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A New Three-Term Hestenes-Stiefel Type Method for Nonlinear Monotone Operator Equations and Image Restoration

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ABSTRACT In this article, a derivative-free method of Hestenes-Stiefel type is proposed for solving system of monotone operator equations with convex constraints. The method proposed is matrix-free, and its sequence of search directions are bounded and satisfies the sufficient descent condition. The global convergence of the proposed approach is established under the assumptions that the underlying operator is monotone and Lipschitz continuous. Numerical experiment results are reported to show the efficiency of the proposed method. Furthermore, to illustrate the applicability of the proposed method, it is used in restoring blurred images.

INDEX TERMS Derivative-free algorithm, Monotone operator equations, projection technique, image restoration.

I. INTRODUCTION

In this article, the problem of finding $v \in \Omega$ for which

$$G(v) = 0, \tag{1}$$

is considered. The set Ω is a nonempty closed and convex subset of $\mathbb{R}^n$ and $G : \Omega \to \mathbb{R}^n$ is continuous and monotone.

The nonlinear monotone operator equations with convex constraint (1) have a wide range of application in various areas such as the power flow equation [1], chemical equilibrium systems [2], economic equilibrium problems [3] and several others. This has inspired so many researchers to explore efficient methods for solving (1). Among the various methods developed for solving (1), Newton method, quasi-Newton method, Gauss-Newton method, Levenberg-Marquardt method, trust region method and its variants are very prominent due to their fast local superlinear convergence properties [4]–[9]. However, these methods are not suitable choice for solving nonlinear equations of large-scale, as they need to solve a linear equation using the Jacobian matrix or its approximation per-iteration.

Motivated by the Solodov and Svaiter projection scheme [10], some researchers have exploited the simplicity and low storage of some of the methods used to solve large-scale unconstrained optimization problems such as conjugate gradient methods, spectral gradient methods, and spectral conjugate gradient methods to solve large-scale nonlinear equations. For instance, Cruz and Raydan [11] popularized the spectral gradient approach for solving the unconstrained version of problem (1) by developing a spectral algorithm (SANE). Subsequently, a complete derivative-free SANE algorithm was studied by La Cruz *et al.* [12] which works well for a class of unconstrained monotone nonlinear equations. Cheng [13] extended the well known PRP method

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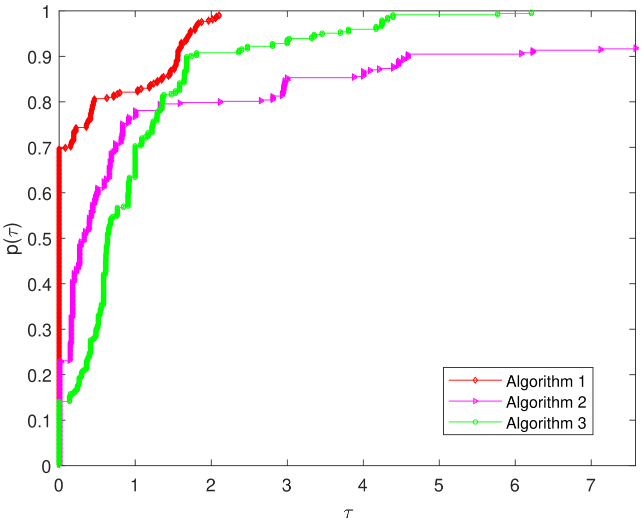

FIGURE 1. **Performance profile for number of iterations.**

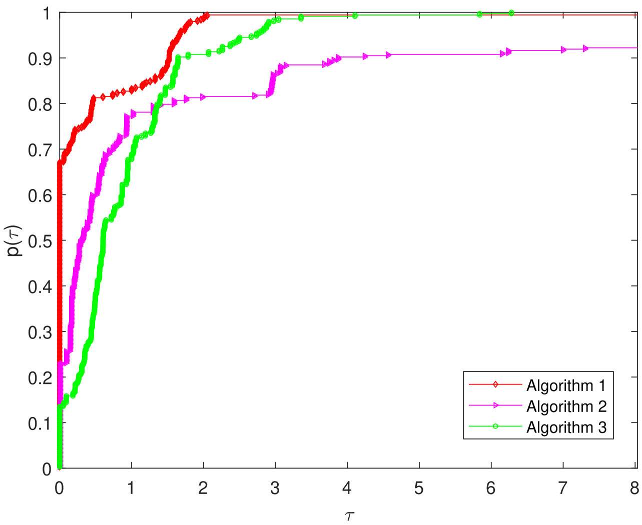

FIGURE 2. **Performance profile for the number of function evaluations.**

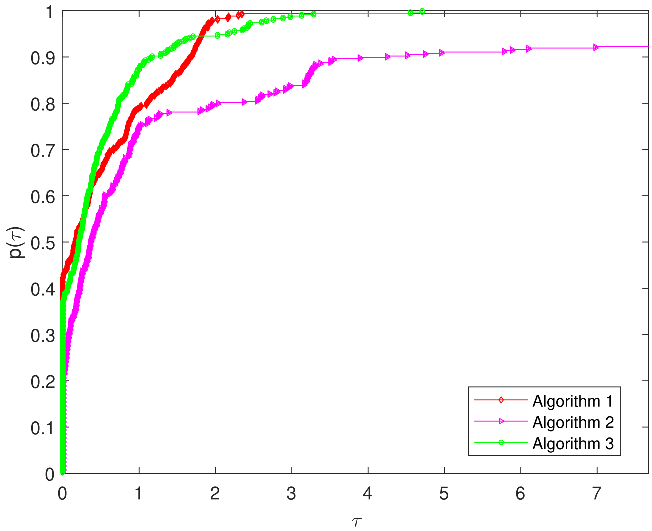

FIGURE 3. **Performance profile for time in seconds.**

FIGURE 4. **The original images: Lenna (top left), Tiffany (top right), Barbara (bottom left) and Airline (bottom right).**

[14], [15] and proposed a hyperplane projection type method to solve unconstrained nonlinear monotone equations. Similarly, Zhang and Zhou [16] combined the spectral gradient method [17] with the projection technique [10]. The Wei-Yao-Liu conjugate gradient projection algorithm for solving (1) was also as a result of the projection technique and the proposed method by Hu and Wei [18]. Also, Liu and Feng [19] introduced a derivative-free projection method which converges to the solution of the convex constraint problem (1). The proposed scheme involves only one projection per iteration with a monotone and Lipschitz continuity assumption imposed on the underlying mapping. Their proposed method can be viewed as a modification of the well known Dai-Yuan conjugate gradient method for unconstrained optimization. Besides, just quite recent, Djordjević [20] proposed a hybrid conjugate gradient method for solving unconstrained optimization problems. The proposed method combines the well known Liu-Storey and Fletcher Reeves conjugate gradient parameter using a convex combination. The reported numerical performance indicates that the method is efficient for solving unconstrained optimization problem. Motivated by [20], Ibrahim *et al.* [21] extended the hybrid conjugate gradient method of Djordjević [20] to solve (1) using the hyperplane projection technique. Kaelo and Koorapetse [22] introduced a derivative-free conjugate gradient-based projection method for solving (1). The proposed method in [22] was developed by combining the projection technique with the family of conjugate gradient methods introduced by Li *et al.* [23]. In addition, Koorapetse *et al.* [24] introduced a three-term derivative-free method for solving (1). The proposed method was based on the Zheng and Zheng [25] conjugate gradient method for the unconstrained optimization problems. To achieve the boundedness of their proposed direction, they modified one of the directions defined in [25]. For more

FIGURE 5. The blurred image (top left), the restored image by Algorithm 1 (top right) ((SNR = 16.77, PSNR = 22.11, SSIM = 0.9138)), by Algorithm 2 (bottom left) ((SNR = 16.74, PSNR = 22.07, SSIM = 0.9131)) and by Algorithm 3 (bottom right) (SNR = 16.77, PSNR = 22.10, SSIM = 0.9137).

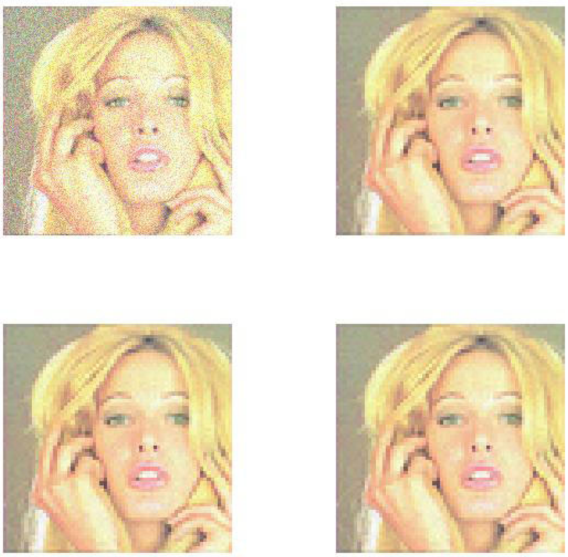

FIGURE 6. The blurred image (top left), the restored image by Algorithm 1 (top right) (SNR = 21.02, PSNR = 22.86, SSIM = 0.9159), by Algorithm 2 (bottom left) (SNR = 21.00, PSNR = 22.83, SSIM = 0.9152) and by Algorithm 3 (bottom right) (SNR = 21.01, PSNR = 22.85, SSIM = 0.9157).

FIGURE 7. The blurred image (top left), the restored image by Algorithm 1 (SNR= 13.67, PSNR= 20.09, SSIM = 0.6289) (top right), by Algorithm 2 (SNR = 13.65, PSNR = 20.07, SSIM = 0.6277) (bottom left) with and by Algorithm 3 (SNR = 13.66, PSNR = 20.08, SSIM = 0.6285) (bottom right).

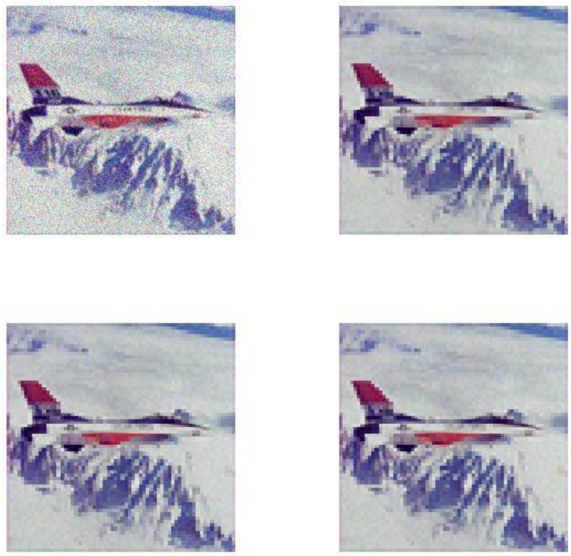

FIGURE 8. The blurred image (top left), the restored image by Algorithm 1 (top right) (SNR = 18.43, PSNR = 21.11, SSIM = 0.6803), by Algorithm 2 (bottom left) (SNR = 18.39, PSNR = 21.08, SSIM = 0.6773) and by Algorithm 3 (bottom right) (SNR = 18.42, PSNR = 21.10, SSIM = 0.6795).

recent articles on derivative-free iterative methods for solving (1), readers can refer to [26]–[42] and references therein.

