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A NEW TYPE OF CONCENTRATION SOLUTIONS FOR A SINGULARLY PERTURBED ELLIPTIC PROBLEM

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ABSTRACT. We prove the existence of positive solutions concentrating on some higher dimensional manifolds near the boundary of the domain for a nonlinear singularly perturbed elliptic problem.

1. INTRODUCTION

The aim of this paper is to construct solutions concentrating on some higher dimensional manifolds for the following singularly perturbed elliptic problem:

$$\begin{cases} -\varepsilon^2\Delta u + u = u^{p-1},\ u>0, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small number.

We assume that Ω is a domain in R^N, whose boundary is Lipschitz continuous, and satisfies the following condition:

(Ω_1): There is an integer m, $1 < m \le N$, such that $y \in \Omega$, if and only if $(|y'|, y'') \in D$, where $y = (y', y'')$, $y' \in R^m$, $y'' \in R^{N-m}$, D is a relatively open domain in R_+^{N-m+1}, and

$$R_+^{N-m+1} = \big\{z = (z_1, \cdots, z_{N-m+1}) : z_1 \ge 0\big\}.$$

Let us emphasize here that we do not assume that Ω is bounded. The domain Ω can be a bounded domain, or an exterior domain in R^N, or many other unbounded domains.

We assume that p satisfies $p \in (2, 2(N-m+1)/(N-m-1))$ if $m < N-1$, $p \in (0, +\infty)$ if $m \ge N-1$.

In view of the assumption on Ω, we will work on the following subspace of $H_0^1(\Omega)$:

$$H_s = \big\{u : u \in H_0^1(\Omega), u(y) = u(|y'|, y'')\big\}.$$

Let U be the unique solution of the following problem:

$$\begin{cases} -\Delta v + v = v^{p-1},\ v>0, & \text{in } R^{N-m+1}, \\ v(0) = \max_{z\in R^{N-m+1}} v(z), \\ v \in H^1(R^{N-m+1}). \end{cases}$$

Then $U(z) = U(|z|)$, $U' < 0$, and

$$|z|^{(N-m)/2} e^{|z|} U(|z|) \to c > 0$$

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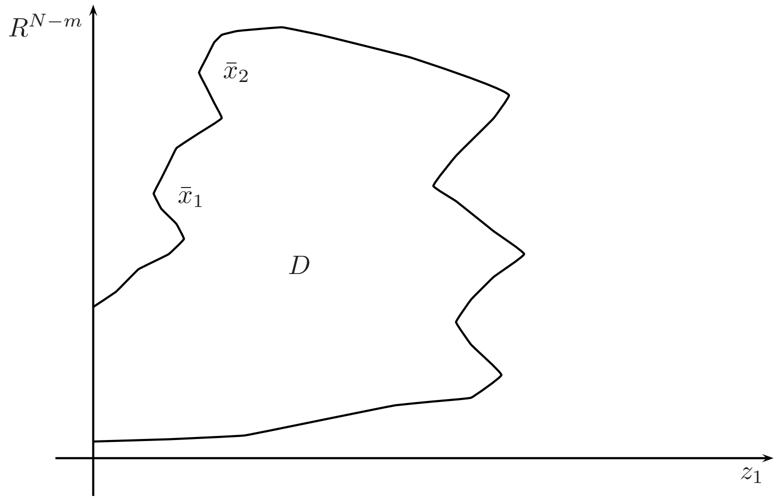

FIGURE 1. D is a bounded domain

as $|z| \to +\infty$. Moreover, $U(z)$ is nondegenerate. That is, the kernel of the operator $-\Delta w + w - (p-1)U^{p-2}w$ in $H^1(R^{N-m+1})$ is spanned by $\{\frac{\partial U(z)}{\partial z_i}, i = 1, \cdots, N-m+1\}$. See [11, 15].

For any $y = (y', y'') \in R^N$, $y' \in R^m$, $y'' \in R^{N-m}$, we denote $\tilde{y} = (|y'|, y'') \in R^{N-m+1}$. Let $\bar{W}(y) = U(\tilde{y})$. For any $\bar{x} \in D$, let $\bar{W}_{\varepsilon,\bar{x}}(y) = U\big(\frac{|\tilde{y}-\bar{x}|}{\varepsilon}\big)$. Then, $\bar{W}_{\varepsilon,\bar{x}}$ satisfies

$$(1.2) \qquad -\varepsilon^2 \Delta \bar{W}_{\varepsilon,\bar{x}} + \bar{W}_{\varepsilon,\bar{x}} = \bar{W}_{\varepsilon,\bar{x}}^{p-1} - \varepsilon \frac{m-1}{|y'|} \frac{|y'| - \bar{x}_1}{|\tilde{y} - \bar{x}|} U'\big(\frac{|\tilde{y}-\bar{x}|}{\varepsilon}\big), \quad \text{in } \Omega.$$

In this paper, we assume that Ω also satisfies the following condition:

(Ω_2): There are k different points $\bar{x}_j = (\bar{x}_{j,1}, \bar{x}_j'') \in \partial D$, $j = 1, \cdots, k$, such that for $j = 1, \cdots, k$,

(i) there is a C^2 function $\psi_j(z'')$ in R^{N-m} such that

$$D \cap B_\delta(\bar{x}_j) = \{z = (z_1, z'') : z_1 > \psi_j(z'')\} \cap B_\delta(\bar{x}_j),$$

and

$$\partial D \cap B_\delta(\bar{x}_j) = \{z = (z_1, z'') : z_1 = \psi_j(z'')\} \cap B_\delta(\bar{x}_j),$$

where $\delta > 0$ is a small constant;

(ii) $\bar{x}_{j,1} = \psi_j(\bar{x}_j'') = \min_{z'' \in B_\delta(\bar{x}_j'')} \psi_j(z'') > 0$, and $\bar{x}_{j,1} < \min_{z'' \in \partial B_\delta(\bar{x}_j'')} \psi_j(z'')$.

By (Ω_2), we can deduce that for each $j = 1, \cdots, k$, there is constant $\delta' \in (0, \delta)$, such that $\min_{z'' \in B_\delta(\bar{x}_j) \setminus B_{\delta'}(\bar{x}_j)} \psi_j(z'') > \min_{z'' \in B_\delta(\bar{x}_j)} \psi_j(z'')$.

We will prove that (1.1) has a solution u_ε, which is close to $\bar{W}_{\varepsilon,\bar{x}_j}$ in a small neighbourhood of $|y'| = \bar{x}_{j,1}$, $j = 1, \cdots, k$, and is close to zero elsewhere. Since the right-hand side of (1.2) has a singularity, we truncate $\bar{W}_{\varepsilon,\bar{x}_j}$ as follows.

Let $\xi_j \in C_0^\infty(B_\delta(\bar{x}_j))$ be a function such that $\xi_j = 1$ in $B_{\delta''}(\bar{x}_j)$ for some $\delta'' \in (\delta', \delta)$. For any $x_j \in D \cap B_{\delta'}(\bar{x}_j)$, define

$$W_{\varepsilon,x_j}(y) = \xi_j(|y'|, y'') \bar{W}_{\varepsilon,x_j}(y).$$

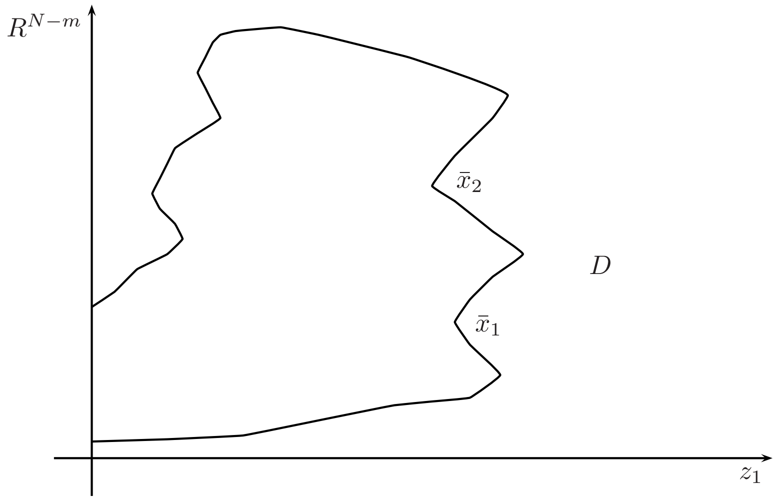

FIGURE 2. D is an exterior domain

Then W_{ε,x_j} satisfies

$$-\varepsilon^2\Delta W_{\varepsilon,x_j} + W_{\varepsilon,\bar{x}_j} = \xi_j \bar{W}^{p-1}_{\varepsilon,x_j} + \tilde{f}_{\varepsilon,x_j}(y) \quad \text{in } \Omega, \tag{1.3}$$

where

$$\tilde{f}_{\varepsilon,x_j}(y) = -\xi_j\varepsilon\frac{m-1}{|y'|}\frac{|y'|-x_{j,1}}{|\tilde{y}-x_j|}U'\Big(\frac{|\tilde{y}-x_j|}{\varepsilon}\Big) - 2\varepsilon D\xi_j DU\Big(\frac{|\tilde{y}-x_j|}{\varepsilon}\Big) - \varepsilon^2 U\Big(\frac{|\tilde{y}-x_j|}{\varepsilon}\Big)\Delta\xi_j.$$

Since $\frac{z-x_{j,1}}{|z-x_j|}U'\big(\frac{|z-x_j|}{\varepsilon}\big) = \varepsilon\frac{\partial}{\partial z_1}U\big(\frac{|z-x_j|}{\varepsilon}\big)$ and $\xi_j = 0$ in a neighbourhood of $|y'| = 0$, it is easy to see that $\tilde{f}_{\varepsilon,x_j}$ is a smooth function in both y and x_j, and satisfies

$$|\tilde{f}_{\varepsilon,x_j}| \le C\varepsilon U\Big(\frac{|\tilde{y}-x_j|}{\varepsilon}\Big).$$

Let $P_{\varepsilon,\Omega}W_{\varepsilon,x_j}$ be the solution of

$$\begin{cases} -\varepsilon^2\Delta v + v = \xi_j\bar{W}^{p-1}_{\varepsilon,x_j} + \tilde{f}_{\varepsilon,x_j}(y), & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

By the uniqueness, we know that $P_{\varepsilon,\Omega}W_{\varepsilon,x_j} \in H_s$.

Let

$$\langle u, v\rangle_\varepsilon = \int_\Omega (\varepsilon^2 DuDv + uv), \quad \|v\|_\varepsilon = \langle u, v\rangle_\varepsilon^{1/2}.$$

The main result of this paper is the following.

Theorem 1.1. *Assume that $1 < m < N$. Suppose that Ω satisfies (Ω_1) and (Ω_2). Then, there is an $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$, (1.1) has at least one solution of the form*

$$u_\varepsilon = \sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}} + \omega_\varepsilon, \tag{1.5}$$

where $\omega_\varepsilon \in H_s$, $x_{\varepsilon,j} = (x_{\varepsilon,j,1}, x''_{\varepsilon,j}) \in D$, *and as* $\varepsilon \to 0$,

$$\frac{d(x_{\varepsilon,j}, \partial D)}{\varepsilon} \to +\infty,$$

$$x_{\varepsilon,j} \to \hat{x}_j \in \partial D \cap B_\delta(\bar{x}_j), \quad \text{with } \hat{x}_{j,1} = \psi_j(\hat{x}''_j) = \min_{z'' \in B_\delta(\bar{x}''_j)} \psi_j(z''),$$

and

$$\|\omega_\varepsilon\|_\varepsilon^2 = o(\varepsilon^{N-m+1}).$$

Our assumption on the boundary implies that $\bar{x}_j$ is the closest point to the subspace $z_1 = 0$ in $D \cap B_\delta(\bar{x}_j)$. If we interpret the assumption on the boundary in this way, we can also include the case $m = N$ in Theorem 1.1. If $m = N$, then (i) and (ii) imply that D is an interval $[r_1, r_2]$ in R^1 with $r_1 > 0$. That is, Ω is an annulus or the exterior domain of a ball. Theorem 1.1 states that (1.1) has a solution concentrating near the inner boundary of the annulus. This is the result in [4].

