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A NEW WEIGHTED GOMPERTZ DISTRIBUTION WITH APPLICATIONS TO RELIABILITY DATA

Hassan S. Bakouch, Ahmed M. T. Abd El-Bar, Tanta

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Abstract. A new weighted version of the Gompertz distribution is introduced. It is noted that the model represents a mixture of classical Gompertz and second upper record value of Gompertz densities, and using a certain transformation it gives a new version of the two-parameter Lindley distribution. The model can be also regarded as a dual member of the log-Lindley-X family. Various properties of the model are obtained, including hazard rate function, moments, moment generating function, quantile function, skewness, kurtosis, conditional moments, mean deviations, some types of entropy, mean residual lifetime and stochastic orderings. Estimation of the model parameters is justified by the method of maximum likelihood. Two real data sets are used to assess the performance of the model among some classical and recent distributions based on some evaluation goodness-of-fit statistics. As a result, the variance-covariance matrix and the confidence interval of the parameters, and some theoretical measures have been calculated for such data for the proposed model with discussions.

Keywords: continuous distribution; distributional properties; weight function; estimation; estimated survival function

MSC 2010: 60E05, 60E99, 62E15

1. Introduction

Selection of random samples with equal probabilities is preferred in order to obtain valid estimators of the parameters. Although, there are situations where it is not possible to do this selection. In such situations recorded data are biased because of no well sampling frames and hence such data do not follow the usual probability distributions. Therefore, modeling such data needs a family of probability distributions denoted as the family of weighted distributions, where the probability of a data value is reweighted by a weight function to guarantee equal representation of all data. The weighted distributions are found to be very useful in reliability, survival, forestry

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and ecological applications. The paper [38] was the first paper to apply them in connection with sampling of wood cell, while [19] applied a weighted distribution to analyze horizontal point sampling diameter increment data. Weighted distributions were also used in [20] to recover the distribution of canopy heights from air-borne laser scanner measurement. The weighted distribution is defined in [29] as follow. For a nonnegative random variable X with density function $f(x)$ and a nonnegative weight function $w(x)$ with finite nonzero expectation, the weighted random variable X has the density function

$$f_w(x) = w(x)f(x)/E[w(x)],$$

where $E[w(x)]$ is the expected value of $w(x)$ and represents a normalizing constant. There are many weighted distributions introduced in the statistical literature, among them the recent ones by the following authors. The paper [12] constructed a new class of weighted exponential distribution using $w(x) = 1 - \mathrm{e}^{-\alpha\lambda x}$, $\alpha, \lambda > 0$, [8] proposed a two parameter weighted Lindley distribution with the size-bias weight function $w(x) = x^{c-1}$, $c > 0$, [14] introduced a class of weighted gamma distributions using the same weight function as in [12], and [3] introduced a new weighted version of the Lindley distribution. In this paper, we introduce a new weighted version of the Gompertz distribution. The Gompertz distribution has the density function

$$g(x) = \lambda\sigma\mathrm{e}^{\lambda x - \sigma(\mathrm{e}^{\lambda x} - 1)}, \quad x > 0;\ \lambda, \sigma > 0, \tag{1.1}$$

and its survival function is

$$\overline{G}(x) = \mathrm{e}^{-\sigma(\mathrm{e}^{\lambda x} - 1)}.$$

The Gompertz distribution plays an important role in modeling reliability, survival times, human mortality and actuarial data that have hazard rate with exponential increase. Therefore, it has received considerable attention from demographers and actuaries. Some applications of the Gompertz distribution can be found in [30] and [24]. The monograph [22] discussed the Gompertz distribution with a negative rate of aging parameter. The paper [39] provided connections between the Gompertz and other related distributions such as the Weibull and type I extreme value distributions. Another version of Gompertz is the shifted Gompertz distribution discussed by [5]. Recently, [17] provided an explicit expression of the expectation and variance with limit distributions of extreme order statistics for the shifted Gompertz, and [16] studied its parameter estimation via the least squares, maximum likelihood and moments methods. The paper [6] proposed a generalization of the Gompertz distribution that admits increasing, decreasing and bathtub hazard rate shapes by raising the cumulative distribution function of the classical Gompertz by an exponent. In what follows we introduce a new weighted version of the Gompertz distribution, the so-called weighted Gompertz (WGo) distribution.

Definition 1.1. A random variable X is said to follow a weighted Gompertz distribution, if its probability density function (pdf) has the form

$$f(x) = \frac{\lambda\sigma^2}{1+\lambda\sigma}(\lambda + \mathrm{e}^{\lambda x} - 1)\mathrm{e}^{\lambda x - \sigma(\mathrm{e}^{\lambda x}-1)}, \quad x > 0;\ \lambda, \sigma > 0. \tag{1.2}$$

Density function plots of the WGo distribution are displayed in Figure 1 for different values of the parameters λ and σ.

The cumulative distribution function (cdf) of the WGo distribution is

$$F(x) = 1 - \Big(1 + \frac{\sigma(\mathrm{e}^{\lambda x} - 1)}{1+\lambda\sigma}\Big)\mathrm{e}^{-\sigma(\mathrm{e}^{\lambda x}-1)}. \tag{1.3}$$

As a result of (1.2) and (1.3), the survival function and the hazard rate function of the WGo distribution can be written as

$$S(x) = \Big(1 + \frac{\sigma(\mathrm{e}^{\lambda x} - 1)}{1+\lambda\sigma}\Big)\mathrm{e}^{-\sigma(\mathrm{e}^{\lambda x}-1)} \tag{1.4}$$

and

$$h(x) = \frac{\lambda\sigma^2(\lambda + \mathrm{e}^{\lambda x} - 1)\mathrm{e}^{\lambda x}}{1 + (\lambda + \mathrm{e}^{\lambda x} - 1)\sigma}.$$

Now, we give some motivations for the new WGo distribution:

Motivation 1.1. The pdf of the WGo distribution defined by (1.2) is obtained by taking $f(x)$ to be the density of the classical Gompertz distribution defined by (1.1) with the weight function $w(x) = \lambda + \mathrm{e}^{\lambda x} - 1$, $x > 0$, $\lambda > 0$, and the expectation of $w(x)$ is $E[w(x)] = (1 + \lambda\sigma)/\sigma$.

Motivation 1.2. The pdf (1.2) can be rewritten as

$$f(x) = \frac{\lambda\sigma}{1+\lambda\sigma}g(x) + \frac{1}{1+\lambda\sigma}k(x),$$

where $g(x)$ is the density defined by (1.1) and $k(x)$ is the pdf of the 2nd upper record value of the classical Gompertz distribution given by

$$k(x) = \lambda\sigma^2(\mathrm{e}^{\lambda x} - 1)\mathrm{e}^{\lambda x - \sigma(\mathrm{e}^{\lambda x}-1)}, \quad x > 0;\ \lambda, \sigma > 0.$$

Thus, the density in (1.2) reveals that the WGo distribution is a two-component mixture of the classical Gompertz and the 2nd upper record value of Gompertz densities with mixing proportions $\lambda\sigma/(1 + \lambda\sigma)$ and $1/(1 + \lambda\sigma)$, respectively.