Inspired by the works in [43] and [44] as well as the good numerical performance of the Hestenes-Stiefel method, we modify and extend the three-term Hestenes-Stiefel method with sufficient descent property for unconstrained minimization problems proposed in [45]. The aim of the modification on the proposed search direction is to achieve the sufficient descent property and boundedness independent of the line search. In addition, such kind of modification have been shown to have a significant impact on the numerical efficiency of a method. Moreover, the global convergence is proved without requirement of the operator to be differentiable. Preliminary numerical results are given to show the efficiency of the method.

II. ALGORITHM

In this section, a derivative-free projection based algorithm is proposed to find approximate solutions to problem (1). The search direction generated by the algorithm is of three term and does not require the derivative of the operator. To have a good understanding of the motivation, the algorithm proposed by Baluch *et al.* [45] for finding solution to the unconstrained optimization problem is recalled. Consider the unconstrained optimization problem:

$$\min_{v\in\mathbb{R}^n} f(v), \tag{2}$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is a continuously differentiable real valued function with gradient at $v^{(t)}$ denoted by $g^{(t)}:=\nabla f(v^{(t)})$. The algorithm proposed in [45] produces a sequence $\{v^{(t)}\}_{t\geq 0}$ via the following iterative formula

$$v^{(t+1)}:=v^{(t)}+\alpha^{(t)}d^{(t)},\quad t=0,1,\cdots, \tag{3}$$

where $\{v^{(t)}\}_{t\geq 0}$ is the previous point, $\{v^{(t+1)}\}_{t\geq 0}$ is the current point, $\alpha^{(t)}$ is a positive step size and $d^{(t)}$ is the search direction defined as:

$$d^{(t)}:=\begin{cases}-g^{(t)}, & \text{if } t=0,\\ -g^{(t)}+\beta^{(t)}d^{(t-1)}-\theta^{(t)}y^{(t-1)}, & \text{if } t\geq 1.\end{cases} \tag{4}$$

$\beta^{(t)}$ and $\theta^{(t)}$ are defined as follows:

$$\beta^{(t)}:=\beta^{(t)}_{BZA}:=\frac{\langle g^{(t)},\ y^{(t-1)}\rangle}{\langle d^{(t-1)},\ y^{(t-1)}\rangle+\gamma|\langle g^{(t)},\ d^{(t-1)}\rangle|}, \tag{5}$$

$$\theta^{(t)}:=\theta^{(t)}_{BZA}:=\frac{\langle g^{(t)},\ d^{(t-1)}\rangle}{\langle d^{(t-1)},\ y^{(t-1)}\rangle+\gamma|\langle g^{(t)},\ d^{(t-1)}\rangle|},\ \gamma>1, \tag{6}$$

$y^{(t-1)}:=g^{(t)}-g^{(t-1)}$. Inspired by $\beta^{(t)}$ and $\theta^{(t)}$ defined in (5)-(6), we propose a derivative-free projection based algorithm to find approximate solutions to (1). For $G^{(t)}\neq 0$ and $d^{(t-1)}\neq 0$, the propose search direction is defined as

$$d^{(t)}:=\begin{cases}-G^{(t)}, & \text{if } t=0,\\ -G^{(t)}+\beta^{(t)}d^{(t-1)}-\theta^{(t)}y^{(t)}, & \text{if } t\geq 1,\end{cases} \tag{7}$$

$$\beta^{(t)}:=\frac{\langle G^{(t)},\ y^{(t-1)}\rangle}{\langle d^{(t-1)},\ x^{(t-1)}\rangle+\gamma\|G^{(t)}\|\|d^{(t-1)}\|} \tag{8}$$

and

$$\theta^{(t)}:=\frac{\langle G^{(t)},\ d^{(t-1)}\rangle}{\langle d^{(t-1)},\ x^{(t-1)}\rangle+\gamma\|G^{(t)}\|\|d^{(t-1)}\|},\quad \gamma>0, \tag{9}$$

$$\begin{aligned} y^{(t-1)}&:=G^{(t)}-G^{(t-1)},\\ x^{(t-1)}&:=y^{(t-1)}+u^{(t-1)}d^{(t-1)}. \end{aligned} \tag{10}$$

$$u^{(t-1)}:=1+\max\left\{0,-\frac{\langle d^{(t-1)},y^{(t-1)}\rangle}{\|d^{(t-1)}\|^2}\right\}. \tag{11}$$

The parameters $\beta^{(t)}$ and $\theta^{(t)}$ defined in (8)-(9) were defined such that the search direction defined by (7) is sufficiently descent and bounded. This is achieved by replacing $\langle d^{(t-1)},y^{(t-1)}\rangle$ and $\gamma|\langle g^{(t)},d^{(t-1)}\rangle|$ in (5)-(6) by $\langle d^{(t-1)},x^{(t-1)}\rangle$ and $\gamma\|G^{(t)}\|\|d^{(t-1)}\|$ respectively.

Remark 1: From the definition of $x^{(t-1)}$, $u^{(t-1)}$ in (10)-(11) with $d^{(t-1)}\neq 0$,

$$\begin{aligned}\langle d^{(t-1)},\ x^{(t-1)}\rangle &\geq \langle d^{(t-1)},\ y^{(t-1)}+\|d^{(t-1)}\|^2\\ &-\langle d^{(t-1)},\ y^{(t-1)}\rangle=\|d^{(t-1)}\|^2>0.\end{aligned} \tag{12}$$

The above remark guarantees that $\beta^{(t)}$ and $\theta^{(t)}$ defined by (8)-(9) are well defined. Before introducing the propose algorithm, the following definition of the projection map is given.

Definition 2: Suppose $\Omega\subset\mathbb{R}^n$ is nonempty, closed and convex set. Then the projection of every $v\in\mathbb{R}^n$ onto Ω, denoted by $P_\Omega(v)$, is defined by

$$P_\Omega(v):=\arg\min\{\|v-y\|\ :y\in\Omega\}.$$

P_Ω is nonexpansive, that is for all $v,y\in\mathbb{R}^n$,

$$\|P_\Omega(v)-P_\Omega(y)\|\leq\|v-y\|. \tag{13}$$

Below is a step by step implementation of the derivative-free projection based algorithm.

Algorithm 1

Step 0. Select $v^{(0)}\in\Omega$, parameters $\sigma>0$, $\mu>0$, $0<\lambda<2$, $0<\rho<1$, $\gamma>0$, $Tol>0$ and set $t:=0$.
Step 1. If $\|G^{(t)}\|\leq Tol$, stop, otherwise go to **Step 2**.
Step 2. Compute $d^{(t)}$ by (7)–(9).
Step 3. Compute the step size $\alpha^{(t)}:=\mu\rho^i$ where i is the smallest non-negative integer such that

$$-\langle F(v^{(t)}+\alpha^{(t)}d^{(t)}),\ d^{(t)}\rangle\geq\sigma\alpha^{(t)}\|d^{(t)}\|^2. \tag{14}$$

Step 4. Set $w^{(t+1)}:=v^{(t)}+\alpha^{(t)}d^{(t)}$. If $w^{(t+1)}\in\Omega$ and $\|G(w^{(t+1)})\|\leq Tol$, stop. Else compute

$$v^{(t+1)}:=P_\Omega[v^{(t)}-\lambda\zeta^{(t)}G(w^{(t+1)})] \tag{15}$$

where

$$\zeta^{(t)}:=\frac{\langle G(w^{(t+1)}),\ v^{(t)}-w^{(t+1)}\rangle}{\|G(w^{(t+1)})\|^2}.$$

Step 5. Let $t=t+1$ and go to **Step 1**.

III. CONVERGENCE ANALYSIS

To establish the global convergence of Algorithm 1, we assume the following:

(*A*1) The operator G is monotone, that is for all $v,y\in\mathbb{R}^n$,

$$\langle G(v)-G(y),\quad v-y\rangle\geq 0.$$

(*A*2) The operator G is Lipschitz continuous, that is there exists a constant $L>0$ such that for all $v,y\in\mathbb{R}^n$

$$\|G(v)-G(y)\|\leq L\|v-y\|.$$

(*A*3) The solution set of (1) denoted by Ω' is nonempty.
(*A*4) $G^{(t)}\neq 0$ unless at the solution.

Lemma 3: Let $d^{(t)}$ be defined by (7)-(9), then $d^{(t)}$ satisfies the sufficient descent condition. That is

$$\langle G^{(t)},\quad d^{(t)}\rangle=-\|G^{(t)}\|^2. \tag{16}$$

Proof: For $t=0$, we have from (7) that $\langle G^{(0)},d^{(0)}\rangle=-\|G^{(0)}\|^2$.

TABLE 1. Efficiency comparison for Algorithm 1, Algorithm 2 and Algorithm 3 based on SNR, PSNR and SSIM.

	Algorithm 1			Algorithm 2			Algorithm 3		
Images	SNR	PSNR	SSIM	SNR	PSNR	SSIM	SNR	PSNR	SSIM
Lenna	16.77	22.11	0.9138	16.74	22.07	0.9131	16.77	22.10	0.9137
Barbara	13.67	20.09	0.6289	13.65	20.07	0.6277	13.66	20.08	0.6285
Tiffany	21.02	22.86	0.9159	21.00	22.83	0.9152	21.01	22.85	0.9157
Airline	18.43	21.11	0.6803	18.39	21.08	0.6773	18.42	21.10	0.6795

TABLE 2. Numerical results of Problem 1.

