A new upper bound on the total domination number of a graph

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Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a total dominating set of G is the total domination number of G. Let G be a connected graph of order n with minimum degree at least two and with maximum degree at least three. We define a vertex as large if it has degree more than 2 and we let $\mathcal{L}$ be the set of all large vertices of G. Let P be any component of $G-\mathcal{L}$; it is a path. If $|P| \equiv 0 \pmod 4$ and either the two ends of P are adjacent in G to the same large vertex or the two ends of P are adjacent to different, but adjacent, large vertices in G, we call P a 0-path. If $|P| \geq 5$ and $|P| \equiv 1 \pmod 4$ with the two ends of P adjacent in G to the same large vertex, we call P a 1-path. If $|P| \equiv 3 \pmod 4$, we call P a 3-path. For $i \in \{0, 1, 3\}$, we denote the number of i-paths in G by p_i. We show that the total domination number of G is at most $(n + p_0 + p_1 + p_3)/2$. This result generalizes a result shown in several manuscripts (see, for example, J. Graph Theory 46 (2004), 207–210) which states that if G is a graph of order n with minimum degree at least three, then the total domination of G is at most $n/2$. It also generalizes a result by Lam and Wei stating that if G is a graph of order n with minimum degree at least two and with no degree-2 vertex adjacent to two other degree-2 vertices, then the total domination of G is at most $n/2$.

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1 Introduction

In this paper, we continue the study of total domination in graphs which was introduced by Cockayne, Dawes, and Hedetniemi [5]. A *total dominating set*, abbreviated TDS, of a graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Every graph without isolated vertices has a TDS, since $S = V(G)$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Total domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [7, 8].

For notation and graph theory terminology we in general follow [7]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v\in S}N(v)$. If $Y \subseteq V$, then the set S is said to *totally dominate* the set Y if $Y \subseteq N(S)$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. We denote the degree of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). We denote a path on n vertices by P_n and a cycle on n vertices by C_n.

2 Known bounds on the total domination number

The decision problem to determine the total domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the total domination number of a graph. In particular, for a connected graph G with minimum degree $\delta \geq 1$ and order n, the problem of finding upper bounds on $\gamma_t(G)$ in terms of δ and n has been studied. The known upper bounds on $\gamma_t(G)$ in terms of δ and n are summarized in Table 1.

$\delta(G) \geq 1$	$\Rightarrow$	$\gamma_t(G)$	$\leq$	$\frac{2}{3}n$	if $n \geq 3$ and G is connected
$\delta(G) \geq 2$	$\Rightarrow$	$\gamma_t(G)$	$\leq$	$\frac{4}{7}n$	if $G \notin \{C_3, C_5, C_6, C_{10}\}$ and G is connected
$\delta(G) \geq 3$	$\Rightarrow$	$\gamma_t(G)$	$\leq$	$\frac{1}{2}n$	
$\delta(G) \geq 4$	$\Rightarrow$	$\gamma_t(G)$	$\leq$	$\frac{3}{7}n$	
$\delta(G)$ large	$\Rightarrow$	$\gamma_t(G)$	$\leq$	$\left(\frac{1+\ln\delta}{\delta}\right)n$	

Table 1. Upper bounds on the total domination number of a graph G.

The result in Table 1 when δ is large is found using probabilistic methods in graph theory. It can easily be deduced from results of Alon [1] that this upper bound for large δ is nearly optimal. But what happens when δ is small? The problem then becomes more difficult.

The result in Table 1 when $\delta \geq 1$ is due to Cockayne et al. [5] and the graphs achieving this upper bound are characterized by Brigham, Carrington, and Vitray [3].

The result in Table 1 when $\delta \geq 2$ can be found in [9]. A characterization of the connected graphs of large order with total domination number exactly four-sevenths their order is also given in [9].

