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A NEW VERSION OF q-HERMITE-HADAMARD'S MIDPOINT AND TRAPEZOID TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish a new variant of q-Hermite-Hadamard inequality for convex functions via left and right q-integrals. Moreover, we prove some new q-midpoint and q-trapezoid type inequalities for left and right q-differentiable functions. To illustrate the newly established inequalities, we give some particular examples of convex functions.

1. Introduction

A function $\Upsilon\colon I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex, if it satisfies the inequality

$$\Upsilon(tx + (1-t)y) \le t\Upsilon(x) + (1-t)\Upsilon(y),$$

where $x, y \in I$ and $t \in [0,1]$.

It is also well known that Υ is convex if and only if it satisfies the Hermite-Hadamard inequality, stated below (see [13]):

$$\Upsilon\Big(\frac{\lambda_1+\lambda_2}{2}\Big) \le \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\mathrm{d}x \le \frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}, \tag{1.1}$$

where $\Upsilon\colon I \to \mathbb{R}$ is a convex function and $\lambda_1, \lambda_2 \in I$ with $\lambda_1 < \lambda_2$.

On the other hand, in [6], Alp et al. proved the following version of quantum Hermite-Hadamard type inequality for convex functions using the left quantum integrals:

$$\Upsilon\Big(\frac{q\lambda_1+\lambda_2}{[2]_q}\Big) \le \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x \le \frac{q\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{[2]_q}. \tag{1.2}$$

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Recently, Bermudo et al. [8] used the right quantum integrals and proved the following variant of Hermite-Hadamard type inequality for convex functions:

$$\Upsilon\Big(\frac{\lambda_1+q\lambda_2}{[2]_q}\Big)\leq\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\leq\frac{\Upsilon(\lambda_1)+q\Upsilon(\lambda_2)}{[2]_q}. \tag{1.3}$$

For the left and right estimates of inequalities (1.2) and (1.3), one can consult [3,4,7,9,12,16,20,21]. In [22], Noor et al. established a generalized version of (1.2). In [1, 5, 10, 18], the authors used convexity and coordinated convexity to prove Simpson's and Newton's type inequalities via q-calculus. For the study of Ostrowski's inequalities, one can consult [2, 11].

Inspired by the ongoing studies, we prove a new version of q-Hermite-Hadamard inequalities for convex functions and prove some new midpoint type inequalities for q-differentiable convex functions. We also prove that the newly established inequalities are the generalization of existing Hermite-Hadamard inequality and midpoint inequalities.

The structure of this paper is as follows: The fundamentals of q-calculus, as well as other relevant topics in this field, are briefly discussed in Section 2. In Section 3, we provide a new variant of the q-Hermite-Hadamard inequality for convex functions and use an example to demonstrate the new inequality. In Sections 4 and 5, some q-midpoint and q-trapezoid type inequalities for q-differentiable functions are studied using q-integrals. It is also taken into account the relationship between the findings given here and similar findings in the literature. We provide some mathematical examples in Section 6 to demonstrate the validity of the newly developed inequalities. Section 7 concludes with some research suggestions for the future.

2. Basics of q-calculus

In this section, we first present the definitions and some properties of quantum derivatives and quantum integrals. We also mention some well known inequalities for quantum integrals. Throughout this paper, let $0<q<1$ be a constant.

The q-number or q-analogue of $n\in\mathbb{N}$ is given by

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\cdots+q^{n-1}. \tag{2.1}$$

The q-Jackson integral for the function Υ over $[0,\lambda_1]$ is defined as (see [15]):

$$\int_0^{\lambda_2}\Upsilon(x)\mathrm{d}_qx=(1-q)\,\lambda_2\sum_{n=0}^{\infty}q^n\Upsilon(\lambda_2q^n) \tag{2.2}$$

and q-Jackson integral for a function Υ over $[\lambda_1,\lambda_2]$ is as follows (see [15]):

$$\int_{\lambda_1}^{\lambda_2}\Upsilon(x)\mathrm{d}_qx=\int_0^{\lambda_2}\Upsilon(x)\mathrm{d}_qx-\int_0^{\lambda_1}\Upsilon(x)\mathrm{d}_qx. \tag{2.3}$$

Definition 1 ([26])**.** Let $\Upsilon\colon[\lambda_1,\lambda_2]\to\mathbb{R}$ be a continuous function. Then the left q-derivative of function Υ at $x\in[\lambda_1,\lambda_2]$ is defined by

$${}_{\lambda_1}D_q\Upsilon(x)=\begin{cases}\dfrac{\Upsilon(x)-\Upsilon(qx+(1-q)\lambda_1)}{(1-q)(x-\lambda_1)}, & \text{if } x\neq\lambda_1;\\ \lim\limits_{x\to\lambda_1}\ {}_{\lambda_1}D_q\Upsilon(x), & \text{if } x=\lambda_1.\end{cases} \tag{2.4}$$

The function Υ is said to be q-differentiable function on $[\lambda_1, \lambda_2]$ if ${}_{\lambda_1}D_q\Upsilon(x)$ exists for all $x \in [\lambda_1, \lambda_2]$.

Note that if $\lambda_1 = 0$ and ${}_0D_q\Upsilon(x) = D_q\Upsilon(x)$, then (2.4) reduces to

$$D_q\Upsilon(x) = \begin{cases} \frac{\Upsilon(x)-\Upsilon(qx)}{(1-q)x}, & \text{if } x \neq 0; \\ \lim\limits_{x\to 0} D_q\Upsilon(x), & \text{if } x = 0, \end{cases}$$

which is the q-Jackson derivative (see [15, 17, 26] for more details).