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	1	6	0.005497	0	16	50	0.054452	6.27E-06	14	44	0.009897	7.52E-06
	v_2	8	26	0.012527	9.27E-06	14	44	0.01078	9.50E-06	13	40	0.005693	4.75E-06
	v_3	9	30	0.009818	8.97E-06	18	57	0.005821	7.62E-06	27	83	0.005154	9.24E-06
	v_4	14	45	0.007545	5.53E-06	17	53	0.00598	8.58E-06	44	134	0.007182	9.22E-06
	v_5	14	45	0.008371	5.56E-06	17	53	0.006672	8.19E-06	44	134	0.008066	8.82E-06
	v_6	14	44	0.005148	5.02E-06	22	68	0.005725	6.98E-06	36	110	0.004718	8.85E-06
	v_7	14	45	0.005582	6.40E-06	17	53	0.009165	7.80E-06	44	134	0.005902	8.98E-06
5000	v_1	1	6	0.004222	0	16	50	0.014908	5.24E-06	14	43	0.010255	3.67E-06
	v_2	9	29	0.009581	3.65E-06	15	47	0.009663	8.28E-06	12	37	0.006652	6.08E-06
	v_3	9	30	0.010208	8.97E-06	18	57	0.01258	7.62E-06	27	83	0.013143	9.24E-06
	v_4	15	47	0.014737	4.09E-06	18	56	0.012962	7.35E-06	47	143	0.025478	7.46E-06
	v_5	15	47	0.014227	4.10E-06	18	56	0.01271	7.26E-06	47	143	0.02656	7.40E-06
	v_6	14	44	0.014748	5.02E-06	22	68	0.015404	7.00E-06	36	110	0.016856	8.86E-06
	v_7	15	47	0.014999	3.98E-06	18	56	0.016004	7.26E-06	47	143	0.023839	7.49E-06
10000	v_1	1	6	0.005103	0	16	50	0.020864	5.63E-06	13	41	0.013625	8.53E-06
	v_2	9	29	0.014318	5.16E-06	16	50	0.021126	4.62E-06	12	37	0.011568	6.47E-06
	v_3	9	30	0.01659	8.97E-06	18	57	0.026575	7.62E-06	27	83	0.023926	9.24E-06
	v_4	15	47	0.023392	5.80E-06	19	59	0.02642	4.09E-06	48	146	0.044716	7.57E-06
	v_5	15	47	0.024388	5.80E-06	19	59	0.024932	4.06E-06	48	146	0.047783	7.54E-06
	v_6	14	44	0.023109	5.02E-06	22	68	0.031726	7.00E-06	36	110	0.035531	8.86E-06
	v_7	15	47	0.022708	6.00E-06	19	59	0.021983	4.02E-06	48	146	0.044045	7.62E-06
50000	v_1	1	6	0.012442	0	16	50	0.076169	9.30E-06	10	31	0.037985	3.58E-06
	v_2	9	30	0.049141	6.49E-06	17	53	0.072592	4.07E-06	10	32	0.036513	8.86E-06
	v_3	9	30	0.050577	8.97E-06	18	57	0.079282	7.62E-06	27	83	0.094733	9.24E-06
	v_4	16	51	0.082627	4.80E-06	19	59	0.085077	9.09E-06	50	152	0.16699	8.75E-06
	v_5	16	51	0.082727	4.80E-06	19	59	0.086996	9.08E-06	50	152	0.18053	8.75E-06
	v_6	14	44	0.070774	5.02E-06	22	68	0.090376	7.00E-06	36	110	0.12606	8.87E-06
	v_7	16	51	0.080748	4.78E-06	19	59	0.082245	9.13E-06	50	152	0.18177	8.78E-06
100000	v_1	1	6	0.021127	0	17	53	0.19324	4.97E-06	11	34	0.10928	7.86E-06
	v_2	9	30	0.085809	9.18E-06	17	53	0.17268	5.76E-06	10	31	0.10326	5.69E-06
	v_3	9	30	0.088752	8.97E-06	18	57	0.17532	7.62E-06	27	83	0.19706	9.24E-06
	v_4	16	51	0.15614	6.79E-06	20	62	0.24514	5.08E-06	51	155	0.49389	8.91E-06
	v_5	16	51	0.14635	6.79E-06	20	62	0.17841	5.07E-06	51	155	0.3808	8.91E-06
	v_6	14	44	0.12638	5.02E-06	22	68	0.29518	7.00E-06	36	110	0.25365	8.87E-06
	v_7	16	51	0.14795	6.79E-06	20	62	0.24514	5.08E-06	51	155	0.36108	8.95E-06

As for $t \geq 1$, using (7)-(9),

$$\begin{aligned}
&\langle G^{(t)}, d^{(t)}\rangle \\
&\quad = -\|G^{(t)}\|^2 + \beta^{(t)}\langle G^{(t)}, \ d^{(t-1)}\rangle - \theta^{(t)}\langle G^{(t)}, \ y^{(t-1)}\rangle \\
&\quad = -\|G^{(t)}\|^2 + \frac{\langle G^{(t)}, \ y^{(t-1)}\rangle\langle G^{(t)}, \ d^{(t-1)}\rangle}{\langle d^{(t-1)}, \ x^{(t-1)}\rangle + \gamma\|G^{(t)}\|\|d^{(t-1)}\|} \\
&\qquad - \frac{\langle G^{(t)}, \ d^{(t-1)}\rangle\langle G^{(t)}, \ y^{(t-1)}\rangle}{\langle d^{(t-1)}, \ x^{(t-1)}\rangle\gamma\|G^{(t)}\|\|d^{(t-1)}\|} \\
&\quad = -\|G^{(t)}\|^2. \qquad (17)
\end{aligned}$$

Remark 4: From (16) and applying the Cauchy-Schwartz inequality, it can be deduced that for all $t \geq 0$,

$$\|d^{(t)}\| \geq \|G^{(t)}\|. \qquad (18)$$

■

TABLE 3. Numerical results of Problem 2.

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	8	23	0.003979	9.39E-06	9	26	0.040657	9.20E-06	8	23	0.002744	7.25E-06
	v_2	7	21	0.003495	7.45E-06	7	20	0.00232	2.66E-06	6	17	0.001737	3.28E-06
	v_3	22	54	0.008316	4.29E-06	9	26	0.002849	4.43E-06	26	77	0.005355	8.89E-06
	v_4	16	44	0.007476	4.10E-06	28	83	0.007185	3.94E-06	36	107	0.007533	9.20E-06
	v_5	16	44	0.007542	4.10E-06	28	83	0.007676	3.94E-06	36	107	0.008508	9.20E-06
	v_6	19	50	0.007256	4.61E-06	21	62	0.005354	4.81E-06	30	89	0.006058	7.73E-06
	v_7	16	44	0.00756	4.44E-06	26	77	0.009721	3.69E-06	36	107	0.009293	9.26E-06
5000	v_1	9	25	0.010519	6.19E-06	10	29	0.010355	3.18E-06	9	26	0.007147	1.79E-06
	v_2	8	23	0.009303	4.91E-06	7	20	0.006959	5.70E-06	6	17	0.005504	7.00E-06
	v_3	19	48	0.022112	7.81E-06	9	26	0.008118	3.98E-06	26	77	0.023401	9.23E-06
	v_4	16	44	0.018801	9.14E-06	30	89	0.028605	3.86E-06	39	116	0.029377	8.19E-06
	v_5	16	44	0.018822	9.14E-06	30	89	0.029244	3.86E-06	39	116	0.030446	8.19E-06
	v_6	16	43	0.018872	7.61E-06	21	62	0.016694	6.36E-06	30	89	0.020779	8.15E-06
	v_7	16	44	0.022119	8.98E-06	30	89	0.032119	3.84E-06	39	116	0.03704	8.20E-06
10000	v_1	9	25	0.0181	8.73E-06	10	29	0.01497	4.48E-06	9	26	0.013713	2.52E-06
	v_2	8	23	0.015841	6.93E-06	7	20	0.01171	8.03E-06	6	17	0.008775	9.84E-06
	v_3	17	44	0.03165	3.88E-06	10	29	0.015993	4.40E-06	26	77	0.033849	9.28E-06
	v_4	17	47	0.03513	4.98E-06	30	89	0.051201	5.48E-06	40	119	0.06067	8.40E-06
	v_5	17	47	0.036997	4.98E-06	30	89	0.051709	5.48E-06	40	119	0.060916	8.40E-06
	v_6	16	43	0.031925	7.74E-06	21	62	0.034224	6.51E-06	30	89	0.041577	8.21E-06
	v_7	17	47	0.044311	4.91E-06	30	89	0.05861	5.51E-06	40	119	0.072277	8.36E-06
50000	v_1	10	28	0.070771	3.12E-06	10	29	0.080475	9.97E-06	9	26	0.048124	5.61E-06
	v_2	8	24	0.060886	8.25E-06	8	23	0.046791	2.86E-06	7	20	0.037062	2.53E-06
	v_3	17	44	0.10813	4.12E-06	11	32	0.072213	8.67E-06	26	77	0.13006	9.31E-06
	v_4	18	50	0.1259	4.30E-06	32	95	0.20369	4.59E-06	42	125	0.23914	9.82E-06
	v_5	18	50	0.12764	4.30E-06	32	95	0.22415	4.59E-06	42	125	0.23893	9.82E-06
	v_6	16	43	0.11026	7.84E-06	21	62	0.13369	6.63E-06	30	89	0.15595	8.25E-06
	v_7	18	50	0.18116	4.42E-06	32	95	0.26326	4.69E-06	42	125	0.30048	9.81E-06
100000	v_1	10	28	0.13466	4.41E-06	11	32	0.13577	2.26E-06	9	26	0.094019	7.93E-06
	v_2	9	26	0.12184	3.50E-06	8	23	0.092716	4.04E-06	7	20	0.068493	3.57E-06
	v_3	17	44	0.20969	4.15E-06	13	38	0.14133	3.15E-06	26	77	0.23996	9.32E-06
	v_4	18	50	0.23803	6.08E-06	32	95	0.40636	6.49E-06	44	131	0.4697	7.21E-06
	v_5	18	50	0.26247	6.08E-06	32	95	0.38614	6.49E-06	44	131	0.46523	7.21E-06
	v_6	16	43	0.19554	7.85E-06	21	62	0.22924	6.65E-06	30	89	0.29389	8.26E-06
	v_7	18	50	0.31139	6.04E-06	30	89	0.52776	7.13E-06	44	131	0.60872	7.23E-06

Lemma 5: If Assumption $(A2)$ hold and the sequence $\{v^{(t)}\}_{t\geq 0}$ is obtained via Algorithm 1, then

$$\alpha^{(t)} \geq \max\left\{1, \frac{\rho\|G(v^{(t)})\|^2}{(L+\sigma)\|d^{(t)}\|^2}\right\}. \tag{19}$$

Proof: Suppose by (14) $\alpha^{(t)} \neq 1$. Then for $\alpha_{\prime}^{(t)} := \alpha^{(t)}\rho^{-1}$, (14) is not true,

$$-\langle G(v^{(t)} + \alpha_{\prime}^{(t)} d^{(t)}), \quad d^{(t)}\rangle < \sigma \alpha_{\prime}^{(t)}\|d^{(t)}\|^2.$$

By (16) and Assumption $(A2)$,

$$\begin{aligned}\|G^{(t)}\|^2 &\leq -\langle G^{(t)}, \quad d^{(t)}\rangle \\ &= \langle (G(v^{(t)} + \alpha_{\prime}^{(t)} d^{(t)}) - G^{(t)}), \quad d^{(t)}\rangle \\ &\quad - \langle G(v^{(t)} + \alpha_{\prime}^{(t)} d^{(t)}), \quad d^{(t)}\rangle \\ &\leq \alpha_{\prime}^{(t)}(L+\sigma)\|d^{(t)}\|^2.\end{aligned}$$

After making $\alpha_{\prime}^{(t)}$ the subject, the result is obtained. ■

Lemma 6: If Assumptions $(A1)$ and $(A3)$ hold, $\{v^{(t)}\}_{t\geq 0}$ and $\{w^{(t+1)}\}_{t\geq 0}$ obtained via Algorithm 1, then $\{v^{(t)}\}_{t\geq 0}$ and $\{w^{(t+1)}\}_{t\geq 0}$ are bounded. Furthermore,

$$\lim_{k\to\infty} \|v^{(t)} - w^{(t+1)}\| = 0, \tag{20}$$

and

$$\lim_{k\to\infty} \|v^{(t+1)} - v^{(t)}\| = 0. \tag{21}$$

Proof: Suppose $\tilde{v} \in \Omega'$, then by assumption $(A1)$ and (14), we have

$$\langle G(w^{(t+1)}), v^{(t)} - \tilde{v}\rangle \geq \langle G(w^{(t+1)}), v^{(t)} - w^{(t+1)}\rangle, \tag{22}$$

and

$$\langle G(w^{(t+1)}), v^{(t)} - w^{(t+1)}\rangle \geq \sigma\alpha_k^2\|d^{(t)}\|^2 \geq 0. \tag{23}$$

Now,

$$\begin{aligned}&\|v^{(t+1)} - \tilde{v}\|^2 \\ &\quad = \|P_\Omega[v^{(t)} - \lambda\zeta^{(t)}G(w^{(t+1)})] - P_\Omega(\tilde{v})\|^2 \\ &\quad \leq \|v^{(t)} - \lambda\zeta^{(t)}G(w^{(t+1)}) - \tilde{v}\|^2\end{aligned}$$

TABLE 4. Numerical results of Problem 3.

