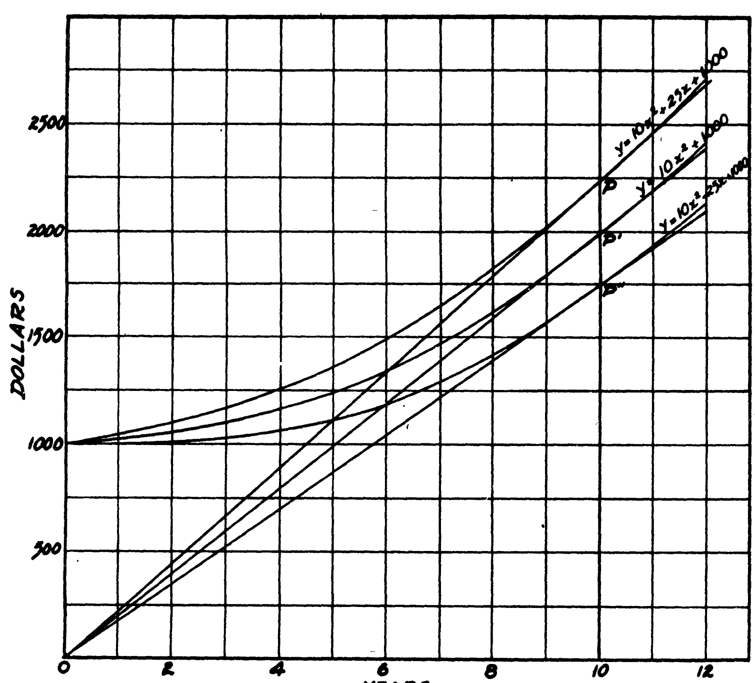


FIGURE 2

(Fig. 2) shows the construction when $T(x)$ has the forms

$$10x^2 + 25x + 1000,$$
$$10x^2 + 1000, \text{ and}$$
$$10x^2 - 25x + 1000$$

respectively. In each case the life-time is found to be 10 years, which verifies the theoretical conclusion of 2: that the life-time is independent of the coefficient of x.

We have seen from graphical considerations that the uniform charge to production will be a minimum when its graph is tangent to the total outlay graph. This condition is precisely the condition expressed by equation (8), which asserts that when x is the life-time of the asset $T'(x)$, the slope of the tangent to the total outlay graph, is numerically the same as the yearly charge to production.

6. We now come to consider the problem of finding the life-time of an asset when interest at a specified rate is to be taken into account. In this case, the various items that make up the total outlay, as well as the component charges to production, must be replaced by their present values at some arbitrarily chosen epoch, as say, the epoch zero.

Let us attempt an analytical solution of the problem. Let Δt represent a small interval of time. The outlay during the interval from t to $t+\Delta t$ is $T(t+\Delta t) - T(t)$. If the specified rate of interest is, i, and if we represent the discount factor by $1/(1+i)$ the conventional symbol v, then the present value of

$$T(t+\Delta t) - T(t)$$

at the epoch O has some value between $[T(t+\Delta t) - T(t)]v^t$ and $[T(t+\Delta t) - T(t)]v^{t+\Delta t}$, let us say

$$[T(t+\Delta t) - T(t)]v^{t+\theta\Delta t}$$

where θ has some value between O and 1. The total outlay during the time t, evaluated for the epoch O, is therefore

(9) $$C + \Sigma[T(t+\Delta t) - T(t)]v^{t+\theta\Delta t}$$

the sum extending over all the time intervals between O and t.

Now $v^t > v^{t+\theta.\Delta t} > v^{t+\theta_M \Delta t}$ where θ_M is the greatest among all the fractions θ. We have, therefore,

$$(10)\qquad \left[T(t+\Delta t)-T(t)\right]v^t > \left[T(t+\Delta t)-T(t)\right]v^{t+\theta\cdot\Delta t} > \left[T(t+\Delta t)-T(t)\right]v^{t+\theta_M\cdot\Delta t}$$