Chvátal and McDiarmid [4] and Tuza [13] independently established that every hypergraph on n vertices and m edges where all edges have size at least three has a transversal T such that $4|T| \leq m+n$. As a consequence of this result about transversals in hypergraphs, we have the result in Table 1 for the case when $\delta \geq 3$. We remark that Archdeacon et al. [2] recently found an elegant one page graph theoretic proof of this upper bound of $n/2$ when $\delta \geq 3$. Two infinite families of connected cubic graphs with total domination number one-half their orders are constructed in [6]. Using transversals in hypergraphs, the connected graphs with minimum degree at least three and with total domination number exactly one-half their order are characterized in [10].

The result when $\delta \geq 3$ has recently been strengthened by Lam and Wei [11].

Theorem 1 (Lam, Wei [11]) *If G is a graph of order n with $\delta(G) \geq 2$ such that every component of the subgraph of G induced by its set of degree-2 vertices has size at most one, then $\gamma_t(G) \leq n/2$.*

The result in Table 1 when $\delta \geq 4$ is due to Thomasse and Yeo [12]. Their proof uses transversals in hypergraphs. Yeo [14] showed that for connected graphs G with minimum degree at least four equality is only achieved in this bound if G is the relative complement of the Heawood graph (or, equivalently, the incidence bipartite graph of the complement of the Fano plane).

3 Main Result

Our aim in this paper is to present a new upper bound on the total domination number of a graph with minimum degree two. For this purpose, we introduce some additional notation.

We call a component of a graph a path-component if it is isomorphic to a path. A path-component isomorphic to a path P_i on i vertices we call a P_i-component.

We define a vertex as *small* if it has degree 2, and *large* if it has degree more than 2. Let G be a connected graph with minimum degree at least two and maximum degree at least three. Let $\mathcal{S}$ be the set of all small vertices of G and $\mathcal{L}$ the set of all large vertices of G. Consider the graph $G - \mathcal{L} = G[\mathcal{S}]$ induced by the small vertices. Let P be any component of $G - \mathcal{L}$; it is a path. If $|P| \equiv 0 \,(\mathrm{mod}\, 4)$ and either the two ends of P are adjacent in G to the same large vertex or the two ends of P are adjacent to different, but adjacent, large vertices in G, we call P a 0-*path*. If $|P| \geq 5$ and $|P| \equiv 1 \pmod 4$ with the two ends of P adjacent in G to the same large vertex, we call P a 1-*path*. If $|P| \equiv 3 \pmod 4$, we call P a 3-*path*. For $i \in \{0, 1, 3\}$, we denote the number of i-paths in G by $p_i(G)$, or simply by p_i if the graph G is clear from context. If G' is a graph, then for $i \in \{0, 1, 3\}$ we denote $p_i(G')$ simply by p'_i. For notational convenience, for a graph G of order n and a graph G' of order n' we let

$$\psi(G) = \frac{1}{2}(n + p_0 + p_1 + p_3) \quad \text{and} \quad \psi(G') = \frac{1}{2}(n' + p'_0 + p'_1 + p'_3).$$

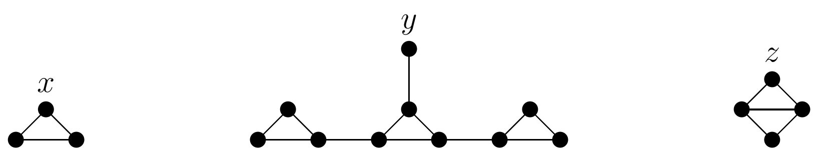
We shall prove:

Theorem 2 *If G is a connected graph of order n with $\delta(G) \geq 2$ and $\Delta(G) \geq 3$, then $\gamma_t(G) \leq \psi(G)$.*

Note that Theorem 2 generalizes Theorem 1 (see [11]) and the result from Table 1 for $\delta(G) \geq 3$ (see [4] and [13]).

3.1 Preliminary Results and Observations

Before presenting a proof of Theorem 2, we define three graphs which we call X, Y and Z shown in Figures 1(a), (b) and (c), respectively. The vertices named x, y and z in Figure 1 we call the *link vertices* of the graphs X, Y and Z, respectively.



(a) X (b) Y (c) Z

Figure 1: The three graphs X, Y and Z.