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	8	24	0.003603	4.27E-06	9	27	0.037976	3.80E-06	8	24	0.002002	2.81E-06
	v_2	7	22	0.003207	8.41E-06	8	24	0.0026	3.38E-06	7	21	0.002102	3.71E-06
	v_3	7	21	0.002951	3.61E-06	F	F	F	F	36	109	0.006108	8.29E-06
	v_4	40	122	0.013247	6.78E-06	221	668	0.037747	9.48E-06	29	88	0.004786	6.41E-06
	v_5	40	122	0.012008	6.78E-06	221	668	0.038538	9.48E-06	29	88	0.005495	6.41E-06
	v_6	19	58	0.006986	8.22E-06	F	F	F	F	29	88	0.005125	8.20E-06
	v_7	26	79	0.009674	5.49E-06	F	F	F	F	31	94	0.007927	6.38E-06
5000	v_1	8	24	0.010055	9.54E-06	9	27	0.006887	8.49E-06	8	24	0.00534	6.27E-06
	v_2	8	24	0.009929	5.64E-06	8	24	0.006311	7.56E-06	7	21	0.005565	8.30E-06
	v_3	7	21	0.008845	3.61E-06	F	F	F	F	37	112	0.026578	8.70E-06
	v_4	41	125	0.045605	8.93E-06	233	704	0.16915	9.41E-06	30	91	0.018617	7.99E-06
	v_5	41	125	0.042326	8.93E-06	233	704	0.1716	9.41E-06	30	91	0.018058	7.99E-06
	v_6	20	61	0.021563	7.46E-06	F	F	F	F	30	91	0.016923	9.42E-06
	v_7	41	125	0.043735	8.88E-06	F	F	F	F	30	91	0.023975	7.99E-06
10000	v_1	8	25	0.016286	7.20E-06	10	30	0.015061	1.92E-06	8	24	0.010638	8.87E-06
	v_2	8	24	0.01552	7.98E-06	9	27	0.012271	1.71E-06	8	24	0.010377	1.35E-06
	v_3	7	21	0.014178	3.61E-06	F	F	F	F	38	115	0.046243	5.95E-06
	v_4	42	128	0.072292	7.43E-06	238	719	0.34033	9.49E-06	31	94	0.038141	6.30E-06
	v_5	42	128	0.071992	7.43E-06	238	719	0.34421	9.49E-06	31	94	0.037598	6.30E-06
	v_6	20	62	0.034154	8.41E-06	F	F	F	F	31	94	0.039455	7.23E-06
	v_7	42	128	0.08857	7.43E-06	238	719	0.49173	9.52E-06	31	94	0.050056	6.30E-06
50000	v_1	9	27	0.055889	4.83E-06	10	30	0.058462	4.30E-06	9	27	0.040132	2.29E-06
	v_2	8	25	0.053744	9.52E-06	9	27	0.051396	3.83E-06	8	24	0.037723	3.03E-06
	v_3	7	21	0.047435	3.61E-06	F	F	F	F	39	118	0.19319	6.35E-06
	v_4	43	131	0.2714	9.78E-06	250	755	1.4401	9.44E-06	32	97	0.1487	7.86E-06
	v_5	43	131	0.27792	9.78E-06	250	755	1.4206	9.44E-06	32	97	0.15047	7.86E-06
	v_6	41	125	0.23145	7.19E-06	F	F	F	F	32	97	0.16098	5.88E-06
	v_7	43	131	0.36828	9.77E-06	250	755	1.903	9.45E-06	32	97	0.20614	7.87E-06
100000	v_1	9	27	0.10271	6.83E-06	10	30	0.10372	6.08E-06	9	27	0.080073	3.24E-06
	v_2	9	27	0.10133	4.04E-06	9	27	0.091185	5.41E-06	8	24	0.10462	4.28E-06
	v_3	7	21	0.080423	3.61E-06	F	F	F	F	39	118	0.43383	7.59E-06
	v_4	44	134	0.52143	8.13E-06	255	770	2.6101	9.52E-06	33	100	0.28277	6.20E-06
	v_5	44	134	0.49357	8.13E-06	255	770	2.6381	9.52E-06	33	100	0.28232	6.20E-06
	v_6	41	126	0.50145	9.37E-06	F	F	F	F	32	97	0.28287	7.79E-06
	v_7	44	134	0.87903	8.14E-06	255	770	3.6893	9.48E-06	33	100	0.41164	6.20E-06

$$
\begin{aligned}
&= \|v^{(t)} - \tilde{v}\|^2 - 2\lambda\zeta^{(t)}\langle G(w^{(t+1)}), v^{(t)} - \tilde{v}\rangle \\
&\quad + \|\lambda\zeta^{(t)}G(w^{(t+1)})\|^2 \\
&= \|v^{(t)} - \tilde{v}\|^2 \\
&\quad - 2\lambda\frac{\langle G(w^{(t+1)}), v^{(t)} - w^{(t+1)}\rangle}{\|G(w^{(t+1)})\|^2}\langle G(w^{(t+1)}), v^{(t)} - \tilde{v}\rangle \\
&\quad +\lambda^2\left(\frac{\langle G(w^{(t+1)}), v^{(t)} - w^{(t+1)}\rangle}{\|G(w^{(t+1)})\|}\right)^2 \\
&\leq \|v^{(t)} - \tilde{v}\|^2 \\
&\quad - 2\lambda\frac{\langle G(w^{(t+1)}), v^{(t)} - w^{(t+1)}\rangle}{\|G(w^{(t+1)})\|^2}\langle G(w^{(t+1)}), v^{(t)} - w^{(t+1)}\rangle \\
&\quad +\lambda^2\left(\frac{\langle G(w^{(t+1)}), v^{(t)} - w^{(t+1)}\rangle}{\|G(w^{(t+1)})\|}\right)^2 \\
&\leq \|v^{(t)} - \tilde{v}\|^2 - \lambda(2-\lambda)\left(\frac{\langle G(w^{(t+1)}), v^{(t)} - w^{(t+1)}\rangle}{\|G(w^{(t+1)})\|}\right)^2 \\
&= \|v^{(t)} - \tilde{v}\|^2 - \lambda(2-\lambda)\frac{\sigma^2\|v^{(t)} - w^{(t+1)}\|^4}{\|G(w^{(t+1)})\|^2}.
\end{aligned}
$$

Thus,

$$
\|v^{(t+1)} - \tilde{v}\|^2 \leq \|v^{(t)} - \tilde{v}\|^2 - \lambda(2-\lambda)\frac{\sigma^2\|v^{(t)} - w^{(t+1)}\|^4}{\|G(w^{(t+1)})\|^2}. \tag{24}
$$

The inequality (24) implies that $\{\|v^{(t)} - \tilde{v}\|\}_{t\geq 0}$ is non-increasing and therefore, $\{v^{(t)}\}_{t\geq 0}$ is bounded. That is,

$$
\|v^{(t)}\| \leq b_1, \quad b_1 > 0. \tag{25}
$$

Also, it can be recursively deduce from (24) that for all $t \geq 0$,

$$
\|v^{(t)} - \tilde{v}\|^2 \leq \|v^{(0)} - \tilde{v}\|^2.
$$

Hence by Assumption $(A2)$ and letting $L\|v^{(0)} - \tilde{v}\| = c_1$, we obtain that

$$
\|G^{(t)}\| = \|G(v^{(t)}) - G(\tilde{v})\| \leq L\|v^{(t)} - \tilde{v}\| \leq L\|v^{(0)} - \tilde{v}\| = c_1.
$$

This implies that for all $t \geq 0$,

$$
\|G^{(t)}\| \leq c_1. \tag{26}
$$

TABLE 5. Numerical results of Problem 4.

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	1	5	0.001588	0	21	64	0.009527	6.57E-06	7	21	0.001653	4.42E-06
	v_2	7	22	0.002804	5.96E-06	8	24	0.001505	2.78E-06	7	21	0.001856	2.83E-06
	v_3	10	31	0.003349	6.96E-06	25	76	0.003618	8.99E-06	32	97	0.004038	7.61E-06
	v_4	16	49	0.004124	6.93E-06	22	67	0.004025	5.38E-06	24	73	0.00319	8.29E-06
	v_5	16	49	0.005647	6.93E-06	22	67	0.003651	5.38E-06	24	73	0.003147	8.29E-06
	v_6	13	40	0.004158	9.55E-06	17	52	0.002874	9.63E-06	20	61	0.002564	7.34E-06
	v_7	16	49	0.005525	6.91E-06	22	67	0.003625	5.42E-06	24	73	0.003877	8.03E-06
5000	v_1	1	5	0.001836	0	22	67	0.010761	7.28E-06	7	21	0.003231	9.89E-06
	v_2	8	24	0.011429	4.00E-06	8	24	0.004044	6.21E-06	7	21	0.003208	6.33E-06
	v_3	10	31	0.007237	6.96E-06	25	76	0.011368	8.99E-06	32	97	0.01128	7.61E-06
	v_4	17	52	0.01361	6.39E-06	23	70	0.010637	5.99E-06	26	79	0.010798	5.77E-06
	v_5	17	52	0.012796	6.39E-06	23	70	0.01071	5.99E-06	26	79	0.010458	5.77E-06
	v_6	13	40	0.016525	9.55E-06	17	52	0.010516	9.63E-06	20	61	0.009189	7.34E-06
	v_7	17	52	0.013338	6.40E-06	23	70	0.011029	5.85E-06	26	79	0.009503	5.77E-06
10000	v_1	1	5	0.003424	0	23	70	0.023621	5.11E-06	8	24	0.005474	1.61E-06
	v_2	8	24	0.011131	5.65E-06	8	24	0.006845	8.79E-06	7	21	0.006116	8.95E-06
	v_3	10	31	0.014106	6.96E-06	25	76	0.022913	8.99E-06	32	97	0.023214	7.61E-06
	v_4	17	52	0.022024	9.04E-06	23	70	0.022879	8.47E-06	26	79	0.021947	8.16E-06
	v_5	17	52	0.019773	9.04E-06	23	70	0.021128	8.47E-06	26	79	0.019797	8.16E-06
	v_6	13	40	0.016566	9.55E-06	17	52	0.016187	9.63E-06	20	61	0.015095	7.34E-06
	v_7	17	52	0.022825	8.97E-06	23	70	0.030557	8.72E-06	26	79	0.018082	8.18E-06
50000	v_1	1	5	0.008602	0	24	73	0.097839	5.67E-06	8	24	0.022207	3.61E-06
	v_2	8	25	0.031443	6.74E-06	9	27	0.033145	3.14E-06	8	24	0.023428	2.31E-06
	v_3	10	31	0.042567	6.96E-06	25	76	0.084879	8.99E-06	32	97	0.080877	7.61E-06
	v_4	18	55	0.066015	8.32E-06	24	73	0.083713	9.40E-06	28	85	0.072611	5.68E-06
	v_5	18	55	0.070378	8.32E-06	24	73	0.12724	9.40E-06	28	85	0.10527	5.68E-06
	v_6	13	40	0.048315	9.55E-06	17	52	0.080423	9.63E-06	20	61	0.060785	7.34E-06
	v_7	18	55	0.06568	8.30E-06	24	73	0.086357	9.45E-06	28	85	0.075831	5.66E-06
100000	v_1	1	5	0.014915	0	24	73	0.17453	8.01E-06	8	24	0.040972	5.10E-06
	v_2	8	25	0.064395	9.54E-06	9	27	0.063608	4.45E-06	8	24	0.040663	3.26E-06
	v_3	10	31	0.074095	6.96E-06	25	76	0.15924	8.99E-06	32	97	0.20307	7.61E-06
	v_4	18	56	0.14652	9.51E-06	25	76	0.25144	6.59E-06	28	85	0.2212	8.03E-06
	v_5	18	56	0.13284	9.51E-06	25	76	0.18203	6.59E-06	28	85	0.20071	8.03E-06
	v_6	13	40	0.090853	9.55E-06	17	52	0.11237	9.63E-06	20	61	0.14914	7.34E-06
	v_7	18	56	0.12286	9.51E-06	25	76	0.22543	6.57E-06	28	85	0.21823	8.04E-06