There are many works in the case $m = 1$ since the pioneering works [16]. See for example [3, 5, 6, 7, 8, 9, 10, 12, 14, 16, 17, 18]. Except for [8], where the exterior domain problem was studied, all the other papers consider the problem in a bounded domain. To obtain the results mentioned above for the case $m = 1$, no symmetry condition is imposed on the domain Ω. In the case $m > 1$, we use the solution U of a lower dimensional problem as an approximate solution for problem (1.1). So, there is no control in some directions for the corresponding linear operator $L_\varepsilon v =: -\varepsilon^2 \Delta v + v - (p-1)\bar{W}_{\varepsilon,\bar{x}}^{p-2} v$ in $H_0^1(\Omega)$. As a consequence, $L_\varepsilon v = \lambda v$, $v \in H_0^1(\Omega)$, will have many small eigenvalues. By imposing some partial symmetry conditions on Ω, we can get rid of the small eigenvalues if we work on the subspace H_s.

As far as we know, Theorem 1.1 is the first result on the existence of solution for (1.1), concentrating on an $(m-1)$-dimensional sphere. In [1, 2], Ambrosetti, Malchiodi and Ni studied the existence of solutions concentrating on spheres for some elliptic problems, assuming that the domain is either a ball or an annulus. But for (1.1), neither Theorem 1.1 nor the results in [2] gives the existence of a solution concentrating on an $(m-1)$-dimensional sphere, if the domain is an annulus and $1 < m < N$.

Solutions concentrating on a connected component of the boundary were constructed in [13] for the singularly perturbed Neumann problem. The techniques to prove Theorem 1.1 can also be used to study the following Neumann problem:

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^{p-1},\ u > 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where n is the outward unit normal of $\partial\Omega$ at $y \in \partial\Omega$.

We assume that Ω is an open connected set in R^N, satisfying (Ω_1) and

(Ω_3): there exists $\bar{x} = (\bar{x}_1, \bar{x}'') \in \partial D$ such that

(i) there is a C^2 function $\psi(z'')$ in R^{N-m} such that

$$D \cap B_\delta(\bar{x}) = \{z = (z_1, z'') : z_1 < \psi(z'')\} \cap B_\delta(\bar{x}),$$

and

$$\partial D \cap B_\delta(\bar{x}) = \{z = (z_1, z'') : z_1 = \psi(z'')\} \cap B_\delta(\bar{x}),$$

where $\delta > 0$ is a constant;

(ii) $\bar{x}_1 = \psi(\bar{x}'') = \max_{z''\in B_\delta(\bar{x}'')} \psi(z'') > 0$ and $\bar{x}_1 > \max_{z''\in \partial B_\delta(\bar{x}'')} \psi(z'')$.

For any $\tilde{x} \in D$, let $P_{\varepsilon,\Omega,N} W_{\varepsilon,\tilde{x}}$ be the solution of

$$\begin{cases} -\varepsilon^2 \Delta v + v = \xi \bar{W}^{p-1}_{\varepsilon,\tilde{x}} + \tilde{f}_{\varepsilon,\tilde{x}}(y), & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\xi \in C_0^\infty(B_\delta(\bar{x}))$ with $0 \le \xi \le 1$; $\xi = 1$ in $B_{\delta/2}(\bar{x})$). Then, we have

Theorem 1.2. *Assume that $1 < m \le N$. Suppose that Ω satisfies (Ω_1) and (Ω_3). Then, for any positive integer k, there is an $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$,* (1.6) *has at least one solution of the form*

$$u_\varepsilon = \sum_{j=1}^k P_{\varepsilon,\Omega,N} W_{\varepsilon,x_{\varepsilon,j}} + \omega_\varepsilon,$$

where $\omega_\varepsilon \in H_s$, $x_{\varepsilon,j} = (x_{\varepsilon,j,1}, x''_{\varepsilon,j}) \in D$, and as $\varepsilon \to 0$,

$$\frac{d(x_{\varepsilon,j}, \partial D)}{\varepsilon} \to +\infty, \quad \frac{|x_{\varepsilon,j} - x_{\varepsilon,i}|}{\varepsilon} \to +\infty, \ \forall\, j \ne i,$$
$$x_{\varepsilon,j} \to \hat{x}_j \in \partial D \cap B_\delta(\bar{x}), \quad \textit{with } \hat{x}_{j,1} = \psi_j(\hat{x}''_j) = \max_{z''\in B_\delta(\bar{x}'')} \psi_j(z''),$$

and

$$\|\omega_\varepsilon\|^2_\varepsilon = o(\varepsilon^{N-m+1}).$$

The solutions obtained in Theorem 1.2 concentrate near the boundary but not on the boundary. Our next result shows that (1.6) has a solution concentrating on several manifolds on the boundary.

Theorem 1.3. *Assume that $1 < m < N$. Suppose that Ω satisfies (Ω_1) and (Ω_3). Then, for any positive integer k, there is an $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$,* (1.6) *has at least one solution of the form*

$$u_\varepsilon = \sum_{j=1}^k P_{\varepsilon,\Omega,N} W_{\varepsilon,x_{\varepsilon,j}} + \omega_\varepsilon,$$

where $\omega_\varepsilon \in H_s$, $x_{\varepsilon,j} = (x_{\varepsilon,j,1}, x''_{\varepsilon,j}) \in \partial D$, and as $\varepsilon \to 0$,

$$\frac{|x_{\varepsilon,j} - x_{\varepsilon,i}|}{\varepsilon} \to +\infty, \ \forall\, j \ne i,$$
$$x_{\varepsilon,j} \to \hat{x}_j \in \partial D \cap B_\delta(\bar{x}), \quad \textit{with } \hat{x}_{j,1} = \psi_j(\hat{x}''_j) = \max_{z''\in B_\delta(\bar{x}'')} \psi_j(z''),$$

and

$$\|\omega_\varepsilon\|^2_\varepsilon = o(\varepsilon^{N-m+1}).$$

Condition (Ω_3) implies that $\bar{x}$ has the largest distance to the subspace $z_1 = 0$ in $D \cap B_\delta(\bar{x})$. Unlike the Dirichlet problem, Theorem 1.2 shows that the Neumann problem (1.6) has solutions with several peaks clustering near the manifold $|y'| = \hat{x}_1$, where $\hat{x} = (\hat{x}_1, \hat{x}'')$ is a maximum point of the distance function of $x \in D$ to $z_1 = 0$. In the case that Ω is an annulus, the results here show that the Dirichlet problem (1.1) has a solution concentrating near the inner boundary; while the Neumann problem (1.6) has a solution with several peaks clustering near the outer boundary.

We will use the reduction method, together with the comparison of the energy, to prove the theorems. In this paper, we only give the proof of Theorem 1.1. For the Neumann problem, we can follow [7, 19] to make the necessary modifications. In Section 2, we will estimate the energy of the approximate solution $P_{\varepsilon,\Omega}W_{\varepsilon,x_j}$ and thus lay the foundation for the proof of Theorem 1.1. In [16], Ni and Wei used the viscosity solution method to obtain the estimate of the energy of the approximate solutions when $m = 1$ and Ω is bounded. But it seems that the viscosity solution method cannot be applied to treat the present case, due to the possible unboundedness of the domain, and/or the occurrence of a singularity of the corresponding problem in D. In this paper, we will modify the techniques developed in [7, 8] to obtain the desired estimates.

The functional corresponding to (1.1) may not be well defined in H_s, because the exponent p may be supercritical. Our objective is to construct solutions concentrating near the $(m-1)$-dimensional manifolds $|y'| = \bar{x}_{j,1}$. So we can modify the nonlinear term u^{p-1} in such a way that corresponding to the modified problem, the functional is well defined in H_s and the modified problem has a solution concentrating near $|y'| = \bar{x}_{j,1}$, which is also a solution of the original problem. To this aim, we define

$$f(y,t) = \sum_{j=1}^{k} 1_{B_j} t_+^{p-1} + \big(1 - \sum_{j=1}^{k} 1_{B_j}\big)\bar{f}(t), \tag{1.7}$$

where $B_j = \{y : y \in \Omega, (|y'|, y'') \in D \cap B_\delta(\bar{x}_j)\}$, $1_{B_j} = 1$ in B_j, and is zero otherwise, and

$$\bar{f}(t) = \begin{cases} t_+^{p-1}, & t \le 1, \\ 1 + (p-1)(t-1), & t > 1. \end{cases}$$

Now we consider the following problem:

$$\begin{cases} -\varepsilon^2 \Delta u + u = f(y,u),\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

The functional corresponding to (1.8) is

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \big(\varepsilon^2 |Du|^2 + u^2\big) - \int_\Omega F(y,u),$$

where $F(y,t) = \int_0^t f(y,\tau)\,d\tau$. For any $y \in B_j$, we have $|y'| \ge \bar{x}_{j,1} > 0$. We see that $I_\varepsilon(u)$ is well defined in H_s if $p \in (2, 2(N-m+1)/(N-m-1))$.

2. Basic estimates

In this section, we give some basic estimates needed in the proof of the main result.

By our assumption on Ω, we can deduce that for each $j = 1, \cdots, k$, there is a constant $\delta' \in (0, \delta)$ such that $\min_{z'' \in B_\delta(\bar{x}_j)\setminus B_{\delta'}(\bar{x}_j)} \psi_j(z'') > \min_{z'' \in B_\delta(\bar{x}_j)} \psi_j(z'')$. Define

$$D_j = \{z = (z_1, z'') : z_1 \in (\psi_j(z''), \psi_j(z'') + \gamma), z'' \in B_{\delta'}(\bar{x}_j)\},$$

where $\gamma > 0$ is a small constant. We choose $\gamma > 0$ so small that if $\bar{x} \in D_j$, then

$$d(\bar{x}, \partial(D \cap B_\delta(\bar{x}_j))) = d(\bar{x}, \partial D). \tag{2.1}$$

In this paper, we always assume that $x_j \in D_j$, and $e^{-d(x_j,\partial D)/\varepsilon} \le \varepsilon^{1-\tilde{\theta}}$ for a fixed small $\tilde{\theta} > 0$.

Let $\varphi_{\varepsilon,x_j} = W_{\varepsilon,x_j} - P_{\varepsilon,\Omega}U_{\varepsilon,x_j}$. Then $\varphi_{\varepsilon,x_j}$ satisfies

$$\begin{cases} -\varepsilon^2\Delta\varphi_{\varepsilon,x_j} + \varphi_{\varepsilon,x_j} = 0, & \text{in } \Omega, \\ \varphi_{\varepsilon,x_j} = W_{\varepsilon,x_j}, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Since $\varphi_{\varepsilon,x_j} \in H_0^1(\Omega)$, by the maximum principle on bounded or unbounded domains, we have

$$0 < \varphi_{\varepsilon,x_j} \le Ce^{-d(x_j,\partial D)/\varepsilon}.$$

Thus, we see

$$W_{\varepsilon,x_j} - Ce^{-d(x_j,\partial D)/\varepsilon} < P_{\varepsilon,\Omega}W_{\varepsilon,x_j} < W_{\varepsilon,x_j}.$$

In this section, we will estimate $I_\varepsilon\big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)$. For each fixed j, noting that $\bar W_{\varepsilon,x_j}$ is exponentially small if $(|y,|,y'') \notin B_{\delta''}(\bar x_j)$, we deduce

$$\begin{aligned} &I_\varepsilon\big(P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big) \\ =&\frac{1}{2}\Big(\int_\Omega \xi_j \bar W^{p-1}_{\varepsilon,x_j} P_{\varepsilon,\Omega}W_{\varepsilon,x_j} + \int_\Omega \tilde f_{\varepsilon,x_j} P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) \\ &-\frac{1}{p}\int_\Omega (P_{\varepsilon,\Omega}W_{\varepsilon,x_j})_+^p \\ =&\frac{1}{2}\int_\Omega \xi_j \bar W^{p-1}_{\varepsilon,x_j} W_{\varepsilon,x_j} - \frac{1}{2}\int_\Omega \xi_j \bar W^{p-1}_{\varepsilon,x_j}\varphi_{\varepsilon,x_j} + O\big(\varepsilon^{N-m+2}\big) \\ &-\frac{1}{p}\int_\Omega W^p_{\varepsilon,x_j} + \int_\Omega W^{p-1}_{\varepsilon,x_j}\varphi_{\varepsilon,x_j} + O\Big(\int_\Omega W^{p-2}_{\varepsilon,x_j}\varphi^2_{\varepsilon,x_j}\Big) \\ =&\frac{p-2}{2p}\int_\Omega \bar W^p_{\varepsilon,x_j} + \frac{1}{2}\int_\Omega \xi_j \bar W^{p-1}_{\varepsilon,x_j}\varphi_{\varepsilon,x_j} + O\Big(\int_\Omega W^{p-2}_{\varepsilon,x_j}\varphi^2_{\varepsilon,x_j} + \varepsilon^{N-m+2}\Big). \end{aligned} \tag{2.3}$$

So, we see that to estimate $I_\varepsilon\big(P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)$, we need to estimate

$$\tau_{\varepsilon,x_j} = \int_\Omega \xi_j \bar W^{p-1}_{\varepsilon,x_j}\varphi_{\varepsilon,x_j},$$

and $\int_\Omega W^{p-2}_{\varepsilon,x_j}\varphi^2_{\varepsilon,x_j}$.

