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Authors

Zhao, XY
Lang, ZQ
Park, G
et al.

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A New Transmissibility Analysis Method for Detection and Location of Damage via Nonlinear Features in MDOF Structural Systems

X. Y. Zhao, Z. Q. Lang, G. Park, C. R. Farrar, M. D. Todd, Z. Mao and K. Worden

***Abstract*—In this paper, a new transmissibility analysis method is proposed for the detection and location of damage via nonlinear features in Multi-Degree-Of-Freedom (MDOF) structural systems. The method is derived based on the transmissibility of Nonlinear Output Frequency Response Functions (NOFRFs), a concept recently proposed to extend the traditional transmissibility concept to the nonlinear case. The implementation of the method is only based on measured system output responses and by evaluating and analyzing the transmissibility of these system responses at super-harmonics. This overcomes the problems with available techniques which assume there is one damaged component with nonlinear features in the system, and require specific testing or assume loading on inspected structural systems is measurable. Both numerical simulation studies and experimental data analysis have been conducted to verify the effectiveness and demonstrate the potential practical applications of the new method.**

***Index Terms*—Damage detection and location, Nonlinear Output Frequency Response Functions (NOFRFs), Transmissibility analysis**

I. Introduction

IN engineering practice the behaviors of many mechanical and civil structural systems, such as, rotary machineries [1-3], multi-storey buildings [4-6] and multi-span bridges [7, 8], should be described by more than one set of coordinates and can, therefore, be modeled by multi-degree-of-freedom (MDOF) systems. All such structural systems are prone to suffering certain damage due to long service time, improper use or hostile working environments. Therefore, more and more efforts have been made by researchers to address the problems of damage detection and location in MDOF systems [1, 2, 9-11]. One class of the most popular techniques is the transmissibility based damage detection and location methods [5, 12-15].

The transmissibility is traditionally defined as the ratio of the spectra of two different system outputs, has been comprehensively studied, and is widely used for damage detection and fault diagnosis. For example, Zhu [16] investigated the sensitivity of the transmissibility and concluded that both mass and stiffness damage could induce a significant change in the system transmissibility. Cao [17] investigated the rate of change of both the system transmissibility and the system frequency response functions (FRF) when a damage occurred, and found that the transmissibility was much more sensitive to the damage than the FRF. Maia [8] conducted a comprehensive research on a transmissibility based fault diagnosis technique, and proposed a DRQ (Detection and Relative Damage Quantification) Indicator, which was the correlation between the measured system response and the response estimated from the undamaged transmissibility function. He also proposed the concept of TDI (Transmissibility Damage Indicator) in [12][13], which was defined as the correlation between the transmissibility of an undamaged system and the transmissibility of a damaged system. The performance of these transmissibility based indicators in fault diagnosis has been verified by experimental studies. In addition to damage detection, the transmissibility has also been used for damage location; Zhang [18] studied the influence of damage on the transmissibility, and found that the transmissibility near the damaged area could incur a more significant change. Consequently, he proposed several damage indicators based on translation transmissibility and curvature transmissibility and verified that these damage indicators could help to find the location of damage correctly by both simulation studies and experimental tests. Jonson [13, 15, 19] analyzed the characteristics of the transmissibility response function and concluded that transmissibility response function was entirely independent of the poles but solely dependent on the zeros of the system transfer function so that the damage could be trapped and identified. Sampaio and Maia [14] pointed out that the summation of the difference between the damaged and the undamaged transmissibility may mask the true damage location if the frequency range was inappropriate. This is because the transmissibility difference near the resonances and anti-resonances was much larger than that in the other

X.Y. Zhao and Z.Q. Lang are with the Department of Automatic Control and Systems Engineering, the University of Sheffield, Mappin Street, Sheffield, S1 3JD, UK (corresponding author: Z.Q. Lang, phone: (+44) (0)114 222 5623; fax: (+44) (0)114 222 5683; e-mail: z.lang@ sheffield.ac.uk).

G. Park is with the School of Mechanical Engineering, Chonnam National University, Gwangiu, S. Korea, 500-757.

C. R. Farrar is with the Engineering Institute, Los Alamos National Laboratory, MST-001, Los Alamos, NM 87545, USA.

M. D. Todd and Z. Mao are with Department of Structural Engineering, University of California San Diego, La Jolla, CA 92093-0085, USA.

K. Worden is with Department of Mechanical Engineering, University of Sheffield, Mappin Street, Sheffield, S1 3JD, UK.

frequency ranges. They counted the occurrences of maximum transmissibility difference at different frequencies and considered the result as a damage indicator. But if the location of operational forces changes, the transmissibility between responses at two fixed points will also change making such techniques become invalid. Devriendt [20] found that the transmissibility around the natural resonance frequencies changed slightly when the location of operational forces changed. So he considered the occurrence times of maximum transmissibility around the resonance frequencies as a damage location indicator and demonstrated its effectiveness by simulation and experimental studies.

Because the transmissibility is basically a linear system concept, all the techniques above assume the systems behave linearly. However, in MDOF structural systems, certain types of damage often manifest themselves as the introduction of non-linearity into an otherwise linear system. Examples include post-buckled structures (Duffing non-linearity), rattling joints (impacting system with discontinuities), or breathing cracks (bilinear stiffness) etc. and such damage has been referred to as damage with nonlinear features [21]. Han [22, 23] studied rub-impact faults in rotor systems, and found that super-harmonic components appeared in both dual-disk rotor systems and dual-rotor systems when rub-impact damage occurred; and the more serious the damage is, the more abundant harmonic components can be observed. Reference [24] also indicated that high order harmonic components appear when a bolt on a pedestal became loose and such harmonic responses change with the variation of looseness clearance. Furthermore, if a cracked object is excited by a harmonic loading, the existence of super-harmonic components can be discovered [25-29]. In addition, cracks in a beam [10] can introduce nonlinear stiffness [30] and may therefore induce nonlinear behaviours into the whole system [26, 29, 31].

In order to extend the transmissibility based damage detection and location approaches to MDOF structural systems which can behave nonlinearly due to the occurrences of damage with nonlinear features, several methods have recently been developed [5, 25, 26, 32-37]. These methods are based on the concept of Non-linear Output Frequency Response Functions (NOFRFs) [38, 39] and use the system response signals to deterministic inputs including sinusoids to detect and locate such damage in the systems. Moreover, Lang et al. [5] proposed the concept of transmissibility of the NOFRFs. The transmissibility of the NOFRFs systematically extends the transmissibility concept to the nonlinear case, and has been used to develop a technique that can detect and locate damage of linear and/or non-linear features in MDOF structural systems. The effectiveness of the technique has been verified by both numerical simulation studies and experimental tests [5]. However there are two limitations in these recently developed techniques. First, these methods all assume that when damage occurs in a MDOF system and makes the system behave nonlinearly, there is only one nonlinear component in the system. In addition, these methods either require specific testing on inspected structures or assume the loading on the structural systems is measurable.

The present study is concerned with the development of a new and more general transmissibility analysis method for the detection and location of damage via nonlinear features in MDOF structural systems. By evaluating and analyzing the transmissibility at super-harmonics - a concept that will be introduced in the paper for MDOF nonlinear structural systems, the method can deal with more than one nonlinearly damaged component in the system, does not need specific tests, and does not require that the loading on inspected structural systems is measurable. The objectives are to extend the basic principles of the NOFRF transmissibility based damage detection and location to more practical situations to enable these ideas to be literally applied in engineering practice.

The paper is organized as follows. Section II provides a brief introduction of the basic concepts of NOFRFs and the transmissibility of the NOFRFs for single-input multi-output (SIMO) nonlinear systems. In Section III, some important properties of the NOFRFs transmissibility for a class of MDOF nonlinear structural systems are described; the concept of the transmissibility at super-harmonics is introduced; and the relationship between the NOFRFs transmissibility and the transmissibility at super-harmonics is derived. Based on these results, a novel method is developed in Section IV for the detection and location of damage via nonlinear features in MDOF structural systems. In Sections V and VI, the effectiveness of the new method is verified by numerical simulation and experimental studies, respectively. Finally, the conclusions are presented in Section VII.

II. The NOFRFs and NOFRFs Transmissibility of Single-Input Multi-Output Nonlinear Systems

For the SIMO (Single-Input-Multiple-Output) nonlinear systems which are stable at zero equilibrium, the system outputs around the equilibrium can be represented by the Volterra series [40]

$$x_i(t) = \sum_{\bar{n}=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{(i,\bar{n})}(\tau_1, \dots, \tau_n) \prod_{i=1}^{\bar{n}} u(t-\tau_i)\, d\tau_i \quad i = 1,2,\dots,n \tag{1}$$

where $x_i(t)$ and $u(t)$ are the ith output and the input of the system, respectively; n is the number of the system output; N is the maximum order of the system nonlinearity; $h_{(i,\bar{n})}(\tau_1, \dots, \tau_n)$ is the $\bar{n}$th order Volterra kernel associated with the ith system output.