Moreover, using the Cauchy-Schwartz inequality, Assumption $(A1)$ and (23),

$$\begin{aligned}\sigma\|v^{(t)}-w^{(t+1)}\| &= \frac{\sigma\|\alpha^{(t)}d^{(t)}\|^2}{\|v^{(t)}-w^{(t+1)}\|}\\ &\leq \frac{\langle G(w^{(t+1)}), v^{(t)}-w^{(t+1)}\rangle}{\|v^{(t)}-w^{(t+1)}\|}\\ &\leq \frac{\langle G^{(t)}, v^{(t)}-w^{(t+1)}\rangle}{\|v^{(t)}-w^{(t+1)}\|}\\ &\leq \|G^{(t)}\|. \end{aligned} \tag{27}$$

Since $\{v^{(t)}\}_{t\geq 0}$ and $\{G^{(t)}\}_{t\geq 0}$ are bounded, then from (27), the sequences $\{w^{(t+1)}\}_{t\geq 0}$ and $\{w^{(t+1)}-\tilde{v}\}_{t\geq 0}$ are also bounded. That is, there exists $d_1 > 0$ such that for all $t \geq 0$

$$\|w^{(t+1)}-\tilde{v}\| \leq d_1.$$

This combined with Assumption $(A2)$ implies that

$$\|G(w^{(t+1)})\| = \|G(w^{(t+1)})-G(\tilde{v})\| \leq L\|w^{(t+1)}-\tilde{v}\| \leq Ld_1.$$

Again from (24),

$$\lambda(2-\lambda)\frac{\sigma^2}{(L\nu)^2}\|v^{(t)}-w^{(t+1)}\|^4 \leq \|v^{(t)}-\tilde{v}\|^2-\|v^{(t+1)}-\tilde{v}\|^2,$$

which implies

$$\begin{aligned}\lambda(2-\lambda)\frac{\sigma^2}{(L\nu)^2}\sum_{k=0}^{\infty}\|v^{(t)}-w^{(t+1)}\|^4 &\leq \sum_{k=0}^{\infty}(\|v^{(t)}-\tilde{v}\|^2-\|v^{(t+1)}-\tilde{v}\|^2)\\ &\leq \|v^{(0)}-\tilde{v}\| < \infty. \end{aligned} \tag{28}$$

Inequality (28) implies that

$$\lim_{k\to\infty}\|v^{(t)}-w^{(t+1)}\| = 0.$$

In addition, utilizing (13) and the Cauchy-Schwartz inequality with $v^{(t)} \in \Omega$,

$$\|v^{(t+1)}-v^{(t)}\| = \|P_{\Omega}[v^{(t)}-\lambda\zeta^{(t)}G(w^{(t+1)})]-P_{\Omega}(v^{(t)})\|$$

TABLE 6. Numerical results of Problem 5.

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	9	27	0.003298	6.98E-06	16	48	0.031637	6.49E-06	22	66	0.004363	8.41E-06
	v_2	9	28	0.005355	5.67E-06	16	48	0.004742	6.84E-06	23	69	0.004126	8.57E-06
	v_3	9	28	0.003673	5.89E-06	15	45	0.004655	6.34E-06	23	69	0.004435	8.70E-06
	v_4	9	27	0.004882	9.09E-06	15	45	0.00438	8.75E-06	23	69	0.004213	9.48E-06
	v_5	9	27	0.005995	9.09E-06	15	45	0.00454	8.75E-06	23	69	0.004524	9.48E-06
	v_6	9	28	0.00338	5.87E-06	15	45	0.004436	6.98E-06	23	69	0.004172	8.57E-06
	v_7	9	27	0.003728	9.08E-06	16	48	0.004882	4.44E-06	25	75	0.00447	9.88E-06
5000	v_1	9	28	0.014131	8.35E-06	12	36	0.012473	2.77E-06	18	54	0.015132	9.37E-06
	v_2	10	30	0.013646	3.82E-06	12	36	0.012391	4.56E-06	19	57	0.01202	9.82E-06
	v_3	10	30	0.015289	3.96E-06	12	36	0.014812	3.25E-06	19	57	0.012741	9.49E-06
	v_4	10	30	0.014506	3.26E-06	12	36	0.013238	5.18E-06	20	60	0.01363	8.47E-06
	v_5	10	30	0.012126	3.26E-06	12	36	0.014431	5.18E-06	20	60	0.013595	8.47E-06
	v_6	10	30	0.013442	3.96E-06	12	36	0.013236	1.75E-06	19	57	0.01523	8.94E-06
	v_7	10	30	0.012737	3.26E-06	11	33	0.012987	7.69E-06	22	66	0.01501	9.78E-06
10000	v_1	10	30	0.063705	3.54E-06	11	33	0.025007	9.22E-06	15	45	0.021485	8.81E-06
	v_2	10	30	0.025924	5.40E-06	12	36	0.02761	8.05E-06	17	51	0.026274	6.52E-06
	v_3	10	30	0.023995	5.60E-06	12	36	0.027255	2.72E-06	17	51	0.02634	6.19E-06
	v_4	10	30	0.022813	4.61E-06	15	45	0.031644	2.15E-06	20	60	0.030613	8.51E-06
	v_5	10	30	0.023696	4.61E-06	15	45	0.043441	2.15E-06	20	60	0.03076	8.51E-06
	v_6	10	30	0.025283	5.60E-06	12	36	0.023506	3.33E-06	17	51	0.027144	5.86E-06
	v_7	10	30	0.026703	4.61E-06	12	36	0.024924	9.01E-06	21	63	0.032875	7.90E-06
50000	v_1	10	30	0.089058	7.92E-06	11	33	0.12213	4.97E-06	12	36	0.072394	5.60E-06
	v_2	10	31	0.082726	6.44E-06	11	33	0.087858	7.74E-06	12	36	0.075453	8.34E-06
	v_3	10	31	0.084789	6.68E-06	11	33	0.093097	7.43E-06	12	36	0.07018	8.52E-06
	v_4	10	31	0.092978	5.50E-06	12	36	0.09661	2.81E-06	17	51	0.10021	8.32E-06
	v_5	10	31	0.080507	5.50E-06	12	36	0.10058	2.81E-06	17	51	0.092856	8.32E-06
	v_6	10	31	0.087035	6.68E-06	11	33	0.088632	6.89E-06	12	36	0.067786	8.48E-06
	v_7	10	31	0.087928	5.50E-06	13	39	0.10279	2.57E-06	17	51	0.094417	9.58E-06
100000	v_1	10	31	0.17859	5.97E-06	11	33	0.18331	2.69E-06	11	33	0.1618	5.21E-06
	v_2	10	31	0.15979	9.10E-06	11	33	0.21467	4.13E-06	11	33	0.125	7.15E-06
	v_3	10	31	0.17232	9.45E-06	11	33	0.20707	4.20E-06	11	33	0.15231	6.58E-06
	v_4	10	31	0.16436	7.78E-06	11	33	0.18466	6.59E-06	14	42	0.19211	9.32E-06
	v_5	10	31	0.17487	7.78E-06	11	33	0.18005	6.59E-06	14	42	0.17592	9.32E-06
	v_6	10	31	0.16177	9.45E-06	11	33	0.17802	4.13E-06	11	33	0.12982	6.04E-06
	v_7	10	31	0.17116	7.77E-06	14	42	0.24247	6.50E-06	18	54	0.21267	6.52E-06

$$
\begin{aligned}
&= \|v^{(t)} - \lambda\zeta^{(t)}G(w^{(t+1)}) - v^{(t)}\| \\
&= \|\lambda\zeta^{(t)}G(w^{(t+1)})\| \\
&\leq \lambda\|v^{(t)} - w^{(t+1)}\|, \quad \forall t \geq 0. \qquad (29)
\end{aligned}
$$

It follows that

$$\lim_{k\to\infty} \|v^{(t+1)} - v^{(t)}\| = 0.$$

■

Remark 7: From equation (20), we have

$$\lim_{k\to\infty} \alpha^{(t)}\|d^{(t)}\| = 0. \qquad (30)$$

Lemma 8: Let $d^{(t)}$ be defined by (7), then for all $t \geq 0$,

$$\|d^{(t)}\| \leq M_0, \quad M_0 > 0. \qquad (31)$$

Proof: For $t = 0$, $\|d^{(0)}\| = \|G^{(0)}\| \leq c_1$.

Now for $t \geq 1$, by (7)-(9) and (25)-(26), we have

$$
\begin{aligned}
\|d^{(t)}\| &= \left\| -G^{(t)} + \beta^{(t)}d^{(t-1)} - \theta^{(t)}y^{(t-1)} \right\| \\
&= \left\| -G^{(t)} + \frac{\langle G^{(t)}, y^{(t-1)}\rangle}{\langle d^{(t-1)}, x^{(t-1)}\rangle + \gamma\|G^{(t)}\|\|d^{(t-1)}\|} d^{(t-1)} - \frac{\langle G^{(t)}, d^{(t-1)}\rangle}{\langle d^{(t-1)}, x^{(t-1)}\rangle + \gamma\|G^{(t)}\|\|d^{(t-1)}\|} y^{(t-1)} \right\| \\
&\leq \|G^{(t)}\| + \frac{\|G^{(t)}\|\|y^{(t-1)}\|\|d^{(t-1)}\|}{\gamma\|G^{(t)}\|\|d^{(t-1)}\|} + \frac{\|G^{(t)}\|\|y^{(t-1)}\|\|d^{(t-1)}\|}{\gamma\|G^{(t)}\|\|d^{(t-1)}\|} \\
&= \|G^{(t)}\| + 2\frac{\|G^{(t)}\|\|y^{(t-1)}\|\|d^{(t-1)}\|}{\gamma\|G^{(t)}\|\|d^{(t-1)}\|} \\
&= \|G^{(t)}\| + 2\frac{\|y^{(t-1)}\|}{\gamma} \\
&\leq \|G^{(t)}\| + 2L\frac{(\|v^{(t)}\| + \|v^{(t-1)}\|)}{\gamma} \\
&\leq c_1 + \frac{4Lb_1}{\gamma}. \qquad (32)
\end{aligned}
$$

Choose $M_0 = c_1 + \frac{4Lb_1}{\gamma}$, the result is obtained. ■

Theorem 9: If Assumptions $(A1)$-$(A4)$ hold and the sequence $\{v^{(t)}\}_{t\geq 0}$ is obtained via Algorithm 1, then

$$\liminf_{k\to\infty} \|G^{(t)}\| = 0. \qquad (33)$$

TABLE 7. Numerical results of Problem 6.