First, we have the following estimate for τ_{ε,x_j}.

Lemma 2.1. *Suppose that $x_j \in D_j$. Then for any small $\theta > 0$, there are $C_2 > C_1 > 0$, such that*

$$\begin{aligned} &C_1\varepsilon^{N-m+1}e^{-(2+\theta)d_j/\varepsilon} + \varepsilon^{N-m+1}O\big(\varepsilon e^{-d_j/\varepsilon}\big) \\ \le&\tau_{\varepsilon,x_j} \le C_2\varepsilon^{N-m+1}e^{-(2-\theta)d_j/\varepsilon} + \varepsilon^{N-m+1}O\big(\varepsilon e^{-d_j/\varepsilon}\big), \end{aligned}$$

where $d_j = d(x_j,\partial D)$.

Proof. Since

$$\Big|\int_{B_j}\tilde f_{\varepsilon,x_j}(y)\varphi_{\varepsilon,x_j}\Big| \le Ce^{-d_j/\varepsilon}\varepsilon\int_\Omega U\big(\frac{|\tilde y - x_j|}{\varepsilon}\big) \le Ce^{-d_j/\varepsilon}\varepsilon^{N-m+2},$$

we have

$$
\begin{aligned}
\tau_{\varepsilon,x_j} &= \int_{B_j} \xi_j \bar{W}_{\varepsilon,x_j}^{p-1} \varphi_{\varepsilon,x_j} \\
&= \int_{B_j} \Big(-\varepsilon^2 \Delta W_{\varepsilon,x_j} + W_{\varepsilon,x_j} - \tilde{f}_{\varepsilon,x_j}(y) \Big) \varphi_{\varepsilon,x_j} \\
&= \int_{\partial B_j} \big(-\varepsilon^2 \frac{\partial W_{\varepsilon,x_j}}{\partial n} \varphi_{\varepsilon,x_j} + \varepsilon^2 \frac{\partial \varphi_{\varepsilon,x_j}}{\partial n} W_{\varepsilon,x_j} \big) \\
&\quad + \int_{B_j} \Big(-\varepsilon^2 \Delta \varphi_{\varepsilon,x_j} + \varphi_{\varepsilon,x_j} \Big) W_{\varepsilon,x_j} + O\big(e^{-d_j/\varepsilon} \varepsilon^{N-m+2} \big) \\
&= \int_{\partial B_j} \big(-\varepsilon^2 \frac{\partial W_{\varepsilon,x_j}}{\partial n} \varphi_{\varepsilon,x_j} + \varepsilon^2 \frac{\partial \varphi_{\varepsilon,x_j}}{\partial n} W_{\varepsilon,x_j} \big) + O\big(e^{-d_j/\varepsilon} \varepsilon^{N-m+2} \big),
\end{aligned} \tag{2.4}
$$

where n is the outward unit normal of ∂B_j at y.

We are now ready to get an upper bound for τ_{ε,x_j}.

We can deduce from (2.1) that

$$
\Big| \int_{\partial B_j} \varepsilon^2 \frac{\partial W_{\varepsilon,x_j}}{\partial n} \varphi_{\varepsilon,x_j} \Big| \le C e^{-d_j/\varepsilon} \int_{\partial B_j} \varepsilon U \big(\frac{|\tilde{y} - x_j|}{\varepsilon} \big) \le C e^{(2-\theta) d_j/\varepsilon} \varepsilon^{N-m+1}. \tag{2.5}
$$

To estimate $\int_{\partial B_j} \varepsilon^2 \frac{\partial \varphi_{\varepsilon,x_j}}{\partial n} W_{\varepsilon,x_j}$, we need to estimate $\frac{\partial \varphi_{\varepsilon,x_j}}{\partial n}$ on ∂B_j.

Let $\bar{z}_j = (x_{j,1}, 0, \cdots, 0, x_j'') \in \Omega$, and let $V_\varepsilon(y) = P_{\varepsilon,\Omega} W_{\varepsilon,x_j}(\varepsilon y + \bar{z}_j)$. Then, V_ε satisfies

$$
-\Delta V_\varepsilon + V_\varepsilon = \eta_\varepsilon(y), y \in B_{\varepsilon,j}, \tag{2.6}
$$

where $\eta_\varepsilon(y) = \tilde{\eta}_\varepsilon(\varepsilon y + \bar{z}_j)$ and $\tilde{\eta}_\varepsilon(y) = \xi_j \bar{W}_{\varepsilon,x_j}^{p-1}(y) + \tilde{f}_{\varepsilon,x_j}(y)$. By the L^p estimate for the elliptic equations, we obtain that for any $\theta > 0$ small and $x_0 \in \partial B_{\varepsilon,j}$,

$$
\begin{aligned}
&\|V_\varepsilon\|_{W^{2,q}(B_\theta(x_0))} \le C |\eta_\varepsilon|_{L^q(B_{2\theta}(x_0))} + C |V_\varepsilon|_{L^q(B_{2\theta}(x_0))} \\
\le & C' |\eta_\varepsilon|_{L^\infty(B_{2\theta}(x_0))} + C' |V_\varepsilon|_{L^\infty(B_{2\theta}(x_0))} \le C e^{-(1-\theta) d_j/\varepsilon},
\end{aligned} \tag{2.7}
$$

since $|P_{\varepsilon,\Omega} W_{\varepsilon,z}| \le W_{\varepsilon,z} + C e^{-d_j/\varepsilon}$, where $C > 0$ is a constant depending on θ. So, for $q > N$, we obtain from (2.7) that

$$
\|V_\varepsilon\|_{C^1(B_\theta(x_0))} \le C e^{-(1-\theta) d_j/\varepsilon}, \tag{2.8}
$$

which implies

$$
\Big| \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial n} \Big| \le C \varepsilon^{-1} e^{-(1-\theta) d_j/\varepsilon}, \quad \text{on } \partial B_j. \tag{2.9}
$$

So we obtain

$$
\Big| \frac{\partial \varphi_{\varepsilon,x_j}}{\partial n} \Big| \le \Big| \frac{\partial P_{\varepsilon,\Omega} W_{\varepsilon,x_j}}{\partial n} \Big| + \Big| \frac{\partial W_{\varepsilon,x_j}}{\partial n} \Big| \le C \varepsilon^{-1} e^{-(1-\theta) d_j/\varepsilon}, \quad \text{on } \partial B_j. \tag{2.10}
$$

Using (2.10), we find

$$
\Big| \int_{\partial B_j} \varepsilon^2 \frac{\partial \varphi_{\varepsilon,x_j}}{\partial n} W_{\varepsilon,x_j} \Big| \le C \varepsilon^{N-m+1} e^{-2(1-\theta) d_j/\varepsilon}. \tag{2.11}
$$

Combining (2.4), (2.5) and (2.10), we obtain

$$
\tau_{\varepsilon,x_j} \le C \varepsilon^{N-m+1} e^{-2(1-\theta) d_j/\varepsilon} + C \varepsilon^{N-m+1} O\big(\varepsilon e^{-d_j/\varepsilon} \big).
$$

Next, we get a lower bound for τ_{ε,x_j}.

Since $W_{\varepsilon,x_j}=0$ on $\partial B_j\setminus\partial\Omega$, using (2.10), we find

$$\tau_{\varepsilon,x_j}=\int_{\partial B_j\cap\partial\Omega}\big(-\varepsilon^2\frac{\partial W_{\varepsilon,x_j}}{\partial n}W_{\varepsilon,x_j}+\varepsilon^2\frac{\partial\varphi_{\varepsilon,x_j}}{\partial n}W_{\varepsilon,x_j}\big)+\varepsilon^{N-m+1}O\big(\varepsilon e^{-d_j/\varepsilon}\big). \tag{2.12}$$

For $x_j=(x_{j,1},x_j'')\in D$, $y=(y',y'')\in\Omega$, let $\tilde y=\big((|y'|-x_{j,1})|y'|^{-1}y',y''-x''\big)\in R^N$. Then

(2.13)

$$-\int_{\partial B_j\cap\partial\Omega}\varepsilon^2\frac{\partial W_{\varepsilon,x_j}}{\partial n}W_{\varepsilon,x_j}=-\int_{\partial B_j\cap\partial\Omega}\varepsilon U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)U'\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\big\langle\frac{1}{|\tilde y-x_j|}\tilde y,n\big\rangle$$
$$=-\,c_{m-1}\int_{\partial D_j\cap\partial D}\varepsilon|y'|^{m-1}U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)U'\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\big\langle\frac{\tilde y-x_j}{|\tilde y-x_j|},n_D\big\rangle,$$

where n_D is the outward unit normal of ∂D at $\tilde y$, and $c_{m-1}>0$ is the area of the unit sphere in R^m.

Let $x_j^*\in\partial D$ be the point such that $|x_j^*-x_j|=d_j$. Let $\sigma>0$ be a small number such that, if $\tilde y\in B_{(1+\sigma)d_j}(x^*)\cap\partial D$, then

$$\big\langle\frac{\tilde y-x_j}{|\tilde y-x_j|},n_D\big\rangle\ge\frac12.$$

As a result,

$$\begin{aligned}&-\,c_{m-1}\int_{\partial D\cap B_{(1+\sigma)d_j}(x_j^*)}\varepsilon|y'|^{m-1}U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)U'\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\big\langle\frac{\tilde y-x_j}{|\tilde y-x_j|},n_D\big\rangle\\ &\ge-\frac12c_{m-1}\int_{\partial D\cap B_{(1+\sigma)d_j}(x_j^*)}\varepsilon|y'|^{m-1}U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)U'\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\\ &\ge c'\int_{\partial D\cap B_{(1+\sigma)d_j}(x_j^*)}\varepsilon|y'|^{m-1}U^2\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\\ &\ge c'\int_{\partial D\cap B_{(1+\theta)d_j}(x_j^*)}\varepsilon|y'|^{m-1}U^2\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\\ &\ge c''e^{-(2+\theta)d_j/\varepsilon}\int_{\partial D\cap B_{(1+\theta)d_j}(x_j^*)}\varepsilon U^\theta\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\\ &\ge c_1\varepsilon^{N-m+1}e^{-(2+\theta)d_j/\varepsilon},\end{aligned}\tag{2.14}$$

where c', c'' and c_1 are some positive constants.

On the other hand, we have

$$\begin{aligned}&\Big|\int_{(\partial D_j\cap\partial D)\setminus B_{(1+\sigma)d_j}(x_j^*)}\varepsilon|y'|^{m-1}U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)U'\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\big\langle\frac{\tilde y-x_j}{|\tilde y-x_j|},n_D\big\rangle\Big|\\ &\le Ce^{-(1+\sigma)(2-\theta)d_j/\varepsilon}\int_{(\partial D_j\cap\partial D)\setminus B_{(1+\sigma)d_j}(x_j^*)}\varepsilon U^\theta\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\\ &\le Ce^{-(1+\sigma)(2-\theta)d_j/\varepsilon}\varepsilon\int_{\partial D}U^\theta\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\\ &\le Ce^{-(1+\sigma)(2-\theta)d_j/\varepsilon}\varepsilon^{N-m+1}.\end{aligned}\tag{2.15}$$

Combining (2.13), (2.14) and (2.15), we are led to

$$-\int_{\partial B_j\cap\partial\Omega}\varepsilon^2\frac{\partial W_{\varepsilon,x_j}}{\partial n}W_{\varepsilon,x_j}\ge c'\varepsilon^{N-m+1}e^{-(2+\theta)d_j/\varepsilon} \tag{2.16}$$

for some $c'>0$. Multiplying (2.2) by $\varphi_{\varepsilon,x_j}$ and integrating by parts, noting that $W_{\varepsilon,x_j}=0$ in $\partial\Omega\setminus\partial B_j$, we find

(2.17)

$$\int_{\partial B_j\cap\partial\Omega}\varepsilon^2\frac{\partial\varphi_{\varepsilon,x_j}}{\partial n}W_{\varepsilon,x_j}=\int_{\partial\Omega}\varepsilon^2\frac{\partial\varphi_{\varepsilon,x_j}}{\partial n}W_{\varepsilon,x_j}=\int_\Omega\big(\varepsilon^2|D\varphi_{\varepsilon,x_j}|^2+\varphi^2_{\varepsilon,x_j}\big)>0.$$

Let us emphasize that we can use the integration by parts to obtain (2.17) because $\partial\Omega$ is C^1 at y if $W_{\varepsilon,x_j}(y)\ne0$.