The output frequency responses of system (1) to a general input can be described by [41]

$$\begin{cases} X_i(j\omega) = \sum_{\bar{n}=1}^{N} X_{(i,\bar{n})}(j\omega) \quad for\ \forall\ \omega \\ X_{(i,\bar{n})}(j\omega) = \frac{1/\sqrt{\bar{n}}}{(2\pi)^{\bar{n}-1}} \int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} H_{(i,\bar{n})}(j\omega_1, \dots, j\omega_{\bar{n}}) \prod_{i=1}^{\bar{n}} U(j\omega_i)\, d\sigma_{\bar{n}\omega} \end{cases} \tag{2}$$

Here $X_i(j\omega)$ and $U(j\omega)$ are the spectra of the ith system output and the system input, respectively; $X_{(i,\bar{n})}(j\omega)$ denotes the $\bar{n}$th order frequency response of the system's ith output, and

$$H_{(i,\bar{n})}(j\omega_1, \dots, j\omega_{\bar{n}}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{(i,\bar{n})}(\tau_1, \dots, \tau_n) e^{-(\omega_1\tau_1+\cdots+\omega_{\bar{n}}\tau_{\bar{n}})}\, d\tau_1 \dots d\tau_{\bar{n}} \tag{3}$$

is known as the $\bar{n}$th order Generalized Frequency Response Function(GFRF) associated with the ith system output, which is the extension of the frequency response functions of a SIMO linear system to the $\bar{n}$th order nonlinear case. In (2),

$$\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} H_{(i,\bar{n})}(j\omega_1,\ldots,j\omega_{\bar{n}}) \prod_{i=1}^{\bar{n}} U(j\omega_i)\, d\sigma_{\bar{n}\omega}$$

represents the integration of $H_{(i,\bar{n})}(j\omega_1,\ldots,j\omega_{\bar{n}}) \prod_{i=1}^{\bar{n}} U(j\omega_i)$ over the $\bar{n}$-dimensional hyper-plane $\omega_1+\cdots+\omega_{\bar{n}}=\omega$.

The concept of NOFRFs was firstly proposed by Lang and Billings [38]. For the SIMO nonlinear system (1), the NOFRFs are defined as

$$G_{(i,\bar{n})}(j\omega) = \frac{\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} H_{(i,\bar{n})}(j\omega_1,\ldots,j\omega_{\bar{n}}) \prod_{i=1}^{\bar{n}} U(j\omega_i) d\sigma_{\bar{n}\omega}}{\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} \prod_{i=1}^{\bar{n}} U(j\omega_i) d\sigma_{\bar{n}\omega}}, \bar{n} = ,1,\ldots,N, i=1,\ldots,n \quad (4)$$

under the condition that

$$\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} \prod_{i=1}^{\bar{n}} U(j\omega_i)\, d\sigma_{\bar{n}\omega} \neq 0 \quad (5)$$

From (2) and (4), it can be shown that the output frequency response of SIMO nonlinear systems can be represented using the NOFRFs as

$$\begin{cases} X_i(j\omega) = \sum_{\bar{n}=1}^{N} X_{(i,\bar{n})}(j\omega) \quad i=1,\ldots,n \\ X_{(i,\bar{n})}(j\omega) = G_{(i,\bar{n})}(j\omega) U_{\bar{n}}(j\omega) \end{cases} \quad (6)$$

where

$$U_{\bar{n}}(j\omega) = \frac{1/\sqrt{\bar{n}}}{(2\pi)^{\bar{n}-1}} \int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} \prod_{i=1}^{\bar{n}} U(j\omega_i)\, d\sigma_{\bar{n}\omega} \quad (7)$$

which is the Fourier Transform of $u^{\bar{n}}(t)$.

The concept of the transmissibility of the NOFRFs between the ith and kth outputs of system (1) was introduced in [5] as

$$T_{i,k}^{NL}(j\omega) = \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} \quad (8)$$

where $i,k \in \{1,\ldots,n\}$. It can be observed that that when $N=1$, the transmissibility of the NOFRFs as defined in (8) reduces to the traditional concept of transmissibility for linear systems. In addition, as the NOFRFs are independent of the change of the system input amplitude [33, 37], the NOFRF transmissibility also does not change with the system input amplitude. This property is the same as the input amplitude independent property with the traditional transmissibility concept.

III. The Transmissibility of MDOF Nonlinear Structural Systems at super-harmonics

A. MDOF nonlinear structural systems

Consider the typical MDOF system shown in Fig. 1 where the motion of all masses is one-dimensional and the input force u is applied on the Lth mass.

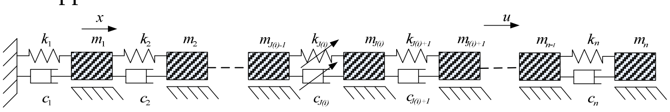


Fig. 1. The MDOF nonlinear structural system considered in the present study

If all the springs and dampers in the system in Fig.1 are linear, then the system is a linear MDOF system with governing equation

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F}(t) \quad (9)$$

where $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{K}, \boldsymbol{F}$ and $\boldsymbol{x}$ are the system mass matrix, damping matrix, stiffness matrix, force vector and displacement vector, respectively.

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & m_n \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} c_1+c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2+c_3 & -c_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -c_{n-1} & c_{n-1}+c_n & -c_n \\ 0 & 0 & 0 & -c_n & c_n \end{bmatrix},$$

$$\boldsymbol{K} = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -k_{n-1} & k_{n-1}+k_n & -k_n \\ 0 & 0 & 0 & -k_n & k_n \end{bmatrix},$$

$$\boldsymbol{F}(t) = [\overbrace{0 \ \ldots \ 0}^{L-1} \quad u(t) \quad \overbrace{0 \ \ldots \ 0}^{n-L}]^T, \boldsymbol{x} = [x_1,\ldots,x_n]^T.$$

When there are $\bar{J}(\bar{J} \geq 1)$ nonlinear springs/dampers in the system which are located between the $J(i)-1$ and $J(i)$th masses, namely $J(i)$th springs/dampers, $(i=1,2,\ldots,\bar{J})$, and the first spring and damper, which are connected to the fixed ground, are not nonlinear, that is, $J(1)>1$, the restoring forces of these nonlinear springs/dampers are the nonlinear functions of the deformation/the deformation derivative. Assume these nonlinear functions can be approximated by a polynomial of the form,

$$\begin{cases} f_s(i) = \sum_{\bar{n}=2}^{\bar{N}} r_{(J(i),\bar{n})} \left(x_{J(i)}(t) - x_{J(i)-1}(t)\right)^{\bar{n}} \\ f_d(i) = \sum_{\bar{n}=2}^{\bar{N}} w_{(J(i),\bar{n})} \left(\dot{x}_{J(i)}(t) - \dot{x}_{J(i)-1}(t)\right)^{\bar{n}} \end{cases} i=1,\ldots,\bar{J} \quad (10)$$

where $f_s(i)$ and $f_d(i)$ are extra nonlinear terms produced by nonlinear components; $r_{(J(i),\bar{n})}$ and $w_{(J(i),\bar{n})}$ are coefficients of the polynomial, and denote

$$nf(i) = [\overbrace{0 \ \ldots \ 0}^{J(i)-2} \quad -(f_s(i)+f_d(i)) \quad f_s(i)+f_d(i) \quad \overbrace{0 \ \ldots \ 0}^{n-J(i)}]^T \quad (11)$$

and

$$\boldsymbol{NF}(t) = \sum_{i=1}^{\bar{J}} nf(i) \quad (12)$$

Then, the motion of the MDOF system in Fig.1 can be described by

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F}(t) + \boldsymbol{NF}(t) \quad (13)$$

Equation (13) represents a class of SIMO nonlinear systems with the input and outputs being $u(t)$ and $x_i(t), i=1,\ldots,2$, respectively.

The basic issues to be addressed in the present study are to detect whether there exist nonlinear components in system (13) and then, if this is the case, to determine the location of these nonlinear components only from the system output responses measured on the masses. Because, in many practical situations, the existence of nonlinear components indicates the existence of structural damage with nonlinear features such as breathing crack, pedestal looseness and rub-impact etc, the detection and location of nonlinear components in system (13) is equivalent to detecting and locating a wide class of damage in the system and therefore has significant implications in engineering practices.

B. The properties of the NOFRF transmissibility

According to [37], it is known that if the outputs of system (13) can also be represented by the Volterra series (1), the transmissibility of the NOFRFs of the system as defined in (8) has the following properties.

(i) When $\bar{J} > 1$, that is, there are more than one nonlinear components in the system,

$$\begin{cases} T_{i,k}^{NL}(j\omega) = \frac{G_{(i,\bar{n})}(j\omega)}{G_{(k,\bar{n})}(j\omega)} = \bar{Q}_{i,k}(j\omega), \bar{n} \in \{2, \dots N-1\}; \\ \quad if\ 1 \le i < k \le J(1)-1\ or\ J(\bar{J}) \le i < k \le n \\ T_{i,k}^{NL}(j\omega) \ne \frac{G_{(i,\bar{n})}(j\omega)}{G_{(k,\bar{n})}(j\omega)}, \bar{n} \in \{2, \dots N-1\}. \quad \text{Otherwise} \end{cases} \tag{14}$$

where $\bar{Q}_{i,k}(j\omega)$ is only dependent on the M, C, and K, that is, the linear characteristic parameters of system (13).