Let $H \in \{X, Y, Z\}$. By *attaching a copy of H* to a vertex v in a graph G we mean adding a copy of H to the graph G and joining v with an edge to the link vertex of H. We call v an *attached vertex* in the resulting graph. We will frequently use the following observations in the proof of Theorem 2.

Observation 1 *If G' is obtained from a graph G with no isolated vertex by attaching a copy of X with link vertex x to a vertex x' of G, then there exists a $\gamma_t(G')$-set S such that $S \cap (V(X) \cup \{x'\}) = \{x, x'\}$.*

Observation 2 *If G' is obtained from a graph G with no isolated vertex by attaching a copy of Y with link vertex y to a vertex y' of G, then there exists a $\gamma_t(G')$-set S that contains exactly four vertices of Y, namely the two vertices of Y at distance 2 from y and the two vertices of Y at distance 3 from y (and so, y' belongs to S to totally dominate y while a neighbor of y' in G belongs to S to totally dominate y').*

Observation 3 *If G' is obtained from a graph G with no isolated vertex by attaching a copy of Z with link vertex z to a vertex z' of G, then there exists a $\gamma_t(G')$-set S that contains exactly two vertices of Z, namely z and a neighbor of z in Z (and so, z totally dominates z' in G').*

We define an *elementary* 4*-subdivision* of a nonempty graph G as a graph obtained from G by subdividing some edge four times. We shall need the following lemma from [9].

Lemma 1 ([9]) *Let G be a nontrivial graph and let G' be obtained from G by an elementary* 4*-subdivision. Then $\gamma_t(G') = \gamma_t(G) + 2$.*

We will refer to a graph G as a *reduced graph* if G has no induced path on six vertices, the internal vertices of which have degree 2 in G. Hence if u, v_1, v_2, v_3, v_4, v is a path in a reduced graph G, then $d_G(v_i) \geq 3$ for at least one i, $1 \leq i \leq 4$, or $uv \in E(G)$.

3.2 Proof of Theorem 2

We proceed by induction on the lexicographic sequence $(p_0+p_1+p_3, n)$, where $p_0+p_1+p_3 \geq 0$ and $n \geq 4$. For notational convenience, for a graph G of order n and a graph G' of order n', we denote the sequence $(p_0+p_1+p_3, n)$ by $s(G)$ and the sequence $(p'_0+p'_1+p'_3, n')$ by $s(G')$. Further, we denote the set of small vertices of G and G' by $\mathcal{S}$ and $\mathcal{S}'$, respectively, and the set of large vertices of G and G' by $\mathcal{L}$ and $\mathcal{L}'$, respectively.

By Lemma 1, we may assume that G is a reduced graph. Thus since G is a connected graph with $\Delta(G) \geq 3$, every component of $G[\mathcal{S}]$ is a path P_i for some i where $1 \leq i \leq 5$.

When $p_0+p_1+p_3 = 0$, every component of $G[\mathcal{S}]$ is either P_1 or P_2 and the desired result follows from Theorem 1. This establishes the base case. Assume, then, that $p_0+p_1+p_3 \geq 1$ and $n \geq 4$ and that for all connected graphs G' of order n' with $\delta(G') \geq 2$ and $\Delta(G') \geq 3$ that have lexicographic sequence $s(G')$ smaller than s, $\gamma_t(G') \leq \psi(G')$. Let $G = (V, E)$ be a connected graph of order n with $\delta(G) \geq 2$ and $\Delta(G) \geq 3$ and with lexicographic sequence $s(G) = s$.

Observation 4 *We may assume that $p_0 = 0$.*

Proof. Suppose that $p_0 \geq 1$. Let $P\colon v_1, v_2, v_3, v_4$ be a P_4-component of $G[\mathcal{S}]$. Let u be the neighbor of v_1 not on P and let v be the neighbor of v_4 not on P.