If the intervals Δt are all equal and their number n, then $\Delta t = t/n$, and as n is increased indefinitely Δt approaches O Then $t+\theta_M.\Delta t$ approaches t, and we see from (10) that (9) must have the same limit as

$$(11)\qquad C+\sum\left[T(t+\Delta t)-T(t)\right]v^t$$

To determine this limit we write

$$T(t+\Delta t)-T(t) = \frac{T(t+\Delta t)-T(t)}{\Delta t}\Delta t$$

where the first factor on the right represents the difference quotient which approaches $T'(t)$ as a limit as Δt approaches O as a limit. With this relation introduced into (11), we obtain for the present value at the epoch O of all the increments of outlay during the time t, the intervals of time being infinitesimal,

$$(12)\qquad \overline{T}(t) = C + \underset{\Delta t\to O}{Limit}\left[\sum\frac{T(t+\Delta t)-T(t)}{\Delta t}\cdot v^t\Delta t\right] = C+\int_0^t v^t\cdot T'(t)\cdot dt.$$

In a like manner we may derive an expression for $\overline{D}(t)$, the limit of the sum of the present values at epoch O of all the charges to production during the time t apportioned at some uniform rate U to each of the intervals Δt. The charge apportioned to the interval from t to $t+\Delta t$ is $U.\Delta t$, its present value at epoch O is $U.v^{t+\theta.\Delta t}.\Delta t$. The present value of the sum of these amounts for all the intervals Δt between O and t is

$$\Sigma\, U.v^{t+\theta.\Delta t}.\Delta t$$

which for infinitesimal values of Δt has the same limit as $\Sigma\, U.v^t.\Delta t$, so that finally

$$\bar{D}(t) = \underset{\Delta t \to O}{\text{Limit}}\, \Sigma\, U.v^t.\Delta t = U\int_0^t v^t.dt. \tag{13}$$

Let $U(x)$ be the value which must be assigned to U in order to recover $\bar{T}(x)$, the total outlay for x years through a uniform charge to production, interest considered. Then $\bar{D}(x)$ must equal $\bar{T}(x)$, that is,

$$U(x)\cdot\int_0^x v^t.dt = C + \int_0^x v^t.T'(t).dt,$$

from which

$$U(x) = \frac{C + \int_0^x v^t.T'(t).dt}{\int_0^x v^t.dt} \tag{14}$$

The life-time of the asset is that value of x in (14) which will make $U(x)$ a minimum. The derivative of $U(x)$ with respect to x must therefore vanish. Differentiating (14) with respect to x and setting the result equal to O, we find

$$v^x.T'(x)\int_0^x v^t.dt - \left[C + \int_0^x v^t.T'(t).dt\right] v^x = O,$$

from which

$$T'(x) = \frac{C + \int_0^x v^t.T'(t).dt}{\int_0^x v^t.dt}, \tag{15}$$

which is the life-equation of the asset, interest considered.

7. In deriving equation (13) we apportioned the charges to production for an interval Δt and found the sum of the present values at epoch O. $\bar{D}(t)$ is the limiting value of this sum as the intervals Δt are indefinitely diminished. If, as is customary, no charge is made to production until the end of the year, this single charge will be the aggregate amount of the constituent portions for the separate intervals Δt, accumulated with interest to the end of the year. The charge for the interval from t to $t+\Delta t$ is $U \cdot \Delta t$, its amount at rate i to the end of the year is $U.(1+i)^t.\Delta t$, where t is the time to the end of the year, and the equivalent single charge at the end of the year is

$$\bar{U} = \underset{\Delta t \to O}{limit} \Sigma\, U.(1+i)^t.\Delta t = U.\underset{\Delta t \to O}{limit} \Sigma (1+i)^t.\Delta t$$

$$(16) \qquad = U \int_0^i (1+i)^t.\,dt = \frac{iU}{\log(1+i)}.$$

8. As an example let us again take $T(t) = at^2 + bt + c$, then

$$T'(t) = 2at + b \quad \int_0^t v^t dt = v^t/\log v \Big|_0^t = (v^t - 1)/\log v,$$

$$\int_0^t v^t.\,T(t).dt = \int_0^t v^t(2at+b)dt$$

$$= \left[2atv^t + b(v^t - 1)\right]/\log v - 2a(v^t - 1)/(\log v)^2$$

$$(17) \qquad \bar{T}(t) = c + \left[2atv^t + b(v^t - 1)\right]/\log v - 2a(v^t - 1)/(\log v)^2,$$

$$(18) \qquad \bar{D}(t) = U.(v^t - 1)/\log v,$$

and (15) reduces to

(19) $$v^{x} - \log v^{x} = 1 + \frac{c(\log v)^2}{2a}.$$

While the life-equation (19) cannot be solved algebraically, it is evident that an approximate solution for could be obtained from a list of tabulated values of the function $v^{x} - \log v^{x}$. When such a table is not available, an approximate solution to any desired degree of accuracy may be obtained as follows:

We may write for v, $e^{\log v}$ where e is the base of the natural system of logarithms. (19) then takes the form

(20) $$e^{x \log v} - x \log v = 1 + c(\log v)^2/2a.$$

On expanding the first term of this equation into a power series in x, and simplifying the result, we have

$$x^2 + x^3(\log v)/3 + x^4(\log v)^2/12 + x(\log v)^3/60 + \cdots = c/a,$$

whence

$$x^2 = \frac{c/a}{1 + x(\log v)/3 + x^2(\log v)^2/12 + x^3(\log v)^3/60 + \cdots}.$$

Now for all ordinary rates of interest log v is necessarily very small, so that if we denote successive approximations of x by x_1, x_2, x_3, etc.,

$$x_1 = (c/a)^{1/2}, \quad x = \left[\frac{c/a}{1 + x_1(\log v)/3}\right]^{1/2}$$

$$x_3 = \left[\frac{c/a}{1 + x_2(\log v)/3 + x_2^2(\log v)^2/12}\right]^{1/2}$$

$$x_4 = \left[\frac{c/a}{1 + x_3(\log v)/3 + x_3^2(\log v)^2 12 + x_3^3(\log v)^3/60}\right]^{1/2}$$

, etc.

Let us take the special case, previously considered in 5., when $c = 1000$, $b = 0$, $a = 10$, and let the assumed rate of interest be 6 percent. Then

$$\log v = -0.058\,269, \quad (\log v)^2 = 0.003\,395,$$

$$1/\log v = -17.161\,788, \quad 1/(\log v)^2 = 294.526\,967,$$

and we find

$$T(t) = 10t^2 + 1000,$$

$$\bar{T}(t) = 1000 - 343.24\,t v^t - 5890.54(v^t - 1),$$

$$\bar{D}(t) = 17.162\,U(1 - v^t),$$

and the life-equation is

$$v^x - \log v^x = 1.169\,764$$

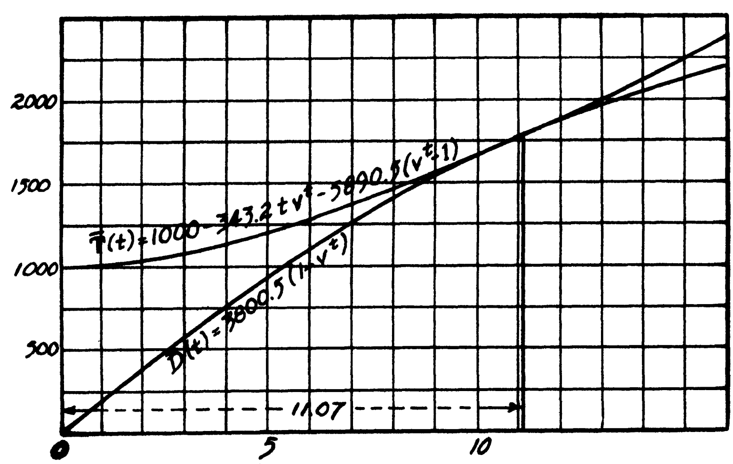


FIGURE 3

The first four successive approximations for x give

$$x_1 = 10, \quad x_2 = 11.14, \quad x_3 = 11.05$$
$$x_4 = 11.07$$

The value $x = 11.07$ substituted in (14) and (16) give us

$U(x)=221.45$ and $\bar{U}(x)= 228.03$

This value of $U(x)$ substituted for U in the expression for $\bar{D}(t)$ gives

$$\bar{D}(t) = 3800.5(1-v^t),$$

which represents the present value at epoch O of the aggregate momentary charges during a period t at a rate such as to recover the total outlay 11.07 years, the theoretical life-time of the asset. The momentary rate is 221.45 per year, the equivalent single charge to production at the end of each year is 228.03.

(Fig. 3) shows the graphs of the two equations.

$$\bar{T}(t) = 1000 - 343.24\,t\,v^t - 5890$$

and

$$\bar{D}(t) = 3800.5(1-v^t).$$

The abscissa of the common ordinate of the two curves represents the life-time of the asset.

9. It appears from (Fig. 3) that at the point common to the two graphs, the graphs have a common tangent as well as a common ordinate. To see whether or not this is a general property let us trace the changes in the total outlay and total charge to production functions when interest is taken into account.