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	7	22	0.00314	3.52E-06	6	19	0.048938	3.52E-06	8	25	0.001826	2.86E-06
	v_2	6	19	0.002454	6.58E-06	6	19	0.002628	6.62E-06	9	28	0.001906	2.22E-06
	v_3	26	79	0.007118	7.92E-06	20	62	0.004783	9.01E-06	24	73	0.004041	9.67E-06
	v_4	55	166	0.015803	9.62E-06	20	62	0.005952	4.39E-06	28	85	0.006052	7.62E-06
	v_5	55	166	0.015687	9.62E-06	20	62	0.005615	4.39E-06	28	85	0.005907	7.62E-06
	v_6	44	133	0.01268	9.06E-06	17	53	0.005252	6.74E-06	26	79	0.005101	6.24E-06
	v_7	55	166	0.020052	9.58E-06	20	62	0.006681	4.48E-06	28	85	0.006959	7.61E-06
5000	v_1	7	22	0.008807	7.86E-06	6	19	0.007336	7.86E-06	8	25	0.005882	6.39E-06
	v_2	7	22	0.007431	1.48E-06	7	22	0.006962	8.34E-07	9	28	0.006313	4.95E-06
	v_3	19	58	0.019829	7.87E-06	17	53	0.015718	4.70E-06	25	76	0.017469	6.40E-06
	v_4	59	178	0.056885	8.90E-06	20	62	0.016301	9.82E-06	30	91	0.01989	6.64E-06
	v_5	59	178	0.055823	8.90E-06	20	62	0.017844	9.82E-06	30	91	0.019987	6.64E-06
	v_6	45	136	0.053487	9.71E-06	19	59	0.015879	6.90E-06	27	82	0.016838	6.73E-06
	v_7	59	178	0.072248	8.88E-06	20	62	0.019357	9.86E-06	30	91	0.022504	6.66E-06
10000	v_1	8	25	0.015739	1.12E-06	7	22	0.011918	6.27E-07	8	25	0.009475	9.03E-06
	v_2	7	22	0.013427	2.10E-06	7	22	0.013501	1.18E-06	9	28	0.010551	7.01E-06
	v_3	19	58	0.035075	7.73E-06	17	53	0.027687	4.92E-06	25	76	0.033161	7.62E-06
	v_4	61	184	0.10308	8.10E-06	21	65	0.032669	6.03E-06	30	91	0.041021	9.40E-06
	v_5	61	184	0.11939	8.10E-06	21	65	0.033162	6.03E-06	30	91	0.040811	9.40E-06
	v_6	43	130	0.074579	9.20E-06	21	65	0.032407	7.80E-06	27	82	0.039473	9.60E-06
	v_7	61	184	0.13726	8.10E-06	21	65	0.039902	6.07E-06	30	91	0.049888	9.37E-06
50000	v_1	8	25	0.049909	2.51E-06	7	22	0.041529	1.40E-06	9	28	0.039712	3.47E-06
	v_2	7	22	0.046088	4.69E-06	7	22	0.044263	2.64E-06	10	31	0.044662	2.69E-06
	v_3	22	67	0.13463	6.53E-06	17	53	0.10092	6.82E-06	26	79	0.11167	9.52E-06
	v_4	65	196	0.39147	7.49E-06	22	68	0.12845	5.85E-06	32	97	0.16218	8.19E-06
	v_5	65	196	0.41315	7.49E-06	22	68	0.12725	5.85E-06	32	97	0.17438	8.19E-06
	v_6	41	124	0.2329	9.73E-06	18	56	0.11439	7.51E-06	28	85	0.14897	8.46E-06
	v_7	65	196	0.49917	7.50E-06	22	68	0.16875	5.86E-06	32	97	0.20541	8.18E-06
100000	v_1	8	25	0.088766	3.55E-06	7	22	0.087935	1.98E-06	9	28	0.075942	4.91E-06
	v_2	7	22	0.079304	6.63E-06	7	22	0.08016	3.73E-06	10	31	0.094628	3.81E-06
	v_3	18	55	0.2137	8.69E-06	17	53	0.19025	8.73E-06	27	82	0.2442	6.91E-06
	v_4	67	202	0.84255	6.82E-06	22	68	0.2388	8.27E-06	33	100	0.34147	7.15E-06
	v_5	67	202	0.83831	6.82E-06	22	68	0.24632	8.27E-06	33	100	0.31176	7.15E-06
	v_6	41	124	0.44601	9.77E-06	18	56	0.20831	7.57E-06	27	82	0.24911	9.07E-06
	v_7	67	202	0.95306	6.82E-06	22	68	0.33241	8.28E-06	33	100	0.46283	7.13E-06

Proof: Suppose equation (33) is false. Then there exists $q > 0$ such that for all $t \geq 0$,

$$\|G^{(t)}\| \geq q. \tag{34}$$

Combining the inequality (34) with (18), we get

$$\|d^{(t)}\| \geq q \quad \forall t \geq 0. \tag{35}$$

Multiplying both sides of (19) with $\|d^{(t)}\|$,

$$\alpha^{(t)}\|d^{(t)}\| \geq \max\left\{1, \frac{\rho\|G^{(t)}\|^2}{(L+\sigma)\|d^{(t)}\|^2}\right\}\|d^{(t)}\|$$
$$\geq \max\left\{q, \frac{\rho q^2}{(L+\sigma)M_0}\right\}.$$

This contradicts with (30) and hence (33) must hold. ■

IV. NUMERICAL EXPERIMENT

In this section, the numerical behavior of the proposed algorithm (Algorithm 1) in comparison with two existing methods is examined. We compare the performance of Algorithm 1 with a conjugate gradient projection method for solving nonlinear equations with convex constraints by Zheng *et al.* [46] denoted as Algorithm 2 and a new three-term conjugate gradient-based projection method for solving large-scale nonlinear monotone equations by Koorapetse *et al.* [24] denoted as Algorithm 3. Algorithm 1 is implemented using the following parameters: $\sigma = 0.001, \mu = 1$ $\rho = 0.7, \gamma = 1.7$ and $\lambda = 1.2$. The selected parameters for Algorithm 2 and Algorithm 3 are chosen as reported in their respective papers. The metrics used for evaluating the results of the numerical experiments are the number of iteration (ITER), number of function evaluations (FVAL) and the time in seconds (TIME). In order to test the performance and robustness of the methods, we use the following initial points

$$v^{(1)} = (1, \ldots, 1)^T, v^{(2)} = (0.1, \ldots, 0.1)^T,$$
$$v^{(3)} = \left(\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}\right)^T, v^{(4)} = \left(1 - \frac{1}{n} \ldots, n-1\right)^T,$$

TABLE 8. Numerical results of Problem 7.

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	7	20	0.003119	3.71E-06	431	1302	0.07026	9.96E-06	9	26	0.00186	4.50E-06
	v_2	7	20	0.002351	3.71E-06	12	39	0.002383	2.95E-06	9	26	0.00155	4.50E-06
	v_3	F	F	F	F	F	F	F	F	F	F	F	F
	v_4	395	1184	0.060121	9.98E-06	155	472	0.023549	9.79E-06	543	1628	0.066887	1.00E-05
	v_5	395	1184	0.078078	9.98E-06	155	472	0.023526	9.79E-06	543	1628	0.065683	1.00E-05
	v_6	7	20	0.002144	3.71E-06	473	1418	0.066563	1.00E-05	382	1145	0.050367	9.99E-06
	v_7	7	20	0.002295	3.71E-06	155	472	0.02273	9.71E-06	519	1556	0.059964	9.98E-06
5000	v_1	7	23	0.008641	7.26E-06	F	F	F	F	7	21	0.003843	3.28E-06
	v_2	7	23	0.007373	7.26E-06	44	137	0.033395	9.60E-06	7	21	0.00572	3.28E-06
	v_3	F	F	F	F	547	1644	0.29756	9.99E-06	538	1616	0.2866	9.81E-06
	v_4	13	39	0.010421	7.18E-06	975	2936	0.61526	9.91E-06	133	399	0.076998	9.90E-06
	v_5	13	39	0.012831	7.18E-06	975	2936	0.70442	9.91E-06	133	399	0.070297	9.90E-06
	v_6	7	23	0.006829	7.26E-06	103	309	0.065006	9.91E-06	57	170	0.032461	9.85E-06
	v_7	7	23	0.008689	7.26E-06	975	2936	0.59495	9.91E-06	132	396	0.075979	9.81E-06
10000	v_1	6	20	0.014897	5.11E-06	F	F	F	F	6	18	0.009029	4.18E-06
	v_2	6	20	0.013685	5.11E-06	111	340	0.16039	9.51E-06	6	18	0.007034	4.18E-06
	v_3	F	F	F	F	330	993	0.47037	9.70E-06	407	1223	0.47079	9.66E-06
	v_4	51	155	0.065896	9.70E-06	F	F	F	F	31	93	0.035059	9.76E-06
	v_5	51	155	0.07858	9.70E-06	F	F	F	F	31	93	0.036222	9.76E-06
	v_6	6	20	0.012538	5.11E-06	26	79	0.041988	9.93E-06	9	27	0.010839	6.08E-06
	v_7	6	20	0.013995	5.11E-06	F	F	F	F	31	93	0.035435	9.38E-06
50000	v_1	3	11	0.028206	9.71E-06	F	F	F	F	7	23	0.032694	5.61E-06
	v_2	3	11	0.027526	9.71E-06	575	1736	3.4677	9.97E-06	7	23	0.032434	5.61E-06
	v_3	148	448	0.79172	9.28E-06	185	559	1.9143	9.26E-06	249	750	1.0544	9.93E-06
	v_4	4	14	0.033269	7.58E-06	F	F	F	F	10	32	0.043975	8.03E-06
	v_5	4	14	0.034212	7.58E-06	F	F	F	F	10	32	0.044574	8.03E-06
	v_6	3	11	0.02693	9.71E-06	9	29	0.063761	8.41E-06	9	28	0.041147	8.04E-06
	v_7	3	11	0.028158	9.71E-06	F	F	F	F	10	32	0.045068	8.21E-06
100000	v_1	5	19	0.075391	3.56E-06	F	F	F	F	8	26	0.08094	5.51E-06
	v_2	5	19	0.078529	3.56E-06	F	F	F	F	8	26	0.071135	5.51E-06
	v_3	207	626	2.2266	6.76E-06	F	F	F	F	119	360	0.96017	9.74E-06
	v_4	5	19	0.080213	5.20E-06	F	F	F	F	10	32	0.085737	5.75E-06
	v_5	5	19	0.070545	5.20E-06	F	F	F	F	10	32	0.090554	5.75E-06
	v_6	5	19	0.088286	3.56E-06	7	24	0.10001	8.24E-06	8	26	0.073764	3.15E-06
	v_7	5	19	0.078947	3.56E-06	F	F	F	F	10	32	0.089156	5.45E-06

$$v^{(5)} = \left(0, \frac{1}{n}, \dots, \frac{n-1}{n}\right)^T, v^{(6)} = \left(1, \frac{1}{2}, \dots, \frac{1}{n}\right)^T,$$
$$v^{(7)} = \text{rand}(0, 1).$$

We tested ten different problems with the dimension

$$n \in \{10^3, 5 \times 10^3, 10^4, 5 \times 10^4, 10^5\}.$$

The three solvers were coded in MATLAB R2019a and run on a PC with Intel(R) Core(TM) i7-7100U processor with 8 GB RAM and CPU 2.40 GHz. The iteration process is terminated whenever the inequality $\|G^{(t)}\| \leq 10^{-5}$ or $\|G(w^{(t+1)})\| \leq 10^{-5}$ is satisfied. If this condition is not satisfied after 1000 iterations, failure is declared.