Suppose firstly that $u \neq v$. Since G is a reduced graph, $uv \in E(G)$. Let $G' = G - V(P)$. Then, G' is a connected graph of order n' with $\delta(G') \geq 2$. Suppose G' is a cycle. Then, $G' \in \{C_3, C_4, C_5, C_6\}$. If $G' = C_3$, then $\gamma_t(G) = 4$ and $\psi(G) = 4$. If $G' = C_4$, then $\gamma_t(G) = 4$ and $\psi(G) = 4\frac{1}{2}$. If $G' = C_5$, then $\gamma_t(G) = 5$ and $\psi(G) = 5\frac{1}{2}$. If $G' = C_6$, then $\gamma_t(G) = 6$ and $\psi(G) = 6$. In all cases, $\gamma_t(G) \leq \psi(G)$. Hence we may assume that $\Delta(G') \geq 3$. We remark that it is possible that the graph G' has an induced path on six vertices containing u and v with the internal vertices on this path having degree 2 in G', in which case G' is not a reduced graph, but then it is not a problem to reduce it. Since $p'_0 + p'_1 + p'_3 \leq p_0 + p_1 + p_3$ and $n' = n - 4$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma_t(G') \le \psi(G') \le \psi(G) - 2$. Every $\gamma_t(G')$-set can be extended to a TDS of G by adding to it the vertices $\{v_2, v_3\}$, and so $\gamma_t(G) \le \gamma_t(G') + 2 \le \psi(G)$.

Suppose secondly that $u = v$. Then, $C\colon v, v_1, v_2, v_3, v_4, v$ is a cycle in G. Let G' be the graph obtained from $G - V(C)$ by attaching the same copy of Z to each vertex in $N_G(v) \setminus \{v_1, v_4\}$. Then, G' is a connected (reduced) graph of order $n' = n - 1$ with $\delta(G') \ge 2$ and $\Delta(G') \ge 3$ (as v was a large vertex, z is attached to at least one vertex and $\Delta(Z) = 3$). The components of $G'[\mathcal{S}']$, other than the P_1-component consisting of the degree-2 vertex in the copy of Z, are precisely the components of $G[\mathcal{S}]$ minus the path-component P. Hence, $p_0' = p_0 - 1$, $p_1' = p_1$ and $p_3' = p_3$. The lexicographic sequence $s(G')$ is therefore smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma_t(G') \le \psi(G') = \psi(G) - 1$. By Observation 3, there exists a $\gamma_t(G')$-set S that contains the link vertex and a neighbor of the link vertex (distinct from the attached vertex) from the attached copy of Z. Deleting these two vertices in the attached copy of Z from the set S and adding to the resulting set the three vertices v, v_1, v_2 produces a TDS of G. Hence, $\gamma_t(G) \le |S| + 1 = \gamma_t(G') + 1 \le \psi(G)$. □

Observation 5 *We may assume that* $p_1 = 0$*.*

Proof. Suppose that $p_1 \ge 1$. Let $P\colon v_1, v_2, \ldots, v_5$ be a P_5-component of $G[\mathcal{S}]$. Since G is a reduced graph, v_1 and v_5 have a common neighbor v in G. Let G' be obtained from G by deleting the vertices v_3, v_4 and v_5 and adding the edge vv_2; that is, $G' = (G - \{v_3, v_4, v_5\}) \cup \{vv_2\}$. Then, G' is a reduced connected graph of order n' with $\delta(G') \ge 2$ and $\Delta(G') = \Delta(G) \ge 3$. Further, $p_0' = p_0$, $p_1' = p_1 - 1$, $p_3' = p_3$, and $n' = n - 3$. Hence the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma_t(G') \le \psi(G') = \psi(G) - 2$. Let S' be a $\gamma_t(G')$-set that contains neither v_1 nor v_2 (if there is a $\gamma_t(G')$-set S' that contains both v_1 and v_2, simply replace these two vertices in S' by v and a neighbor of v in $G - V(P)$, while if there is a $\gamma_t(G')$-set S' that contains exactly one of v_1 and v_2, simply replace this vertex in S' by a neighbor of v in $G - V(P)$). Then, $S' \cup \{v_3, v_4\}$ is a TDS of G, and so $\gamma_t(G) \le |S'| + 2 = \gamma_t(G') + 2 \le \psi(G)$. □

By Observations 4 and 5, we have $p_0 = p_1 = 0$ and $p_3 \ge 1$. Thus, since G is a reduced graph, every component of $G[\mathcal{S}]$ is a path P_i for some i where $1 \le i \le 3$. Let $P\colon v_1, v_2, v_3$ be a P_3-component of $G[\mathcal{S}]$. Let u be the neighbor of v_1 not on P and let v be the neighbor of v_3 not on P.