(ii) When $\bar{J} = 1$, that is, there is only one nonlinear component in system (13),

$$T_{i,k}^{NL}(j\omega) = \frac{G_{(i,\bar{n})}(j\omega)}{G_{(k,\bar{n})}(j\omega)} = \bar{\bar{Q}}_{i,k}(j\omega), \bar{n} \in \{2, \dots N-1\}; if\ i,k \in \{1, \dots n\} \text{ and } i < k \tag{15}$$

In addition, if $L \ge J(1)$,

$$\begin{cases} T_{i,k}^{L}(j\omega) = \frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} = Q_{i,k}(j\omega) = \frac{G_{(i,\bar{n})}(j\omega)}{G_{(k,\bar{n})}(j\omega)} = T_{i,k}^{NL}(j\omega), \bar{n} \in \{2, \dots N-1\}; \\ \quad if\ 1 \le i < k \le J(1)-1\ or\ L \le i < k \le n \\ T_{i,k}^{L}(j\omega) = \frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} = Q_{i,k}(j\omega) \ne \frac{G_{(i,\bar{n})}(j\omega)}{G_{(k,\bar{n})}(j\omega)} = T_{i,k}^{NL}(j\omega), \bar{n} \in \{2, \dots N-1\}; \\ \quad \text{otherwise} \end{cases} \tag{16}$$

and if $L < J(1)$,

$$\begin{cases} T_{i,k}^{L}(j\omega) = \frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} = Q_{i,k}(j\omega) = \frac{G_{(i,\bar{n})}(j\omega)}{G_{(k,\bar{n})}(j\omega)} = T_{i,k}^{NL}(j\omega), \bar{n} \in \{2, \dots N-1\}; \\ \quad if\ 1 \le i < k \le L\ or\ J(1) \le i < k \le n \\ T_{i,k}^{L}(j\omega) = \frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} = Q_{i,k}(j\omega) \ne \frac{G_{(i,\bar{n})}(j\omega)}{G_{(k,\bar{n})}(j\omega)} = T_{i,k}^{NL}(j\omega), \bar{n} \in \{2, \dots N-1\}; \\ \quad \text{otherwise} \end{cases} \tag{17}$$

In equations (15) - (17), $T_{i,k}^{L}(j\omega)$ is the traditional transmissibility; $Q_{i,k}(j\omega)$ and $\bar{\bar{Q}}_{i,k}(j\omega)$ are also only dependent on M, C, and K.

In [5], Property (ii) of the NOFRF transmissibility has been exploited to develop a method for detection and location of damage in MDOF system (13) when there is only one nonlinear component in the system, and the system input $u(t)$ is measurable so can be used for damage detection and location purpose. The present study is motivated by the need to extend the results achieved in [5] to more practical situations including the case where there are more than one nonlinear components in the system; Property (i) of the NOFRF transmissibility is an important basis for addressing these more complicated problems.

C. Transmissibility at super-harmonics

In order to more clearly explain the basic ideas, the present study assumes that the input force $u(t)$ to system (13) is a sinusoidal and the location L where the input is applied is known a priori. Under this assumption, the system output frequency response can be represented using the system NOFRF as described in Proposition 1 below.

Proposition 1

Under the condition that the outputs of system (13) can be represented by the Volterra series (1) and the input to the system is a harmonic input

$$u(t) = Acos(\omega_F t + \beta) \tag{18}$$

the range of the system output frequencies are $\Omega = \{0, \pm 1\omega_F, \pm 2\omega_F, \dots, \pm N\omega_F\}$ and the system output responses at the harmonic frequencies can be determined by

$$\begin{cases} X_i(j\omega_F) = G_{(i,1)}(j\omega_F)U_1(j\omega_F) + G_{(i,3)}(j\omega_F)U_3(j\omega_F) + \\ \quad \dots + G_{(i,N)}(j\omega_F)U_N(j\omega_F) \\ X_i(j2\omega_F) = G_{(i,2)}(j2\omega_F)U_2(j2\omega_F) + G_{(i,4)}(j2\omega_F)U_4(2j\omega_F) + \\ \quad \dots + G_{(i,N-1)}(j2\omega_F)U_{N-1}(j2\omega_F) \\ \quad \dots\ \dots\ \dots \\ X_i(jN\omega_F) = G_{(i,N)}(jN\omega_F)U_N(jN\omega_F) \end{cases} \tag{19}$$

for $i = 1, \dots, n$ when N is odd or

$$\begin{cases} X_i(j\omega_F) = G_{(i,1)}(j\omega_F)U_1(j\omega_F) + G_{(i,3)}(j\omega_F)U_3(j\omega_F) + \\ \quad \dots + G_{(i,N-1)}(j\omega_F)U_{N-1}(j\omega_F) \\ X_i(j2\omega_F) = G_{(i,2)}(j2\omega_F)U_2(j2\omega_F) + G_{(i,4)}(j2\omega_F)U_4(2j\omega_F) + \\ \quad \dots + G_{(i,N)}(j2\omega_F)U_N(j2\omega_F) \\ \quad \dots\ \dots\ \dots \\ X_i(jN\omega_F) = G_{(i,N)}(jN\omega_F)U_N(jN\omega_F) \end{cases} \tag{20}$$

when N is even.

Proof: See Appendix A1.

From Proposition 1, it is known that, in the cases considered in the present study, the output frequency responses of system (13) include not only the component at the base frequency ω_F but also the components at $2\omega_F$, $3\omega_F$,… etc super-harmonics. The transmissibility at super-harmonics is defined as the ratio of the super-harmonic responses on two consecutive masses, that is

$$ST^{i,i+1}(jk\omega_F) = \frac{X_i(jk\omega_F)}{X_{i+1}(jk\omega_F)} \quad k = 2, \dots, N\ and\ i = 1, \dots, n-1 \tag{21}$$

From the properties of the NOFRF transmissibility described in B above and Proposition 1, the relationship between the transmissibility at super-harmonics as defined in (21) and the NOFRFs transmissibility can be derived. The result is summarized in Proposition 2 as follows.

Proposition 2

Under the same condition of Proposition 1,

(i) When there are more than one nonlinear components in system (13), that is $\bar{J} > 1$, if two consecutive masses of the system are all on the left or right side of the nonlinear components, namely, $1 \le i \le J(1)-2$ or $J(\bar{J}) \le i \le n-1$, then

$$ST^{i,i+1}(jk\omega_F) = T_{i,i+1}^{NL}(jk\omega_F) = \bar{Q}_{i,i+1}(jk\omega_F), k = 2, \dots, N \tag{22}$$

If at least one mass is within the range of nonlinear components, namely, $J(1)-1 \le i \le J(\bar{J})-1$, then

$$ST^{i,i+1}(jk\omega_F) \ne T_{i,i+1}^{NL}(jk\omega_F) \quad k = 2, \dots, N \tag{23}$$

(ii) When there is only one nonlinear component in the system, that is $\bar{J} = 1$, then

$$ST^{i,i+1}(jk\omega_F) = T_{i,i+1}^{NL}(jk\omega_F) = \bar{\bar{Q}}_{i,i+1}(jk\omega_F), i = 1, \dots, n-1, k = 2, \dots, N \tag{24}$$

and

$$\begin{cases} ST^{i,i+1}(j\omega_F) = T_{i,i+1}^{L}(j\omega_F) = Q_{i,i+1}(j\omega_F), \\ \quad \text{if } 1 \le i \le J(1)-2\ or\ L \le i < n-1 \text{ when } L \ge J(1); \\ \quad or\ if\ 1 \le i \le L-1\ or\ J(1) \le i < n-1 \text{ when } L < J(1) \\ ST^{i,i+1}(j\omega_F) \ne T_{i,i+1}^{L}(j\omega_F) = Q_{i,i+1}(j\omega_F) \\ \quad \text{otherwise} \end{cases} \tag{25}$$

(iii) Results (i) and (ii) above hold for $k = 2,4 \dots, N$ if k and N are all even; for $k = 3,5, \dots, N$, if k and N are all odd; for $k = 2,4, \dots, N-1$ if k is even but N is odd; and for $k = 3,5, \dots, N-1$ if k is odd but N is even.

Proof: See Appendix A2.

Result (i) of Proposition 2 indicates that if there are more than one nonlinear components in the system and the two consecutive masses involved in the transmissibility evaluation are located both on the same side of the nonlinear components, then the transmissibility at super-harmonics only depend on the system linear characteristic parameters and is, therefore, independent from the system input. Otherwise, that is, when the two masses involved in the transmissibility evaluation are located inside the area of system nonlinear components, the transmissibility at super-harmonics may be dependent on the system input. On the other hand, Result (ii) of Proposition 2 indicates that if there is only one nonlinear component in the system, the transmissibility at super-harmonics is completely dependent on the system linear characteristic parameters and independent from the system input. In addition, if there is only one nonlinear component in the system, and the two consecutive masses involved in the transmissibility evaluation are not located between this nonlinear component and the mass where an input excitation is applied, the transmissibility at driving frequency also only depends on the system linear characteristic parameters and is independent from the system input. These significant properties of the transmissibility at either super-harmonics or driving frequency with system (13) will be exploited to propose a more general approach for the detection and location of damage with nonlinear features for MDOF structural system (13) in the next section.

IV. Detection and Location of Damage via Nonlinear Features Using a New Transmissibility Analysis Method

A. Basic ideas

According to [23, 24, 26], the damage with nonlinear features in MDOF systems can make the whole system behave nonlinearly and, particularly, produce super-harmonics. So the higher order harmonics can be used to determine whether there exists such damage in the system.