In the first place it is evident that the increments of the ordinates of both of the graphs in (Fig. 1) must be replaced by their present values at the chosen epoch. If this epoch is the effect in question will be to shorten progressively the ordinates of both graphs. The charge to production graph will then be no longer a straight line but some convex curve, while the total outlay graph will go over into another graph which is less concave than the original graph. But both graphs will continue to rise indefinitely as we proceed from left to right because the increments of their ordinates, while decreasing indefinitely remain positive.

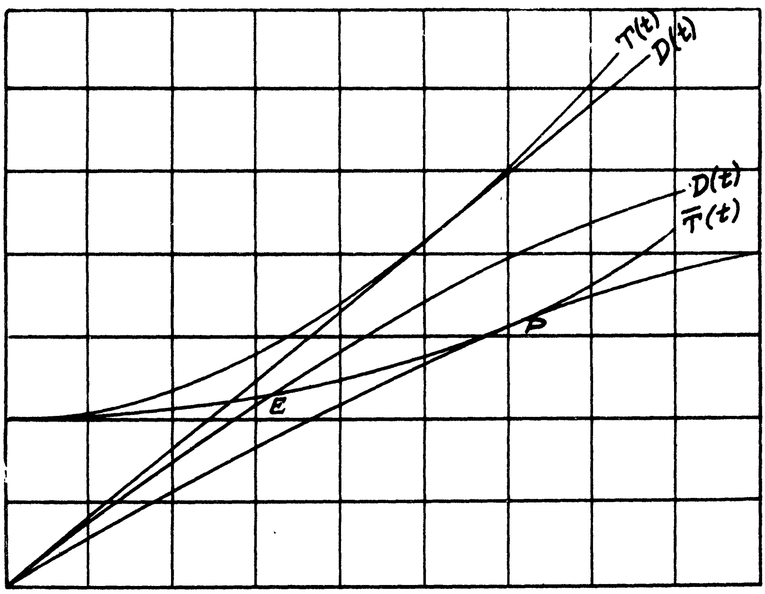


FIG. 4

In (Fig. 4) let $T(t)$ and $D(t)$ represent respectively the total outlay graph and the charge to production graph, interest disregarded. $\bar{T}(t)$ the total outlay graph, interest considered, and $\bar{D}(t)$ the charge to production graph, interest considered, through any point E on $\bar{T}(t)$. The ordinates on $\bar{D}(t)$ represent the present values at epoch O of the momentary charges to production during time t at a rate such as to recover the entire outlay during the time corresponding to the abscissa of the point E. This rate is measured by the initial slope of $\bar{D}(t)$, the slope of $\bar{D}(t)$ when $t=O$.

Let us follow the changes in this slope for the various positions of the point E as it moves along $\bar{T}(t)$ from left to right. It is evident that this slope at first decreases, also that it cannot keep on decreasing indefinitely, it is therefore plausible that it will ultimately increase, reaching a minimum value at the point P where the $\bar{T}(t)$ curve and the $\bar{D}(t)$ curve have a common tangent. The abscissa of the point of contact, P, is then the lifetime of the asset under discussion.

10. The foregoing considerations, however plausible, are open to objections, because we have reasoned from graphs resulting from the assumption of a special law governing the repair function. Different assumptions might give rise to essentially different graphs. We shall, therefore establish the conclusions above arrived at, by an analytical proof, which is independent of any assumptions regarding the nature of the outlay function. We shall prove the

Theorem: If the rate U of a uniform charge to production curve is a minimum, this curve is tangent to the corresponding total outlay curve, and the abscissa of the point of contact represents the life-time of the asset. Conversely,

If a uniform charge to production curve is tangent to the corresponding total outlay curve, U is a minimum.

To prove this theorem, let $y = \bar{T}(t)$ be the equation of the total outlay curve, $y = \bar{D}(t) = U\int_0^t v^t dt$, the equation of the uniform charge to production curve, and x the abscissa of a point common to the two curves.

Then $\bar{T}(x) = \bar{D}(x) = U\int_0^x v^t dt$ from which

$$U = \bar{T}(x) / \int_0^x v^t dt. \tag{21}$$

Since by hypothesis U is a minimum, its derivative with respect to x must vanish, that is

$$\bar{T}'(x)\int_0^x v^t dt - v^x \bar{T}(x) = 0 \tag{22}$$

From (22) and (21) follows

$$\bar{T}'(x) = v^x \bar{T}(x) / \int_0^x v^t dt = v^x U = \bar{D}'(x). \tag{23}$$

This shows that at the point common to the two curves their slopes are equal, they have therefore a common tangent, and since

U has a minimum value, x must represent the life-time of the asset.