We consider the following test problems where the operator G is $G(v) = (g_1(v), g_2(v), \dots, g_n(v))^T$ and $v = (v_1, v_2, \dots, v_n)^T$.

Problem 1: The Exponential Function [12].

$$g_1(v) = e^{v_1} - 1,$$
$$g_i(v) = e^{v_i} + v_i - 1, \quad \text{for } i = 2, 3, \dots, n, \quad \text{and} \quad \Omega = \mathbb{R}^n_+.$$

Problem 2 [12]: Modified Logarithmic Function.

$$g_i(v) = \ln(v_i + 1) - \frac{v_i}{n}, \quad \text{for } i = 2, 3, \dots, n,$$

$$\text{and } \Omega = \{v \in \mathbb{R}^n : \sum_{i=1}^{n} v_i \leq n, v_i > -1, i = 1, 2, \dots, n\}.$$

Problem 3 [7]: Nonsmooth Function I.

$$g_i(v) = 2v_i - \sin|v_i|, \quad i = 1, 2, 3, \dots, n,$$

$$\text{and } \Omega = \{v \in \mathbb{R}^n : \sum_{i=1}^{n} v_i \leq n, v_i \geq 0, i = 1, 2, \dots, n\}.$$

It is clear that Problem 3 is nonsmooth at $v = 0$.

Problem 4: Strictly Convex Function I [12].

$$g_i(v) = e^{v_i} - 1, \quad \text{for } i = 1, 2, \dots, n, \text{ and } \Omega = \mathbb{R}^n_+.$$

Problem 5 [47]: Tridiagonal Exponential Function.

$$g_1(v) = v_1 - e^{\cos(h(v_1+v_2))},$$
$$g_i(v) = v_i - e^{\cos(h(v_{i-1}+v_i+v_{i+1}))}, \quad \text{for } i = 2, \dots, n-1,$$

TABLE 9. Numerical results of Problem 8.

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	8	26	0.002473	5.55E-06	9	29	0.021606	4.32E-06	16	50	0.002084	6.76E-06
	v_2	8	26	0.001999	2.18E-06	9	29	0.001829	1.69E-06	15	47	0.001786	7.09E-06
	v_3	8	26	0.002815	3.03E-06	9	29	0.001997	2.36E-06	15	47	0.001779	9.87E-06
	v_4	8	26	0.001949	2.78E-06	9	29	0.001696	2.16E-06	15	47	0.001768	9.04E-06
	v_5	8	26	0.002416	2.78E-06	9	29	0.002028	2.16E-06	15	47	0.002055	9.04E-06
	v_6	8	26	0.001677	2.99E-06	9	29	0.001664	2.33E-06	15	47	0.001743	9.73E-06
	v_7	8	26	0.001886	2.78E-06	9	29	0.002012	2.12E-06	15	47	0.001685	9.13E-06
5000	v_1	9	29	0.005801	2.04E-06	9	29	0.004756	9.66E-06	17	53	0.006639	5.66E-06
	v_2	8	26	0.004964	4.87E-06	9	29	0.004886	3.79E-06	16	50	0.005428	5.93E-06
	v_3	8	26	0.004612	6.78E-06	9	29	0.004898	5.28E-06	16	50	0.004939	8.27E-06
	v_4	8	26	0.005857	6.21E-06	9	29	0.00519	4.84E-06	16	50	0.004858	7.57E-06
	v_5	8	26	0.00603	6.21E-06	9	29	0.004183	4.84E-06	16	50	0.004814	7.57E-06
	v_6	8	26	0.005151	6.76E-06	9	29	0.004207	5.26E-06	16	50	0.004913	8.24E-06
	v_7	8	26	0.006216	6.16E-06	9	29	0.006128	4.83E-06	16	50	0.005137	7.54E-06
10000	v_1	9	29	0.011138	2.88E-06	10	32	0.009805	1.98E-06	17	53	0.011051	8.01E-06
	v_2	8	26	0.010343	6.88E-06	9	29	0.009463	5.36E-06	16	50	0.011192	8.39E-06
	v_3	8	26	0.009623	9.60E-06	9	29	0.008733	7.47E-06	17	53	0.011892	4.38E-06
	v_4	8	26	0.010367	8.79E-06	9	29	0.009425	6.84E-06	17	53	0.011032	4.01E-06
	v_5	8	26	0.009623	8.79E-06	9	29	0.009237	6.84E-06	17	53	0.011505	4.01E-06
	v_6	8	26	0.009442	9.58E-06	9	29	0.009022	7.46E-06	17	53	0.01032	4.37E-06
	v_7	8	26	0.010682	8.76E-06	9	29	0.007668	6.84E-06	17	53	0.01053	4.02E-06
50000	v_1	9	29	0.033466	6.44E-06	10	32	0.03221	4.42E-06	18	56	0.042762	6.71E-06
	v_2	9	29	0.033582	2.53E-06	10	32	0.035575	1.73E-06	17	53	0.047633	7.03E-06
	v_3	9	29	0.034933	3.52E-06	10	32	0.034807	2.42E-06	17	53	0.040036	9.80E-06
	v_4	9	29	0.028516	3.23E-06	10	32	0.032695	2.21E-06	17	53	0.046819	8.97E-06
	v_5	9	29	0.031213	3.23E-06	10	32	0.040047	2.21E-06	17	53	0.051814	8.97E-06
	v_6	9	29	0.030315	3.52E-06	10	32	0.035587	2.42E-06	17	53	0.044096	9.79E-06
	v_7	9	29	0.032401	3.23E-06	10	32	0.033417	2.21E-06	17	53	0.040316	8.96E-06
100000	v_1	9	29	0.064535	9.11E-06	10	32	0.061762	6.25E-06	18	56	0.083787	9.49E-06
	v_2	9	29	0.059653	3.57E-06	10	32	0.062715	2.45E-06	17	53	0.088593	9.94E-06
	v_3	9	29	0.061734	4.98E-06	10	32	0.064769	3.42E-06	18	56	0.099384	5.19E-06
	v_4	9	29	0.059696	4.56E-06	10	32	0.063566	3.13E-06	18	56	0.099158	4.75E-06
	v_5	9	29	0.060738	4.56E-06	10	32	0.06153	3.13E-06	18	56	0.087448	4.75E-06
	v_6	9	29	0.059943	4.98E-06	10	32	0.062881	3.42E-06	18	56	0.091639	5.19E-06
	v_7	9	29	0.066378	4.55E-06	10	32	0.061708	3.12E-06	18	56	0.082491	4.75E-06

$$g_n(v) = v_n - e^{\cos(h(v_{n-1}+v_n))},$$
$$h = \frac{1}{n+1} \text{ and } \Omega = \mathbb{R}^n_+.$$

Problem 6 [48]: Nonsmooth Function II.

$$g_i(v) = v_i - \sin|v_i - 1|, \quad i = 1, 2, 3, \ldots, n.$$

$$\text{and } \Omega = \{v \in \mathbb{R}^n : \sum_{i=1}^{n} v_i \leq n, v_i \geq -1, i = 1, 2, \ldots, n\}.$$

Problem 7 [49]: Penalty Function 1.

$$t_i = \sum_{i=1}^{n} v_i^2, \quad c = 10^{-5}$$
$$g_i(v) = 2c(v_i - 1) + 4(t_i - 0.25)v_i, \quad i = 1, 2, 3, \ldots, n.$$
and $\Omega = \mathbb{R}^n_+$.

Problem 8 [21]: Pursuit-Evasion problem.

$$g_i(v) = \sqrt{8}v_i - 1, \quad i = 1, 2, 3, \ldots, n.$$
and $\Omega = \mathbb{R}^n_+$.

Problem 9 [7]:

$$g_1(v) = 2v_1 + \sin v_1 - 1,$$
$$g_i(v) = -v_{i-1} + 2v_i + \sin v_i - 1, \quad \text{for } i = 2, \ldots, n-1,$$
$$g_n(v) = 2v_n + \sin v_n - 1,$$
and $\Omega = \mathbb{R}^n_+$.

Problem 10 [50]:

$$g_i(v) = e^{v_i^2} + 3\sin v_1 \cos v_i - 1, \text{ for } i = 1, 2, \ldots, n$$
and $\Omega = \mathbb{R}^n_+$.

A detail report of the numerical experiments are presented in Table 2-11 of the appendix section. The columns of the presented tables have the following definitions:
DIM: denotes the dimension of the problem
INP: denotes the initial points
ITER: denotes the number of iterations
FVAL: denotes the number of function evaluations

TABLE 10. Numerical results of Problem 9.

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	66	201	0.021603	8.77E-06	20	63	0.021953	4.17E-06	25	77	0.004248	8.01E-06
	v_2	45	138	0.01354	8.19E-06	22	69	0.006178	6.60E-06	26	80	0.006082	9.03E-06
	v_3	63	192	0.018492	9.46E-06	22	69	0.006682	9.23E-06	29	89	0.005133	4.31E-06
	v_4	66	201	0.017244	8.34E-06	19	60	0.00819	6.72E-06	27	83	0.004442	9.98E-06
	v_5	65	198	0.01804	9.31E-06	24	75	0.007479	5.19E-06	31	95	0.005206	6.96E-06
	v_6	68	207	0.023017	9.95E-06	23	72	0.005736	8.34E-06	28	86	0.004671	7.02E-06
	v_7	66	201	0.018136	8.55E-06	18	57	0.005276	9.34E-06	27	83	0.006519	8.77E-06
5000	v_1	66	201	0.060832	9.24E-06	22	69	0.018376	4.21E-06	27	83	0.016243	8.43E-06
	v_2	54	165	0.048319	9.30E-06	22	69	0.025083	6.15E-06	26	80	0.013754	7.67E-06
	v_3	67	204	0.060916	9.33E-06	22	69	0.025888	7.85E-06	28	86	0.015848	9.29E-06
	v_4	69	210	0.058481	9.50E-06	24	75	0.029353	4.67E-06	28	86	0.015666	5.29E-06
	v_5	68	207	0.064125	9.29E-06	21	66	0.017702	6.09E-06	30	92	0.017177	9.33E-06
	v_6	72	219	0.062083	8.59E-06	22	69	0.019432	9.47E-06	30	92	0.015217	4.96E-06
	v_7	75	228	0.066708	9.70E-06	18	57	0.014894	9.96E-06	29	89	0.019227	7.65E-06
10000	v_1	66	201	0.11472	9.35E-06	20	63	0.038598	8.52E-06	29	89	0.034928	9.27E-06
	v_2	64	195	0.10065	7.63E-06	22	69	0.038999	5.60E-06	26	80	0.033014	8.09E-06
	v_3	61	186	0.096951	9.49E-06	22	69	0.045268	7.70E-06	29	89	0.035836	5.79E-06
	v_4	67	204	0.10783	9.87E-06	21	66	0.034675	9.72E-06	30	92	0.046983	7.25E-06
	v_5	67	204	0.11027	9.54E-06	21	66	0.036945	4.98E-06	29	89	0.033912	5.34E-06
	v_6	71	216	0.117	8.65E-06	22	69	0.035698	5.66E-06	31	95	0.039641	4.05E-06
	v_7	77	234	0.12135	9.95E-06	19	60	0.037798	5.91E-06	30	92	0.036551	6.49E-06
50000	v_1	69	210	0.3947	8.58E-06	23	72	0.1638	5.49E-06	30	92	0.14858	5.86E-06
	v_2	69	210	0.40104	8.32E-06	20	63	0.18313	5.71E-06	26	80	0.11959	9.06E-06
	v_3	69	210	0.37217	9.30E-06	22	69	0.14792	7.38E-06	27	83	0.12681	8.98E-06
	v_4	82	249	0.47636	7.02E-06	21	66	0.16443	7.31E-06	32	98	0.14871	7.69E-06
	v_5	81	246	0.4551	9.17E-06	23	72	0.16816	6.74E-06	33	101	0.16592	7.42E-06
	v_6	73	222	0.38263	8.14E-06	22	69	0.15737	7.55E-06	34	104	0.1558	7.86E-06
	v_7	84	255	0.46779	8.76E-06	20	63	0.12934	4.62E-06	31	95	0.14279	9.55E-06
100000	v_1	60	183	0.66996	9.91E-06	26	81	0.3299	8.15E-06	30	92	0.28028	7.58E-06
	v_2	60	183	0.68421	9.66E-06	21	66	0.2692	4.84E-06	27	83	0.264	6.03E-06
	v_3	65	198	0.71363	9.44E-06	22	69	0.27706	9.02E-06	27	83	0.26202	9.14E-06
	v_4	84	255	0.9838	9.62E-06	24	75	0.33167	4.01E-06	32	98	0.29531	6.01E-06
	v_5	82	249	0.99822	7.55E-06	23	72	0.32984	5.80E-06	32	98	0.31747	8.00E-06
	v_6	76	231	0.95289	9.80E-06	22	69	0.38733	7.97E-06	29	89	0.26857	7.91E-06
	v_7	86	261	1.1855	8.13E-06	20	63	0.26303	6.72E-06	32	98	0.30907	8.41E-06