So, from (2.12), (2.16) and (2.17), we obtain

$$\tau_{\varepsilon,x_j}\ge c_1\varepsilon^{N-m+1}e^{-(2+\theta)d_j/\varepsilon}+\varepsilon^{N-m+1}O\big(\varepsilon e^{-d_j/\varepsilon}\big).$$

□

Lemma 2.2. *Let $q\in(1,p]$. Suppose that $x_j\in D_j$. There is a $\sigma>0$, such that*

$$\int_\Omega\varphi^q_{\varepsilon,x_j}W^{p-q}_{\varepsilon,x_j}=O\big(e^{-(q-1)\sigma d_j/\varepsilon}\tau_{\varepsilon,x_j}\big). \tag{2.18}$$

Proof. Write

$$\int_\Omega\varphi^q_{\varepsilon,x_j}W^{p-q}_{\varepsilon,x_j}=\int_{\tilde B_j}\varphi^q_{\varepsilon,x_j}W^{p-q}_{\varepsilon,x_j}+\int_{\Omega\setminus\tilde B_j}\varphi^q_{\varepsilon,x_j}W^{p-q}_{\varepsilon,x_j}, \tag{2.19}$$

where $\tilde B_j=\{y:y\in\Omega:(|y'|,y'')\in B_{(1-2\sigma)d_j}(x_j)\}$, $\sigma>0$ is a small constant.

It is easy to see that

$$\begin{aligned}&\int_{\Omega\setminus\tilde B_j}\varphi^q_{\varepsilon,x_j}W^{p-q}_{\varepsilon,x_j}\le Ce^{-qd_j/\varepsilon}\int_{\Omega\setminus\tilde B_j}W^{p-q}_{\varepsilon,x_j}\\ \le&Ce^{-(q+(1-2\sigma)(p-q-\theta))d_j/\varepsilon}\int_{\Omega\setminus\tilde B_j}W^\theta_{\varepsilon,x_j}\le C\varepsilon^{N-m+1}e^{-(2+\sigma)d_j/\varepsilon},\end{aligned} \tag{2.20}$$

since $p>2$.

On the other hand, if $y\in\tilde B_j$, then

$$U\Big(\frac{|\tilde y-x_j|}{\varepsilon}\Big)\ge c'e^{-(1+\theta)|\tilde y-x_j|/\varepsilon}\ge c'e^{-(1+\theta)(1-2\sigma)d_j/\varepsilon}.$$

Thus

$$\frac{\varphi_{\varepsilon,x_j}(y)}{U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)}\le\frac{Ce^{-d_j/\varepsilon}}{e^{-(1+\theta)(1-2\sigma)d_j/\varepsilon}}\le Ce^{-\sigma d_j/\varepsilon}.$$

As a result,

$$\begin{aligned}&\int_{\tilde B_j}\varphi^q_{\varepsilon,x_j}W^{p-q}_{\varepsilon,x_j}=\int_{\tilde B_j}\varphi_{\varepsilon,x_j}W^{p-1}_{\varepsilon,x_j}\Big(\frac{\varphi_{\varepsilon,x_j}(y)}{U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)}\Big)^{q-1}\\ \le&Ce^{-\sigma(q-1)d_j/\varepsilon}\int_\Omega\varphi_{\varepsilon,x_j}W^{p-1}_{\varepsilon,x_j}\le Ce^{-\sigma(q-1)d_j/\varepsilon}\tau_{\varepsilon,z_j}.\end{aligned} \tag{2.21}$$

Combining (2.19), (2.20) and (2.21), we obtain (2.18). □

We are ready to prove the main result of this section.

Proposition 2.3. *Suppose that $x_j \in D_j$, $j = 1, \cdots, k$. Then*

$$I_\varepsilon\Big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big) = A\varepsilon^{N-m+1}\sum_{j=1}^k x_{j,1}^{m-1} + \frac{1}{2}\sum_{j=1}^k \tau_{\varepsilon,x_j} + \varepsilon^{N-m+1} O\Big(e^{-(2+\sigma)d_j/\varepsilon} + \varepsilon\Big),$$

where $A = \frac{p-2}{2p} c_{m-1}\int_{R^{N-m+1}} U^p$, c_{m-1} is the area of the unit sphere in R^m and $\sigma > 0$ is a small constant.

Proof. By (2.3) and Lemma 2.2, we have

$$\begin{aligned} &I_\varepsilon\Big(P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big) \\ =&\frac{p-2}{2p}\int_\Omega W_{\varepsilon,x_j}^p + \frac{1}{2}\tau_{\varepsilon,x_j} + O\Big(\int_\Omega W_{\varepsilon,x_j}^{p-2}\varphi_{\varepsilon,x_j}^2 + \varepsilon^{N-m+2}\Big) \\ =&A\varepsilon^{N-m+1} x_{j,1}^{m-1} + \frac{1}{2}\tau_{\varepsilon,x_j} + \varepsilon^{N-m+1} O\Big(e^{-(2+\sigma)d_j/\varepsilon} + \varepsilon\Big). \end{aligned} \tag{2.22}$$

On the other hand,

$$\begin{aligned} &I_\varepsilon\Big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big) \\ =&\sum_{j=1}^k I_\varepsilon\Big(P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big) + \sum_{i<j}\Big\langle P_{\varepsilon,\Omega} W_{\varepsilon,x_i}, P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big\rangle_\varepsilon \\ &+ \frac{1}{p}\int_\Omega\Big(\Big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big)_+^p - \sum_{j=1}^k (P_{\varepsilon,\Omega} W_{\varepsilon,x_j})_+^p\Big) \\ =&\sum_{j=1}^k I_\varepsilon\Big(P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big) \\ &+ \sum_{i<j}\Big(\int_\Omega \xi_j \bar{W}_{\varepsilon,x_i}^{p-1} P_{\varepsilon,\Omega} W_{\varepsilon,x_j} + \int_\Omega \tilde{f}_{\varepsilon,x_i}(y) P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big) \\ &+ \frac{1}{p}\int_\Omega\Big(\Big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big)_+^p - \sum_{j=1}^k (P_{\varepsilon,\Omega} W_{\varepsilon,x_j})_+^p\Big). \end{aligned}$$

By Lemma A.1, we have

$$|P_{\varepsilon,\Omega} W_{\varepsilon,x_i}||P_{\varepsilon,\Omega} W_{\varepsilon,x_j}| \le Ce^{-\tilde{\sigma}/\varepsilon},$$

for $i \neq j$. As a result,

$$\begin{aligned} &\Big|\int_\Omega\Big(\Big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big)_+^p - \sum_{j=1}^k (P_{\varepsilon,\Omega} W_{\varepsilon,x_j})_+^p\Big)\Big| \\ \le& C\sum_{j\neq i}\int_\Omega |P_{\varepsilon,\Omega} W_{\varepsilon,x_i}|^{p-1}|P_{\varepsilon,\Omega} W_{\varepsilon,x_j}| \\ \le& Ce^{-\sigma'/\varepsilon}\sum_{j\neq i}\int_\Omega |P_{\varepsilon,\Omega} W_{\varepsilon,x_i}|^{p-1-\sigma}|P_{\varepsilon,\Omega} W_{\varepsilon,x_j}|^{1-\sigma} \le Ce^{-\sigma'/\varepsilon}, \end{aligned} \tag{2.23}$$

for some $\sigma' > 0$ small.

Similarly, we can prove

$$\Big|\sum_{i<j}\Big(\int_\Omega W_{\varepsilon,x_i}^{p-1}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}+\int_\Omega \tilde f_{\varepsilon,x_i}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)\Big|\le Ce^{-\sigma/\varepsilon}.$$

So, we obtain

$$I_\varepsilon\Big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)=\sum_{j=1}^k I_\varepsilon\Big(P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)+O\big(e^{-\tilde\sigma/\varepsilon}\big), \tag{2.24}$$

where $\tilde\sigma>0$ is a constant.

The result follows from (2.24) and (2.22). □

3. Proof of the main result

First we define

$$D_\varepsilon=\{x=(x_1,\cdots,x_k): x_j\in D_j,\ j=1,\cdots,k,\ e^{-2d_j/\varepsilon}\le\varepsilon^{1-\tilde\theta}\}, \tag{3.1}$$

where $\tilde\theta>0$ is a fixed small constant, $d_j=d(x_j,\partial D)$.

We also define

$$J(x,\omega)=I\Big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}+\omega\Big),\quad \forall x\in D_\varepsilon,\ \omega\in H_s. \tag{3.2}$$

Let

$$E_{\varepsilon,x,k}=\Big\{\omega:\omega\in H_s,\big\langle\omega,\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{j,l}}\big\rangle_{D,\varepsilon}=0, j=1,\cdots,k, l=1,\cdots,N-m+1\Big\},$$

where

$$\big\langle u,v\big\rangle_{D,\varepsilon}=\int_D z_1^{m-1}\big(\varepsilon^2 DuDv+uv\big)\,dz.$$

First, we will prove that for each $x\in D_\varepsilon$, there is an $\omega_{\varepsilon,x}\in E_{\varepsilon,x,k}$ such that

$$\frac{\partial J(x,\omega_{\varepsilon,x})}{\partial\omega}=\sum_{j=1}^k\sum_{l=1}^{N-m+1}G_{jl}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial x_{jl}}, \tag{3.3}$$

for some constants $G_{jl}\in R,\ j=1,\cdots,k,\ l=1,\cdots,N-m+1$. Then, we will choose $x\in D_\varepsilon$ such that $G_{jl}=0$, for all $j=1,\cdots,k,\ l=1,\cdots,N-m+1$.

We expand $J(x,\omega)$ near $\omega=0$ as follows:

$$J(x,\omega)=J(x,0)+l_{\varepsilon,x}(\omega)+\frac12 Q_{\varepsilon,x}(\omega)+R_{\varepsilon,x}(\omega),$$

where

(3.4)

$$l_{\varepsilon,x}(\omega)=\sum_{j=1}^k\int_\Omega\big(\varepsilon^2 DP_{\varepsilon,\Omega}W_{\varepsilon,x_j}D\omega+P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\omega\big)-\int_\Omega\big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\omega,$$

$$Q_{\varepsilon,x}(\omega)=\int_\Omega\big(\varepsilon^2|D\omega|^2+\omega^2\big)-(p-1)\int_\Omega\big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-2}\omega^2, \tag{3.5}$$

and

$$R_{\varepsilon,x}(\omega) = -\int_\Omega F\Big(y, P_{\varepsilon,\Omega}W_{\varepsilon,x_j}+\omega\Big) + \int_\Omega F\Big(y, \sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) + \int_\Omega \big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\omega + \frac{1}{2}(p-1)\int_\Omega \big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-2}\omega^2. \tag{3.6}$$

Lemma 3.1. *There is a constant $C>0$ and $\sigma>0$ such that*

$$|l_{\varepsilon,x}(\omega)| \le C\varepsilon^{(N-m+1)/2}\big(\sum_{j=1}^k e^{-(1+\sigma)d_j/\varepsilon}+\varepsilon\big)\|\omega\|_\varepsilon.$$

Proof. We have

$$l_{\varepsilon,x}(\omega) = \sum_{j=1}^k \int_\Omega \Big(\xi_j \bar W_{\varepsilon,x_j}^{p-1} + \tilde f_{\varepsilon,x_j}(y)\Big)\omega - \int_\Omega \big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\omega. \tag{3.7}$$