Motivation 1.3. Using the log-Lindley family of distributions proposed by [9] and the integral transformation

$$S_{\overline{G}}(x;\lambda,\sigma) = \frac{\sigma^2}{1+\lambda\sigma}\int_0^{\overline{G}(x)} (\lambda - \log t)t^{\sigma-1}\,\mathrm{d}t,$$

where $0 < t < 1$ and $\overline{G}(x)$ is the survival function of a distribution with density function $g(x)$, we get the log-Lindley-X family of distributions with the density function

$$f(x;\lambda,\sigma) = \frac{\sigma^2}{1+\lambda\sigma}(\lambda - \log\overline{G}(x))(\overline{G}(x))^{\sigma-1}g(x). \tag{1.5}$$

The density defined by (1.5) can be viewed as a result of $T - X$ family introduced in [2].

Let $\overline{G}(x)$ be the survival function of the classical Gompertz with density (1.1), then equation (1.5) gives the WGo density defined by (1.2). Therefore, the WGo distribution can be regarded as a dual member of the family of log-Lindley-X family.

Motivation 1.4. Based on relations of reliability measures of weighted distributions investigated by [13], we conclude the following. As the weight function $w(x) = \lambda + \mathrm{e}^{\lambda x} - 1$ is increasing and concave up, and the classical Gompertz with density (1.1) has an increasing hazard rate, hence the hazard rate of WGo distribution defined by (1.5) is increasing. Also, Figure 2 confirms this result again.

Motivation 1.5. Under X having the WGo distribution defined by (1.2) and the transformation $Y = \mathrm{e}^{\lambda X} - 1$, the distribution of Y follows the two-parameter Lindley distribution with the cdf and pdf

$$F(y) = 1 - \Big(1 + \frac{\sigma y}{1+\lambda\sigma}\Big)\mathrm{e}^{-\sigma y}, \quad y > 0;\ \lambda,\sigma > 0,$$

and

$$f(y) = \frac{\sigma^2(\lambda + y)}{1+\lambda\sigma}\mathrm{e}^{-\sigma y},$$

respectively, which was not proposed before as only [33] and [4] introduced different forms of the Lindley distribution with two parameters. Further, in the two equations above, when $\lambda = 1$ we get the classical Lindley distribution that is used to study stress strength reliability modeling (see [1]).

The rest of this paper is organized as follows. In Section 2, we present various properties of the WGo distribution such as moment generating function, quantile function, skewness, kurtosis, conditional moments and mean deviations. Three popular entropies are investigated in Section 3, namely Shannon entropy, Rényi entropy and Mathai-Haubold entropy and we get some numerical values for each one. Some measures of residual lifetime and reversed residual lifetime of the WGo distribution are obtained in Section 4, such as density, survival and hazard rate functions with mean and variance. In Section 5, stochastic ordering is used to compare WGo random variables (rvs) and classical Gompertz rvs. Estimation of the distribution parameters with the observed information matrix are verified in Section 6. Also, the applicability of the WGo distribution is shown by considering two reliability data sets, and related measures are obtained for both the data sets under the WGo distribution.

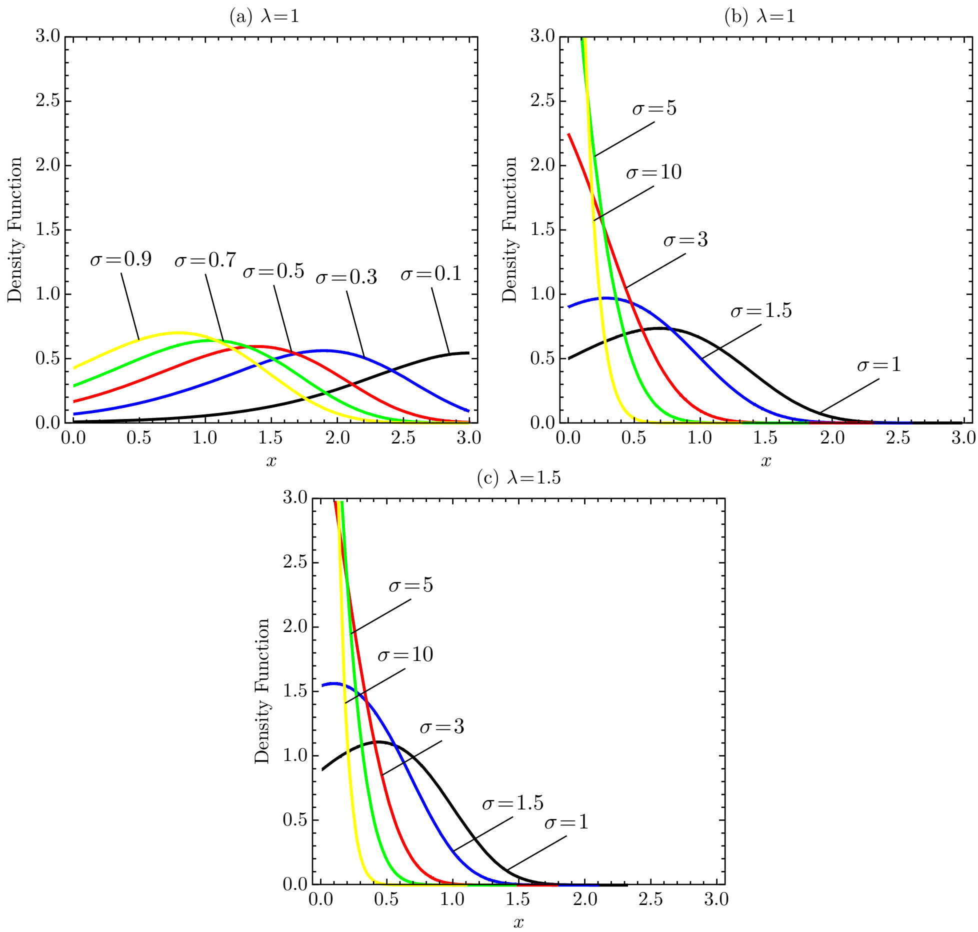

Figure 1. Plots of the density function (1.2) for some values of the parameters.

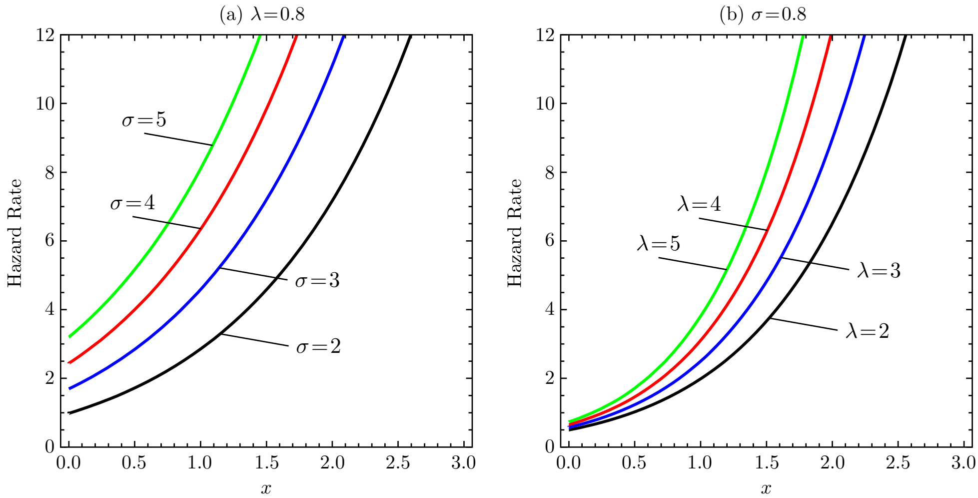

Figure 2. Plots of the WGo hazard function for some values of the parameters.

