

# A new upper bound for the cardinality of 2-distance sets in Euclidean space

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A NEW UPPER BOUND FOR THE CARDINALITY OF  
2-DISTANCE SETS IN EUCLIDEAN SPACE

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Abstract

It is proved that the cardinality of a 2-distance set  $S$  in Euclidean  $d$ -dimensional space satisfies

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2) .$$

Introduction

A set  $S$  in Euclidean  $d$ -space  $E^d$  is called a 2-distance set if the distance between distinct points of  $S$  assumes only two values.

The maximum size of such a set is 5 in  $E^2$  (Kelly), and 6 in  $E^3$  (Croft).

Delsarte, Goethals and Seidel [1] treated the case where the points of  $S$  lie on a sphere. Their argument can be modified to obtain the bound  $\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 4)$  for general 2-distance sets as was established by Larman, Rogers and Seidel [2]. E. and E. Bannai [3] showed that equality doesn't occur in this case. The proof of Larman, Rogers and Seidel can be modified again to obtain  $\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2)$ .

Theorem.

Let  $S$  be a 2-distance set in  $E^d$ , then

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2) .$$

Proof.

Let  $a$  and  $b$  the distances in  $S$ . For each point  $s$  in  $S$  and  $x \in E^d$  we define

$$F_s(x) = \frac{1}{a^2 b^2} (\|x - s\|^2 - a^2)(\|x - s\|^2 - b^2) .$$

These functions form an independent set of functions since  $F_s(t) = \delta_{s,t}$  for all  $s, t \in S$ . They are linear combinations of the following functions:

$$\|x\|^4 ; \|x\|^2 x_i ; x_i x_j ; x_i ; 1 ; \quad \text{where } 1 \leq i \leq j \leq d .$$

Hence the total number of functions  $F_s$  cannot exceed

$$1 + d + \frac{1}{2}d(d + 1) + d + 1 = \frac{1}{2}(d + 1)(d + 4) .$$

We proceed to show that in fact the set

$$\{F_s(x) , x_i , 1 \mid s \in S , 1 \leq i \leq d\}$$

is linearly independent, which implies

$$\text{card}(S) + d + 1 \leq \frac{1}{2}(d + 1)(d + 4)$$

and hence

$$\text{card}(S) \leq \frac{1}{2}(d + 1)(d + 2) .$$

Now suppose we have

$$(1) \quad \sum_{s \in S} c_s F_s(x) + \sum_{i=1}^d c_i x_i + c = 0 .$$

Inserting  $s$  in relation (1) we get

$$(2) \quad c_s + \sum_i c_i s_i + c = 0 .$$

Inserting  $ke_i$  in (1), where  $e_i$  is the  $i$ -th column of the unit matrix, we get

$$(3) \quad \frac{1}{a^2 b^2} \sum_s c_s (k^2 - 2ks_i + \|s\|^2 - a^2)(k^2 - 2ks_i + \|s\|^2 - b^2) + kc_i + c = 0 .$$

Comparing the coefficients of  $k^4$  and of  $k^3$  we obtain

$$(4) \quad \sum_s c_s = 0 \quad \text{and} \quad \sum_s c_s s_i = 0$$

for  $i = 1, \dots, d$ .

Multiply relation (2) by  $c_s$  and sum over all  $s \in S$ :

$$(5) \quad \sum_s c_s^2 + \sum_i c_i \sum_s c_s s_i + c \sum_s c_s = 0.$$

Now (4) and (5) yield  $c_s = 0$  for all  $s \in S$ , whence also  $c = c_i = 0$  for  $i = 1, \dots, d$ . This completes the proof of the theorem.

#### References

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