A New Transformation for the Lotka-Volterra Problem

C. M. Evans and G. L. Findley

Department of Chemistry, Northeast Louisiana University, Monroe, LA 71209

The Lotka-Volterra dynamical system [$\dot{x}_1 = a\,x_1 - b\,x_1\,x_2$; $\dot{x}_2 = -c\,x_2 + b\,x_1\,x_2$] is reduced to a single second-order autonomous ordinary differential equation by means of a new variable transformation. Formal analytic solutions are presented for this latter differential equation.

The Lotka-Volterra (LV) problem consists of the following pair of first-order autonomous ordinary differential equations,

$$\dot{x}_1 = a x_1 - b x_1 x_2$$
$$\dot{x}_2 = -c x_2 + b x_1 x_2 , \tag{1}$$

where $x_1(t)$ and $x_2(t)$ are real functions of time; $\dot{x}_i := dx_i/dt$; and a, b, c are positive real constants. This system was originally introduced by Lotka [1] in 1920 as a model of undamped oscillations in autocatalytic chemical reactions, and was later applied by Volterra [2] to treat predator-prey interactions in ecology. Other applications have followed in the intervening years in physics [3], chemistry [4], population biology [5] and epidemiology [6]. Indeed, the LV dynamical system is today a standard textbook example in the theory of nonlinear ordinary differential equations [7,8].

Since the original publication by Lotka [1], it has been known that eq. (1) possesses a dynamical invariant, namely,

$$\Lambda = b x_1 + b x_2 - c \ln x_1 - a \ln x_2 . \tag{2}$$

By means of a logarithmic transformation, Kerner [9] showed that Λ serves to reduce eq. (1) to a Hamiltonian system. This has recently sparked a resurgence of interest in the LV problem (including a rediscovery of some previously known results [10,11]), particularly with regard to dynamical invariants of generalizations of eq. (1) [3,12].

Although the existence of a dynamical invariant for eq. (1) implies that this system is solvable, very little is known about the analytic form of these solutions, with the exception of a Lie series analysis [13]. The purpose of the present Paper is to present a new transformation that reduces the LV system to a single second-order autonomous ordinary differential equation, and to reduce the solution of this equation to an integral quadrature. Thus, formal analytic solutions to eq. (1) will be presented.

We begin by defining new coordinates $z_1(t)$ and $z_2(t)$ as follows:

$$z_1 = \Lambda^{-1/2} (b x_1 + b x_2)^{1/2},$$
$$z_2 = \Lambda^{-1/2} (- c \ln x_1 - a \ln x_2)^{1/2} . \tag{3}$$

From eqs. (1-3) we find

$$x_1 = \frac{\Lambda}{b(a+c)} (c z_1^2 - 2 z_2 \dot{z}_2),$$
$$x_2 = \frac{\Lambda}{b(a+c)} (a z_1^2 + 2 z_2 \dot{z}_2) , \tag{4}$$

and

$$z_1^2 + z_2^2 = 1. \tag{5}$$

Eq. (5) permits the definition of an angle ϕ such that

$$z_1 = \sin \phi , \quad z_2 = \cos \phi . \tag{6}$$

From eqs. (2, 4 and 6), then, we find

$$\ddot{\phi} + \left[\cot \phi - \tan \phi - \frac{2\Lambda}{a+c} \cos \phi \sin \phi \right] \dot{\phi}^2$$

$$+ (c-a)\left(1 - \frac{\Lambda}{a+c} \sin^2 \phi \right) \dot{\phi} - \frac{1}{2} ac \left(1 - \frac{\Lambda}{a+c} \right) \tan \phi \qquad (7)$$

$$- \frac{1}{2} \frac{ac\Lambda}{a+c} \sin \phi \cos \phi = 0.$$

Making the substitution

$$w = \frac{\Lambda}{2(a+c)} (1 - \cos 2\phi), \qquad (8)$$

eq. (7) becomes

$$\ddot{w} - \dot{w}^2 - (c-a)(w-1)\dot{w} \qquad (9)$$

$$+ acw(w-1) = 0.$$

Moreover, from eqs. (4, 6 and 8) it follows that

$$x_1 = \frac{1}{b}(cw + \dot{w}),$$

$$\qquad (10)$$

$$x_2 = \frac{1}{b}(aw - \dot{w}).$$

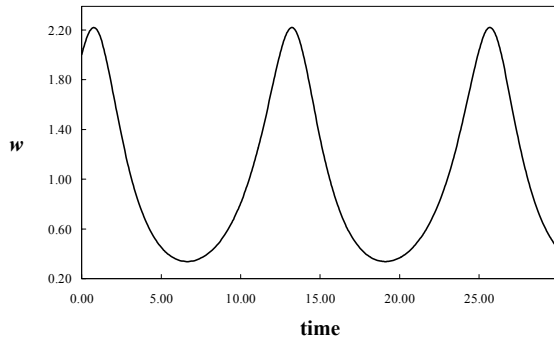Eq. (9) is fully equivalent to the LV dynamical system [i.e., eq. (1)]. A fourth-order Runge-Kutta [14] integration of eq. (9) is shown in Fig. 1 for a typical trajectory, and the usual phase-plane plot of eq. (10) for this same trajectory is given in Fig. 2. (These results are identical with those we have obtained from a direct numerical integration of eq. (1), the plots of which are not shown for the sake of brevity.)