2. Statistical properties of the model

In this section, we obtain some properties of the model, including the moments, moment generating function, quantile function, skewness, kurtosis, conditional moments, and mean deviations.

2.1. Moments and moment generating function.

Proposition 2.1. *Let X be a random variable with the WGo density function (1.2). Then the rth moment about the origin and the moment generating function (mgf) of X are, respectively, given by*

$$E(X^r) = \sum_{j=0}^{\infty} \nu_j \Big(\frac{(\lambda - 1)}{(j+1)^{r+1}} + \frac{1}{(j+2)^{r+1}} \Big), \tag{2.1}$$

where

$$\nu_j = (-1)^{r+j+1} \frac{\mathrm{e}^{\sigma} \sigma^{j+2}}{(1+\lambda\sigma)\lambda^r} \frac{\Gamma(r+1)}{\Gamma(j+1)}, \quad r = 1, 2, \ldots,$$

and

$$M_X(t) = \frac{\mathrm{e}^{\sigma}\sigma^{-t/\lambda}}{1+\lambda\sigma} \Big((\lambda - 1)\sigma\Gamma\Big(\frac{t}{\lambda} + 1, \sigma\Big) + \Gamma\Big(\frac{t}{\lambda} + 2, \sigma\Big) \Big).$$

Proof. Using the definition of the moment about the origin with the series expansion of $\mathrm{e}^{-\sigma \mathrm{e}^{\lambda x}}$ and after some algebraic manipulation, we obtain (2.1). Meanwhile, the mgf of X is

$$M_X(t) = E(\mathrm{e}^{tx}) = \frac{\lambda\sigma^2}{1+\lambda\sigma}\int_0^\infty \mathrm{e}^{tx}(\lambda + \mathrm{e}^{\lambda x} - 1)\mathrm{e}^{\lambda x - \sigma(\mathrm{e}^{\lambda x}-1)}\,\mathrm{d}x.$$

Letting $y = \sigma(\mathrm{e}^{\lambda x} - 1)$ implies

$$M_X(t) = \frac{\sigma}{1+\lambda\sigma}\int_0^\infty \Big(\frac{y+\sigma}{\sigma}\Big)^{t/\lambda}\mathrm{e}^{-y}\Big(\lambda + \frac{y+\sigma}{\sigma} - 1\Big)\,\mathrm{d}y,$$

hence using the transformation $u = (y+\sigma)/\sigma$ completes the proof.

□

In particular, using (2.1), the mean of the WGo distribution follows as

$$E(X) = \frac{\lambda\sigma^2\mathrm{e}^\sigma}{1+\lambda\sigma}\sum_{j=0}^\infty \frac{(-1)^j\sigma^j}{\Gamma(j+1)}\Big(\frac{\lambda-1}{\lambda^2(j+1)^2} + \frac{1}{\lambda^2(j+2)^2}\Big).$$

Some numerical values for the mean and variance of the WGo distribution are displayed in Table 1 for some arbitrary choices of the distribution parameters. It is observed that both of them decrease as the values of the parameters increase.

Parameters	$\sigma = 0.1$	
$\lambda \downarrow$	Mean	Variance
1	2.74058	0.59728
1.5	1.80601	0.279711
2	1.34004	0.16425
2.5	1.06139	0.108953
3	0.8763	0.779791
3.5	0.744614	0.0587793
$\sigma \downarrow$	$\lambda = 0.8$	
0.2	2.62243	0.810304
0.4	1.85985	0.642433
0.6	1.45995	0.517256
0.8	1.20476	0.423683
1	1.02575	0.352562
1.2	0.892672	0.297501

Table 1. Mean and variance for some arbitrary parameter values.

2.2. Quantile function. Quantiles are fundamental for estimation and simulation of a distribution parameter, so we provide them for the WGo distribution in the next proposition. In it we give an explicit expression for quantile function $Q(u)$ in terms of the Lambert W function. For more details on W see [18].

Proposition 2.2. *For a nonnegative continuous random variable X that follows the WGo distribution, the quantile function $Q(u)$ is given by*

$$X = Q(u) = \frac{1}{\lambda}\ln\Big(1-\lambda-\frac{1}{\sigma}-\frac{1}{\sigma}W[(u-1)(1+\lambda\sigma)\mathrm{e}^{-1-\lambda-\sigma}]\Big),\ 0<u<1, \tag{2.2}$$

where $W(\cdot)$ is the Lambert function.

Proof. For any $0<u<1$, we solve $F(x)=u$, $x>0$, with respect to x, that is

$$\Big(1+\frac{\sigma(\mathrm{e}^{\lambda x}-1)}{1+\lambda\sigma}\Big)\mathrm{e}^{-\sigma(\mathrm{e}^{\lambda x}-1)} = 1-u.$$

This equation can be written as

$$\ln\Big(1+\frac{\sigma(\mathrm{e}^{\lambda x}-1)}{1+\lambda\sigma}\Big)-\sigma(\mathrm{e}^{\lambda x}-1) = \ln(1-u),$$

hence, from [18], we get the required proof. □

In particular, the median can be written as

$$Q(0.5) = \frac{1}{\lambda}\ln\Big(1-\lambda-\frac{1}{\sigma}-\frac{1}{\sigma}W\Big[-\frac{(1+\lambda\sigma)}{2}\mathrm{e}^{-1-\lambda-\sigma}\Big]\Big).$$

2.3. Skewness and kurtosis based on quantiles. Skewness measures the degree of the long tail while kurtosis is a measure of the degree of tail heaviness. Based on quantile function $Q(\cdot)$. In [7] and [25] the skewness and kurtosis are defined, respectively, as

$$S_G = \frac{Q(\frac{3}{4})-2Q(\frac{1}{2})+Q(\frac{1}{4})}{Q(\frac{3}{4})-Q(\frac{1}{4})},$$

and

$$K_M = \frac{Q(\frac{7}{8})-Q(\frac{5}{8})-Q(\frac{3}{8})+Q(\frac{1}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})}.$$

Therefore, Galton's skewness and Moors' kurtosis of the quantile function defined by (2.2) can be obtained easily. Plots of those skewness and kurtosis for selected values of σ as functions of λ, and for selected values of λ as functions of σ are shown in Figures 3 and 4, respectively. These plots indicate that both measures increase when σ increases and decrease when λ increases.