Theorem 2.1 ([26]). *If $\Upsilon, g\colon J \to \mathbb{R}$ are q-differentiable functions, then the following identities hold:*

(i) *The product $\Upsilon g\colon [\lambda_1, \lambda_2] \to \mathbb{R}$ is q-differentiable on $[\lambda_1, \lambda_2]$ with*

$$\begin{aligned} {}_{\lambda_1}D_q(\Upsilon g)(x) &= \Upsilon(x)_{\lambda_1}D_q g(x) + g(qx + (1-q)\lambda_1)_{\lambda_1}D_q(x) \\ &= g(x)_{\lambda_1}D_q\Upsilon(x) + \Upsilon(qx + (1-q)\lambda_1)_{\lambda_1}D_q g(x) \end{aligned}$$

(ii) *If $g(x)g(qx + (1-q)\lambda_1) \neq 0$, then Υ/g is q-differentiable on $[\lambda_1, \lambda_2]$ with*

$${}_{\lambda_1}D_q\Big(\frac{\Upsilon}{g}\Big)(x) = \frac{g(x)_{\lambda_1}D_q\Upsilon(x) - \Upsilon(x)_{\lambda_1}D_q g(x)}{g(x)g(qx + (1-q)\lambda_1)}$$

Definition 2 ([26]). Let $\Upsilon\colon [\lambda_1, \lambda_2] \to \mathbb{R}$ be a continuous function. Then the left q-integral of function Υ at $z \in [\lambda_1, \lambda_2]$ is defined by

$$\int_{\lambda_1}^{z} \Upsilon(x)_{\lambda_1}\mathrm{d}_q x = (1-q)(z-\lambda_1)\sum_{n=0}^{\infty} q^n\Upsilon(q^n z + (1-q^n)\lambda_1). \tag{2.5}$$

The function Υ is said to be q-integrable function on $[\lambda_1, \lambda_2]$ if $\int_{\lambda_1}^{z} \Upsilon(x)_{\lambda_1}\mathrm{d}_q x$ exists for all $z \in [\lambda_1, \lambda_2]$.

Note that if $\lambda_1 = 0$, then (2.5) reduces to

$$\int_0^z \Upsilon(x)_0\mathrm{d}_q x = \int_0^z \Upsilon(x)\mathrm{d}_q x = (1-q)z\sum_{n=0}^{\infty} q^n\Upsilon(q^n z),$$

which is the q-Jackson integral (see [15, 17, 26] for more details).

Theorem 2.2 ([26]). *If $\Upsilon\colon [\lambda_1, \lambda_2] \to \mathbb{R}$ is a continuous function and $z \in [\lambda_1, \lambda_2]$, then the following identities hold:*

(i) ${}_{\lambda_1}D_q \int_{\lambda_1}^{z} \Upsilon(x)_{\lambda_1}\mathrm{d}_q x = \Upsilon(z)$;

(ii) $\int_{c}^{z} {}_{\lambda_1}D_q\Upsilon(x)_{\lambda_1}\mathrm{d}_q x = \Upsilon(z) - \Upsilon(c)$ *for $c \in (\lambda_1, z)$.*

On the other hand, Bermudo et al. defined the following new quantum derivative and quantum integral which are called right q-derivative and right q-integral:

Definition 3 ([8]). The right q-derivative of mapping $\Upsilon\colon [\lambda_1, \lambda_2] \to \mathbb{R}$ is defined as:

$${}^{\lambda_2}D_q\Upsilon(x) = \frac{\Upsilon(qx + (1-q)\lambda_2) - \Upsilon(x)}{(1-q)(\lambda_2 - x)}, \quad x \neq \lambda_2.$$

If $x = \lambda_2$, we define ${}^{\lambda_2}D_q\Upsilon(\lambda_2) = \lim\limits_{x\to\lambda_2} {}^{\lambda_2}D_q\Upsilon(x)$ if it exists and it is finite.

Definition 4 ([8]). The right q-definite integral of mapping $\Upsilon\colon [\lambda_1,\lambda_2]\to\mathbb{R}$ on $[\lambda_1,\lambda_2]$ is defined as:

$$\int_{\lambda_1}^{\lambda_2}\Upsilon(x)^{\lambda_2}\mathrm{d}_q x=(1-q)(\lambda_2-\lambda_1)\sum_{k=0}^{\infty}q^k\Upsilon\left(q^k\lambda_1+(1-q^k)\lambda_2\right).$$

Theorem 2.3 ([8]). *If $\Upsilon\colon [\lambda_1,\lambda_2]\to\mathbb{R}$ is a continuous function and $z\in[\lambda_1,\lambda_2]$, then the following identities hold:*

(i) ${}_{\lambda_1}D_q\int_z^{\lambda_2}\Upsilon(x)_{\lambda_1}\mathrm{d}_q x=-\Upsilon(z)$;

(ii) $\int_c^{\lambda_2}{}_{\lambda_1}D_q\Upsilon(x)_{\lambda_1}\mathrm{d}_q x=\Upsilon(\lambda_2)-\Upsilon(z)$.

Lemma 2.1 ([24]). *For continuous functions $\Upsilon, g\colon [\lambda_1,\lambda_2]\to\mathbb{R}$, the following equality holds:*

$$\begin{aligned}&\int_0^c g(t)\ {}^{\lambda_2}D_q\Upsilon\left(t\lambda_1+(1-t)\lambda_2\right)\mathrm{d}_q t\\&=\frac{1}{\lambda_2-\lambda_1}\int_0^c D_q g(t)\Upsilon\left(qt\lambda_1+(1-qt)\lambda_2\right)\mathrm{d}_q t-\left.\frac{g(t)\Upsilon\left(t\lambda_1+(1-t)\lambda_2\right)}{\lambda_2-\lambda_1}\right|_0^c.\end{aligned}$$

Lemma 2.2 ([25]). *For continuous functions $\Upsilon, g\colon [\lambda_1,\lambda_2]\to\mathbb{R}$, the following equality holds:*

$$\begin{aligned}&\int_0^c g(t)\ {}_{\lambda_1}D_q\Upsilon\left(t\lambda_2+(1-t)\lambda_1\right)\mathrm{d}_q t\\&=\left.\frac{g(t)\Upsilon\left(t\lambda_2+(1-t)\lambda_1\right)}{\lambda_2-\lambda_1}\right|_0^c-\frac{1}{\lambda_2-\lambda_1}\int_0^c D_q g(t)\Upsilon\left(qt\lambda_2+(1-qt)\lambda_1\right)\mathrm{d}_q t.\end{aligned}$$

3. q-Hermite-Hadamard inequality

Theorem 3.1. *Let $\Upsilon\colon [\lambda_1,\lambda_2]\to\mathbb{R}$ be a convex mapping, then we have the following inequality:*

$$\Upsilon\Big(\frac{\lambda_1+\lambda_2}{2}\Big)\le\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_q x\right]\le\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}. \tag{3.1}$$