When damage with nonlinear features has been detected in system (13), the results of Proposition 2 can be used to find out whether there is only one or more than one damage with nonlinear features in the system and the locations of the damage. This is based on the following observations.

First, equation (24) in (ii) , Proposition 2 indicates that when there is only one nonlinear component in MDOF system (13), the transmissibility at super-harmonics depends only on the system parameters M, C, K and does not change with the system input. This is a very distinctive feature and can be used, if damage with nonlinear features has been detected in system (13), to determine whether there is only one nonlinear component in the system or not.

Secondly, if there is only one nonlinear component in the system, equation (25) in (ii), Proposition 2 indicates that whether the transmissibility at base frequency ω_F varies with a change in the system input depends on the location of the two masses involved in the transmissibility evaluation. This property can be exploited to find the location of the only nonlinear component in the system.

Finally, if there are more than one nonlinear components in system (13), equations (22) and (23) in (i), Proposition 2, indicate that whether the transmissibility at super-harmonics varies with a change in the system input depends on the location of the two masses involved in the transmissibility evaluation. This can be used to find the locations of nonlinear components in the system.

B. The method

From the super-harmonic analysis based damage detection idea, and the above observations from Proposition 2, a new transmissibility analysis method for the detection and location of damage with nonlinear features in system (13) can be proposed under the following two assumptions.

a) The system output responses to two different sinusoidal inputs

$$u_1(t) = A_1 sin(w_F t + \beta_1) \text{ and } u_2(t) = A_2 sin(w_F t + \beta_2) \quad (26)$$

can be obtained so that two sets of transmissibility analysis results

$$\begin{cases} ST1^{i,i+1}(jk\omega_F) = \frac{X_i^1(jk\omega_F)}{X_{i+1}^1(jk\omega_F)} = \frac{X_i(jk\omega_F)}{X_{i+1}(jk\omega_F)}\Big|_{u(t)=u_1(t)=A_1\sin(\omega_F t+\beta_1)} \\ ST2^{i,i+1}(jk\omega_F) = \frac{X_i^2(jk\omega_F)}{X_{i+1}^2(jk\omega_F)} = \frac{X_i(jk\omega_F)}{X_{i+1}(jk\omega_F)}\Big|_{u(t)=u_2(t)=A_2 sin(\omega_F t+\beta_2)} \end{cases} \quad (27)$$

and their differences

$$S\delta^{i,i+1}(jk\omega_F) = \left|ST1^{i,i+1}(jk\omega_F) - ST2^{i,i+1}(jk\omega_F)\right| \quad (28)$$

can be determined. Here, $i = 1, \ldots, n-1$; $k = 1,2, \ldots N$. In (27), $X_i^1(jk\omega_F)$ and $X_i^2(jk\omega_F)$ are the spectra of the kth harmonic responses of the system to input $u_1(t)$ and $u_2(t)$, respectively, and $A_1 \neq A_2$.

b) The location where the input force $u(t)$ is applied to the system, that is, mass number L is known a priori.

The detailed procedures of the new method can be described as follows.

(1). Evaluate the spectra of the output responses of system (13) to inputs $u_1(t)$ and $u_2(t)$, respectively, and determine the amplitudes of these spectra at all the harmonics, that is, $X_i^1(jk\omega_F)$ and $X_i^2(jk\omega_F)$, for $i = 1, \ldots, n$ and $k = 2, \ldots, N$. Here, N can be determined as the highest order at which the harmonics are observed in the system outputs. Determine the value of index IND_1 as defined below to represent the strength of higher order harmonics in the system output responses

$$IND_1 = \max\left\{\left|\frac{X_i^1(jk\omega_F)}{X_i^1(j\omega_F)}\right|, \left|\frac{X_i^2(jk\omega_F)}{X_i^2(j\omega_F)}\right|, i = 1, \ldots, n, \text{and } k = 2, \ldots, N\right\} \quad (29)$$

If

$$IND_1 \geq \varepsilon_1 \quad (30)$$

then it can be concluded that there exists damage with nonlinear feature in the system. Otherwise, there is no such damage in the system. In (30), ε_1 *is* a threshold to be determined a priori.

(2). If Step (1) indicates there is damage with nonlinear features in the system, select a $\bar{k} \in \{2, \ldots, N\}$ such that both $X_i^1(j\bar{k}\omega_F)$ $i = 1, \ldots, n$ and $X_i^2(j\bar{k}\omega_F)$ $i = 1, \ldots, n$ have significant amplitudes. Calculate $ST1^{i,i+1}(j\bar{k}\omega_F)$, $ST2^{i,i+1}(j\bar{k}\omega_F)$, and $S\delta^{i,i+1}(j\bar{k}\omega_F)$ for $i = 1, \ldots, n-1$ using (27) and (28). Then, evaluate

$$S\delta_{\max}(\bar{k}) = \max\{S\delta^{i,i+1}(j\bar{k}\omega_F), i \in \{1,2, \ldots, n-1\}\} \quad (31)$$

to see whether

$$S\delta_{\max}(\bar{k}) \leq \varepsilon_2 \quad (32)$$

where ε_2 *is* another a priori determined threshold. If (32) holds, it can be concluded that there exist only one damaged component with nonlinear features in the system. Otherwise, there are more than one damaged components with nonlinear features.

(3). If Step (2) indicates there exist only one damaged component with nonlinear features, calculate $ST1^{i,i+1}(j\omega_F)$, $ST2^{i,i+1}(j\omega_F)$, and $S\delta^{i,i+1}(j\omega_F)$ for $i = 1, \dots, n-1$ using (27) and (28). Then evaluate

$$S\delta_{\max}(1) = \max\left\{S\delta^{i,i+1}(j\omega_F), i \in \{1,2,\dots,n-1\}\right\} \quad (33)$$

and

$$\overline{S\delta}^{i,i+1}(j\omega_F) = \frac{S\delta^{i,i+1}(j\omega_F)}{S\delta_{\max}(1)} \ \ for\ i = 1, \dots, n-1 \quad (34)$$

to find those i's such that

$$\overline{S\delta}^{i,i+1}(j\omega_F) \geq \varepsilon_3 \quad (35)$$

where ε_3 is again a priori determined threshold. Denote those i's such that (35) holds are

$$i', i'+1, \dots, i'+m'-1$$

where $m' \geq 1$.

Then, there are only two possibilities which are $L = i'$ or $L = i' + m'$. If $L = i'$, it can be concluded that the only nonlinear component is located between mass $(i' + m' - 1)$ and mass $(i' + m')$. Otherwise, $L = i' + m'$, and it can be concluded that the only nonlinear component is located between mass i' and mass $(i' + 1)$.

(4). If Step (2) indicates there exist more than one damaged components with nonlinear features in the system, evaluate

$$\overline{S\delta}^{i,i+1}(j\bar{k}\omega_F) = \frac{S\delta^{i,i+1}(j\bar{k}\omega_F)}{S\delta_{\max}(\bar{k})} \text{ for } i = 1, \dots, n-1 \quad (36)$$

to find those i's such that

$$\overline{S\delta}^{i,i+1}(j\bar{k}\omega_F) \geq \varepsilon_4 \quad (37)$$

where ε_4 is also a priori determined threshold. Denote those i's such that (37) hold are

$$i'', i''+1, \dots, i''+m''-1$$

where $m'' > 1$.

Then, it can be concluded that these nonlinear components are located between mass i'' and mass $i'' + m''$.

C. Remarks

For the new method described above, following remarks can be made regarding the theoretical basis of relevant steps and the choice of the parameters that are required to be determined a priori.

a) Step 1) of the method is based on the well-known fact that nonlinearity will generate harmonics in the system output response. Step 2) exploits the property of system (13) described in the first point of Result (ii), Proposition 2, which indicates if there is only one nonlinear component in the system, the transmissibility at super-harmonics is completely determined by the system linear characteristic parameters and, therefore , independent of the system input. The theoretical basis of Step 3) is the second point of Result (ii), Proposition 2, which reveals an important relationship between the transmissibility at base frequency and the location of the only nonlinear component in the system. Step 4) makes use of the property of the transmissibility at super-harmonics of system (13) described by Result (i), Proposition 2, which shows where the transmissibility at super-harmonics is only dependent on the system linear characteristic parameters and, therefore, independent of the system input and where this is not the case.

b) $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are four parameters in the method. Theoretically, these parameters are zeros. But, in practice they are thresholds that should be determined a priori from experimental data using statistical analyses. This allows the effects of noise, un-modeled dynamics, and inherent but less significant system non-linearity to be omitted when the method is used in practice. For example, ε_1 should be a small number associated with a noise threshold in a case where the system basically behaves linearly. So ε_1 can be determined from the statistics (such as mean and standard deviation etc) of the values of IND_1 in the situations when there is no damage with nonlinear features in the system.

c) The determination of N and $\bar{k}$ can be achieved by observing the spectra of the system outputs. The details will be demonstrated in Section VI.

In the next sections, simulation and experimental studies will be conducted to demonstrate the performance of the proposed method and its potential in practical applications.