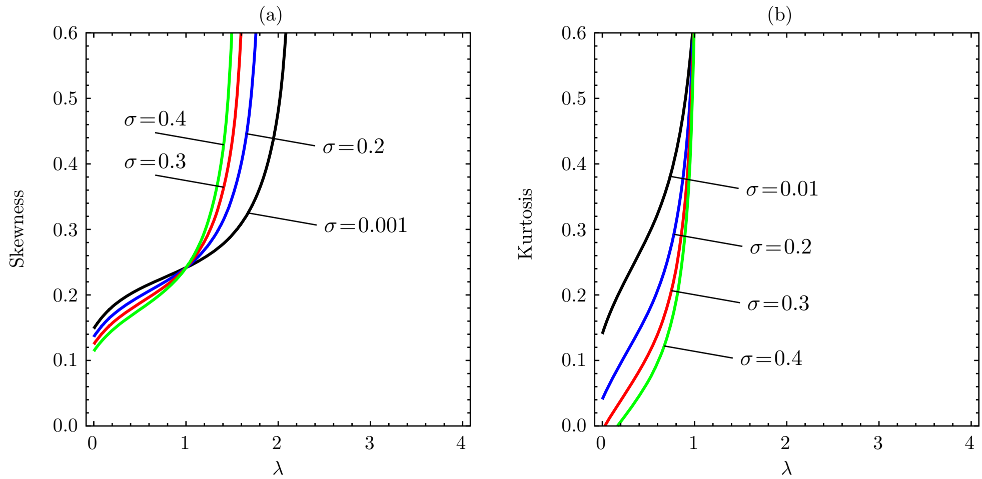

Figure 3. Skewness and kurtosis for the WGo distribution as a function of λ, for some values of σ.

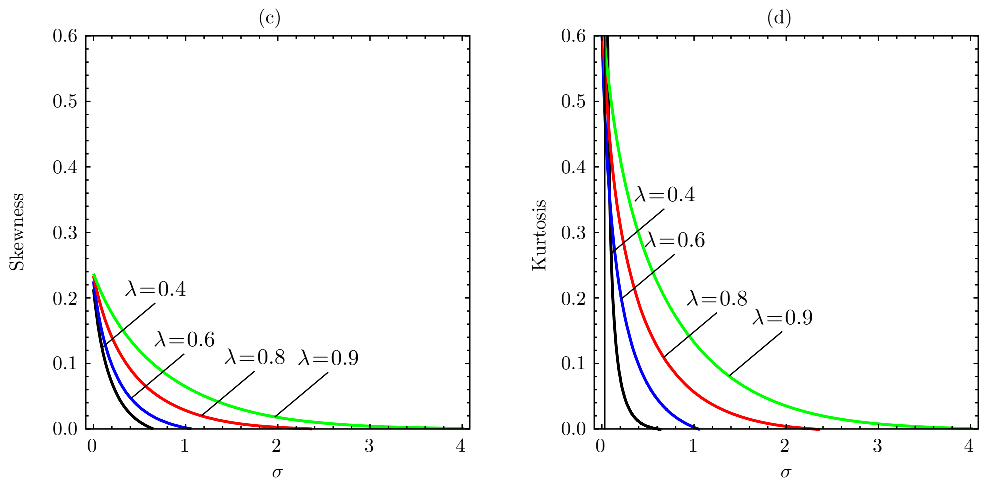

Figure 4. Skewness and kurtosis for the WGo distribution as a function of σ, for some values of λ.

At some places of the next section, we will make use of the following lemma.

Lemma 2.1. *Let*

$$\begin{aligned} J(z; r, \lambda, \sigma) &= \int_0^z x^r f(x)\,\mathrm{d}x \\ &= \frac{\lambda\sigma^2 \mathrm{e}^{\sigma}}{1+\lambda\sigma} \sum_{j=0}^{\infty} \frac{(-1)^j \sigma^j}{j!} \int_0^z x^r (\lambda - 1 + \mathrm{e}^{\lambda x}) \mathrm{e}^{\lambda(j+1)x}\,\mathrm{d}x, \quad r = 1, 2, \ldots. \end{aligned}$$

Then we have

$$J(z;r,\lambda,\sigma)=\frac{\sigma^2\mathrm{e}^{\sigma}}{1+\lambda\sigma}\sum_{j=0}^{\infty}\frac{(-1)^{j-r-1}\sigma^j}{\lambda^r\Gamma(j+1)}$$
$$\times\Big(\frac{1}{(j+1)^{r+1}}(\Gamma(r+1)-\Gamma(r+1,-(j+1)z\lambda))$$
$$+\frac{1}{(j+2)^{r+1}}(\Gamma(r+1)-\Gamma(r+1,-(j+2)z\lambda))\Big).$$

2.4. Mode, conditional moments, and mean deviations. By differentiating the pdf of the WGo distribution with respect to x as

$$f'(x)=f(x)\Big(\lambda-\sigma\lambda\mathrm{e}^{\lambda x}+\frac{\lambda\mathrm{e}^{\lambda x}}{\lambda+\mathrm{e}^{\lambda x}-1}\Big),$$

and since $f(x)\succ 0$, the mode is the solution of the equation with respect to x

$$\lambda(\lambda+\mathrm{e}^{\lambda x}-1)+\lambda\mathrm{e}^{\lambda x}(1-\sigma(\lambda+\mathrm{e}^{\lambda x}-1))=0.$$

The above expression can be solved numerically via Mathematica Package.

In connection with lifetime distribution, it is important to determine the conditional moments $E(X^r\mid X>t)$, $r=1,2,\ldots$, which are of interest in the predictive inference.

Proposition 2.3. *The conditional moment of the WGo distribution is*

$$E(X^r\mid X>t)=\frac{1+\lambda\sigma}{1+\lambda\sigma+\sigma(\mathrm{e}^{\lambda t}-1)}\mathrm{e}^{\sigma(\mathrm{e}^{\lambda t}-1)}(E(X^r)-J(t;r,\lambda,\sigma)),$$

where $E(X^r)$ is defined by (2.1) *and $J(t;r,\lambda,\sigma)$ is given by Lemma* 2.1.

Proof. The proof follows by applying the identity

$$E(X^r\mid X>t)=\frac{1}{S(t)}\Big(E(X^r)-\int_0^t x^r f(x)\,\mathrm{d}x\Big).$$

□

The mean deviations provide useful information about the characteristics of a population, namely the amount of dispersion, and it can be obtained from the first incomplete moment. The mean deviations of X about the mean $\mu=E(X)$ and about the median M can be expressed as $\delta=2\mu F(\mu)-2m(\mu)$ and $\mu=\mu-2m(M)$, where $F(\mu)$ is obtained from (1.3) and $m(z)=J(z;1,\lambda,\sigma)$, where $J(z;1,\lambda,\sigma)$ can be obtained from Lemma 2.1 for $r=1$.