Proof. Since Υ is a convex function, therefore

$$2\Upsilon\Big(\frac{x+y}{2}\Big)\le\Upsilon(x)+\Upsilon(y).$$

By letting $x=\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2$ and $y=\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2$, we obtain

$$2\Upsilon\Big(\frac{\lambda_1+\lambda_2}{2}\Big)\le\Upsilon\Big(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\Big)+\Upsilon\Big(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\Big). \tag{3.2}$$

q-integrating the inequality (3.2) over $[0,1]$, we have

$$2\Upsilon\Big(\frac{\lambda_1+\lambda_2}{2}\Big) \le \int_0^1 \Upsilon\Big(\lambda_1 + t\Big(\frac{\lambda_1+\lambda_2}{2} - \lambda_1\Big)\Big) \mathrm{d}_q t + \int_0^1 \Upsilon\Big(\lambda_2 + t\Big(\frac{\lambda_1+\lambda_2}{2} - \lambda_2\Big)\Big) \mathrm{d}_q t$$
$$= \frac{2}{\lambda_2-\lambda_1}\Bigg[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}} \Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x + \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_q x\Bigg],$$

hence, the first inequality proved. We again use the convexity of Υ to prove the second inequality, we have

$$\Upsilon\Big(\frac{2-t}{2}\lambda_1 + \frac{t}{2}\lambda_2\Big) + \Upsilon\Big(\frac{t}{2}\lambda_1 + \frac{2-t}{2}\lambda_2\Big) \le \Upsilon(\lambda_1) + \Upsilon(\lambda_2). \tag{3.3}$$

q-integrating the inequality (3.3) over $[0,1]$, we have

$$\frac{2}{\lambda_2-\lambda_1}\Bigg[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}} \Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x + \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_q x\Bigg] \le \Upsilon(\lambda_1) + \Upsilon(\lambda_2).$$

Thus, we obtain the required inequality. □

Remark 1. In Theorem 3.1, if we set $q \to 1^-$, then we obtain the classical Hermite-Hadamard inequality (1.1).

***Example* 1.** For a convex function $\Upsilon(x) = x^2$. From Theorem 3.1 with $\lambda_1 = 0$, $\lambda_2 = 1$ and $q = \frac{1}{2}$, we have

$$\Upsilon\Big(\frac{\lambda_1+\lambda_2}{2}\Big) = \Big(\frac{0+1}{2}\Big)^2 = 0.25,$$

$$\frac{1}{\lambda_2-\lambda_1}\Bigg[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}} \Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x + \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_q x\Bigg]$$
$$= \frac{1}{4}\sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^n\Big(\Big(\frac{1}{2}\Big)^n\Big(\frac{1}{2}\Big)\Big)^2 + \frac{1}{4}\sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^n\Big(\Big(\frac{1}{2}\Big)^n\Big(\frac{1}{2}\Big) + \Big(1-\Big(\frac{1}{2}\Big)^n\Big)1\Big)^2$$
$$= 0.30$$

and

$$\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2} = \frac{0+1}{2} = 0.5.$$

Thus, it is clear that the obtained inequality in Theorem 3.1 is valid.

4. Midpoint inequalities

In this section, we prove some left-estimates of the newly proved Hermite-Hadamard inequality (3.1) using the q-differentiability of the function.

Lemma 4.1. *Let $\Upsilon\colon [\lambda_1,\lambda_2]\subset\mathbb{R}\to\mathbb{R}$ be a function. If the functions ${}_{\lambda_1}D_q\Upsilon$ and ${}^{\lambda_2}D_q\Upsilon$ are continuous and integrable over $[\lambda_1,\lambda_2]$, then we have the following new equality:*

$$\begin{aligned}&\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\\&=\frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1 qt\ {}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\mathrm{d}_qt-\int_0^1 qt\ {}_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\mathrm{d}_qt\right].\end{aligned}\tag{4.1}$$

Proof. Let

$$\begin{aligned}&\frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1 qt\ {}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\mathrm{d}_qt-\int_0^1 qt\ {}_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\mathrm{d}_qt\right]\\&=\frac{(\lambda_2-\lambda_1)}{4}\left[I_1-I_2\right].\end{aligned}\tag{4.2}$$

From Lemma 2.1, we have

$$\begin{aligned}I_1&=\int_0^1 qt\ {}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\mathrm{d}_qt\\&=-\frac{2q}{(\lambda_2-\lambda_1)}\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)+\frac{2q}{(\lambda_2-\lambda_1)}\int_0^1\Upsilon\left(\lambda_2+qt\left(\frac{\lambda_1+\lambda_2}{2}-\lambda_2\right)\right)\mathrm{d}_qt\\&=-\frac{2q}{(\lambda_2-\lambda_1)}\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)+\frac{2q}{(\lambda_2-\lambda_1)}(1-q)\sum_{n=0}^{\infty}q^n\Upsilon\left(\lambda_2+q^{n+1}\left(\frac{\lambda_1+\lambda_2}{2}-\lambda_2\right)\right)\\&=-\frac{2q}{(\lambda_2-\lambda_1)}\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)+\frac{2}{(\lambda_2-\lambda_1)}(1-q)\sum_{n=0}^{\infty}q^{n+1}\Upsilon\left(\lambda_2+q^{n+1}\left(\frac{\lambda_1+\lambda_2}{2}-\lambda_2\right)\right)\\&=-\frac{2q}{(\lambda_2-\lambda_1)}\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)+\frac{2}{(\lambda_2-\lambda_1)}(1-q)\sum_{n=1}^{\infty}q^{n}\Upsilon\left(\lambda_2+q^{n}\left(\frac{\lambda_1+\lambda_2}{2}-\lambda_2\right)\right)\\&=-\frac{2q}{(\lambda_2-\lambda_1)}\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\\&\quad+\frac{2}{(\lambda_2-\lambda_1)}\left[(1-q)\sum_{n=0}^{\infty}q^{n}\Upsilon\left(\lambda_2+q^{n}\left(\frac{\lambda_1+\lambda_2}{2}-\lambda_2\right)\right)-(1-q)\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right]\\&=-\frac{2}{\lambda_2-\lambda_1}\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)+\frac{4}{(\lambda_2-\lambda_1)^2}\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx.\end{aligned}\tag{4.3}$$

Similarly, from Lemma 2.2, we obtain the following equality:

$$I_2=\int_0^1 qt\ {}_{\lambda_1}D_q\Upsilon\Big(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\Big)\mathrm{d}_qt=\frac{2}{\lambda_2-\lambda_1}\Upsilon\Big(\frac{\lambda_1+\lambda_2}{2}\Big)-\frac{4}{(\lambda_2-\lambda_1)^2}\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx.\tag{4.4}$$

Thus, we obtain the resultant equality by putting (4.3) and (4.4) in (4.2). The proof is completed. □

Remark 2. In Lemma 4.1, if we take the limit as $q \to 1^-$, then we obtain [23: Corollary 1].