TIME: denotes the CPU time in seconds
NORM: denotes the norm of the operator at the solution

From Tables 2–11, it is clear that Algorithm 1 obtained the solutions of virtually all the test problems with least number of ITER, FVAL and TIME. These information is further illustrated in Figures 1-3 based on the Dolan and Morè [51] performance profile. The performance profile tells the percentage of win by each solver. In all the experiments, we can see from the Figures 1-3, that the proposed algorithm performs better with higher percentage win in all the 3 metrics, i.e., ITER, FVAL and TIME. The reasons behind the good performance of Algorithm 1 are; the search direction is of three-term and a good selection of the control parameters σ, ρ, γ and λ.

A. IMAGE RESTORATION

Fundamental theory of compressed sensing come up with the possibility of obtaining the sparsest solution of the linear system which involves finding solution to an l_0-norm regularized minimization problem [52]. The minimization problem belongs to combinatorial optimization and so becomes difficult to find an efficient process to find the most sparsest solution. On this basis, another process which replaces the l_0-norm with l_1-norm [53] was introduced. That is, finding solution to the following continuous optimization problem:

$$\min_{v}\{\|v\|_1 : Av = b\}. \tag{36}$$

It was shown that (36) possesses high possibility of finding the most useful result for image restoration problems [53]. However, problem (36) can be reformulated as follows when noise is taken into account:

$$\min_{v}\left\{\omega\|v\|_1 + \frac{1}{2}\|b - Av\|_2^2\right\}, \tag{37}$$

where ω is called a balance parameter.

A number of methods have been proposed for finding solution to problem (37). Interested readers are referred to (Refs. [54]–[56]). Not so long, Figueiredo *et al.* [57] made a breakthrough by showing that problem (37) can be written as

TABLE 11. **Numerical results of Problem 10.**

		Algorithm 1				Algorithm 2				Algorithm 3			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	v_1	1	9	0.001478	0	21	67	0.016787	6.78E-06	18	57	0.014424	5.42E-06
	v_2	1	8	0.001176	0	7	24	0.007137	2.85E-06	18	57	0.009273	6.13E-06
	v_3	12	42	0.003312	6.52E-06	17	56	0.007911	4.63E-06	16	52	0.005987	4.74E-06
	v_4	14	47	0.003957	4.46E-06	21	67	0.007754	7.65E-06	21	66	0.007162	4.69E-06
	v_5	14	47	0.005066	4.46E-06	21	67	0.013234	7.65E-06	21	66	0.010573	4.69E-06
	v_6	12	42	0.00334	8.90E-06	19	61	0.007346	5.68E-06	18	57	0.005149	6.34E-06
	v_7	14	47	0.003879	4.52E-06	21	67	0.007173	8.13E-06	21	66	0.006436	4.71E-06
5000	v_1	1	9	0.003802	0	22	70	0.024855	6.91E-06	19	60	0.016949	5.42E-06
	v_2	1	8	0.002946	0	7	24	0.010193	6.37E-06	19	60	0.015698	6.13E-06
	v_3	12	42	0.011937	6.52E-06	17	56	0.016794	4.63E-06	16	52	0.015366	4.74E-06
	v_4	14	47	0.013373	1.00E-05	22	70	0.024347	7.83E-06	22	69	0.019665	4.70E-06
	v_5	14	47	0.011366	1.00E-05	22	70	0.024633	7.83E-06	22	69	0.02219	4.70E-06
	v_6	12	42	0.009675	8.90E-06	19	61	0.020804	5.68E-06	18	57	0.016674	6.34E-06
	v_7	14	48	0.014224	7.38E-06	22	70	0.022787	8.05E-06	22	69	0.019273	4.73E-06
10000	v_1	1	9	0.005851	0	22	70	0.043913	9.77E-06	19	60	0.026559	7.66E-06
	v_2	1	8	0.004666	0	7	24	0.017763	9.01E-06	19	60	0.029641	8.67E-06
	v_3	12	42	0.017395	6.52E-06	17	56	0.030149	4.63E-06	16	52	0.024738	4.74E-06
	v_4	15	50	0.027093	4.16E-06	23	73	0.047703	5.05E-06	22	69	0.030872	6.65E-06
	v_5	15	50	0.024969	4.16E-06	23	73	0.049013	5.05E-06	22	69	0.032393	6.65E-06
	v_6	12	42	0.020858	8.90E-06	19	61	0.038463	5.68E-06	18	57	0.028964	6.34E-06
	v_7	14	48	0.024232	9.97E-06	23	73	0.046756	5.05E-06	22	69	0.031703	6.69E-06
50000	v_1	1	9	0.019995	0	23	73	0.15542	9.96E-06	20	63	0.10775	7.66E-06
	v_2	1	8	0.017208	0	8	27	0.064103	1.87E-06	20	63	0.10476	8.67E-06
	v_3	12	42	0.066015	6.52E-06	17	56	0.11196	4.63E-06	16	52	0.079069	4.74E-06
	v_4	15	50	0.084946	9.31E-06	24	76	0.15429	5.15E-06	23	72	0.11585	6.65E-06
	v_5	15	50	0.080223	9.31E-06	24	76	0.16436	5.15E-06	23	72	0.11778	6.65E-06
	v_6	12	42	0.069871	8.90E-06	19	61	0.13224	5.68E-06	18	57	0.085279	6.34E-06
	v_7	15	50	0.085089	9.41E-06	24	76	0.16658	5.16E-06	23	72	0.11909	6.67E-06
100000	v_1	1	9	0.040488	0	24	76	0.30559	6.42E-06	21	66	0.20684	4.84E-06
	v_2	1	8	0.030606	0	8	27	0.11949	2.65E-06	21	66	0.20701	5.48E-06
	v_3	12	42	0.12066	6.52E-06	17	56	0.21088	4.63E-06	16	52	0.15965	4.74E-06
	v_4	15	51	0.15788	9.40E-06	24	76	0.30218	7.28E-06	23	72	0.24567	9.41E-06
	v_5	15	51	0.15638	9.40E-06	24	76	0.30387	7.28E-06	23	72	0.23736	9.41E-06
	v_6	12	42	0.12795	8.90E-06	19	61	0.24309	5.68E-06	18	57	0.17444	6.34E-06
	v_7	15	51	0.17258	9.40E-06	24	76	0.30278	7.26E-06	23	72	0.2232	9.42E-06

a bound-constrained quadratic problem. Subsequently, Xiao *et al.* [58] converted the bound-constrained quadratic problem into the following nonlinear convex constrained equation:

$$G(p) = \min\{p, Bp + h\} = 0. \tag{38}$$

where $p = \begin{bmatrix} q_a \\ q_b \end{bmatrix}, q_a, q_b > 0, B = \begin{bmatrix} A^TA & -A^TA \\ -A^TA & A^TA \end{bmatrix}, A \in \mathbb{R}^{m\times n}(m < n)$ and $h = \tau e_{2n} + \begin{bmatrix} -y \\ y \end{bmatrix}, \tau > 0, e_n$ is an n-dimensional vector with all elements one and $y = A^Tb$. Furthermore, the equivalent nonlinear convex constrained equation (38) was shown to be Lipschitz continuous and monotone. Hence, Algorithm 1 can be used to solve problem (38).

The efficiency of Algorithm 1 in restoring noisy and blurred images is depicted in this experiment. Four colored images of different sizes are considered in the experiment. These images are distorted using a Gaussian noise with standard deviation of 10^{-2} . For Algorithm 1, we select the following control parameters for its implementation: $\rho = 0.1$; $\mu = 0.1$; $\sigma = 0.0001$; $\gamma = 0.1$. $\lambda = 1$. We compare Algorithm 1 with Algorithm 2 and Algorithm 3 proposed in [59] and [58], respectively. All methods were implemented from same initial point $v^{(0)} = A^Tb$ and terminated when $\frac{|f^{(t)} - f^{(t-1)}|}{|f^{(t-1)}|} < 10^{-5}$, where $f(v) = \omega\|v\|_1 + \frac{1}{2}\|b - Av\|_2^2$ and $f^{(t)}$ is the function evaluation of f and $v^{(t)}$. Figure 4 shows the original images, the blurred with noise images and the restored images by the various algorithm are presented in Figure 5, 6, 7 and 8.

The experimental results of Algorithm 1, Algorithm 2 and Algorithm 3 are presented in Table 1. The comparison is based on the signal-to-noise ratio (SNR), Peak signal-to-noise ratio [60] and the Structural Similarity index (SSIM) [61]. From Table 1, it is evident that for all the test images, the restored images by Algorithm 1 are closer to the original than those restored by Algorithm 2 and Algorithm 3. This is reflected by it's bigger value of SNR, PSNR and SSIM in Table 1.

V. CONCLUSION

This article modified and extended the work of Baluch *et al.* [45] to solve nonlinear monotone operator equations. The modification become necessary so as to establish the descent and boundedness property of the search direction without the use of the line search. The algorithm was derivative-free and could handle problems of high dimensions. Using some suitable properties of the projection map as well as some appropriate assumptions, we proved the global convergence of the algorithm. Two types of numerical experiments were conducted and presented in order to show the efficiency of the proposed algorithm. The first was on some benchmark nonlinear monotone operator equations and the second was on image restoration. From the two experiments, the proposed algorithm outperform some existing algorithms in terms of the metrics considered.

APPENDIX

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