3. Entropies

Entropy is a measure of randomness of systems which is widely used in areas like physics, molecular imaging of tumors and sparse kernel density estimation. Three popular entropy measures are the Shannon entropy [34], Rényi entropy [31] and the Mathai-Haubold entropy [23] defined by

$$\eta_x = E(-\log f(x)), \tag{3.1}$$

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\Big(\int_{\mathbb{R}} f^{\gamma}(x)\,\mathrm{d}x\Big), \tag{3.2}$$

and

$$J_{MH}(\delta) = \frac{\int_{\mathbb{R}}(f(x))^{2-\delta}\,\mathrm{d}x - 1}{\delta - 1}, \tag{3.3}$$

respectively, where $\gamma > 0$, $\gamma \neq 1$, $\delta \neq 1$, and $\delta < 2$. The $J_{MH}(\delta)$ entropy is an inaccuracy measure through disturbance or distortion of systems, and recall that $1-\delta$ represents the strength of information in the distribution of interest.

Some recent applications of such entropies are as follows. Shannon entropy is used to classify emergent behavior in a simulation of laser dynamics [10]; Rényi entropy is used to estimate the number of components of a multicomponent nonstationary signal [37] and to identify cardiac autonomic neuropathy in diabetes [15]; and Mathai-Haubold entropy is employed to study the queuing theory [35].

The next three theorems give explicit expressions of those entropies for the WGo distribution.

Theorem 3.1. *The Shannon entropy for the WGo distribution is given by*

$$\eta_x = \log\Big(\frac{1+\lambda\sigma}{\lambda\sigma^2}\Big) + \frac{\lambda\sigma+2}{\lambda\sigma+1} - \frac{1}{\lambda\sigma+1}(1 - \mathrm{e}^{\lambda\sigma}E_i(-\lambda\sigma) + \log\lambda + \lambda\sigma\log\lambda) - \lambda\mu_x,$$

where μ_x is the mean of the WGo distribution and $E_n(z) = \int_1^{\infty}\mathrm{e}^{-zt}t^{-n}\,\mathrm{d}t$ is known as the generalized exponential integral function.

Proof. By the definition of the Shannon entropy, we get

$$\eta_x = \log(1+\lambda\sigma) - \log(\lambda\sigma^2) - \sigma + \sigma E(\mathrm{e}^{\lambda x}) - E(\log(\lambda + \mathrm{e}^{\lambda x} - 1)) - \lambda\mu_x.$$

Making use of $y = \lambda + \mathrm{e}^{\lambda x} - 1$, we conclude that

$$E(\log(\lambda + \mathrm{e}^{\lambda x} - 1)) = \frac{1}{1+\lambda\sigma}(1 - \mathrm{e}^{\lambda\sigma}E_i(-\lambda\sigma) + \log\lambda + \lambda\sigma\log\lambda).$$

Also, substituting $y = \sigma(\mathrm{e}^{\lambda x} - 1)$, we obtain

$$E(\mathrm{e}^{\lambda x}) = \frac{\sigma}{1+\lambda\sigma}\int_0^\infty \Big(\lambda - 1 + \Big(\frac{y+\sigma}{\sigma}\Big)\Big)\Big(\frac{y+\sigma}{\sigma}\Big)\mathrm{e}^{-y}\,\mathrm{d}y = 1 + \frac{\lambda\sigma+2}{\sigma(\lambda\sigma+1)}.$$

Applying all the above results, we get

$$\eta_x = \log\Big(\frac{1+\lambda\sigma}{\lambda\sigma^2}\Big) + \frac{\lambda\sigma+2}{\lambda\sigma+1} - \frac{1}{\lambda\sigma+1}(1 - \mathrm{e}^{\lambda\sigma}E_i(-\lambda\sigma) + \log\lambda + \lambda\sigma\log\lambda) - \lambda\mu_x,$$

which completes the proof. □

Some numerical values for the Shannon entropy are displayed in Table 2. It can be observed that this entropy decreases with increasing λ and σ and can have negative values.

Parameters	$\sigma = 0.5$	$\lambda = 1$	
$\lambda\downarrow$	S. Entropy	$\sigma\downarrow$	S. Entropy
1	0.894545	0.2	1.07895
2	0.202791	0.4	0.957871
3	−0.212271	0.6	0.831738
4	−0.51055	0.8	0.710468
5	−0.743321	1.0	0.5968
6	−0.93398	1.2	0.491146

Table 2. Shannon entropy for several arbitrary parameter values.

Theorem 3.2. *Let X have the pdf given by* (1.2). *Then the Rényi entropy of X is given by*

$$\begin{aligned}I_R(\gamma) &= \frac{(2\gamma-1)}{1-\gamma}\log(\lambda\sigma) - \frac{1}{1-\gamma}\log\gamma - \frac{\gamma}{1-\gamma}\log(1+\lambda\sigma)\\ &\quad+ \frac{1}{1-\gamma}\log\Big(\sum_{j=0}^{\gamma}\sum_{k=0}^{\gamma-1}\binom{\gamma}{j}\binom{\gamma-1}{k}(-1)^{2(j+k)}\Big(\frac{1}{\gamma}\Big)^j\Big(\frac{1}{\sigma\gamma}\Big)^{k+j}\Gamma(j+k+1)\Big).\end{aligned}$$

P r o o f. Since

$$I(\gamma) = \Big(\frac{\lambda\sigma^2}{1+\lambda\sigma}\Big)^\gamma \int_0^\infty (\lambda + \mathrm{e}^{\lambda x} - 1)^\gamma \mathrm{e}^{\lambda\gamma x}\mathrm{e}^{-\sigma\gamma(\mathrm{e}^{\lambda x}-1)}\,\mathrm{d}x,$$

and due to $y = \mathrm{e}^{-\sigma\gamma(\mathrm{e}^{\lambda x}-1)}$, it follows that

$$I(\gamma) = \frac{(\lambda\sigma)^{2\gamma-1}}{\gamma(1+\lambda\sigma)^\gamma}\Big(\frac{\lambda\sigma^2}{1+\lambda\sigma}\Big)^\gamma \int_0^1 \Big(1 - \frac{\log y}{\sigma\gamma\lambda}\Big)^\gamma\Big(1 - \frac{\log y}{\sigma\gamma}\Big)^{\gamma-1}\,\mathrm{d}y.$$

By expanding the binomial terms above, we get the desired proof. □

Some numerical values for the Rényi entropy are given in Table 3. It can be noted that this entropy can take negative values which may be interpreted as loss of information in physical systems. It can also be observed that this entropy decreases with increasing λ and σ.

Parameters	$\sigma = 0.5,\ \gamma = 2$		$\lambda = 1,\ \gamma = 2$
$\lambda\downarrow$	S. Entropy	$\sigma\downarrow$	S. Entropy
1	0.81093	0.2	0.946249
2	0.133531	0.4	0.862895
3	−0.277632	0.6	0.755189
4	−0.575364	0.8	0.638495
5	−0.80866	1.0	0.521297
6	−1.00017	1.2	0.407844

Table 3. Rényi entropy for several arbitrary parameter values.