Theorem 4.1. *We assume that the conditions of* Lemma 4.1 *hold. If the functions* $\left|{}_{\lambda_1}D_q\Upsilon\right|$ *and* $\left|{}^{\lambda_2}D_q\Upsilon\right|$ *are convex, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\Big(\frac{\lambda_1+\lambda_2}{2}\Big)\right|\\&\le\frac{(\lambda_2-\lambda_1)}{8[2]_q\,[3]_q}\left[q[2]_q\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|+q\left([3]_q+q^2\right)\left|{}^{\lambda_2}D_q\Upsilon\left(\lambda_2\right)\right|\right]\\&\quad+\frac{(\lambda_2-\lambda_1)}{8[2]_q\,[3]_q}\left[q\left([3]_q+q^2\right)\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|+q[2]_q\left|{}_{\lambda_1}D_q\Upsilon\left(\lambda_2\right)\right|\right].\end{aligned}\tag{4.5}$$

Proof. On taking modulus in (4.1) and using convexity of $\left|{}_{\lambda_1}D_q\Upsilon\right|$ and $\left|{}^{\lambda_2}D_q\Upsilon\right|$, we have

$$\begin{aligned}&\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon\left(x\right)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon\left(x\right)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|\\&\le\frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1 qt\ \left|{}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\right|\mathrm{d}_qt\right.\\&\quad\left.+\int_0^1 qt\ \left|{}_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\right|\mathrm{d}_qt\right]\\&\le\frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1 qt\ \left(\frac{t}{2}\left|{}^{\lambda_2}D_q\Upsilon\left(\lambda_1\right)\right|+\frac{2-t}{2}\left|{}^{\lambda_2}D_q\Upsilon\left(\lambda_2\right)\right|\right)\mathrm{d}_qt\right.\\&\quad\left.+\int_0^1 qt\ \left(\frac{2-t}{2}\left|{}_{\lambda_1}D_q\Upsilon\left(\lambda_1\right)\right|+\frac{t}{2}\left|{}_{\lambda_1}D_q\Upsilon\left(\lambda_2\right)\right|\right)\mathrm{d}_qt\right]\\&=\frac{(\lambda_2-\lambda_1)}{4}\left[\frac{q}{2\,[3]_q}\left|{}^{\lambda_2}D_q\Upsilon\left(\lambda_1\right)\right|+\frac{1}{2}q\frac{[3]_q+q^2}{[2]_q\,[3]_q}\left|{}^{\lambda_2}D_q\Upsilon\left(\lambda_2\right)\right|\right]\\&\quad+\frac{(\lambda_2-\lambda_1)}{4}\left[\frac{1}{2}q\frac{[3]_q+q^2}{[2]_q\,[3]_q}\left|{}_{\lambda_1}D_q\Upsilon\left(\lambda_1\right)\right|+\frac{q}{2\,[3]_q}\left|{}_{\lambda_1}D_q\Upsilon\left(\lambda_2\right)\right|\right].\end{aligned}$$

Thus, the proof is completed. □

Remark 3. In Theorem 4.1, if we set the limit as $q \to 1^-$, then we obtain [19: Theorem 2.2].

Theorem 4.2. *We assume that the conditions of* Lemma 4.1 *hold. If the functions* $|_{\lambda_1}D_q\Upsilon|^s$ *and* $|^{\lambda_2}D_q\Upsilon|^s$, $s \geq 1$ *are convex, then the following inequality holds:*

$$\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|$$
$$\leq \frac{q(\lambda_2-\lambda_1)}{4[2]_q}\left[\left(\frac{[2]_q\left|^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+([3]_q+q^2)\left|^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2[3]_q}\right)^{\frac{1}{s}}\right. \tag{4.6}$$
$$\left.+\left(\frac{([3]_q+q^2)\left|_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+[2]_q\left|_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s}{2[3]_q}\right)^{\frac{1}{s}}\right].$$

Proof. By taking modulus in (4.1) and using the power mean inequality, we have

$$\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|$$
$$\leq \frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1 qt\left|^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\right|\mathrm{d}_qt+\int_0^1 qt\left|_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\right|\mathrm{d}_qt\right]$$
$$\leq \frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1 qt\mathrm{d}_qt\right)^{1-\frac{1}{s}}\left[\left(\int_0^1 qt\left|^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\right|^s\mathrm{d}_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1 qt\left|_{\lambda_1}D_q\Upsilon\Big(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\Big)\right|^s\mathrm{d}_qt\right)^{\frac{1}{s}}\right].$$

Applying convexity of $|_{\lambda_1}D_q\Upsilon|^s$ and $\left|^{\lambda_2}D_q\Upsilon\right|^s$, we have

$$\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|$$
$$\leq \frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1 qt\mathrm{d}_qt\right)^{1-\frac{1}{s}}\left[\left(\int_0^1 qt\left(\frac{t}{2}\left|^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+\frac{2-t}{2}\left|^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s\right)\mathrm{d}_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1 qt\left(\frac{2-t}{2}\left|_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\frac{t}{2}\left|_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s\right)\mathrm{d}_qt\right)^{\frac{1}{s}}\right]$$
$$=\frac{(\lambda_2-\lambda_1)}{4}\left(\frac{q}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{q}{2[3]_q}\left|^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+\frac{1}{2}q\frac{[3]_q+q^2}{[2]_q[3]_q}\left|^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{1}{2}q\frac{[3]_q+q^2}{[2]_q[3]_q}\left|_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\frac{q}{2[3]_q}\left|_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s\right)^{\frac{1}{s}}\right].$$

Thus, the proof is completed. □

Theorem 4.3. *We assume that the conditions of* Lemma 4.1 *hold. If the functions* $|_{\lambda_1}D_q\Upsilon|^s$ *and* $\left|^{\lambda_2}D_q\Upsilon\right|^s$, $s>1$ *are convex, then the following inequality holds:*

$$\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|$$
$$\le\frac{q(\lambda_2-\lambda_1)}{4\left([r+1]_q\right)^{\frac{1}{r}}}\left[\left(\frac{\left|^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+([2]_q+q)\left|^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{([2]_q+q)\left|_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\left|_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right],\tag{4.7}$$

where $s^{-1}+r^{-1}=1$.