If we now set $c = \alpha a$, eq. (2) can be rearranged to yield

$$bx_1x_2 = -\frac{k^2}{b} x_1^{1-\alpha} e^{b(x_1 + x_2)/a},$$

$$\qquad (11)$$

$$k^2 = -b^2 e^{-\Lambda/a},$$

which reduces eq. (1) to

$$\dot{x}_1 = ax_1 + \frac{k^2}{b} x_1^{1-\alpha} e^{b(x_1 + x_2)/a}$$

$$\qquad (12)$$

$$\dot{x}_2 = -a\alpha x_2 - \frac{k^2}{b} x_1^{1-\alpha} e^{b(x_1 + x_2)/a} \quad .$$

Fig. 1 Fourth-order Runge-Kutta solution to eq. (9) of the text for $a = 0.50$, $b = 1.30$, $c = 0.67$. Initial data for this trajectory are $w\,(t{=}0) = 2.00$, $\dot{w}\,(t{=}0) = 0.50$. The invariant [eq. (2) of text] is $\Lambda = 2.5850$.
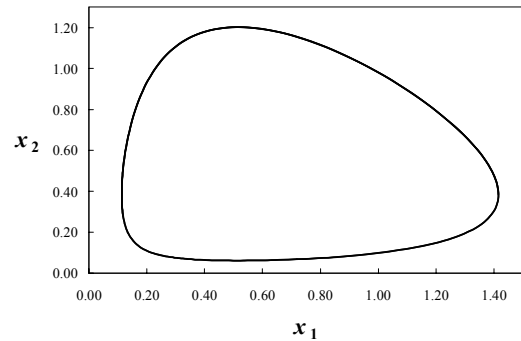
Fig. 2 Phase-plane plot of eq. (10) of the text for the trajectory specified in Fig. 1.

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Substituting eq. (10) into eq. (12) gives

$$\ddot{w} = a^2 \alpha w + a(1 - \alpha)\dot{w}$$
$$+ k^2 \left[\frac{1}{b}(a\alpha w + \dot{w}) \right]^{1-\alpha} e^{(\alpha+1)w} \quad . \tag{13}$$

Folding eq. (13) into eq. (9) [with $c = \alpha a$] yields the following equation for the first integral:

$$\dot{w}^2 + a(\alpha - 1)w\dot{w} - \alpha a^2 w^2$$
$$- k^2 \left[\frac{1}{b}(\alpha a w + \dot{w}) \right]^{1-\alpha} e^{(\alpha+1)w} = 0 . \tag{14}$$

For the case $c = a$ ($\alpha = 1$), eq. (14) reduces to

$$\dot{w}^2 - a^2 w^2 - k^2 e^{2w} = 0 , \tag{15}$$

or

$$\dot{w} = \pm \left[a^2 w^2 + k^2 e^{2w} \right]^{1/2} . \tag{16}$$

Formal integration of eq. (12) leads to the quadrature

$$t - t_0 = \pm \int^{w} \left[a^2 w'^2 + k^2 e^{2w'} \right]^{-1/2} dw'. \tag{17}$$

Eq. (17) represents an analytic solution to the $c = a$ LV problem. Moreover, an analysis of the form of this integral [15] shows its relationship to the family of elliptic integrals, and leads to a new class of LV related differential equations.

In order to provide a similar quadrature for the $c \neq a$ case, an alternative form of the first integral may be found by writing eq. (11) as

$$- \frac{k^2}{b} x_1^{-\alpha} e^{b(x_1 + x_2)/a} = b x_2 \tag{18}$$

or, using eq. (10),

$$- \frac{k^2}{b} \left[\frac{1}{b}(\alpha a w + \dot{w}) \right]^{-\alpha} e^{(\alpha+1)w} \tag{19}$$
$$= a w - \dot{w} .$$

If we now define a function $\rho(w)$ such that

$$\alpha a w + \dot{w} = \alpha a e^{\rho} , \tag{20}$$

and write $a w - \dot{w} = a(\alpha + 1)w - (\alpha a w + \dot{w})$, eq. (19) becomes

$$- \frac{k^2}{b} \left(\frac{\alpha a}{b} \right)^{-\alpha} e^{(\alpha+1)w} e^{-\alpha\rho} \tag{21}$$
$$= a(\alpha + 1)w - \alpha a e^{\rho}$$

or,

$$b a(\alpha + 1)w - b \alpha a e^{\rho}$$
$$+ k^2 \left(\frac{\alpha a}{b} \right)^{-\alpha} e^{(\alpha+1)w} e^{-\alpha\rho} = 0 . \tag{22}$$

Since $\rho(w)$ can be determined from eq. (22), we have from eq. (20) the first integral

$$\dot{w} = \alpha a (e^{\rho} - w) . \tag{23}$$

Formal integration of eq. (23) yields the quadrature

$$t - t_0 = \int^{w} \left[\alpha a (e^{\rho} - w') \right]^{-1} dw'. \tag{24}$$

Eq. (24) represents an analytic solution to the general LV problem. As in the $c = a$ case, an analysis of the form of eq. (24) [15] shows the relationship of this integral to the family of elliptic integrals, and leads to a new class of LV related differential equations.

Finally, it is worth noting that we have integrated eq. (13) directly using a symbolic processor [16], and that this integration results in the same formal solution presented above [i.e., eq. (24), with ρ being given by eq. (22)]. Using the same symbolic processor [16], we have also shown that eq. (24) reduces to eq. (17) under the assumption $\alpha = 1$ ($c = a$), which demonstrates the consistency of these solutions.

In summary, we have presented a transformation for the LV dynamical system that reduces this system to a single second-order autonomous ordinary differential equation. We

have also provided formal analytic solutions to the LV problem by means of integral quadratures. An analytic investigation of these integrals will be presented separately [15].

Acknowledgment

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