Theorem 3.3. *The Mathai-Haubold entropy of the WGo distribution is*

$$J_{MH}(\delta) = \frac{1}{\delta-1}\bigg(\frac{\lambda^{2-\delta}}{1+\lambda\sigma}\sum_{j=0}^{2-\delta}\sum_{k=0}^{1-\delta}\binom{2-\delta}{j}\binom{1-\delta}{k}(-1)^{2(j+k)}\Big(\frac{1}{\lambda}\Big)^{j} \times\Big(\frac{1}{\sigma}\Big)^{j+k-1}\Big(\frac{1}{2-\delta}\Big)^{j+k+1}\Gamma(j+k+1)-1\bigg).$$

P r o o f. Since

$$\int_0^\infty f^{2-\delta}(x)\,\mathrm{d}x = \frac{\lambda\sigma^2}{1+\lambda\sigma}\int_0^\infty(\lambda+\mathrm{e}^{\lambda x}-1)^{2-\delta}\mathrm{e}^{\lambda(2-\delta)x}\mathrm{e}^{-\sigma(2-\delta)x}\mathrm{e}^{-\sigma(2-\delta)(\mathrm{e}^{\lambda x}-1)}\,\mathrm{d}x,$$

and by virtue of the transformation $y = \mathrm{e}^{-\sigma(2-\delta)(\mathrm{e}^{\lambda x}-1)}$, it follows that

$$\int_0^\infty f^{2-\delta}(x)\,\mathrm{d}x = \frac{\sigma\lambda^{2-\delta}}{(1+\lambda\sigma)(2-\delta)}\int_0^1\Big(1-\frac{1}{\sigma\lambda(2-\delta)}\ln y\Big)^{2-\delta}\Big(1-\frac{1}{\sigma(2-\delta)}\ln y\Big)^{1-\delta}\,\mathrm{d}y.$$

By expanding the binomial terms in the above integral, the proof is obtained. □

Some numerical values for the Mathai-Haubold entropy are summarized in Table 4. It is seen that this entropy decreases with increasing λ and σ.

Parameters	$\sigma = 0.5$, $\delta = 1.5$		$\lambda = 1$, $\delta = 1.5$
$\lambda \downarrow$	M-H. Entropy	$\sigma \downarrow$	M-H. Entropy
1	2.26599	0.2	2.65148
2	1.30286	0.4	2.38062
3	0.871031	0.6	2.16228
4	0.61286	0.8	1.98142
5	0.436806	1.0	1.82743
6	0.307142	1.2	1.6968

Table 4. Mathai-Haubold entropy for several arbitrary parameter values.

4. Residual life and reversed residual life functions

Residual life and reversed residual life random variables are used extensively in reliability analysis and the risk theory. Consequently, we investigate some of their related statistical functions, such as the survival function, mean and variance in connection with the WGo distribution.

4.1. Residual lifetime function. The residual life is the period from time until the time of failure and is defined by the conditional random variable $R_{(t)} := X - t \mid X > t$, $t \geqslant 0$.

Proposition 4.1. *The survival function of the residual lifetime $R_{(t)}$ for the WGo distribution is*

$$S_{R_{(t)}}(x) = \frac{(1 + \lambda\sigma + \sigma(\mathrm{e}^{\lambda(x+t)} - 1))}{(1 + \lambda\sigma + \sigma(\mathrm{e}^{\lambda t} - 1))}\mathrm{e}^{-\sigma\mathrm{e}^{\lambda t}(\mathrm{e}^{\lambda x}-1)}, \quad x > 0. \tag{4.1}$$

Proof. The proof follows from the identity $S_{R_{(t)}}(x) = S(x+t)/S(t)$, where $S(\cdot)$ is defined by equation (1.4). □

Corollary 4.1. *Based on* (4.1), *the pdf and the hazard rate function of $R_{(t)}$ are, respectively, given as*

$$f_{R_{(t)}}(x) = \frac{\lambda\sigma^2(\lambda + \mathrm{e}^{\lambda(x+t)} - 1)}{(1 + \lambda\sigma + \sigma(\mathrm{e}^{\lambda t} - 1))}\mathrm{e}^{\lambda(x+t)-\sigma\mathrm{e}^{\lambda t}(\mathrm{e}^{\lambda x}-1)},$$

and

$$h_{R_{(t)}}(x) = \frac{\lambda\sigma^2(\lambda + \mathrm{e}^{\lambda(x+t)} - 1)}{1 + (\lambda + \mathrm{e}^{\lambda(x+t)} - 1)\sigma}\mathrm{e}^{\lambda(x+t)}.$$

In reliability theory, the mean and variance residual lifetime have been studied in recent years (see [11]*).*

Proposition 4.2. *The mean and variance of $R_{(t)}$ for the WGo distribution are*

$$K(t) = \frac{1}{S(t)}(E(X) - J(t; 1, \sigma, \lambda)) - t, \quad t \geqslant 0,$$

and

$$V(t) = \frac{1}{S(t)}(E(x^2) - J(t; 2, \lambda, \sigma)) - t^2 - 2tK(t) - (K(t))^2,$$

respectively, where $E(X)$ and $E(X^2)$ can be obtained using (2.1)*, and $J(t; 2, \sigma, \lambda)$ is defined by Lemma* 2.1 *for $r = 2$.*

Proof. The proof results by using the definitions of $K(t)$ and $V(t)$ given as

$$K(t) = E(R_{(t)}) = \frac{1}{S(t)} \int_t^\infty x f(x)\,\mathrm{d}x - t$$

and

$$V(t) = \mathrm{var}(R_{(t)}) = \frac{2}{S(t)} \int_t^\infty x S(x)\,\mathrm{d}x - 2tK(t) - (K(t))^2,$$

respectively. □

Some numerical values of the mean residual life are displayed in Table 5 for a set of arbitrary choices of the parameters λ and σ at the time points $t = 1, 2, 4, 6, 7$. This table shows that the mean residual life decreases with increasing the time points t, and decreases with increasing λ and σ.

Parameters	$\sigma = 0.01$				
$\lambda \downarrow t \rightarrow$	1	2	4	6	7
1.0	4.0296	3.03717	1.21365	0.239331	0.905942
1.2	3.1905	2.20376	0.57221	0.061932	0.0187301
1.4	2.59143	1.61394	0.247748	0.016054	0.0039607
1.5	2.35192	1.38093	0.159531	0.0082261	0.0018357
1.6	2.14244	1.17954	0.101942	0.0042328	−7
Parameters	$\lambda = 0.5$				
$\sigma \downarrow t \rightarrow$	1	2	4	6	7
0.1	4.57473	3.63642	2.00249	0.907294	0.577058
0.3	2.66164	1.89771	0.837132	0.326663	0.19997
0.4	2.22679	1.5372	0.645554	0.246649	0.150418
0.5	1.91626	1.29066	0.524451	0.197985	0.120506
0.7	1.49803	0.974603	0.380451	0.141855	0.0861845

Table 5. Mean residual life function for several arbitrary parameter values.

4.2. Reversed residual life function. The reversed residual life is the time elapsed from the failure of a component given that its life satisfies $X \leqslant t$, and is defined as the conditional random variable $\overline{R}(t) := t - X \mid X \leqslant t$.