Proof. On taking modulus in (4.1) and applying Hölder's inequality, we have

$$\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|$$
$$\le\frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1 qt\ \left|^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\right|\mathrm{d}_qt+\int_0^1 qt\ \left|_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\right|\mathrm{d}_qt\right]$$
$$\le\frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1(qt)^r\mathrm{d}_qt\right)^{\frac{1}{r}}\left[\left(\int_0^1\left|^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\right|^s\mathrm{d}_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left|_{\lambda_1}D_q\Upsilon\Big(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\Big)\right|^s\mathrm{d}_qt\right)^{\frac{1}{s}}\right].$$

Using convexity of $|_{\lambda_1}D_q\Upsilon|^s$ and $\left|^{\lambda_2}D_q\Upsilon\right|^s$, we have

$$\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|$$
$$\le\frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1(qt)^r\mathrm{d}_qt\right)^{\frac{1}{r}}\left[\left(\int_0^1\left(\frac{t}{2}\left|^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+\frac{2-t}{2}\left|^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s\right)\mathrm{d}_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left(\frac{2-t}{2}\left|_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\frac{t}{2}\left|_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s\right)\mathrm{d}_qt\right)^{\frac{1}{s}}\right]$$
$$=\frac{(\lambda_2-\lambda_1)}{4}\left(\frac{q^r}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{\left|^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+([2]_q+q)\left|^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{([2]_q+q)\left|_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\left|_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right].$$

Hence, the proof is completed. □

5. Trapezoid inequalities

In this section, we prove some right-estimates of the newly proved Hermite-Hadamard inequality (3.1) using the q-differentiability of the function.

Lemma 5.1. *Let $\Upsilon\colon [\lambda_1, \lambda_2] \subset \mathbb{R} \to \mathbb{R}$ be a function. If the functions ${}_{\lambda_1}D_q\Upsilon$ and ${}^{\lambda_2}D_q\Upsilon$ are continuous and integrable over $[\lambda_1, \lambda_2]$, then we have the following new equality:*

$$\begin{aligned}&\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}-\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]\\&=\frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1(1-qt)\ {}^{\lambda_2}D_q\Upsilon\Big(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\Big)\mathrm{d}_qt\right.\\&\left.+\int_0^1(qt-1)\ {}_{\lambda_1}D_q\Upsilon\Big(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\Big)\mathrm{d}_qt\right].\end{aligned}\tag{5.1}$$

P r o o f. Let consider

$$\begin{aligned}\int_0^1(1-qt)\ {}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\mathrm{d}_qt&=\int_0^1{}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\mathrm{d}_qt\\&-\int_0^1qt\ {}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\mathrm{d}_qt.\end{aligned}$$

By the equality (4.3), we have

$$\int_0^1qt\ {}^{\lambda_2}D_q\Upsilon\Big(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\Big)\mathrm{d}_qt=-\frac{2}{\lambda_2-\lambda_1}\Upsilon\Big(\frac{\lambda_1+\lambda_2}{2}\Big)+\frac{4}{(\lambda_2-\lambda_1)^2}\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx.\tag{5.2}$$

On the other hand, by (2.2), Definition 4 and Theorem 2.3 we have

$$\begin{aligned}\int_0^1{}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\mathrm{d}_qt&=(1-q)\sum_{n=0}^{\infty}q^n\ {}^{\lambda_2}D_q\Upsilon\left(\frac{q^n}{2}\lambda_1+\frac{2-q^n}{2}\lambda_2\right)\mathrm{d}_qt\\&=(1-q)\sum_{n=0}^{\infty}q^n\ {}^{\lambda_2}D_q\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}q^n+(1-q^n)\lambda_2\right)\mathrm{d}_qt\\&=\frac{2}{\lambda_2-\lambda_1}\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}{}^{\lambda_2}D_q\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\\&=\frac{2}{\lambda_2-\lambda_1}\left[\Upsilon(\lambda_2)-\Upsilon\Big(\frac{\lambda_1+\lambda_2}{2}\Big)\right].\end{aligned}\tag{5.3}$$

By equalities (5.2) and (5.3), we have

$$\int_0^1 (1-qt)\ {}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\mathrm{d}_qt=\frac{2}{\lambda_2-\lambda_1}\Upsilon(\lambda_2)-\frac{4}{(\lambda_2-\lambda_1)^2}\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx. \quad (5.4)$$

In similarly way, we can write

$$\int_0^1 (qt-1)\ {}_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\mathrm{d}_qt=\frac{2}{\lambda_2-\lambda_1}\Upsilon(\lambda_1)-\frac{4}{(\lambda_2-\lambda_1)^2}\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx.$$

This completes the proof. □

Theorem 5.1. *We assume that the conditions of* Lemma 5.1 *hold. If the functions* $|{}_{\lambda_1}D_q\Upsilon|$ *and* $|{}^{\lambda_2}D_q\Upsilon|$ *are convex, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}-\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]\right|\\&\le\frac{(\lambda_2-\lambda_1)}{8[2]_q[3]_q}\left[\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|+\left([3]_q+q+q^2\right)\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|\right]\\&\quad+\frac{(\lambda_2-\lambda_1)}{8[2]_q[3]_q}\left[\left([3]_q+q+q^2\right)|{}_{\lambda_1}D_q\Upsilon(\lambda_1)|+|{}_{\lambda_1}D_q\Upsilon(\lambda_2)|\right].\end{aligned} \quad (5.5)$$

P r o o f. On taking modulus in (5.1) and using convexity of $|{}_{\lambda_1}D_q\Upsilon|$ and $|{}^{\lambda_2}D_q\Upsilon|$, we have

$$\begin{aligned}&\left|\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}-\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]\right|\\&\le\frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1(1-qt)\left|{}^{\lambda_2}D_q\Upsilon\Big(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\Big)\right|\mathrm{d}_qt\right.\\&\quad\left.+\int_0^1(1-qt)\left|{}_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\right|\mathrm{d}_qt\right]\\&\le\frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1(1-qt)\left(\frac{t}{2}\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|+\frac{2-t}{2}\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|\right)\mathrm{d}_qt\right.\\&\quad\left.+\int_0^1(1-qt)\left(\frac{2-t}{2}|{}_{\lambda_1}D_q\Upsilon(\lambda_1)|+\frac{t}{2}|{}_{\lambda_1}D_q\Upsilon(\lambda_2)|\right)\mathrm{d}_qt\right]\\&=\frac{(\lambda_2-\lambda_1)}{4}\left[\frac{1}{2[2]_q[3]_q}\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|+\frac{[3]_q+q+q^2}{2[2]_q[3]_q}\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|\right]\\&\quad+\frac{(\lambda_2-\lambda_1)}{4}\left[\frac{[3]_q+q+q^2}{2[2]_q[3]_q}|{}_{\lambda_1}D_q\Upsilon(\lambda_1)|+\frac{1}{2[2]_q[3]_q}|{}_{\lambda_1}D_q\Upsilon(\lambda_2)|\right].\end{aligned}$$

Thus, the proof is completed. □

Remark 4. In Theorem 5.1, if we set the limit as $q \to 1^-$, then we obtain [14: Theorem 2.2].