V. Simulation studies

In order to verify the effectiveness of the proposed method, simulation studies are conducted in this section. For this purpose, a 10 DOF system as described by (13) is considered where

$$m_1 = m_2 = \dots = m_{10} = 1,$$
$$k_1 = k_2 = \dots = k_5 = k_{10} = 3.6 \times 10^4, k_6 = k_7 = k_8 = 0.8k_1, k_9 = 0.9k_1,$$
$$\mu = 0.01, C = \mu K,$$

and the parameters of the nonlinear springs and dampers are

$$\bar{N} = 3, r_{(J(i),2)} = 0.8k_1^2, r_{(J(i),3)} = 0.4k_1^3, w_{(J(i),2)} = w_{(J(i),3)} = 0, i = 1, \dots, \bar{J}$$
.

where $\bar{J}$, the number of nonlinear components in the system, are taken as $\bar{J} = 3$ and $\bar{J} = 1$, respectively, in the two cases of simulation studies considered below.

A. Simulation Study Case 1

In this case, there are three ($\bar{J} = 3$) nonlinear components in the system, which are the 3rd, 5th and 6th springs. Two loading conditions are considered where the input forces are

$$u_1(t) = 10sin(40\pi t) \text{ and } u_2(t) = 20sin(40\pi t)$$

respectively, and are applied on the 7th mass, that is, $L = 7$. The new method was applied to the spectra of the output responses of the system under the two loading conditions, that is,

$$X_i^1(jk\omega_F) \text{ and } X_i^2(jk\omega_F),\ i = 1, \dots, 10,\ k = 1, \dots N.$$

where N was determined as 4 and the four threshold parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ in the method all taken as 2%=0.02. The results obtained in each step are given as follows.

Step (1)

In this case, the index IND_1 was evaluated using (29) as

$$IND_1 = \max\left\{\left|\frac{X_i^1(jk\omega_F)}{X_i^1(j\omega_F)}\right|, \left|\frac{X_i^2(jk\omega_F)}{X_i^2(j\omega_F)}\right|, i = 1, \dots, 10, \text{and } k = 2, \dots, 4\right\}$$

$=0.0287 \geq \varepsilon_1 = 0.02$

Therefore, it is concluded that damage with nonlinear feature exists in the system.

Step (2)

At this step, $\bar{k}$ was determined as $\bar{k} = 2$. So $ST1^{i,i+1}(j2\omega_F)$, $ST2^{i,i+1}(j2\omega_F)$, and $S\delta^{i,i+1}(j2\omega_F)$ $i = 1,\ldots,9$ were evaluated using (27) and (28). Then, $S\delta_{\max}(2)$ was determined using (31); the result is

$$S\delta_{\max}(2) = 1.5349 > \varepsilon_2 = 0.02$$

So it is known that there are more than one nonlinear components in the system.

Step (4)

As Step (2) has shown that there are more than one nonlinear components in the system, Step (4) rather than Step (3) of the proposed method are needed in this case. At this step, $\overline{S\delta}^{i,i+1}(j\bar{k}\omega_F) = \overline{S\delta}^{i,i+1}(j2\omega_F)$, $i = 1,\ldots,9$ were evaluated using (36). The results are shown in Table I, in which it can be observed that

$$\overline{S\delta}^{i,i+1}(j2\omega_F) \geq 0.02 = \varepsilon_4,\ i = 2,3,4,5$$

Therefore $i'' = 2$ and $m'' = 4$, and it can be concluded that nonlinear components are located between mass $i'' = 2$ and mass $i'' + m'' = 6$ in the system.

TABLE I

THE VALUE OF $\overline{S\delta}^{i,i+1}(j2\omega_F)$ WHEN THE 3RD, 5TH AND 6TH SPRINGS ARE NONLINEAR

i	$\overline{S\delta}^{i,i+1}(j2\omega_F)$	i	$\overline{S\delta}^{i,i+1}(j2\omega_F)$	i	$\overline{S\delta}^{i,i+1}(j2\omega_F)$
1	5.16×10^{-5}	4	1	7	0.000133
2	0.109899	5	0.340627	8	8.63×10^{-5}
3	0.071504	6	9.63×10^{-5}	9	9.59×10^{-6}

Obviously, the conclusions reached at each step are all consistent with the real situation of the simulated system. So the effectiveness of the proposed method is verified by this simulation study.

B. Simulation Study Case 2

In this case, there is only one ($\bar{J} = 1$) nonlinear component in the system, which is the 8th spring. The same two loading conditions as in Simulation Study Case 1 were considered and the input force was applied on the 3rd mass, that is, $L = 3$. The new method was again applied to the spectra of the output responses of the system under the two loading conditions. Again, N was determined as 4 and the four threshold parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ in the method were taken as 2%=0.02. The results obtained in each step are given as follows.

Step (1)

In this case, the index IND_1 was evaluated by (29) as

$$IND_1 = \max\left\{\left|\frac{X_i^1(jk\omega_F)}{X_i^1(j\omega_F)}\right|, \left|\frac{X_i^2(jk\omega_F)}{X_i^2(j\omega_F)}\right|, i = 1,\ldots,10, \text{and } k = 2,\ldots,4\right\}$$

$=0.0387 \geq \varepsilon_1 = 0.02$

So, damage with nonlinear feature exists in the system.

Step (2)

At this step, $\bar{k}$ is again determined as $\bar{k} = 2$. Therefore, in the same way as in Step (2), Simulation Case Study 1, $S\delta_{\max}(2)$ was determined; the result is

$$S\delta_{\max}(2) = 6.7163 \times 10^{-4} < \varepsilon_2 = 0.02$$

So it is known that there is only one nonlinear component in the system.

Step (3)

Because Step (2) indicates there is only one nonlinear component in the system, Step (3) of the proposed method was followed to evaluate $ST1^{i,i+1}(j\omega_F)$, $ST2^{i,i+1}(j\omega_F)$, and $S\delta^{i,i+1}(j\omega_F)$ for $i = 1,\ldots,9$ using (27) and (28). Then, $\overline{S\delta}^{i,i+1}(j\omega_F)$ for $i = 1,\ldots,9$ were evaluated using (33) and (34). The results are shown in Table II indicating

$$\overline{S\delta}^{i,i+1}(j\omega_F) \geq \varepsilon_3 = 0.02,\ i = 3,\ldots,7$$

So $i' = 3$ and $m' = 5$. As $L = 3 = i'$, it is known that the only nonlinear component is located between mass $(i' + m' - 1) = 7$ and mass $(i' + m') = 8$

TABLE II

THE VALUE OF $\overline{S\delta}^{i,i+1}(j\omega_F)$ WHEN ONLY THE EIGHTH SPRING IS NONLINEAR

i	$\overline{S\delta}^{i,i+1}(j\omega_F)$	i	$\overline{S\delta}^{i,i+1}(j\omega_F)$	i	$\overline{S\delta}^{i,i+1}(j\omega_F)$
1	2.2×10^{-6}	4	0.112423	7	1
2	3.56×10^{-6}	5	0.305335	8	5.79×10^{-6}
3	0.076072	6	0.986996	9	3.44×10^{-6}

Again, the conclusions reached at each step above are all consistent with the real situation of the simulated system. So the effectiveness of the proposed method is further verified by the second simulation study.

VI. EXPERIMENTAL STUDIES

A. Experimental setup

In order to demonstrate the potential of the new transmissibility analysis based damage dectection and location method in practical applications, the method was applied to analyse the experimental data from testing a three-storey building structure shown in Fig.2. The structure consists of aluminum columns and plates, assembled using bolted joints with a rigid base. The structure slides on rails that allow movement in only one direction. At each floor, four columns are connected to the top and bottom aluminum plates, which form a four degree-of-freedom system. Additionally, a center column can be suspended from the top of each floor, which is used to induce nonlinear behaviors when the column contacts a bumper mounted on the next floor. The position of the bumper can be adjusted to vary the extent of the nonlinearity. This source of nonlinearity can, for example, simulate the fatigue cracks that subsequently open and close under operational, environmental, or loading conditions. An electromagnetic shaker provides the excitation to the ground floor of the structure. Four accelerometers are attached to each floor at the opposite side from the excitation source to measure the response from each floor. Fig. 4 shows the spring-damper model of the three-storey building structure which is clearly a specific case of the nDOF model in Fig.1.

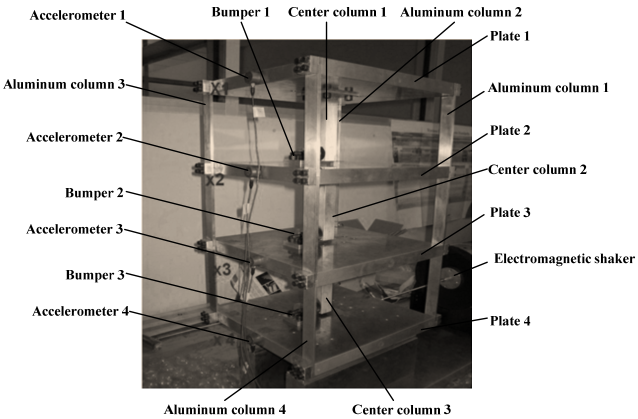


Fig. 2. Three-storey building structure used for the experimental studies

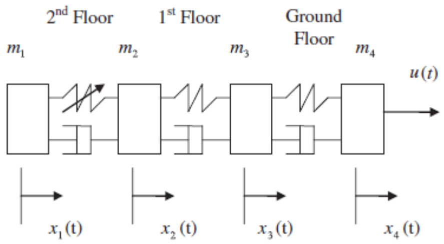


Fig. 3 4DOF system model of the three-storey building structure

B. Experiments and experimental data analyses

Data were collected from four different experiments on the three-story building structure. The details of the experiments are summarized in Table III. Two different state conditions of the structure were investigated. These are the structural state condition under Experiments #1 and #2, and the structural state condition under Experiments #3 and #4, respectively. So data collected from Experiments #1 and #2 were used to determine the situation of state condition 1, and data collected from Experiments #3 and #4 were used to determine the situation of state condition 2. The objectives of the experimental data analysis were to apply the method proposed in the present study for each state condition to detect whether there exists a nonlinear component in the experimental system and, if this is the case, determine the location of the nonlinear component in the system.