We also have

$$\begin{aligned}
&\sum_{j=1}^k \int_\Omega \xi_j \bar W_{\varepsilon,x_j}^{p-1}\omega - \int_\Omega \big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\omega\\
=&\sum_{j=1}^k \int_\Omega \Big(\xi_j \bar W_{\varepsilon,x_j}^{p-1} - W_{\varepsilon,x_j}^{p-1}\Big)\omega + \sum_{j=1}^k \int_\Omega \Big(W_{\varepsilon,x_j}^{p-1} - \big(P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\Big)\omega\\
&+\int_\Omega \Big(\sum_{j=1}^k (P_{\varepsilon,\Omega}W_{\varepsilon,x_j})_+^{p-1} - \big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\Big)\omega\\
=&O\Big(\sum_{j=1}^k \int_\Omega W_{\varepsilon,x_j}^{(p-1)/2}\varphi_{\varepsilon,x_j}^{(p-1)/2}|\omega|\Big) + O\big(e^{-\delta'/\varepsilon}\big)\|\omega\|_\varepsilon\\
&+O\Big(\sum_{j\neq i}\int_\Omega |P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|^{(p-1)/2}|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}|^{(p-1)/2}|\omega|\Big).
\end{aligned} \tag{3.8}$$

From Lemmas A.1 and 2.2, we obtain

(3.9)

$$\begin{aligned}
&\int_\Omega W_{\varepsilon,x_j}^{(p-1)/2}\varphi_{\varepsilon,x_j}^{(p-1)/2}|\omega|\\
=&\int_{B_j} W_{\varepsilon,x_j}^{(p-1)/2}\varphi_{\varepsilon,x_j}^{(p-1)/2}|\omega| + \int_{\Omega\setminus B_j} W_{\varepsilon,x_j}^{(p-1)/2}\varphi_{\varepsilon,x_j}^{(p-1)/2}|\omega|\\
\le&\Big(\int_\Omega W_{\varepsilon,x_j}^{p/2}\varphi_{\varepsilon,x_j}^{p/2}\Big)^{(p-1)/p}\Big(\int_{B_j}|\omega|^p\Big)^{1/p} + Ce^{-\sigma'/\varepsilon}\int_{\Omega\setminus B_j} W_{\varepsilon,x_j}^{(p-1-\sigma)/2}|\omega|\\
\le&\big(C\varepsilon^{N-m+1}e^{-(2+\sigma)d_j/\varepsilon}\big)^{(p-1)/p}\Big(C\varepsilon^{N-m+1-p(N-m+1)/2}\big(\int_{B_j}\big(\varepsilon^2|D\omega|^2+\omega^2\big)\big)^{p/2}\Big)^{1/p}\\
&+Ce^{-\sigma'/\varepsilon}\|\omega\|_\varepsilon\\
\le&C\varepsilon^{(N-m+1)/2}e^{-(2+\sigma)(1-1/p)d_j/\varepsilon}\|\omega\|_\varepsilon.
\end{aligned}$$

Similar to the proof of (2.23), we deduce that for $i \neq j$,

$$\begin{aligned}&\int_\Omega |P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|^{(p-1)/2}|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}|^{(p-1)/2}|\omega|\\ \le&\Big(\int_\Omega |P_{\varepsilon,\Omega}W_{\varepsilon,x_i}|^{p-1}|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}|^{p-1}\Big)^{1/2}\|\omega\|_\varepsilon\\ \le& Ce^{-\sigma'/\varepsilon}\|\omega\|_\varepsilon.\end{aligned}\tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we are led to

$$\begin{aligned}&\sum_{j=1}^k\int_\Omega W_{\varepsilon,x_j}^{p-1}\omega-\int_\Omega\big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\omega\\ =&\varepsilon^{(N-m+1)/2}O\Big(\sum_{j=1}^k e^{-(1+\sigma)d_j/\varepsilon}+e^{-\tilde\sigma/\varepsilon}\Big)\|\omega\|_\varepsilon.\end{aligned}\tag{3.11}$$

On the other hand,

$$\begin{aligned}&\Big|\int_\Omega \tilde f_{\varepsilon,x_j}(y)\omega\Big|\le\big(\int_\Omega|\tilde f_{\varepsilon,x_j}(y)|^2\big)^{1/2}\|\omega\|_\varepsilon\\ \le& C\varepsilon^{1+(N-m+1)/2}\|\omega\|_\varepsilon.\end{aligned}\tag{3.12}$$

Combining (3.7), (3.11) and (3.12), we obtain the result. □

Let $Q_{\varepsilon,x}$ be the bounded linear map $E_{\varepsilon,x,k}$ to $E_{\varepsilon,x,k}$ such that

$$\begin{aligned}\big\langle Q_{\varepsilon,x}\omega_1,\omega_2\big\rangle_\varepsilon=&\int_\Omega\big(\varepsilon^2D\omega_1D\omega_2+\omega_1\omega_2\big)\\ &-(p-1)\int_\Omega\big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-2}\omega_1\omega_2,\quad \omega_1,\ \omega_2\in E_{\varepsilon,x,k}.\end{aligned}$$

Then we have

Lemma 3.2. *There are constants $\varepsilon_0>0$ and $\rho>0$ such that, for each $\varepsilon\in(0,\varepsilon_0]$ and $x\in D_\varepsilon$,*

$$\|Q_{\varepsilon,x}\omega\|_\varepsilon\ge\rho\|\omega\|_\varepsilon,\quad \omega\in E_{\varepsilon,x,k}.$$

Proof. We argue by contradiction. Suppose that there are $\varepsilon_n\to0$, $x_n\in D_{\varepsilon_n}$ and $\omega_n\in E_{\varepsilon_n,x_n,k}$ such that

$$\|Q_{\varepsilon_n,x_n}\omega_n\|_{\varepsilon_n}=o(1)\|\omega_n\|_{\varepsilon_n},\tag{3.13}$$

where $o(1)\to0$ as $n\to+\infty$.

From (3.13), we see

$$\begin{aligned}&\int_\Omega\big(\varepsilon_n^2D\omega_nD\xi+\omega_n\xi\big)-(p-1)\int_\Omega\big(\sum_{j=1}^k P_{\varepsilon_n,\Omega}W_{\varepsilon_n,x_n,j}\big)_+^{p-2}\omega_n\xi\\ =&o(1)\|\omega_n\|_{\varepsilon_n}\|\xi\|_{\varepsilon_n},\quad \xi\in E_{\varepsilon_n,x_n,k}.\end{aligned}\tag{3.14}$$

In (3.14), we assume that

$$\|\omega_n\|_{\varepsilon_n}=\varepsilon_n^{(N-m+1)/2}.\tag{3.15}$$

For each fixed i, let

$$\tilde\omega_{n,i}(\tilde y)=\omega_n(\varepsilon_n\tilde y+x_{n,i}).$$

Since $x_{n,i,1} \geq c' > 0$, from (3.15), we obtain

$$\int_{B_R(0)} \big(|D\tilde{\omega}_{n,i}|^2 + |\tilde{\omega}_{n,i}|^2\big) \leq C, \tag{3.16}$$

for any $R > 0$ large, where $C > 0$ is a constant independent of R, $B_R(0)$ is the ball in R^{N-m+1} with radius R, centred at the origin. So, we may assume that there is an $\omega \in H^1(R^{N-m+1})$ such that, for any $R > 0$,

$$\tilde{\omega}_{n,i} \rightharpoonup \omega, \quad \text{weakly in } H^1(B_R(0)), \tag{3.17}$$

and

$$\tilde{\omega}_{n,i} \to \omega, \quad \text{strongly in } L^2(B_R(0)). \tag{3.18}$$

Now, we prove $\omega = 0$.

From (3.14), we see that $\tilde{\omega}_{n,i}$ satisfies

$$\begin{aligned} &\int_{D_n} |\varepsilon_n z_1 + x_{n,i,1}|^{m-1}\big(D\tilde{\omega}_{n,i} D\xi + \tilde{\omega}_{n,i}\xi\big) \\ &- (p-1)\int_{D_n} |\varepsilon_n z_1 + x_{n,i,1}|^{m-1}\big(\sum_{j=1}^{k} V_{n,j}\big)_+^{p-2}\tilde{\omega}_{n,i}\xi = o(1)\|\xi\|_{\varepsilon_n}, \quad \xi \in \tilde{E}_n, \end{aligned} \tag{3.19}$$

where

$$D_n = \{z : z \in R^{N-m+1}, \varepsilon_n z + x_{n,i} \in D\},$$
$$V_{n,j}(z) = (P_{\varepsilon_n,\Omega} W_{\varepsilon_n, x_{n,j}})(\varepsilon_n z + x_{n,i}),$$

and

$$\tilde{E}_n = \big\{\xi : \xi\big(\frac{\tilde{y} - x_{n,i}}{\varepsilon_n}\big) \in H_s, \int_{D_n} |\varepsilon_n z_1 + x_{n,i,1}|^{m-1}\big(D\xi D\frac{\partial V_{n,j}}{\partial x_{j,h}} + \xi\frac{\partial V_{n,j}}{\partial x_{j,h}}\big)\, dz = 0\big\},$$

for $j = 1, \cdots, k$, $h = 1, \cdots, N-m+1$.

For any $\xi \in C_0^\infty(R^{N-m+1})$, we can choose $a_{n,j,h} \in R^1$, such that

$$\xi_n = \xi - \sum_{j=1}^{k}\sum_{h=1}^{N-m+1} a_{n,j,h}\frac{\partial V_{n,j}}{\partial x_{j,h}} \in \tilde{E}_n.$$

For $j \neq i$, we have

$$\int_{D_n} |\varepsilon_n z_1 + x_{n,i,1}|^{m-1}\big(D\xi D\frac{\partial V_{n,j}}{\partial x_{j,h}} + \xi\frac{\partial V_{n,j}}{\partial x_{j,h}}\big)\, dz = o(1),$$

for any $\xi \in C_0^\infty(R^{N-m+1})$, since the support of $\frac{\partial V_{n,j}}{\partial x_{j,h}}$ tends to infinity as $n \to +\infty$. On the other hand, we have

$$\int_{D_n} |\varepsilon_n z_1 + x_{n,i,1}|^{m-1}\big(D\xi D\frac{\partial V_{n,j}}{\partial x_{j,h}} + \xi\frac{\partial V_{n,j}}{\partial x_{j,h}}\big)\, dz = O(1).$$

So, it is easy to check that $a_{n,j,h} \to 0$ as $n \to +\infty$ for $j \neq i$, while $a_{n,i,h} \to a_{i,h}$ (up to a subsequence).

Putting ξ_n into (3.19) and letting $n \to +\infty$, noting that $x_{n,i,1} \geq c' > 0$, we find

$$\begin{aligned} &\int_{R^{N-m+1}} \big(D\omega D\xi + \omega\xi\big) - (p-1)\int_{R^{N-m+1}} U^{p-2}\omega\xi \\ &+ \sum_{h=1}^{N-m+1} a_{i,h}\Big(\int_{R^{N-m+1}} \Big(D\omega D\frac{\partial U}{\partial x_h} + \omega\frac{\partial U}{\partial x_h}\Big) - (p-1)\int_{R^{N-m+1}} U^{p-2}\omega\frac{\partial U}{\partial x_h}\Big) = 0. \end{aligned}$$

But

$$\int_{R^{N-m+1}}\Big(D\omega D\frac{\partial U}{\partial x_h}+\omega\frac{\partial U}{\partial x_h}\Big)-(p-1)\int_{R^{N-m+1}}U^{p-2}\omega\frac{\partial U}{\partial x_h}=0.$$

So, we obtain

$$\int_{R^{N-m+1}}\big(D\omega D\xi+\omega\xi\big)-(p-1)\int_{R^{N-m+1}}U^{p-2}\omega\xi=0,\quad \xi\in C_0^{\infty}(R^{N-m+1}). \tag{3.20}$$

Since U is nondegenerate, we see from (3.20) that

$$\omega=\sum_{h=1}^{N-m+1}b_h\frac{\partial U}{\partial z_h}, \tag{3.21}$$

for some $b_h\in R^1$.

On the other hand, from $\tilde\omega_{n,j}\in\tilde E_n$, we can deduce

$$\int_{R^{N-m+1}}\big(D\omega D\frac{\partial U}{\partial z_h}+\omega\frac{\partial U}{\partial z_h}\big)=0, \tag{3.22}$$

for $h=1,\cdots,N-m+1$.