THEOREM 5.2. *We assume that the conditions of* Lemma 5.1 *hold. If the functions* $|_{\lambda_1}D_q\Upsilon|^s$ *and* $|^{\lambda_2}D_q\Upsilon|^s$, $s \geq 1$ *are convex, then the following inequality holds:*

$$\left|\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2} - \frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}} \Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x + \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_q x\right]\right|$$
$$\leq \frac{(\lambda_2-\lambda_1)}{4[2]_q}\left[\left(\frac{\left|^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s + \left([3]_q+q+q^2\right)\left|^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2\,[3]_q}\right)^{\frac{1}{s}} + \left(\frac{\left([3]_q+q+q^2\right)|_{\lambda_1}D_q\Upsilon(\lambda_1)|^s + |_{\lambda_1}D_q\Upsilon(\lambda_2)|^s}{2\,[3]_q}\right)^{\frac{1}{s}}\right]. \tag{5.6}$$

P r o o f. By taking modulus in (5.1) and using the power mean inequality, we have

$$\left|\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2} - \frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}} \Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x + \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_q x\right]\right|$$
$$\leq \frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1 (1-qt)\ \left|^{\lambda_2}D_q\Upsilon\Big(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\Big)\right|\mathrm{d}_q t + \int_0^1 (1-qt)\ \left|_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\right|\mathrm{d}_q t\right]$$
$$\leq \frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1 (1-qt)\ \mathrm{d}_q t\right)^{1-\frac{1}{s}}\left[\left(\int_0^1 (1-qt)\ \left|^{\lambda_2}D_q\Upsilon\Big(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\Big)\right|^s\mathrm{d}_q t\right)^{\frac{1}{s}} + \left(\int_0^1 (1-qt)\ \left|_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\right|^s\mathrm{d}_q t\right)^{\frac{1}{s}}\right].$$

Applying convexity of $|_{\lambda_1}D_q\Upsilon|^s$ and $\left|^{\lambda_2}D_q\Upsilon\right|^s$, we have

$$\left|\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2} - \frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}} \Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x + \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_q x\right]\right|$$
$$= \frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1(1-qt)\mathrm{d}_q t\right)^{1-\frac{1}{s}} \times \left[\left(\int_0^1(1-qt)\ \left(\frac{t}{2}\left|^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s + \frac{2-t}{2}\left|^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s\right)\mathrm{d}_q t\right)^{\frac{1}{s}} + \left(\int_0^1 (1-qt)\ \left(\frac{2-t}{2}|_{\lambda_1}D_q\Upsilon(\lambda_1)|^s + \frac{t}{2}|_{\lambda_1}D_q\Upsilon(\lambda_2)|^s\right)\mathrm{d}_q t\right)^{\frac{1}{s}}\right]$$

$$= \frac{(\lambda_2-\lambda_1)}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{1}{2[2]_q[3]_q}\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+\frac{[3]_q+q+q^2}{2[2]_q[3]_q}\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{[3]_q+q+q^2}{2[2]_q[3]_q}\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\frac{1}{2[2]_q[3]_q}\left|{}_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s\right)^{\frac{1}{s}}\right].$$

Thus, the proof is completed. ☐

Theorem 5.3. *We assume that the conditions of* Lemma 5.1 *hold. If the functions* $|{}_{\lambda_1}D_q\Upsilon|^s$ *and* $|{}^{\lambda_2}D_q\Upsilon|^s$, $s>1$ *are convex, then the following inequality holds:*

$$\left|\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}-\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]\right|$$
$$\le\frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1(1-qt)^r\mathrm{d}_qt\right)^{\frac{1}{r}}\left[\left(\frac{\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+([2]_q+q)\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.\tag{5.7}$$
$$\left.+\left(\frac{([2]_q+q)\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\left|{}_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right],$$

where $s^{-1}+r^{-1}=1$.

P r o o f. On taking modulus in (5.1) and applying Hölder's inequality, we have

$$\left|\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}-\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]\right|$$
$$\le\frac{(\lambda_2-\lambda_1)}{4}\left[\int_0^1(1-qt)\ \left|{}^{\lambda_2}D_q\Upsilon\Big(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\Big)\right|\mathrm{d}_qt+\int_0^1(1-qt)\left|{}_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\right|\mathrm{d}_qt\right]$$
$$\le\frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1(1-qt)^r\mathrm{d}_qt\right)^{\frac{1}{r}}\left[\left(\int_0^1\left|{}^{\lambda_2}D_q\Upsilon\left(\frac{t}{2}\lambda_1+\frac{2-t}{2}\lambda_2\right)\right|^s\mathrm{d}_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left|{}_{\lambda_1}D_q\Upsilon\left(\frac{2-t}{2}\lambda_1+\frac{t}{2}\lambda_2\right)\right|^s\mathrm{d}_qt\right)^{\frac{1}{s}}\right].$$

Using convexity of $|{}_{\lambda_1}D_q\Upsilon|^s$ and $\left|{}^{\lambda_2}D_q\Upsilon\right|^s$, we have

$$\left|\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}-\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]\right|$$
$$\le\frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1(1-qt)^r\mathrm{d}_qt\right)^{\frac{1}{r}}\left[\left(\int_0^1\left(\frac{t}{2}\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+\frac{2-t}{2}\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s\right)\mathrm{d}_qt\right)^{\frac{1}{s}}\right.$$

$$+\left(\int_0^1 \left(\frac{2-t}{2}\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\frac{t}{2}\left|{}_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s\right){\rm d}_qt\right)^{\frac{1}{s}}\Bigg]$$

$$=\frac{(\lambda_2-\lambda_1)}{4}\left(\int_0^1(1-qt)^r{\rm d}_qt\right)^{\frac{1}{r}}\left[\left(\frac{\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+([2]_q+q)\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(\frac{([2]_q+q)\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\left|{}_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right].$$

Hence, the proof is completed. □

6. Examples

In this section, we give some examples to support newly established inequalities.