Observation 6 *We may assume that* $u \ne v$*.*

Proof. Suppose that $u = v$. Let G' be the graph obtained from $G - V(P)$ by attaching both a copy of X and a copy of Z to the vertex v. Then, G' is a connected (reduced) graph of order $n' = n + 4$ with $\delta(G') \ge 2$ and $\Delta(G') = \Delta(G) \ge 3$. The degree of the large vertex v is unchanged in G and G'. Since $p_0' = p_0 = 0$, $p_1' = p_1 = 0$ and $p_3' = p_3 - 1$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma_t(G') \le \psi(G') = \psi(G) + 3/2$. By Observations 1 and 3, there exists a $\gamma_t(G')$-set S that contains the vertex v and three vertices from the attached copies of X and Z, namely the link vertex and a neighbor of the link vertex in the attached copy of Z and the link vertex in the attached copy of X. Deleting these three vertices in the attached copies of X and Z from the set S and adding to the resulting set the vertex v_1 produces a TDS of G. Hence, $\gamma_t(G) \le |S| - 2 = \gamma_t(G') - 2 \le \psi(G) - 1/2$. $\Box$

Observation 7 *We may assume that no common neighbor of u and v has degree two.*

Proof. Suppose that u and v have a common neighbor w with $N(w) = \{u, v\}$. Let W be the set of all such degree-2 vertices that are adjacent to both u and v. Let $R = W \cup \{u, v, v_1, v_2, v_3\}$. Let $N_{uv} = (N(u) \cup N(v)) \setminus R$.

Suppose $V = R$. If $|W| = 1$, then $uv \in E$, $n = 6$, $p_3 = 1$, and $\gamma_t(G) = 3 = \psi(G) - 1/2$. If $|W| \ge 2$, then $n \ge 7$, $p_3 = 1$, and $\gamma_t(G) \le 4 \le \psi(G)$. Hence we may assume that $V \ne R$. Thus, $|N_{uv}| \ge 1$. At least one of u and v, say v, is therefore adjacent to a vertex in $V \setminus R$.

If $|W| \ge 2$, then let $G' = G - w$. The graph G' is a connected reduced graph of order $n' = n - 1$ with $\delta(G') \ge 2$ and $\Delta(G') \ge d_G(v) - 1 \ge 3$. If $d_{G'}(u) = 2$, then $p'_0 = p_0$, $p'_1 = p_1 + 1$ and $p'_3 = p_3 - 1$, while if $d_{G'}(u) \ge 3$, then $p'_0 = p_0$, $p'_1 = p_1$ and $p'_3 = p_3$. In both cases, $p'_0 + p'_1 + p'_3 = p_0 + p_1 + p_3$. Applying the inductive hypothesis to G', $\gamma_t(G') \le \psi(G') = \psi(G) - 1/2$. Every $\gamma_t(G')$-set is a TDS of G, and so $\gamma_t(G) \le \gamma_t(G') < \psi(G)$. Hence we may assume that $|W| = 1$, and so $W = \{w\}$ and $R = \{u, v, v_1, v_2, v_3, w\}$.