Combining (3.21) and (3.22), we obtain $\omega=0$. As a result, we have

$$\int_{B_{i,R}}\omega_n^2=o(\varepsilon_n^{N-m+1}),\quad i=1,\cdots,k, \tag{3.23}$$

where $B_{i,R}=\{y\in\Omega:(|y'|,y'')\in B_{\varepsilon_nR}(x_{n,i})\}$. For $y\in\Omega\setminus\bigcup_{i=1}^k B_{i,R}$, by Lemma A.1, we have

$$\sum_{j=1}^k|P_{\varepsilon_n,\Omega}W_{\varepsilon_n,x_{x,j}}(y)|=o_R(1),$$

where $o_R(1)\to0$ as $R\to+\infty$. From (3.14) and (3.23), we find

$$\begin{aligned}o(\varepsilon_n^{N-m+1})=&\|\omega_n\|_{\varepsilon_n}^2-o(\varepsilon_n^{N-m+1})-o_R(1)\varepsilon_n^{N-m+1}\\=&\varepsilon_n^{N-m+1}-o(\varepsilon_n^{N-m+1})-o_R(1)\varepsilon_n^{N-m+1}.\end{aligned}$$

This is a contradiction. □

Let

$$S_\varepsilon=\big\{\omega:\omega\in H_s(\Omega),\ |\omega|\le\sum_{j=1}^k e^{-\alpha|\tilde y-x_j|/\varepsilon}\big\},$$

where $\alpha>0$ is a small constant.

Lemma 3.3. *For any $\omega\in S_\varepsilon$ with $\|\omega\|_\varepsilon\le\varepsilon^{(N-m+1)/2}$, we have*

$$R_{\varepsilon,x}(\omega)=\varepsilon^{N-m+1}O\big(\varepsilon^{-\tilde p(N-m+1)/2}\|\omega\|_\varepsilon^{\tilde p}\big), \tag{3.24}$$

$$\big\langle R'_{\varepsilon,x}(\omega),\xi\big\rangle_\varepsilon=\varepsilon^{(N-m+1)/2}O\big(\varepsilon^{-(\tilde p-1)(N-m+1)/2}\|\omega\|_\varepsilon^{\tilde p-1}\big)\|\xi\|_\varepsilon \tag{3.25}$$

and

$$R''_{\varepsilon,x}(\omega)(\xi_1,\xi_2)=O\big(\varepsilon^{-(\tilde p-2)(N-m+1)/2}\|\omega\|_\varepsilon^{\tilde p-2}\big)\|\xi_1\|_\varepsilon\|\xi_2\|_\varepsilon, \tag{3.26}$$

where $\tilde p$ is a constant with $\tilde p>2$.

Proof. Let $\tilde p \in (2, 2N/(N-2))$ be a constant with $\tilde p < \min(3,p)$. For any $\omega \in S_\varepsilon$, we have $|\omega(y)| \le \frac{1}{2}$ if $y \in \Omega \setminus \bigcup_{j=1}^k B_j$. Since $\left|\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\right| \le \frac{1}{2}$ for any $y \in \Omega \setminus \bigcup_{j=1}^k B_j$, we deduce that

$$\begin{aligned}
&F\Big(y, \sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j} + \omega\Big) - F\Big(y, \sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big)\\
&- \big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\big)_+^{p-1}\omega - \frac{1}{2}(p-1)\big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\big)_+^{p-2}\omega^2\\
=&\frac{1}{p}\Big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j} + \omega\Big)_+^p - \frac{1}{p}\Big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\Big)_+^p\\
&- \big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\big)_+^{p-1}\omega - \frac{1}{2}(p-1)\big(\sum_{j=1}^k P_{\varepsilon,\Omega} W_{\varepsilon,x_j}\big)_+^{p-2}\omega^2.
\end{aligned}$$

By the definition of $R_{\varepsilon,x}(\omega)$, we see that for any $\omega \in S_\varepsilon$,

(3.27)

$$\begin{aligned}
&|R_{\varepsilon,x}(\omega)| \le C\int_\Omega |\omega|^{\min(3,p)} \le C\int_{\bigcup_{j=1}^k B_j} |\omega|^{\min(3,p)} + C\int_{\Omega\setminus\bigcup_{j=1}^k B_j} |\omega|^{\min(3,p)}\\
\le& C\int_{\bigcup_{j=1}^k B_j} |\omega|^{\min(3,p)} + e^{-\bar\theta\alpha/\varepsilon}\int_\Omega |\omega|^{\tilde p}\\
=& C\int_{\bigcup_{j=1}^k B_j} |\omega|^{\min(3,p)} + Ce^{-\bar\theta/\varepsilon}\varepsilon^{-\tilde p}\|\omega\|_\varepsilon^{\tilde p},
\end{aligned}$$

where $\bar\theta > 0$ is a small constant.

For any j, let $\tilde\omega(y) = \omega(\varepsilon y + x_j)$. Then

$$\begin{aligned}
&\int_{B_j} |\omega|^{\min(3,p)} = c_0\int_{D\cap B_\delta(\bar x_j)} |z_1|^{m-1}|\omega|^{\min(3,p)}\,dz\\
\le& C\int_{D\cap B_\delta(\bar x_j)} |\omega|^{\min(3,p)}\,dz = C\varepsilon^{N-m+1}\int_{\tilde B_{j,\varepsilon}} |\tilde\omega|^{\min(3,p)}\\
\le& C\varepsilon^{N-m+1}\Big(\int_{\tilde B_{j,\varepsilon}} \big(|D\tilde\omega|^2 + |\tilde\omega|^2\big)\Big)^{\min(p,3)/2}\\
=& C\varepsilon^{N-m+1}\big(\varepsilon^{-(N-m+1)}\int_{D\cap B_\delta(\bar x_j)} \big(\varepsilon^2|D\omega|^2 + \omega^2\big)\big)^{\min(3,p)/2}\\
\le& C\varepsilon^{N-m+1}\big(\varepsilon^{-\min(3,p)(N-m+1)/2}\|\omega\|_\varepsilon^{\min(3,p)}\big),
\end{aligned} \tag{3.28}$$

where $B_{j,\varepsilon} = \{z : \varepsilon z + x_j \in D \cap B_\delta(\bar x_j)\}$. In the last relation, we use $|y'| \ge c > 0$ for $y \in B_j$.

Combining (3.27) and (3.28), we obtain (3.24).

Now, we prove (3.25). We have

$$\begin{aligned}
&\Big|\big\langle R'_{\varepsilon,x}(\omega), \xi\big\rangle_\varepsilon\Big| \le C\int_\Omega |\omega|^{\min(p-1,2)}|\xi|\\
=& C\int_{\bigcup_{j=1}^k B_j} |\omega|^{\min(p-1,2)}|\xi| + C\int_{\Omega\setminus\bigcup_{j=1}^k B_j} |\omega|^{\min(p-1,2)}|\xi|,
\end{aligned} \tag{3.29}$$

from which we can prove (3.25) by using the same techniques as in the proof of (3.24).

We can prove (3.26) in a similar way. □

Let us point out that (3.24), (3.25) and (3.26) are not true in the whole space H_s. So, we need to carry out the reduction procedure in a closed subset of $E_{\varepsilon,x,k}$.

Proposition 3.4. *There is an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$, there exists a C^1-map $\omega_{\varepsilon,x} : D_\varepsilon \to H_s$ such that $\omega_{\varepsilon,x} \in E_{\varepsilon,x,k}$, (3.3) holds for some constants G_{jl}. Moreover, we have*

$$\|\omega_{\varepsilon,x}\|_\varepsilon \le C\varepsilon^{\sigma+(N-m+2)/2}, \tag{3.30}$$

where $\sigma > 0$ is a constant.

Proof. By Lemma 3.1, we know that there is a $l_{\varepsilon,x} \in E_{\varepsilon,x,k}$, such that

$$\langle l_{\varepsilon,x}, \omega\rangle_\varepsilon = l_{\varepsilon,x}(\omega), \quad \forall\, \omega \in E_{\varepsilon,x,k}.$$

Thus, solving (3.30) is equivalent to solving

$$l_{\varepsilon,x} + Q_{\varepsilon,x}\omega + R'_{\varepsilon,x}(\omega) = 0, \quad \text{in } E_{\varepsilon,x,k}. \tag{3.31}$$

By Lemma 3.2, $Q_{\varepsilon,x}$ is invertible. So we can write (3.31) as

$$\omega = G_{\varepsilon,x}\omega =: -Q^{-1}_{\varepsilon,x} l_{\varepsilon,x} - Q^{-1}_{\varepsilon,x} R'_{\varepsilon,x}(\omega). \tag{3.32}$$

Let

$$\tilde S_\varepsilon = \Big\{\omega : \omega \in H_s(\Omega),\ |\omega| \le \varepsilon^\alpha \sum_{j=1}^k e^{-\alpha|\tilde y - x_j|/\varepsilon}, \|\omega\|_\varepsilon \le \varepsilon^{(N-m+2)/2}\Big\},$$

where $\alpha > 0$ is a small constant.

Now, we prove that $G_{\varepsilon,x}$ is a contraction map from $\tilde S_s$ to $\tilde S_\varepsilon$.

By (3.26), we see that for any $\omega_1, \omega_2 \in \tilde S_\varepsilon$,

$$\|G_{\varepsilon,x}\omega_1 - G_{\varepsilon,x}\omega_2\|_\varepsilon \le C\|R'_{\varepsilon,x}(\omega_1) - R'_{\varepsilon,x}(\omega_2)\|_\varepsilon \le C\varepsilon^{(\tilde p-2)/2}\|\omega_1 - \omega_2\|_\varepsilon. \tag{3.33}$$

Thus, $G_{\varepsilon,x}$ is a contraction map.

For any $\omega \in \tilde S_\varepsilon$, we have

$$\begin{aligned} \|G_{\varepsilon,x}\omega\|_\varepsilon &\le C\|l_{\varepsilon,x}\|_\varepsilon + C\|R'_{\varepsilon,x}(\omega)\|_\varepsilon \\ &\le C\|l_{\varepsilon,x}\|_\varepsilon + C\varepsilon^{(N-m+1)/2}\varepsilon^{(\tilde p-1)/2}. \end{aligned} \tag{3.34}$$

For any $x \in D_\varepsilon$, by Lemma 3.1, we have

$$\|l_{\varepsilon,x}\|_\varepsilon \le C\varepsilon^{N-m+1)/2}\Big(\sum_{j=1}^k e^{-(1+\sigma)d_j/\varepsilon} + \varepsilon\Big) \le C\varepsilon^{(N-m+1)/2}\big(\varepsilon^{(1-\tilde\theta)(1+\sigma)} + \varepsilon\big),$$

which, together with (3.34), gives

$$\begin{aligned} &\|G_{\varepsilon,x}\omega\|_\varepsilon \\ \le &C\varepsilon^{(N-m+1)/2}\big(\varepsilon^{(1-\tilde\theta)(1+\sigma)} + \varepsilon + \varepsilon^{(\tilde p-1)/2}\big) \\ \le &C\varepsilon^{\sigma_1+(N-m+2)/2} \le \varepsilon^{(N-m+2)/2}, \end{aligned} \tag{3.35}$$

since $\tilde p > 2$.

To finish the proof of $G_{\varepsilon,x}\omega \in \tilde{S}_\varepsilon$, we need to prove

$$|G_{\varepsilon,x}\omega| \le \varepsilon^\alpha \sum_{j=1}^k e^{-\alpha|\tilde{y}-x_j|/\varepsilon}.$$

Let $\omega_1 = G_{\varepsilon,x}\omega$. Then, we have

$$Q_{\varepsilon,x}\omega_1 = -l_{\varepsilon,x} - R'(\omega), \quad \text{in } E_{\varepsilon,x,k},$$

which is equivalent to

$$\begin{aligned}&\big\langle Q_{\varepsilon,x}\omega_1, \xi\big\rangle_\varepsilon + \big\langle l_{\varepsilon,x}, \xi\big\rangle_\varepsilon + \big\langle R'(\omega), \xi\big\rangle_\varepsilon \\ =&\sum_{j=1}^k \sum_{h=1}^{N-m+1} G_{jh}\Big\langle \frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{j,h}}, \xi\Big\rangle_\varepsilon,\end{aligned} \tag{3.36}$$

for some $G_{jh} \in R^1$.