***Example* 2.** For a convex functions $\Upsilon\colon [0,2]\to\mathbb{R}$ is defined by $\Upsilon(x)=x+4$. From Theorem 4.1 with $q=\frac{3}{4}$, the left hand side of the inequality (4.5) becomes

$$\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}{\rm d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}{\rm d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|$$

$$=\left|\frac{1}{2}\left[\int_0^1(x+4)\ {}_0{\rm d}_{\frac{3}{4}}x+\int_1^2(x+4)\ {}^2{\rm d}_{\frac{3}{4}}x\right]-5\right|=0$$

and the right side becomes

$$\frac{(\lambda_2-\lambda_1)}{8[2]_q[3]_q}\left[q[2]_q\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|+q\left([3]_q+q^2\right)\left|{}^{\lambda_2}D_q\Upsilon\left(\lambda_2\right)\right|\right]$$

$$+\frac{(\lambda_2-\lambda_1)}{8[2]_q[3]_q}\left[q\left([3]_q+q^2\right)\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|+q[2]_q\left|{}_{\lambda_1}D_q\Upsilon\left(\lambda_2\right)\right|\right]=\frac{1}{2}\left[\frac{3}{7}+\frac{3}{7}\right]$$

$$=0.42.$$

It is clear that $0<0.42$ which demonstrate the inequality (4.5).

***Example* 3.** For a convex functions $\Upsilon\colon [0,2]\to\mathbb{R}$ is defined by $\Upsilon(x)=x+4$. From Theorem 4.2 with $q=\frac{3}{4}$ and $s=\frac{1}{3}$, the left hand side of the inequality (4.6) becomes

$$\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}{\rm d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}{\rm d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|$$

$$=\left|\frac{1}{2}\left[\int_0^1(x+4)\ {}_0{\rm d}_{\frac{3}{4}}x+\int_1^2(x+4)\ {}^2{\rm d}_{\frac{3}{4}}x\right]-5\right|=0$$

and the right side becomes

$$\frac{q(\lambda_2-\lambda_1)}{4[2]_q}\left[\left(\frac{[2]_q\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+\left([3]_q+q^2\right)\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2[3]_q}\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{\left([3]_q+q^2\right)\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+[2]_q\left|{}_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s}{2[3]_q}\right)^{\frac{1}{s}}\right]=0.57$$

It is clear that $0<0.57$ which demonstrate the inequality (4.6).

***Example* 4.** For a convex functions $\Upsilon\colon [0,2]\to\mathbb{R}$ is defined by $\Upsilon(x)=x+4$. From Theorem 4.3 with $q=\frac{3}{4}$, $s=\frac{3}{2}$ and $r=3$, the left hand side of the inequality (4.7) becomes

$$\left|\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]-\Upsilon\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|$$
$$=\left|\frac{1}{2}\left[\int_0^1(x+4)\ {}_0\mathrm{d}_{\frac{3}{4}}x+\int_1^2(x+4)\ {}^2\mathrm{d}_{\frac{3}{4}}x\right]-5\right|=0$$

and the right side becomes

$$\frac{q(\lambda_2-\lambda_1)}{4}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s+([2]_q+q)\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{([2]_q+q)\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s+\left|{}_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right]=0.53.$$

It is clear that $0<0.53$ which demonstrate the inequality (4.7).

***Example* 5.** For a convex functions $\Upsilon\colon [0,2]\to\mathbb{R}$ is defined by $\Upsilon(x)=x+4$. From Theorem 5.1 with $q=\frac{3}{4}$, the left hand side of the inequality (5.5) becomes

$$\left|\frac{\Upsilon(\lambda_1)+\Upsilon(\lambda_2)}{2}-\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_qx+\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_qx\right]\right|$$
$$=\left|5-\frac{1}{2}\left[\int_0^1(x+4)\ {}_0\mathrm{d}_{\frac{3}{4}}x+\int_1^2(x+4)\ {}^2\mathrm{d}_{\frac{3}{4}}x\right]\right|=0$$

and right side becomes

$$\frac{(\lambda_2-\lambda_1)}{8[2]_q[3]_q}\left[\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|+\left([3]_q+q+q^2\right)\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|\right]$$
$$+\frac{(\lambda_2-\lambda_1)}{8[2]_q[3]_q}\left[\left([3]_q+q+q^2\right)\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|+\left|{}_{\lambda_1}D_q\Upsilon(\lambda_2)\right|\right]=0.57.$$

It is clear that $0<0.57$ which demonstrate the inequality (5.5).

***Example* 6.** For a convex functions $\Upsilon\colon [0,2] \to \mathbb{R}$ is defined by $\Upsilon(x) = x + 4$. From Theorem 5.2 with $q = \frac{3}{4}$ and $s = \frac{1}{3}$, the left hand side of the inequality (5.6) becomes

$$\left|\frac{\Upsilon(\lambda_1) + \Upsilon(\lambda_2)}{2} - \frac{1}{\lambda_2 - \lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}} \Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x + \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_q x\right]\right|$$

$$= \left|5 - \frac{1}{2}\left[\int_0^1 (x+4)\ {}_0\mathrm{d}_{\frac{3}{4}} x + \int_1^2 (x+4)\ {}^2\mathrm{d}_{\frac{3}{4}} x\right]\right| = 0$$

and the right side becomes

$$\frac{(\lambda_2 - \lambda_1)}{4[2]_q}\left[\left(\frac{\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s + \left([3]_q + q + q^2\right)\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2\,[3]_q}\right)^{\frac{1}{s}} + \left(\frac{\left([3]_q + q + q^2\right)\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s + \left|{}_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s}{2\,[3]_q}\right)^{\frac{1}{s}}\right] = 0.57$$

It is clear that $0 < 0.57$ which demonstrate the inequality (5.6).