Proposition 4.3. *The survival function of the reversed residual lifetime $\overline{R}(t)$ for the WGo distribution is*

$$S_{\overline{R}(t)}(x) = \frac{1+\lambda\sigma-(1+\lambda\sigma+\sigma(\mathrm{e}^{\lambda(t-x)}-1))\mathrm{e}^{-\sigma(\mathrm{e}^{\lambda(t-x)}-1)}}{1+\lambda\sigma-(1+\lambda\sigma+\sigma(\mathrm{e}^{\lambda t}-1))\mathrm{e}^{-\sigma(\mathrm{e}^{\lambda t-1})}}, \quad 0 \leqslant x < t. \tag{4.2}$$

Proof. The proof follows from the expression $S_{\overline{R}(t)}(x) = \frac{F(t-x)}{F(t)}$, where $F(\cdot)$ is defined by (1.3). $\square$

Corollary 4.2. *Using* (4.2), *the pdf and the hazard rate function of $\overline{R}(t)$ are*

$$f_{\overline{R}(t)}(x) = \frac{\lambda\sigma^2(\lambda+\mathrm{e}^{\lambda(t-x)}-1)\mathrm{e}^{\lambda(t-x)-\sigma(\mathrm{e}^{\lambda(t-x)}-1)}}{1+\lambda\sigma-(1+\lambda\sigma+\sigma(\mathrm{e}^{\lambda t}-1))\mathrm{e}^{-\sigma(\mathrm{e}^{\lambda t}-1)}}$$

and

$$h_{\overline{R}(t)}(x) = \frac{\lambda\sigma^2(\lambda+\mathrm{e}^{\lambda(t-x)}-1)\mathrm{e}^{\lambda(t-x)-\sigma(\mathrm{e}^{\lambda(t-x)}-1)}}{1+\lambda\sigma-(1+\lambda\sigma(\mathrm{e}^{\lambda(t-x)}-1))\mathrm{e}^{-\sigma(\mathrm{e}^{\lambda(t-x)}-1)}},$$

respectively.

Parameters	$\sigma = 0.01$				
$\lambda \downarrow t \rightarrow$	1	2	4	6	7
1.0	0.344729	0.469449	0.562217	1.09024	1.9722
1.2	0.325766	0.417113	0.544483	1.81141	2.81137
1.4	0.307385	0.373143	0.627414	2.41108	3.41108
1.5	0.298445	0.354272	0.729905	2.65097	3.65097
1.6	0.289689	0.337442	0.877506	2.86086	3.86086
Parameters	$\lambda = 0.5$				
$\sigma \downarrow t \rightarrow$	1	2	4	6	7
0.1	0.398499	0.955288	0.993448	1.42368	1.84595
0.3	0.408641	0.703175	1.28671	2.50893	3.45589
0.4	0.41375	0.7278	1.44399	2.9476	3.93636
0.5	0.418882	0.752808	1.60246	3.29467	4.29251
0.7	0.429206	0.803733	1.90635	3.79379	4.79372

Table 6. Mean reversed residual life function for several arbitrary parameter values.

Proposition 4.4. *The mean and the variance of $\overline{R}(t)$ for the WGo distribution are given by*

$$L(t) = t - \frac{J(t; 1, \lambda, \sigma)}{F(t)}$$

and

$$W(t) = 2tL(t) - (L(t))^2 - t^2 + \frac{J(t; 2, \lambda, \sigma)}{F(t)},$$

respectively.

P r o o f. The proof comes directly using the definitions of $L(t)$ and $W(t)$ given by

$$L(t) = E(\overline{R}(t)) = t - \frac{1}{F(t)} \int_0^t x f(x) \, \mathrm{d}x$$

and

$$W(t) = \operatorname{var}(\overline{R}(t)) = 2tL(t) - (L(t))^2 - \frac{2}{F(t)} \int_0^t x F(x) \, \mathrm{d}x,$$

respectively. In Table 6 we give some numerical values for the mean reversed life with arbitrary choices of the parameters λ and σ at the time points $t = 1, 2, 4, 6, 7$. It can be seen that the mean reversed residual life increases with increasing the time points t and increases with increasing λ and σ.

5. Stochastic ordering

Stochastic ordering quantifies the concept of one random variable being smaller than another, that is, a measure to judge the comparative behavior of random variables. In this regard, common practice of comparison is done by cumulative distribution function, hazard function, likelihood ratio function and their related functions. The orders considered here are the likelihood ratio $\leqslant_{\mathrm{lr}}$, stochastic order $\leqslant_{\mathrm{st}}$, hazard rate order $\leqslant_{\mathrm{hr}}$, and mean residual life order $\leqslant_{\mathrm{mrl}}$, which in general imply the implications

$$X \leqslant_{\mathrm{lr}} Y \Rightarrow X \leqslant_{\mathrm{hr}} Y \Rightarrow X \leqslant_{\mathrm{mrl}} \Rightarrow X \leqslant_{\mathrm{st}} Y \tag{5.1}$$

(see [32]) where X and Y are two continuous random variables.

The next theorems give some results on stochastic ordering of the WGo distribution.

Theorem 5.1. *Let* $X \sim \mathrm{WGo}(\lambda, \sigma_1)$ *and* $Y \sim \mathrm{WGo}(\lambda, \sigma_2)$*. If* $\sigma_2 < \sigma_1$*. Then* $X \leqslant_{\mathrm{lr}} Y$ *and hence* $X \leqslant_{\mathrm{hr}} Y$*,* $X \leqslant_{\mathrm{mr}l} Y$ *and* $X \leqslant_{\mathrm{st}} Y$*.*

Proof. The density ratio is given as

$$\frac{f_X(x)}{f_Y(x)} = \frac{\sigma_1^2(1+\lambda\sigma_2)}{\sigma_2^2(1+\lambda\sigma_1)} \mathrm{e}^{-(\mathrm{e}^{\lambda x}-1)(\sigma_1-\sigma_2)}.$$

Taking the derivative of the above expression with respect to x, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{f_X(x)}{f_Y(x)}\Big] = (\sigma_2-\sigma_1)\lambda\mathrm{e}^{\lambda x}\frac{f_X(x)}{f_Y(x)}.$$

Since $\sigma_2 < \sigma_1$, then $\frac{\mathrm{d}}{\mathrm{d}x}\big[\frac{f_X(x)}{f_Y(x)}\big] < 0$. Hence, $\frac{f_X(x)}{f_Y(x)}$ is decreasing in x. That is $X \leqslant_{\mathrm{lr}} Y$. The remaining statements follow from the implication in (5.1), which completes the proof. □

Theorem 5.2. *Let* $X \sim \mathrm{Go}(\lambda, \sigma_1)$ *and* $Y \sim \mathrm{WGo}(\lambda, \sigma_2)$*. If* $\sigma_2 < \sigma_1$*, then* $X \leqslant_{\mathrm{lr}} Y$ *and hence* $X \leqslant_{\mathrm{hr}} Y$*,* $X \leqslant_{\mathrm{mr}l} Y$ *and* $X \leqslant_{\mathrm{st}} Y$*.*

Proof. The density ratio is given as

$$\frac{f_X(x)}{f_Y(x)} = \frac{\sigma_1(1+\lambda\sigma)}{\sigma_2^2(\lambda+\mathrm{e}^{\lambda x}-1)} \mathrm{e}^{-(\mathrm{e}^{\lambda x}-1)(\sigma_1-\sigma_2)}.$$

Taking the derivative with respect to x for the above expression, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{f_X(x)}{f_Y(x)}\Big] = \Big((\sigma_2-\sigma_1)\lambda\mathrm{e}^{\lambda x} - \frac{\lambda\mathrm{e}^{\lambda x}}{\lambda+\mathrm{e}^{\lambda x}-1}\Big)\frac{f_X(x)}{f_Y(x)},$$

since for $\sigma_2 < \sigma_1$ we have $\frac{\mathrm{d}}{\mathrm{d}x}\big[\frac{f_X(x)}{f_Y(x)}\big] < 0$, hence $\frac{f_X(x)}{f_Y(x)}$ is decreasing in x. That is $X \leqslant_{\mathrm{lr}} Y$ and the remaining statements follow from the implication in (5.1). □

The above theorem shows the flexibility of WGo distribution is better than that of the classical Gompertz distribution.