We claim that there is a $\sigma > 0$ such that

$$|G_{jh}| \le C\varepsilon^{\sigma+3/2}, \quad j = 1, \cdots, k,\ h = 1, \cdots, N-m+1. \tag{3.37}$$

In fact, letting $\xi = \frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_i}}{\partial x_{i,\bar{h}}}$ in (3.36), using Lemma A.3, we can solve the linear system to obtain

$$\begin{aligned}|G_{jh}| \le& C\varepsilon^{1-(N-m+1)/2}\big(\|\omega_1\|_\varepsilon + \|l_{\varepsilon,x}\|_\varepsilon + \|R'(\omega)\|_\varepsilon\big)\\ \le& C\varepsilon^{1-(N-m+1)/2}\varepsilon^{\frac{1}{2}+\sigma+(N-m+1)/2} \le C\varepsilon^{\sigma+3/2}.\end{aligned}$$

Rewrite (3.36) as

(3.38)

$$\begin{aligned}&-\varepsilon^2\Delta\omega_1 + \omega_1 - (p-1)\big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-2}\omega_1\\ =&-\sum_{j=1}^k\Big(\xi_j\bar{W}_{\varepsilon,x_j}^{p-1} + \tilde{f}_{\varepsilon,x_j}(y)\Big) + \big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\\ &+\Big(f\Big(y, \sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j} + \omega\Big) - f\Big(y, \sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) - f'\Big(y, \sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)\omega\Big)\\ &+\sum_{j=1}^k\sum_{h=1}^{N-m+1} G_{jh}\frac{\partial g_{\varepsilon,x_j}(y)}{\partial x_{j,h}}\\ =:&G_{\varepsilon,x}(y)\end{aligned}$$

where $f(y,t)$ is the function defined in (1.7), and

$$g_{\varepsilon,x_j}(y) = W_{\varepsilon,x_j}^{p-1} + \tilde{f}_{\varepsilon,x_j}(y).$$

Since $\omega \in \tilde{S}_\varepsilon$, we see $|\omega| \le \frac{1}{2}$ in $\Omega \setminus \bigcup_{j=1}^k B_j$. Thus

(3.39)

$$\Big|f\Big(y, \sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j} + \omega\Big) - f\Big(y, \sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big) - f'\Big(y, \sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)\Big| \le C|\omega|^{\tilde{p}-1},$$

where $\tilde{p} > 2$ is a constant.

On the other hand, we have

$$
\begin{aligned}
&-\sum_{j=1}^{k}\xi_j\bar W_{\varepsilon,x_j}^{p-1}+\big(\sum_{j=1}^{k}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\\
=&-\sum_{j=1}^{k}\Big(W_{\varepsilon,x_j}^{p-1}-\big(P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\Big)+O\Big(e^{-\delta'/\varepsilon}\sum_{j=1}^{k}U^{p-2}\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\Big)\\
&+\big(\sum_{j=1}^{k}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}-\sum_{j=1}^{k}\big(P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)_+^{p-1}\\
=&O\Big(\sum_{j=1}^{k}e^{-d_j/\varepsilon}U^{p-2}\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\Big).
\end{aligned}\tag{3.40}
$$

Direct calculations show that

$$
\Big|\frac{\partial g_{\varepsilon,x_j}(y)}{\partial x_{j,h}}\Big|\le C\varepsilon^{-1}U^{p-1}\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)+CU\big(\frac{|\tilde y-x_j|}{\varepsilon}\big).\tag{3.41}
$$

Combining (3.39), (3.40) and (3.41), we find that

$$
\begin{aligned}
|G_{\varepsilon,x}(y)|\le&C\sum_{j=1}^{k}\Big(e^{-d_j/\varepsilon}U^{p-2}\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)+|\omega|^{\tilde p-1}+\varepsilon^{\sigma+1/2}U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\Big)\\
\le&C\sum_{j=1}^{k}\varepsilon^{(\tilde p-1)\alpha}e^{-(\tilde p-1)\alpha|\tilde y-x_j|/\varepsilon},
\end{aligned}\tag{3.42}
$$

if $\alpha>0$ is small enough.

Since $|y'|\ge c>0$ if $y\in B_j$, we can prove that

$$
|\omega_1|\le C\varepsilon^{(\tilde p-1)\alpha},\quad \text{in }\bigcup_{j=1}^{k}B_j.\tag{3.43}
$$

In fact, let

$$
\tilde g_{\varepsilon,x}(y,t)=-t+(p-1)\big(\sum_{j=1}^{k}P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\big)^{p-2}t.
$$

Then $|\tilde g_{\varepsilon,x}(y,t)|\le C|t|$ and

$$
-\varepsilon^2\Delta\omega_1=G_{\varepsilon,x}(y)+\tilde g_{\varepsilon,x}(y,\omega_1),\quad \text{in } B_j.
$$

Since $\omega_1\in H_s$, we can rewrite the above equation as

$$
-\varepsilon^2\sum_{j=1}^{N-m+1}\frac{\partial}{\partial z_j}\big(z_1^{m-1}\frac{\partial\omega_1}{\partial z_j}\big)=z_1^{m-1}\big(G_{\varepsilon,x}(z)+\tilde g_{\varepsilon,x}(z,\omega_1)\big),\quad \text{in } \tilde B_j,
$$

where $\tilde B_j=D\cap B_\delta(\bar x_j)$.

Let $\tilde{\omega}_1(z) = \omega(\varepsilon z + x_j)$. Then

$$\begin{aligned}&-\sum_{j=1}^{N-m+1} \frac{\partial}{\partial z_j}\big((\varepsilon z_1 + x_{j,1})^{m-1}\frac{\partial \tilde{\omega}_1}{\partial z_j}\big)\\ =&(\varepsilon z_1 + x_{j,1})^{m-1}\big(G_{\varepsilon,x}(\varepsilon z + x_{j,1}) + \tilde{g}_{\varepsilon,x}(\varepsilon z + x_{j,1}, \tilde{\omega}_1)\big), \quad \text{in } B_{j,\varepsilon},\end{aligned}$$

where $B_{j,\varepsilon} = \{z : \varepsilon z + x_j \in \tilde{B}_j\}$.

For any $x_0 \in B_{j,\varepsilon}$, by (3.35), we have

$$\int_{B_1(x_0)} |\tilde{\omega}_1|^2 = \varepsilon^{-(N-m+1)} \int_{B_j} |\omega_1|^2 \le C\varepsilon^{-(N-m+1)}\|\omega_1\|_\varepsilon^2 \le C\varepsilon^{1+2\sigma}.$$

So, by (3.42), we have

$$|\tilde{\omega}_1|_{L^\infty(B_1(x_0))} \le C|\tilde{\omega}_1|_{L^2(B_2(x_0))} + C|\tilde{G}_{\varepsilon,x}|_{L^\infty(B_2(x_0))} \le C\varepsilon^{(\tilde{p}-1)\alpha}.$$

Thus, (3.43) follows.

Let

$$a_\varepsilon(y) = \Big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\Big)^{p-2}\eta,$$

where η is a C^1 function, such that $\eta = 0$ if $y \in \bigcup_{j=1}^k B_j$. It is easy to see that $a_\varepsilon(y) \to 0$ uniformly in Ω as $\varepsilon \to 0$. From (3.43), we have

$$-\varepsilon^2\Delta\omega_1 + (1-(p-1)a_\varepsilon)\omega_1 = G_{\varepsilon,x}(y) + O(\varepsilon^{(\tilde{p}-1)\alpha})\Big(\sum_{j=1}^k W_{\varepsilon,x_j}\Big)^{p-2}. \tag{3.44}$$

Similar to the proof of Lemma A.1, we can prove that

$$v = \varepsilon^\alpha \sum_{j=1}^k e^{-\alpha|\tilde{y}-x_j|/\varepsilon}$$

satisfies

$$\begin{aligned}&-\varepsilon^2\Delta v + (1-(p-1)a_\varepsilon)v\\ =&\big(1-(p-2)a_\varepsilon(y) - \alpha^2\big)\varepsilon^\alpha \sum_{j=1}^k e^{-\alpha|\tilde{y}-x_j|/\varepsilon}\\ \ge&\frac{1}{2}\varepsilon^\alpha \sum_{j=1}^k e^{-\alpha|\tilde{y}-x_j|/\varepsilon} \ge |G_{\varepsilon,x}(y)| + O(\varepsilon^{(\tilde{p}-1)\alpha})\Big(\sum_{j=1}^k W_{\varepsilon,x_j}\Big)^{p-2}.\end{aligned}$$

Since $\omega_1 \in H_s$, by the maximum principle, we obtain

$$|\omega_1| \le v = \varepsilon^\alpha \sum_{j=1}^k e^{-\alpha|\tilde{y}-x_j|/\varepsilon}.$$

Thus, $\omega_1 \in \tilde{S}_\varepsilon$.

We have proved that $G_{\varepsilon,x}$ is a contraction map from $\tilde{S}_\varepsilon$ into itself. By the contraction mapping theorem, we know that there is a $\omega_{\varepsilon,x} \in \tilde{S}_{\varepsilon,x}$, such that

$$\omega_{\varepsilon,x} = G_{\varepsilon,x}\omega_{\varepsilon,x}.$$

Moreover, by (3.35),

$$\|\omega_{\varepsilon,x}\|_\varepsilon \le C\varepsilon^{\sigma+(N-m+2)/2}.$$

□

We will choose $x_\varepsilon \in D_\varepsilon$, such that the corresponding G_{jl} in (3.3) are all zero.

Proof of Theorem 1.1. Let

$$K(x) = J(x, \omega_{\varepsilon,x}), \quad x \in D_\varepsilon.$$

Consider the following problem:

$$\min_{x\in D_\varepsilon} K(x). \tag{3.45}$$

Let $x_\varepsilon \in D_\varepsilon$ be a minimum point of (3.45). We will prove that x_ε is an interior point of D_ε. Thus, x_ε is a critical point of $K(x)$.

It follows from Propositions 3.4 and 2.3 that for any $x \in D_\varepsilon$,

$$\begin{aligned} K(x) &= J(x,0) + O(\varepsilon^{N-m+2+\sigma}) \\ &= A\varepsilon^{N-m+1}\sum_{j=1}^k x_{j,1}^{m-1} + \frac{1}{2}\sum_{j=1}^k \tau_{\varepsilon,x_j} + \varepsilon^{N-m+1}O\big(\sum_{j=1}^k e^{-(2+\sigma)d_j/\varepsilon} + \varepsilon\big), \end{aligned} \tag{3.46}$$

Let

$$\bar{x}_{\varepsilon,j} = (\bar{x}_{\varepsilon,j,1}, x''_{\varepsilon,j}), \qquad \bar{x}_{\varepsilon,j,1} = \bar{x}_{j,1} + L\varepsilon|\ln\varepsilon|\psi_j(\bar{x}''_j),$$
$$\bar{x}''_{\varepsilon,j} = \bar{x}''_j, \qquad \bar{x}_\varepsilon = (\bar{x}_{\varepsilon,1}, \cdots, \bar{x}_{\varepsilon,k}),$$

where $L > 0$ is a large constant. Then $\bar{x}_\varepsilon \in D_\varepsilon$. By Lemma 2.1, we see that

$$\tau_{\varepsilon,\bar{x}_j} = O(\varepsilon^{N-m+3}),$$

if $L > 0$ is large. So, from (3.46), we obtain

$$K(\bar{x}_\varepsilon) = A\varepsilon^{N-m+1}\sum_{j=1}^k \bar{x}_{j,1}^{m-1} + \varepsilon^{N-m+1}O\big(\varepsilon|\ln\varepsilon|\big). \tag{3.47}$$

Suppose that $x_\varepsilon \in \partial D_\varepsilon$. If $e^{-2d(x_{\varepsilon,j},\partial D)/\varepsilon} = \varepsilon^{1-\tilde{\theta}}$ for some j, then, by (3.46),

$$K(x_\varepsilon) \ge A\varepsilon^{N-m+1}\sum_{j=1}^k \bar{x}_{j,1}^{m-1} + \frac{1}{2}\varepsilon^{N-m+1+(1+\theta)(1-\tilde{\theta})} + O(\varepsilon^{N-m+2}) > K(\bar{x}_\varepsilon),$$

since $(1-\tilde{\theta})(1+\theta) < 1$ if $\theta > 0$ is small enough. This is a contradiction.