The results of the experimental data analyses are given in Table IV where $N = 3$ and $\bar{k} = 3$ were determined from observing the spectra of the system outputs in the four experiments shown in Figs 4-7. In these analyses, $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are all taken as 0.02.

Because IND_1=0.3453>$\varepsilon_1 = 0.02$ in state condition 1 and IND_1=0.1714>$\varepsilon_1 = 0.02$ in state condition 2, it was concluded that there exits nonlinear damage in the structural system in both state conditions.

Moreover, because

$$S\delta_{\max}(\bar{k}) = S\delta_{\max}(3) = 0.016 < \varepsilon_2 = 0.02$$

in state condition 1 and,

$$S\delta_{\max}(\bar{k}) = S\delta_{\max}(3) = 0.013 < \varepsilon_2 = 0.02$$

in state condition 2, it was concluded that there is only one nonlinear component in both state conditions. Therefore, Step 3 rather than Step 4 of the proposed method should be used to find the location of the nonlinear component.

The last row of Table IV shows the data analysis results for the two state conditions in Step 3. The analysis results for state condition 1 indicate

$$\overline{S\delta}^{i,i+1}(j\omega_F) \geq \varepsilon_3 = 0.02, \ i = 2,3$$

So $i' = 2$ and $i' + m' - 1 = 3 \rightarrow m' = 2$. Because $L = 4 = i' + m'$ in this case, it is known that the nonlinear component is located between mass $i' = 2$ and mass $i' + 1 = 3$ in state condition 1.

The analysis results for state condition 2 indicate

$$\overline{S\delta}^{i,i+1}(j\omega_F) \geq \varepsilon_3 = 0.02, \ i = 1,2,3$$

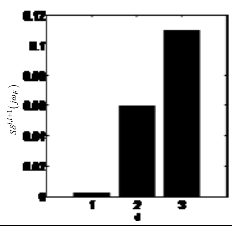
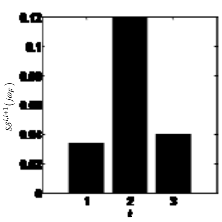
So $i' = 1$ and $i' + m' - 1 = 3 \rightarrow m' = 3$. Because again $L = 4 = i' + m'$ in this case, it is known that the nonlinear component is located between mass $i' = 1$ and mass $i' + 1 = 2$ in state condition 2.

Obviously, the conclusions reached by the analysis of the experimental data from the two state conditions of the experimental system using the proposed method are completely consistent with the real situations of the system. Therefore, the potential of the proposed method in engineering applications have been verified.

TABLE III
DETAILS OF THE EXPERIMENTS

Experiment	Input excitation applied by shaker control computer	Structure state condition under which experiment was conducted
Experiment #1	25 Hz sinusoidal with amplitude 2	**State Condition 1:** A 0.13mm gap was introduced between the column and bumper on the first floor to generate a nonlinear effect.
Experiment #2	25 Hz sinusoidal with amplitude 2.5	
Experiment #3	25 Hz sinusoidal with amplitude 2	**State Condition 2:** A 0.20mm gap was introduced between the column and bumper on the second (top) floor to generate a nonlinear effect.
Experiment #4	25 Hz sinusoidal with amplitude 2.5	

TABLE IV
DETAILS OF THE EXPERIMENTAL DATA ANALYSYS RESUTS

	The experimental data analysis results for the three-story building structure under state condition 1	The experimental data analysis results for the three-story building structure under state condition 2
N	3	3
IND_1	$0.3453 > \varepsilon_1 = 0.02$	$0.1747 > \varepsilon_1 = 0.02$
$\bar{k}$	3	3
$S\delta_{\max}(\bar{k})$	0.016< $\varepsilon_2 = 0.02$	0.013< $\varepsilon_2 = 0.02$
$S\delta^{i,i+1}(j\omega_F)$ for $i = 1,\ldots,3$		

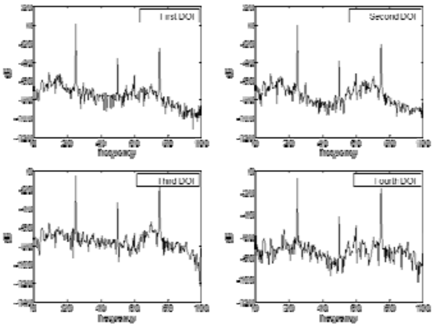


Fig. 4 Output spectra of the outputs of the 3 story building structure in Experiment 1

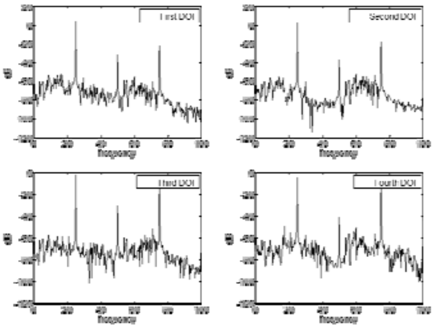


Fig. 5 Output spectra of the outputs of the 3 story building structure in Experiment 2

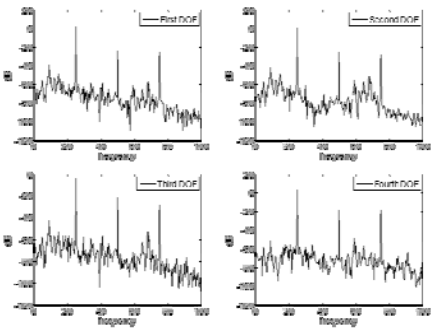


Fig. 6 Output spectra of the outputs of the 3 story building structure in Experiment 3

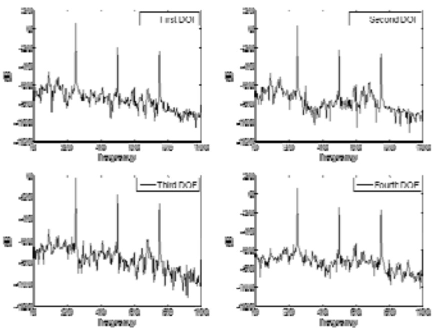


Fig. 7 Output spectra of the outputs of the 3 story building structure in Experiment 4

VII. CONCLUSIONS

Transmissibility analysis is a well-established method and has been widely applied in structural analysis including damage detection and fault diagnosis. But, traditional transmissibility is a linear system concept which cannot be directly applied to the analysis of nonlinear structural systems. Recently, the concept of transmissibility of the Nonlinear Output Frequency Response Functions (NOFRFs) has been introduced to extend the transmissibility concept to the nonlinear case, and the NOFRF transmissibility based/related techniques have been developed to detect and locate damage in MDOF structural systems. However, these techniques assume there is only one nonlinear component in a damaged system, and either require specific testing or assume that the loading on inspected structural systems is measurable. To address these issues so as to enable NOFRF transmissibility based damage detection and location to be applicable in engineering practice, a new transmissibility analysis method has been developed in the present study for the detection and location of damage via nonlinear features in MDOF structural systems. The new method is derived using the NOFRF transmissibility concept and can be implemented by evaluation and analysis of the transmissibility of the system responses at super-harmonics. Both numerical simulation studies and experimental data analysis have been conducted to verify the effectiveness and demonstrate the potential practical applications of the new technique. Although, for convenience of introducing main ideas, sinusoidal loadings are considered in the present study, the method can readily be extended to more general band limited loading cases and, therefore, has potential to be applied in practice to tackle nonlinear damage detection and location problems.