***Example* 7.** For a convex functions $\Upsilon\colon [0,2] \to \mathbb{R}$ is defined by $\Upsilon(x) = x + 4$. From Theorem 5.3 with $q = \frac{3}{4}$, $s = \frac{3}{2}$ and $r = 3$, the left hand side of the inequality (5.7) becomes

$$\left|\frac{\Upsilon(\lambda_1) + \Upsilon(\lambda_2)}{2} - \frac{1}{\lambda_2 - \lambda_1}\left[\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}} \Upsilon(x)\ {}_{\lambda_1}\mathrm{d}_q x + \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \Upsilon(x)\ {}^{\lambda_2}\mathrm{d}_q x\right]\right|$$

$$= \left|5 - \frac{1}{2}\left[\int_0^1 (x+4)\ {}_0\mathrm{d}_{\frac{3}{4}} x + \int_1^2 (x+4)\ {}^2\mathrm{d}_{\frac{3}{4}} x\right]\right| = 0$$

and the right side becomes

$$\frac{(\lambda_2 - \lambda_1)}{4}\left(\int_0^1 (1 - qt)^r \mathrm{d}_q t\right)^{\frac{1}{r}}\left[\left(\frac{\left|{}^{\lambda_2}D_q\Upsilon(\lambda_1)\right|^s + ([2]_q + q)\left|{}^{\lambda_2}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}} + \left(\frac{([2]_q + q)\left|{}_{\lambda_1}D_q\Upsilon(\lambda_1)\right|^s + \left|{}_{\lambda_1}D_q\Upsilon(\lambda_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right] = 0.66.$$

It is clear that $0 < 0.66$ which demonstrate the inequality (5.7).

7. Conclusions

In this work, we proved a new version of q-Hermite-Hadamard inequality for convex functions through left and right q-integrals. We also proved some new midpoint and trapezoid type inequalities for left and right q-differentiable convex functions. It is new and interesting problem that the upcoming researchers can obtain the similar inequalities for co-ordinated convex functions.

REFERENCES

[1] ALI, M. A.—BUDAK, H.—ZHANG, Z.—YILDRIM, H.: *Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus*, Math. Methods Appl. Sci. **44** (2021), 4515–4540.

[2] ALI, M. A.—CHU, Y.-M.—BUDAK, H.—AKKURT, A.—YILDRIM, H.: *Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables*, Adv. Differ. Equ. **2021** (2021), Art. No. 25.

[3] ALI, M. A.—ALP, N.—BUDAK, H.—CHU, Y.-M.—ZHANG, Z.: *On some new quantum midpoint type inequalities for twice quantum differentiable convex functions*, Open Math. **19** (2021), 427–439.

[4] ALI, M. A.—BUDAK, H.—ABBAS, M.—CHU, Y.-M.: *Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second q^b-derivatives*, Adv. Differ. Equ. **2021** (2021), Art. No. 7.

[5] ALI, M. A.—ABBAS, M.—BUDAK, H.—AGARWAL, P.—MURTAZA G.—CHU, Y.-M.: *New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions*, Adv. Differ. Equ. **2021** (2021), Art. No. 64.

[6] ALP, N.—SARIKAYA, M. Z.—KUNT, M.—İŞCAN, İ.: *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, J. King Saud Univ. Sci. **30** (2018), 193–203.

[7] ALP, N.—SARIKAYA, M. Z.: *Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral*, Appl. Math. E-Notes **20** (2020), 341–356.

[8] BERMUDO, S.—KÓRUS, P.—VALDÉS, J. N.: *On q-Hermite-Hadamard inequalities for general convex functions*, xActa Math. Hungar. **162** (2020), 364–374.

[9] BUDAK, H.—ALI, M. A.—TARHANACI, M.: *Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions*, J. Optim. Theory Appl. **186** (2020), 899–910.

[10] BUDAK, H.—ERDEN, S.—ALI, M. A.: *Simpson and Newton type inequalities for convex functions via newly defined quantum integrals*, Math. Methods Appl. Sci. **44** (2020), 378–390.

[11] BUDAK, H.—ALI, M. A.—ALP, N.—CHU, Y.-M.: *Quantum Ostrowski type integral inequalities*, J. Math. Inequal. 2021, in press.

[12] DING, Y.—KALSOOM, H.—WU, S.: *Some new quantum Hermite–Hadamard-type estimates within a class of generalized (s, m)-preinvex functions*, Symmetry **11** (2019), Art. No. 1283.

[13] DRAGOMIR, S. S.—PEARCE, C. E. M.: *Selected Topics on Hermite-Hadamard Inequalities and Applications.* RGMIA Monographs, Victoria University, 2000.

[14] DRAGOMIR, S. S.—AGARWAL, R.: *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), 91–95.

[15] JACKSON, F. H.: *On a q-definite integrals*, Quarterly J. Pure Appl. Math. **41** (1910) 193–203.

[16] JHANTHANAM, S.—TARIBOON, J.—NTOUYAS, S. K.—NONLAOPON, N.: *On q-Hermite-Hadamard inequalities for differentiable convex functions*, Mathematics **7** (2019), Art. No. 632.

[17] KAC, V.—CHEUNG, P.: *Quantum Calculus*, Springer, 2001.

[18] KALSOOM, H.—WU, J.-D.—HUSSAIN, S.—LATIF, M. A.: *Simpson's type inequalities for co-ordinated convex functions on quantum calculus*, Symmetry **11** (2019), Art. No. 768.

[19] KIRMACI, U. S.: *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput. **147** (2004), 137–146.

[20] LIU, W.—HEFENG, Z.: *Some quantum estimates of Hermite-Hadamard inequalities for convex functions*, J. Appl. Anal. Comput. **7** (2016), 501–522.

[21] NOOR, M. A.—NOOR, K. I.—AWAN, M. U.: *Some quantum estimates for Hermite-Hadamard inequalities*, Appl. Math. Comput. **251** (2015), 675–679.

[22] NOOR, M. A.—NOOR, K. I.—AWAN, M. U.: *Some quantum integral inequalities via preinvex functions*, Appl. Math. Comput. **269** (2015), 242–251.

[23] SARIKAYA, M. Z.—YILDRIM, H.: *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Math. Notes **17** (2017), 1049–1059.

[24] SIAL, I. B.—MEI, S.—ALI, M. A.—NANLAOPON, K.: *On some generalized Simpson's and Newton's inequalities for (λ_1, m)-convex functions in q-calculus*, Mathematics **2021** (2021), Art. No. 3266.

[25] SOONTHARANON, J.—ALI, M. A.—BUDAK, H.—NANLAOPON, K.—ABDULLAH, Z.: *Simpson's and Newton's inequalities for (α, m)-convex functions via quantum calculus*, Symmetry **14** (2022), Art. No. 736.

[26] TARIBOON, J.—NTOUYAS, S. K.: *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Differ. Equ. **2013** (2013), Art. No. 282.

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