To prove the converse theorem we observe that if the two curves have a common tangent at the point $t=x$,

$$\bar{T}(x)=\bar{D}(x)=U\int_0^x v^t dt \tag{24}$$

and

$$\bar{T}'(x)=\bar{D}'(x)=v^x U \tag{25}$$

Substituting the value of U from (24) in (25) we find

$$\bar{T}(x)=v^x\,\bar{T}(x)\Big/\int_0^x v^t dt$$

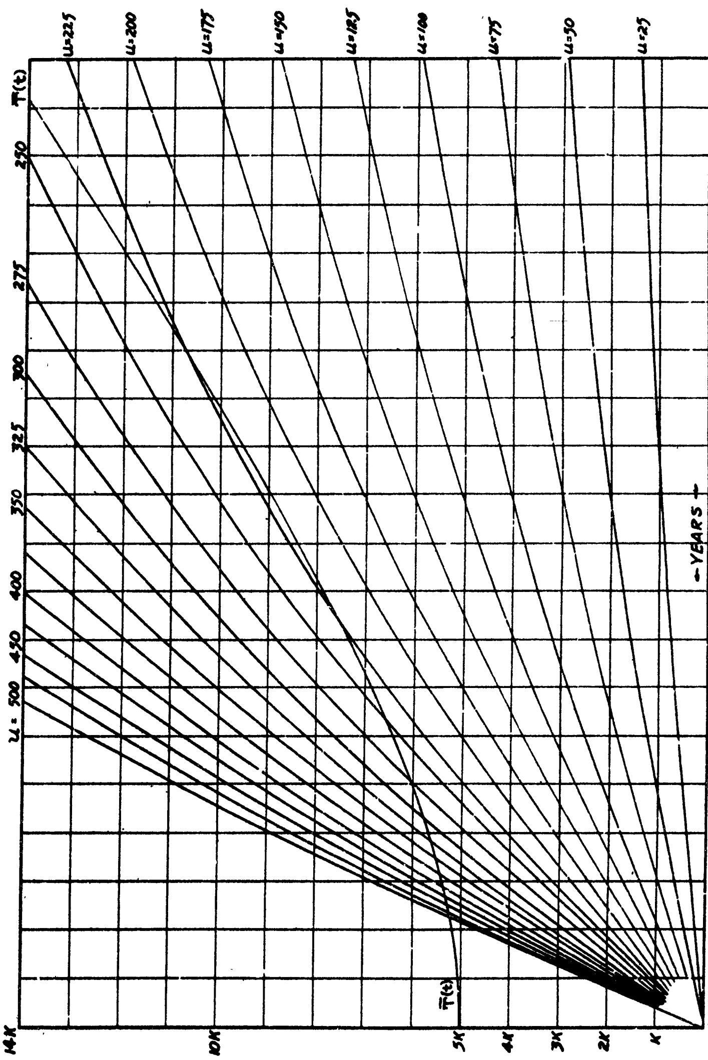
from which

$$\bar{T}(x)\int_0^x v^t dt - v^x\,\bar{T}(x)=0.$$

But by (22) this is precisely the condition that U is a minimum.

11. In most cases which arise in practice the analytical method of finding the life-equation of an asset fails owing to the empirical character of the outlay function. The question suggests itself whether a graphic method, similar to that employed in the simpler case treated in 7, can be devised, which will yield an approximate solution of the problem. The theorems of the preceding article offer the key to such a method.

Let us suppose that the total outlay graph has been constructed on a convenient scale, the scale depending on the magnitude of the quantities involved. Every point on this curve determines a definite uniform charge to production curve. We seek that particular one of these curves which is tangent to the total outlay graph. The abscissa of the point of contact would



K = $200, i = 100
FIG 5

give us the life-time of the asset, and the initial slope of the uniform charge to production curve would give us the rate

Instead of constructing first the total outlay curve, we may first prepare a sheet with rulings, as shown in (Fig. 5), each ruling representing the uniform charge to production curve corresponding to a definite U, the successive values U being chosen at suitable intervals. We may then plot the graph of any given outlay function on this same sheet and from among the various rulings select that on which comes nearest having contact with the outlay graph. This will yield a first approximation of x. A closer approximation may then be obtained by the usual processes of interpolation.

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