APPENDIX A1 PROOF OF PROPOSITION 1

According to [41], the $\bar{n}$th order frequency response of the system's ith output can be expressed as

$$X_{(i,\bar{n})}(j\omega) = \frac{1}{2^{\bar{n}}}\sum_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} A(j\omega_1)\ldots A(j\omega_{\bar{n}})H_{(i,\bar{n})}(j\omega_1,\ldots,j\omega_{\bar{n}}) \qquad \text{(A1.1)}$$

where

$$A(j\omega_k) = \begin{cases} Ae^{j\beta} & if\ \omega_k = \omega_F \\ Ae^{-j\beta} & if\ \omega_k = -\omega_F \\ 0 & otherwise \end{cases} \qquad \text{(A1.2)}$$

Obviously, if b ($b \in \{0,1,2,\ldots,\bar{n}\}$) $\omega_k's$ in $\omega_1,\ldots,\omega_{\bar{n}}$ take the value of ω_F, then the remaining $(\bar{n}-b)$ $\omega_k's$ in $\omega_1,\ldots,\omega_{\bar{n}}$ take the value of $-\omega_F$. Consequently, the possible frequency components in $X_{(i,\bar{n})}(j\omega)$ can be obtained as

$$\Omega_{\bar{n}} = \{(-\bar{n}+2b)\omega_F, b=0,1,\ldots,\bar{n}\} = \{-\bar{n}\omega_F, -(\bar{n}-2)\omega_F,\ldots,(\bar{n}-2)\omega_F,\bar{n}\omega_F\} \qquad \text{(A1.3)}$$

and the possible frequency components of system output are given by [41]

$$\Omega = \bigcup_{\bar{n}=N-1}^{N}\Omega_{\bar{n}} = \{-N\omega_F,-(N-2)\omega_F,\ldots,(N-2)\omega_F,N\omega_F\} \cup \{-(N-1)\omega_F,-(N-3)\omega_F,\ldots,(N-3)\omega_F,(N-1)\omega_F\} = \{0,\pm 1\omega_F,\pm 2\omega_F,\ldots,\pm N\omega_F\} \qquad \text{(A1.4)}$$

From Equation (A1.1), it is known that

$$X_{(i,\bar{n})}(j\omega) = G_{(i,\bar{n})}(j\omega)U_{\bar{n}}(j\omega) \qquad \text{(A1.5)}$$

where

$$G_{(i,\bar{n})}(j\omega) = \frac{\sum_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} A(j\omega_1)\ldots A(j\omega_{\bar{n}})H_{(i,\bar{n})}(j\omega_1,\ldots,j\omega_{\bar{n}})}{\sum_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} A(j\omega_1)\ldots A(j\omega_{\bar{n}})}$$

and

$$U_{\bar{n}}(j\omega) = \frac{1}{2^{\bar{n}}} \sum_{\omega_1 + \cdots + \omega_{\bar{n}} = \omega} A(j\omega_1) \dots A(j\omega_{\bar{n}})$$

in this case. As Equation (A1.3) indicates even order harmonics are produced by even order nonlinearity and odd order harmonics are produced by odd order nonlinearity, Equations (19) and (20) can be obtained from (A1.5) from the case of N is odd and from the case of N is even, respectively. Thus, the proof of Proposition1 is completed.

APPENDIX A2 PROOF OF PROPOSITION 2

Consider k and N are all even first. In this case, it is known from (20) that

$$X_i(jk\omega_F) = G_{(i,k)}(jk\omega_F)U_k(jk\omega_F) + G_{(i,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F) + \cdots + G_{(i,N)}(jk\omega_F)U_N(jk\omega_F) \quad \text{(A2.1)}$$

$$X_{i+1}(jk\omega_F) = G_{(i+1,k)}(jk\omega_F)U_k(jk\omega_F) + G_{(i+1,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F) + \cdots + G_{(i+1,N)}(jk\omega_F)U_N(jk\omega_F) \quad \text{(A2.2)}$$

where $k = 2,4, \dots N-2, N$. It is known from Property (i) of the NOFRF transmissibility given by (14) that if $\bar{J} > 1$, for $1 \le i \le J(1) - 2$ or $J(\bar{J}) \le i \le n-1$,

$$\frac{G_{(i,k)}(jk\omega_F)U_k(jk\omega_F)}{G_{(i+1,k)}(jk\omega_F)U_k(jk\omega_F)} = \frac{G_{(i,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F)}{G_{(i+1,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F)} = \cdots = \frac{G_{(i,N)}(jk\omega_F)U_N(jk\omega_F)}{G_{(i+1,N)}(jk\omega_F)U_N(jk\omega_F)} = T^{NL}_{i,i+1}(jk\omega_F) = \bar{Q}_{i,i+1}(jk\omega_F) \quad \text{(A2.3)}$$

Equations (A2-1)-(A2-3) imply that

$$ST^{i,i+1}(jk\omega_F) = \frac{X_i(jk\omega_F)}{X_{i+1}(jk\omega_F)} = \frac{G_{(i,k)}(jk\omega_F)U_k(jk\omega_F) + G_{(i,k+2)}(jk\omega_F)U_{k+2}(jk\omega_F) + \cdots + G_{(i,N)}(jk\omega_F)U_N(jk\omega_F)}{G_{(i+1,k)}(jk\omega_F)U_k(jk\omega_F) + G_{(i+1,k+2)}(jk\omega_F)U_{k+2}(jk\omega_F) + \cdots + G_{(i+1,N)}(jk\omega_F)U_N(jk\omega_F)} = T^{NL}_{i,i+1}(jk\omega_F) = \bar{Q}_{i,i+1}(jk\omega_F) \quad \text{(A2.4)}$$

Therefore (22) holds.

Also according to Property (i) of the NOFRF transmissibility, it is known that if $\bar{J} > 1$, for $J(1) - 1 \le i \le J(\bar{J}) - 1$, $\frac{G_{(i,k)}(jk\omega_F)U_k(jk\omega_F)}{G_{(i+1,k)}(jk\omega_F)U_k(jk\omega_F)} \ne T^{NL}_{i,i+1}(jk\omega_F)$, $\frac{G_{(i,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F)}{G_{(i+1,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F)} \ne T^{NL}_{i,i+1}(jk\omega_F), \dots,$ and $\frac{G_{(i,N-2)}(jk\omega_F)U_N(jk\omega_F)}{G_{(i+1,N-2)}(jk\omega_F)U_N(jk\omega_F)} \ne T^{NL}_{i,i+1}(jk\omega_F)$, so that

$$ST^{i,i+1}(jk\omega_F) = \frac{X_i(jk\omega_F)}{X_{i+1}(jk\omega_F)} = \frac{G_{(i,k)}(jk\omega_F)U_k(jk\omega_F) + G_{(i,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F) + \cdots + G_{(i,N)}(jk\omega_F)U_N(jk\omega_F)}{G_{(i+1,k)}(jk\omega_F)U_k(jk\omega_F) + G_{(i+1,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F) + \cdots + G_{(i+1,N)}(jk\omega_F)U_N(jk\omega_F)} \ne T^{NL}_{i,i+1}(jk\omega_F) \quad \text{(A2.5)}$$

Therefore (23) holds.

According to Property (ii) of the NOFRF transmissibility given by (15), if $\bar{J} = 1$, for $i = 1, \dots, n-1$

$$\frac{G_{(i,k)}(jk\omega_F)U_k(jk\omega_F)}{G_{(i+1,k)}(jk\omega_F)U_k(jk\omega_F)} = \frac{G_{(i,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F)}{G_{(i+1,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F)} = \cdots = \frac{G_{(i,N)}(jk\omega_F)U_N(jk\omega_F)}{G_{(i+1,N)}(jk\omega_F)U_N(jk\omega_F)} = T^{NL}_{i,i+1}(jk\omega_F) = \bar{\bar{Q}}_{i,i+1}(jk\omega_F) \quad \text{(A2.6)}$$

Equations (A2-1), (A2-2) and (A2-6) imply that

$$ST^{i,i+1}(jk\omega_F) = \frac{X_i(jk\omega_F)}{X_{i+1}(jk\omega_F)} = \frac{G_{(i,k)}(jk\omega_F)U_k(jk\omega_F) + G_{(i,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F) + \cdots + G_{(i,N)}(jk\omega_F)U_N(jk\omega_F)}{G_{(i+1,k)}(jk\omega_F)U_k(jk\omega_F) + G_{(i+1,k+2)}(jk\omega_F)U_{k+2}(kj\omega_F) + \cdots + G_{(i+1,N)}(jk\omega_F)U_N(jk\omega_F)} = T^{NL}_{i,i+1}(jk\omega_F) = \bar{\bar{Q}}_{i,i+1}(jk\omega_F) \quad \text{(A2.7)}$$

Therefore (24) holds.

As N is assumed to be even, for $\bar{J} = 1$, it is known from the first equation in (16) and (17) that when *L≥ J(1)*, if $1 \le i \le J(1) - 2$ *or* $L \le i < n-1$, or when *L<J(1)* if $1 \le i \le L-1$ *or* $J(1) \le i < n-1$,

$$ST^{i,i+1}(j\omega_F) = \frac{X_i(j\omega_F)}{X_{i+1}(j\omega_F)} = \frac{G_{(i,1)}(j\omega_F)U_1(j\omega_F) + G_{(i,3)}(j\omega_F)U_3(j\omega_F) + \cdots + G_{(i,N-1)}(j\omega_F)U_{N-1}(j\omega_F)}{G_{(i+1,1)}(j\omega_F)U_1(j\omega_F) + G_{(i+1,3)}(j\omega_F)U_3(j\omega_F) + \cdots + G_{(i+1,N-1)}(j\omega_F)U_{N-1}(j\omega_F)} = T^{L}_{i,i+1}(j\omega_F) = Q_{i,i+1}(j\omega_F) \quad \text{(A2.8)}$$

that is, the first equation of (25) holds. Otherwise, it is known from the second equation of (16), (17) and the first equation of (20) that

$$ST^{i,i+1}(j\omega_F) = \frac{X_i(j\omega_F)}{X_{i+1}(j\omega_F)} = \frac{G_{(i,1)}(j\omega_F)U_1(j\omega_F) + G_{(i,3)}(j\omega_F)U_3(j\omega_F) + \cdots + G_{(i,N-1)}(j\omega_F)U_{N-1}(j\omega_F)}{G_{(i+1,1)}(j\omega_F)U_1(j\omega_F) + G_{(i+1,3)}(j\omega_F)U_3(j\omega_F) + \cdots + G_{(i+1,N-1)}(j\omega_F)U_{N-1}(j\omega_F)} \ne T^{L}_{i,i+1}(j\omega_F) = Q_{i,i+1}(j\omega_F) \quad \text{(A2.9)}$$

So the second equation of (26) holds.