Suppose that $x_\varepsilon \in \partial D_\varepsilon \setminus \{x : e^{-2d(x_j,\partial D)/\varepsilon} = \varepsilon^{1-\theta}, \text{for some } j\}$. Then $x_{\varepsilon,j,1} \ge \bar{x}_{j,1} + \beta$ for some j, where $\beta > 0$ is a small constant. So, by (3.46),

$$K(x_\varepsilon) \ge \varepsilon^{N-m+1}\sum_{j=1}^k \bar{x}_{j,1}^{m-1} + c'\varepsilon^{N-m+1} + \varepsilon^{N-m+1}O\big(\varepsilon\big) > K(\bar{x}_\varepsilon),$$

where $c' > 0$ is a small constant. This is a contradiction. So x_ε is an interior point of D_ε. As a result,

$$DK(x_\varepsilon) = 0.$$

We claim that for this x_ε, the corresponding $G_{jh} = 0$, $j = 1, \cdots, k$, $h = 1, \ldots, N-m+1$. So $\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}} + \omega_{\varepsilon,x_\varepsilon}$ is a solution of (1.8). Since $f(y,t) = 0$ if $t \le 0$, we see that $\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}} + \omega_{\varepsilon,x_\varepsilon}$ is positive. But $\omega_{\varepsilon,x_\varepsilon} \in S_\varepsilon$. Thus $|\omega_{\varepsilon,x_\varepsilon}| \le \frac{1}{2}$ in $\Omega \setminus \bigcup_{j=1}^k B_j$, which gives

$$f\big(y, \sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}} + \omega_{\varepsilon,x_\varepsilon}\big) = \big(\sum_{j=1}^k P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}} + \omega_{\varepsilon,x_\varepsilon}\big)_+^{p-1}.$$

As a result, $\sum_{j=1}^{k} P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}}+\omega_{\varepsilon,x_\varepsilon}$ is a solution of (1.1).

Now, we prove $G_{jh}=0$.

Since

$$\begin{aligned}\frac{\partial K(x_\varepsilon)}{\partial x_{ih}} =&\Big\langle I'\Big(\sum_{j=1}^{k} P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}}+\omega_{\varepsilon,x_\varepsilon}\Big),\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}+\frac{\partial\omega_{\varepsilon,x_\varepsilon}}{\partial x_{ih}}\Big\rangle_\varepsilon\\ =&\sum_{j=1}^{k}\sum_{l=1}^{N-m+1}G_{jl}\Big\langle\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}}}{\partial x_{jl}},\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}+\frac{\partial\omega_{\varepsilon,x_\varepsilon}}{\partial x_{ih}}\Big\rangle_\varepsilon\\ =&\sum_{j=1}^{k}\sum_{l=1}^{N-m+1}G_{jl}\Big\langle\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}}}{\partial x_{jl}},\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\Big\rangle_\varepsilon\\ &-\sum_{l=1}^{N-m+1}G_{il}\Big\langle\frac{\partial^2 P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}\partial x_{ih}},\omega_{\varepsilon,x_\varepsilon}\Big\rangle_\varepsilon\end{aligned}\tag{3.48}$$

we obtain

$$\begin{aligned}&\sum_{j=1}^{k}\sum_{l=1}^{N-m+1}G_{jl}\Big\langle\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}}}{\partial x_{jl}},\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\Big\rangle_\varepsilon\\ &-\sum_{l=1}^{N-m+1}G_{il}\Big\langle\frac{\partial^2 P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}\partial x_{ih}},\omega_{\varepsilon,x_\varepsilon}\Big\rangle_\varepsilon=0.\end{aligned}\tag{3.49}$$

We claim

$$\Big\|\frac{\partial^2 P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}\partial x_{ih}}\Big\|_\varepsilon=O\big(\varepsilon^{-2+(N-m+1)/2}\big).\tag{3.50}$$

Assume this at the moment. Then

$$\Big\langle\frac{\partial^2 P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}\partial x_{ih}},\omega_{\varepsilon,x_\varepsilon}\Big\rangle_\varepsilon=O\big(\varepsilon^{-2+(N-m+1)/2}\big)\|\omega_{\varepsilon,x_\varepsilon}\|_\varepsilon=o(\varepsilon^{N-m-1}),$$

which, together with Lemma A.3 and (3.49), implies $G_{jl}=0$, $j=1,\cdots,k$, $j=1,\cdots,N-m+1$.

Now we prove (3.50). We have

$$\begin{aligned}&-\varepsilon^2\Delta\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_i}}{\partial x_{il}\partial x_{ih}}+\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_i}}{\partial x_{il}\partial x_{ih}}\\ =&\frac{\partial^2}{\partial x_{il}\partial x_{ih}}\Big(\xi_j\bar W_{\varepsilon,x_i}^{p-1}+\tilde f_{\varepsilon,x_j}(y)\Big),\quad\text{in }\Omega,\end{aligned}\tag{3.51}$$

and

$$\frac{\partial^2 P_{\varepsilon,\Omega}W_{\varepsilon,x_i}}{\partial x_{il}\partial x_{ih}}=0\quad\text{on }\partial\Omega.\tag{3.52}$$

Multiplying (3.51) by $\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_i}}{\partial x_{il}\partial x_{ih}}$ and integrating by parts, we obtain

$$\begin{aligned}\|\frac{\partial^2 P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}\partial x_{ih}}\|_\varepsilon^2 \le C\varepsilon^{-2}\int_\Omega \big(\xi_j \bar W_{\varepsilon,x_i}^{p-1} + W_{\varepsilon,x_j}\big)\big|\frac{\partial^2 P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}\partial x_{ih}}\big| \\ \le C\varepsilon^{-2}\Big(\int_\Omega \big(\xi_j \bar W_{\varepsilon,x_i}^{p-1} + W_{\varepsilon,x_j}\big)^2\Big)^{1/2}\|\frac{\partial^2 P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}\partial x_{ih}}\|_\varepsilon \\ \le C\varepsilon^{-2+(N-m+1)/2}\|\frac{\partial^2 P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}\partial x_{ih}}\|_\varepsilon.\end{aligned}$$

Thus, (3.50) follows. □

Appendix A.

From the choice of x_j, the term $e^{-d(x_j,\partial D)/\varepsilon}$ is only algebraically small. To obtain a solution concentrating near several manifolds, we need to prove that $P_{\varepsilon,\Omega}W_{\varepsilon,x_j}$ is exponentially small outside a small neighbourhood of the set $\{y : |y'| = x_{j,1}, y'' = x_j''\}$.

Lemma A.1. *Let $\theta > 0$ be any small constant. There is a constant $C > 0$ such that*

$$|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}| \le Ce^{-\sqrt{1-\theta}|\tilde y - x_j|/\varepsilon}.$$

Proof. Let $v = Ce^{-\sqrt{1-\theta}|\tilde y - x_j|/\varepsilon}$, where $C > 0$ is a large constant. Then

$$-\varepsilon^2\Delta v + v \ge \frac{1}{2}C\theta e^{-\sqrt{1-\theta}|\tilde y - x_j|/\varepsilon}. \tag{A.1}$$

On the other hand, we have

$$\begin{aligned}&-\varepsilon^2\Delta P_{\varepsilon,\Omega}W_{\varepsilon,x_j} + P_{\varepsilon,\Omega}W_{\varepsilon,x_j}\\ &=\xi_j \bar W_{\varepsilon,x_j}^{p-1} + \tilde f_{\varepsilon,x_j}(y) = \xi_j \bar W_{\varepsilon,x_j}^{p-1} + O\big(\xi_j\varepsilon e^{-|\tilde y - x_j|/\varepsilon}\big).\end{aligned} \tag{A.2}$$

But if $C > 0$ is large enough, then

$$\frac{1}{2}C\theta e^{-\sqrt{1-\theta}|\tilde y - x_j|/\varepsilon} \ge \Big|\xi_j \bar W_{\varepsilon,x_j}^{p-1} + O\big(\xi_j\varepsilon e^{-|\tilde y - x_j|/\varepsilon}\big)\Big|. \tag{A.3}$$

By the maximum principle, we obtain from (A.1), (A.2) and (A.3) that

$$|P_{\varepsilon,\Omega}W_{\varepsilon,x_j}| \le v = Ce^{-\sqrt{1-\theta}|\tilde y - x_j|/\varepsilon}.$$

□

Lemma A.2. *Let $\theta > 0$ be any small constant. There is a constant $C > 0$ such that*

$$\Big|\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{jh}}\Big| \le C\varepsilon^{-1}e^{-\sqrt{1-\theta}|\tilde y - x_j|/\varepsilon}.$$

Proof. We know that $\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{jh}}$ satisfies

$$-\varepsilon^2\Delta\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{jh}} + \frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{jh}} = \frac{\partial}{\partial x_{jh}}\Big(\xi_j \bar W_{\varepsilon,x_j}^{p-1} + \tilde f_{\varepsilon,x_j}(y)\Big), \quad \text{in } \Omega, \tag{A.4}$$

and

$$\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_j}}{\partial x_{jh}} = 0, \quad \text{on } \partial\Omega. \tag{A.5}$$

On the other hand, it is easy to check that

$$\Big|\frac{\partial}{\partial x_{jh}}\Big(W_{\varepsilon,x_j}^{p-1}+\tilde f_{\varepsilon,x_j}(y)\Big)\Big|\le C\varepsilon^{-1}\big(W_{\varepsilon,x_j}^{p-1}+\tilde f_{\varepsilon,x_j}(y)\big). \tag{A.6}$$

Thus we can prove this lemma in a similar way as in Lemma A.1. □

Lemma A.3. *We have*

$$\Big\langle\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}}}{\partial x_{jl}},\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\Big\rangle_\varepsilon=\begin{cases}\tilde c\varepsilon^{N-m-1}, & i=j,l=h,\\ o(\varepsilon^{N-m-1}), & \textit{otherwise},\end{cases}$$

where $\tilde c>0$ is a constant.

Proof. We have

$$\begin{aligned}&\Big\langle\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}}}{\partial x_{jl}},\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\Big\rangle_\varepsilon\\ =&(p-1)\int_\Omega\xi_j\bar W_{\varepsilon,x_j}^{p-2}\frac{\partial W_{\varepsilon,x_{\varepsilon,j}}}{\partial x_{jl}},\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\\ &+\int_\Omega\frac{\partial}{\partial x_{jl}}\Big(\tilde f_{\varepsilon,x_j}(y)\Big)\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\\ =&(p-1)\int_\Omega\xi_j\bar W_{\varepsilon,x_j}^{p-2}\frac{\partial W_{\varepsilon,x_{\varepsilon,j}}}{\partial x_{jl}}\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}+O\Big(\int_\Omega U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\Big|\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\Big|\Big).\end{aligned}\tag{A.7}$$

It follows from Lemma A.2 that

$$\begin{aligned}&\int_\Omega U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)\Big|\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\Big|\\ \le& C\varepsilon^{-1}\int_\Omega U\big(\frac{|\tilde y-x_j|}{\varepsilon}\big)e^{-\sqrt{1-\theta}|\tilde y-x_j|/\varepsilon}\le C\varepsilon^{N-m}.\end{aligned}\tag{A.8}$$

Using Lemma A.2, (A.7) and (A.8), we conclude that if $i\ne j$, then

$$\Big\langle\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,j}}}{\partial x_{jl}},\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\Big\rangle_\varepsilon=o(\varepsilon^{N-m-1}).$$

It remains to study the case $i=j$.

Using (2.2), we deduce

$$\Big|\frac{\partial\varphi_{\varepsilon,x_i}}{\partial x_{ih}}\Big|\le C\varepsilon^{-1}e^{-d_i/\varepsilon}.$$

Thus,

$$\begin{aligned}&\int_\Omega\xi_i\bar W_{\varepsilon,x_i}^{p-2}\frac{\partial\bar W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}}\frac{\partial P_{\varepsilon,\Omega}W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}\\ =&\int_\Omega\bar W_{\varepsilon,x_i}^{p-2}\frac{\partial\bar W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{il}}\frac{\partial\bar W_{\varepsilon,x_{\varepsilon,i}}}{\partial x_{ih}}+O\big(\varepsilon^{N-m-1}e^{-d_i/\varepsilon}\big)\\ =&\begin{cases}\tilde c\varepsilon^{N-m-1}+O\big(\varepsilon^{N-m-1}e^{-d_i/\varepsilon}\big), & l=h,\\ O\big(\varepsilon^{N-m-1}e^{-d_i/\varepsilon}\big), & l\ne h.\end{cases}\end{aligned}\tag{A.9}$$

□

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