Let G' be the connected graph obtained from $G - R$ by attaching the same subgraph X to every vertex in N_{uv}. Let $N^*_{uv} = (N(u) \cap N(v)) \setminus R$ and if $N^*_{uv} \ne \emptyset$ then also attach the same subgraph Z to every vertex in N^*_{uv}. Note that $d_{G'}(x) = d_G(x)$ for every vertex $x \in V(G') \setminus V(X \cup Z)$. Furthermore, $\Delta(G') \ge 3$ as the link vertex in the copy of X has degree at least three. The components of $G'[\mathcal{S}']$, other than the P_2-component consisting of the two degree-2 vertices in the copy of X and, if $N^*_{uv} \ne \emptyset$, the P_1-component consisting of the degree-2 vertex in the copy of Z, are precisely the components of $G[\mathcal{S}]$ minus the path-component P and the P_1-component consisting of the vertex w. Hence, $p'_0 = p_0 = 0$, $p'_1 = p_1 = 0$ and $p'_3 = p_3 - 1$. Thus, $p'_0 + p'_1 + p'_3 = p_0 + p_1 + p_3 - 1$. Applying the inductive hypothesis to G', $\gamma_t(G') \le \psi(G')$. By the construction of X, there exists a $\gamma_t(G')$-set S, such that $S \cap N_{uv} \ne \emptyset$ and $|S \cap X| = 1$. We may assume without loss of generality that v is adjacent in G to a vertex in $S \cap N_{uv}$.

On the one hand, suppose that $N^*_{uv} \ne \emptyset$. Then, $n' = n + 1$ and $\psi(G') = \psi(G)$. Delete from S the vertices in X and Z and add the vertices $\{u, v, v_1\}$. The resulting set has size at most that of S and is a TDS of G. Hence, $\gamma_t(G) \le \gamma_t(G') \le \psi(G') = \psi(G)$.

On the other hand, suppose that $N^*_{uv} = \emptyset$. Then, $n' = n - 3$ and $\psi(G') = \psi(G) - 2$. Now delete from S the vertex in X and add the vertices $\{u, v, v_1\}$. The resulting set has size $|S| + 2$ and is a TDS of G. Hence, $\gamma_t(G) \le \gamma_t(G') + 2 \le \psi(G') + 2 = \psi(G)$. $\Box$

Let $R = \{u, v, v_1, v_2, v_3\}$ and let $N_{uv} = (N(u) \cup N(v)) \setminus R$. Then, $|N_{uv}| \ge 1$. By Observation 7, every vertex in N_{uv} that is adjacent to both u and v has degree at least 3. Hence every vertex in N_{uv} is adjacent to at least one vertex different from u and v.

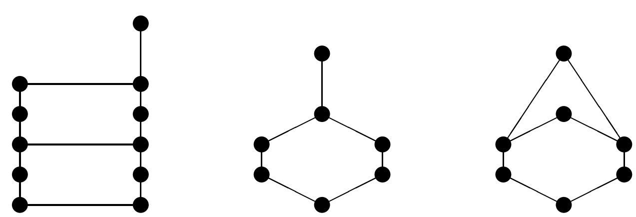
Observation 8 *We may assume that* $|N_{uv}| = 1$.

Proof. Suppose that $|N_{uv}| \geq 2$. Let G' be obtained from $G - V(P)$ by adding all possible edges between the set $\{u, v\}$ and the set N_{uv}, and by adding the edge uv if u and v are not adjacent to G. Then, G' is a connected (reduced) graph of order $n' = n - 3$ with $\delta(G') \geq 2$ and $\Delta(G') \geq 3$. By construction, both u and v are large vertices in G'. Note that some vertices in N_{uv} may be large in G' even though they were not large in G. However as every component in $G[\mathcal{S}]$ is a path containing at most three vertices, we note that $p'_0 + p'_1 + p'_3 \leq p_0 + p_1 + p_3 - 1$. We can therefore apply the inductive hypothesis to G'. Thus, $\gamma_t(G') \leq \psi(G') \leq \psi(G) - 2$. Let S' be a $\gamma_t(G')$-set. If $\{u, v\} \subseteq S'$, let $S = S' \cup \{v_1, v_3\}$. If $|\{u, v\} \cap S'| \leq 1$, then the set S' contains a vertex $u' \in N_{uv}$ to totally dominate u or v in G'. The vertex u' is adjacent in G to at least one of u and v, say to u. If $|\{u, v\} \cap S'| = 1$, let $S = S' \cup \{u, v, v_3\}$. If $\{u, v\} \cap S' = \emptyset$, let $S = S' \cup \{v_2, v_3\}$. In all three cases, S is a TDS of G and $|S| = |S'| + 2$. Hence, $\gamma_t(G) \leq |S| = \gamma_t(G') + 2 \leq \psi(G)$. □