For all the other cases of N and k, i.e., N and k are all odd, or N is odd but k is even, or N is even but k is odd, (22)-(25) can be proved by following the same approach as above. Thus, the proof of Proposition 2 is completed.

REFERENCES

1. Cade, I.S., P.S. Keogh, and M.N. Sahinkaya, *Fault identification in rotor/magnetic bearing systems using discrete time wavelet coefficients.* IEEE-ASME Transactions on Mechatronics, 2005. **10**(6): p. 648-657.
2. Quinn, D.D., et al., *Damage detection of a rotating cracked shaft using an active magnetic bearing as a force actuator - Analysis and experimental verification.* IEEE-ASME Transactions on Mechatronics, 2005. **10**(6): p. 640-647.
3. Sahinkaya, M.N., et al., *Multiple sliding and rolling contact dynamics for a flexible rotor/magnetic bearing system.* IEEE-ASME Transactions on Mechatronics, 2007. **12**(2): p. 179-189.
4. Sakellariou, J.S. and S.D. Fassois, *Stochastic output error vibration-based damage detection and assessment in structures under earthquake excitation.* Journal of Sound and Vibration, 2006. **297**(3-5): p. 1048-1067.
5. Lang, Z.Q., et al., *Transmissibility of non-linear output frequency response functions with application in detection and location of damage in MDOF structural systems.* International Journal of Non-Linear Mechanics, 2011. **46**(6): p. 841-853.
6. Loh, C.H. and C.H. Chao, *Effectiveness of active tuned mass damper and seismic isolation on vibration control of multi-storey building.* Journal of Sound and Vibration, 1996. **193**(4): p. 773-792.
7. Law, S.S. and X.Q. Zhu, *Dynamic behavior of damaged concrete bridge structures under moving vehicular loads.* Engineering Structures, 2004. **26**(9): p. 1279-1293.
8. Marchesiello, S., et al., *Dynamics of multi-span continuous straight bridges subject to multi-degrees of freedom moving vehicle excitation.* Journal of Sound and Vibration, 1999. **224**(3): p. 541-561.
9. Farrar, C.R. and K. Worden, *An introduction to structural health monitoring.* Philosophical Transactions of the Royal Society a-Mathematical Physical and Engineering Sciences, 2007. **365**(1851): p. 303-315.
10. Khatkhate, A., et al., *Symbolic time-series analysis for anomaly detection in mechanical systems.* IEEE-ASME Transactions on Mechatronics, 2006. **11**(4): p. 439-447.
11. Schubel, P.J., et al., *Review of structural health and cure monitoring techniques for large wind turbine blades.* Renewable Energy, 2013. **51**: p. 113-123.
12. Maia, N.M.M., et al., *Damage detection and quantification using transmissibility.* Mechanical Systems and Signal Processing, 2011. **25**(7): p. 2475-2483.
13. Johnson, T., D.E. Adams, and M. Schiefer, *An analytical and experimental study to assess structural damage and integrity using dynamic transmissibility.* Proceedings of Imac-Xx: Structural Dynamics Vols I and Ii, 2002. **4753**: p. 472-476.

14. Sampaio, R.P.C., et al., *Transmissibility techniques for damage detection.* Proceedings of Imac-Xix: A Conference on Structural Dynamics, Vols 1 and 2, 2001. **4359**: p. 1524-1527.
15. Johnson, T.J. and D.E. Adams, *Transmissibility as a differential indicator of structural damage.* Journal of Vibration and Acoustics-Transactions of the ASME, 2002. **124**(4): p. 634-641.
16. Zhu, D.P., X.H. Yi, and Y. Wang, *Sensitivity Analysis of Transmissibility Functions for Structural Damage Detection.* Nondestructive Characterization for Composite Materials, Aerospace Engineering, Civil Infrastructure, and Homeland Security 2011, 2011. **7983**.
17. Cao, M.S., Q.W. Ren, and P.Z. Qiao, *Novel structural damage indices: Vibration transmissibility and its curvatures.* Fracture and Damage Mechanics V, Pts 1 and 2, 2006. **324-325**: p. 343-346.
18. Zhang, H., et al., *Structural health monitoring using transmittance functions.* Mechanical Systems and Signal Processing, 1999. **13**(5): p. 765-787.
19. Johnson, T.J., *Analysis of dynamic transmissibility as a feature for structural damage detection*, in *Mechanical Engineering*2002, Purdue University: Purdue.
20. Devriendt, C., et al., *Structural health monitoring in changing operational conditions using tranmissibility measurements.* Shock and Vibration, 2010. **17**(4-5): p. 651-675.
21. Chanpheng T, Y.H., Katsuchi H, *Nonlinear features for damage detectin on large civil structures due to earthquakes.* Structural Health Monitoring-an International Journal, 2012. **11**(4): p. 482-488
22. Han, Q.K., H.T. Luo, and B.C. Wen, *Simulations of a Dual-Rotor System with Local Rub-Impacts Based on Rigid-Flexible Multi-body Model.* Damage Assessment of Structures Viii, 2009. **413-414**: p. 677-682.
23. Han, Q., Z. Zhang, and B. Wen, *Periodic motions of a dual-disc rotor system with rub-impact at fixed limiter.* Proceedings of the Institution of Mechanical Engineers Part C-Journal of Mechanical Engineering Science, 2008. **222**(10): p. 1935-1946.
24. Ma, H., et al., *Analysis of dynamic characteristics for a rotor system with pedestal looseness.* Shock and Vibration, 2011. **18**(1-2): p. 13-27.
25. Peng, Z.K., Z.Q. Lang, and S.A. Billings, *Crack detection using nonlinear output frequency response functions.* Journal of Sound and Vibration, 2007. **301**(3-5): p. 777-788.
26. Peng, Z.K., Z.Q. Lang, and F.L. Chu, *Numerical analysis of cracked beams using nonlinear output frequency response functions.* Computers & Structures, 2008. **86**(17-18): p. 1809-1818.
27. Saavedra, P.N. and L.A. Cuitino, *Crack detection and vibration behavior of cracked beams.* Computers & Structures, 2001. **79**(16): p. 1451-1459.
28. Sundermeyer, J.N. and R.L. Weaver, *On Crack Identification and Characterization in a Beam by Nonlinear Vibration Analysis.* Journal of Sound and Vibration, 1995. **183**(5): p. 857-871.
29. Bovsunovsky, A. and C. Surace, *Considerations regarding superharmonic vibrations of a cracked beam and the variation in damping caused by the presence of the crack.* Journal of Sound and Vibration, 2005. **288**(4-5): p. 865-886.
30. Abraham, O.N.L. and J.A. Brandon, *The Modeling of the Opening and Closure of a Crack.* Journal of Vibration and Acoustics-Transactions of the Asme, 1995. **117**(3): p. 370-377.
31. Andreaus, U., P. Casini, and F. Vestroni, *Non-linear dynamics of a cracked cantilever beam under harmonic excitation.* International Journal of Non-Linear Mechanics, 2007. **42**(3): p. 566-575.
32. Chen, M.Y., et al., *A nonlinear frequency analysis based approach for power cable insulation fault detection.* Compel-the International Journal for Computation and Mathematics in Electrical and Electronic Engineering, 2012. **31**(2): p. 369-386.
33. Lang, Z.Q. and Z.K. Peng, *A novel approach for nonlinearity detection in vibrating systems.* Journal of Sound and Vibration, 2008. **314**(3-5): p. 603-615.
34. Lang, Z.Q. and L. Zhao, *Detection and location of nonlinearities in MDOF structural systems.* Acta Montanistica Slovaca, 2010. **15**(1): p. 28-32.
35. Peng, Z.K. and Z.Q. Lang, *Detecting the position of non-linear component in periodic structures from the system responses to dual sinusoidal excitations.* International Journal of Non-Linear Mechanics, 2007. **42**(9): p. 1074-1083.
36. Peng, Z.K., et al., *Locating Nonlinear Components in Periodic Structures using Nonlinear Effects.* Structural Health Monitoring-an International Journal, 2010. **9**(5): p. 401-411.
37. Peng, Z.K., Z.Q. Lang, and S.A. Billings, *Non-linear output frequency response functions of MDOF systems with multiple non-linear components.* International Journal of Non-Linear Mechanics, 2007. **42**(7): p. 941-958.
38. Lang, Z.Q. and S.A. Billings, *Energy transfer properties of non-linear systems in the frequency domain.* International Journal of Control, 2005. **78**(5): p. 345-362.
39. Peng, Z.K., Z.Q. Lang, and S.A. Billings, *Non-linear output frequency response functions for multi-input non-linear Volterra systems.* International Journal of Control, 2007. **80**(6): p. 843-855.
40. Sandberg, I.W., *A Perspective on System-Theory.* IEEE Transactions on Circuits and Systems, 1984. **31**(1): p. 88-103.
41. Lang, Z.Q. and S.A. Billings, *Output frequency characteristics of nonlinear systems.* International Journal of Control, 1996. **64**(6): p. 1049-1067.