By Observation 8, $|N_{uv}| = 1$, implying that $uv \in E$. Let $N_{uv} = \{w\}$. Let $G' = G - V(P)$. Then, G' is a connected (reduced) graph of order $n' = n - 3$ with $\delta(G') \geq 2$ and $\Delta(G') = \Delta(G) \geq 3$. Since $p'_0 + p'_1 + p'_3 = p_0 + p_1 + p_3 - 1$, we can apply the inductive hypothesis to G'. Thus, $\gamma_t(G') \leq \psi(G') = \psi(G) - 2$. Let S' be a $\gamma_t(G')$. Then, $S' \cup \{v_1, v_2\}$ is a TDS of G, and so $\gamma_t(G) \leq |S'| + 2 = \gamma_t(G') + 2 = \psi(G)$. □

3.3 Sharpness of Theorem 2

To illustrate that the bound in Theorem 2 is sharp, we introduce a family $\mathcal{G}$ of graphs. For this purpose, we define three types of graphs which we call *units*.



(i) Type-0 (ii) Type-1 (iii) Type-3

Figure 2: The three types of units

We define a *type*-0 *unit* to be the graph obtained from a 10-cycle by adding a chord joining two vertices at maximum distance 5 apart on the cycle and then adding a pendant edge to a resulting vertex that has no degree-3 neighbor. We define a *type*-1 *unit* to be the graph obtained from a 6-cycle by adding to this cycle a pendant edge. We define a *type*-3 *unit* to be the graph obtained from a 6-cycle by adding to this cycle a new vertex and joining it to two vertices at distance 2 on this cycle. The three types of units are shown in Figure 2.

Next we define a *link vertex* in each unit as follows. In a type-0 unit and type-1 unit, we call the degree-1 vertex in the unit the *link vertex* of the unit, while in a type-3 unit we select one of the two degree-2 vertices with both its neighbors of degree 3 and call it the *link vertex* of the unit.

Let $\mathcal{G}$ denote the family of all graphs G that are obtained from the disjoint union of at least two units, each of which is of type-0, type-1 or type-3, in such a way that G is connected and every added edge joins two link vertices. A graph G in the family $\mathcal{G}$ is illustrated in Figure 3 (here the subgraph of G induced by the link vertices is a cycle C_4).

The graph G in Figure 3 has order $n = 32$, $p_0 = 1$, $p_1 = 1$, $p_3 = 2$, and $\gamma_t(G) = 18 = \psi(G)$. In general, if $G \in \mathcal{G}$ and $i \in \{0, 1, 3\}$, then each type-i unit in G contains an i-path and contributes one to p_i. Thus if $G \in \mathcal{G}$ has a type-0 units, b type-1 units, and c type-3 units, then $n = 11a + 7(b + c)$, $p_0 = a$, $p_1 = b$, $p_3 = c$ and $\gamma_t(G) = 6a + 4(b + c) = \psi(G)$.

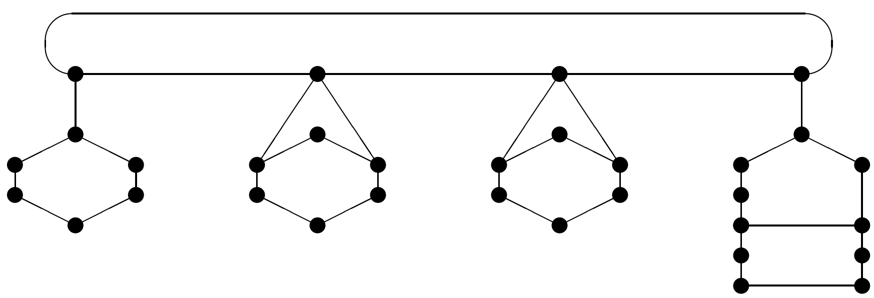


Figure 3: A graph G in the family $\mathcal{G}$.

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