6.1. Estimation with inference. In this section, the method of maximum likelihood is considered to estimate the unknown parameters of the WGo distribution. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the WGo distribution with parameters λ and σ. Then the corresponding log-likelihood function is

$$\begin{aligned} l = {} & n\log(\lambda) + 2n\log(\sigma) - n\log(1+\lambda\sigma) \\ & + \lambda\sum_{i=1}^{n} x_i - \sigma\sum_{i=1}^{n}(\mathrm{e}^{\lambda x_i} - 1) + \sum_{i=1}^{n}\log(\lambda + \mathrm{e}^{\lambda x_i} - 1). \end{aligned} \tag{6.1}$$

Differentiating (6.1) with respect to λ and σ, respectively, we have

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \frac{n\sigma}{1+\lambda\sigma} + \sum_{i=1}^{n} x_i - \sigma\sum_{i=1}^{n} x_i \mathrm{e}^{\lambda x_i} + \sum_{i=1}^{n}\frac{1 + x_i\mathrm{e}^{\lambda x_i}}{\lambda + \mathrm{e}^{\lambda x_i} - 1}, \tag{6.2}$$

$$\frac{\partial l}{\partial \sigma} = \frac{2n}{\sigma} - \frac{n\lambda}{1+\lambda\sigma} - \sum_{i=1}^{n}(\mathrm{e}^{\lambda x_i} - 1). \tag{6.3}$$

The maximum likelihood estimators (MLEs) $\widehat{\lambda}$ and $\widehat{\sigma}$ of the parameters λ and σ, respectively, can be obtained by solving the above nonlinear equations numerically for λ and σ using the statistical software Mathematica package. For the interval estimation and hypotheses tests on the model parameters, we require the observed information matrix. The corresponding 2×2 observed information matrix $I_n = I_n(\lambda, \sigma)$ is

$$I_n = -\begin{pmatrix} I_{\lambda\lambda} & I_{\lambda\sigma} \\ I_{\sigma\lambda} & I_{\sigma\sigma} \end{pmatrix}.$$

The elements of I_n are given by

$$I_{\lambda\lambda} = -\frac{n}{\lambda^2} + \frac{n\sigma^2}{(1+\lambda\sigma)^2} - \sigma\sum_{i=1}^{n} x_i^2\mathrm{e}^{\lambda x_i} + \sum_{i=1}^{n}\frac{x_i^2\mathrm{e}^{\lambda x_i}(\lambda + \mathrm{e}^{\lambda x_i} - 1) - (1 + x_i\mathrm{e}_i^{\lambda x})^2}{(\lambda + \mathrm{e}^{\lambda x_i} - 1)^2},$$

$$I_{\lambda\sigma} = I_{\sigma\lambda} = -\frac{n}{(1+\lambda\sigma)^2} - \sum_{i=1}^{n} x_i\mathrm{e}^{\lambda x_i},$$

and

$$I_{\sigma\sigma} = -\frac{2n}{\sigma^2} + \frac{n\lambda^2}{(1+\lambda\sigma)^2},$$

hence the variance covariance matrix would be $I_n^{-1}(\lambda, \sigma)$, where $I_n^{-1}(\lambda, \sigma)$ is the inverse of $I_n(\lambda, \sigma)$. The approximate $(1-\delta)100\%$ confidence intervals (CIs) for the parameters λ and σ are $\widehat{\lambda} \pm Z_{\delta/2}\sqrt{\mathrm{var}(\widehat{\lambda})}$ and $\widehat{\sigma} \pm Z_{\delta/2}\sqrt{\mathrm{var}(\widehat{\sigma})}$, respectively, where

$\mathrm{var}(\widehat{\lambda})$ and $\mathrm{var}(\widehat{\sigma})$ are the variances of $\widehat{\lambda}$ and $\widehat{\sigma}$, which are given by the diagonal elements of $I_n^{-1}(\lambda,\sigma)$, and $Z_{\delta/2}$ is the upper $(\delta/2)$ percentile of the standard normal distribution.

6.2. Real data applications. Two reliability data sets are used to assess the performance of the WGo distribution among some classical and recent continuous distributions based on a set of goodness-of-fit tests. The compared distributions are

▷ Muth (Mu) distribution [26] with density function

$$g(x) = (\mathrm{e}^{\alpha x} - \alpha)\mathrm{e}^{\alpha x - 1/\alpha(\mathrm{e}^{\alpha x}-1)}, \quad x > 0,\ \alpha \in (0,1].$$

▷ Generalized Lindley (GL) distribution [27] with density function

$$g(x) = \frac{\gamma\alpha^2}{1+\alpha}(1+x)\Big(1 - \frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}\Big)^{\gamma-1}\mathrm{e}^{-\alpha x}, \quad x > 0,\ \alpha > 0,\ \gamma > 0.$$

▷ Exponentiated exponential (EE) distribution with density function

$$g(x) = \alpha\lambda\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{\alpha-1}, \quad x > 0,\ \alpha > 0,\ \lambda > 0.$$

▷ Linear exponential (LE) distribution with density function

$$g(x) = (\alpha + \beta x)\mathrm{e}^{-\alpha x - x^2\beta/2}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

▷ Gompertz (Go) distribution with density function

$$g(x) = \lambda\sigma\mathrm{e}^{\lambda x - \sigma(\mathrm{e}^{\lambda x}-1)}, \quad x > 0,\ \sigma > 0,\ \lambda > 0.$$

▷ Shifted Gompertz (SGo) distribution [5] with density function

$$g(x) = \beta\mathrm{e}^{-\beta x - \alpha\mathrm{e}^{-\beta x}}(1 + \alpha(1-\mathrm{e}^{-\beta x})), \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

▷ Gamma (Ga) distribution with density function

$$g(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}\mathrm{e}^{-x/\beta}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

▷ Gomperz-Makeham (GK) distribution [21] with density function

$$g(x) = (\alpha\mathrm{e}^{\beta x} + \lambda)\mathrm{e}^{-\lambda x - (\alpha/\beta)(\mathrm{e}^{\beta x}-1)}, \quad x > 0,\ \alpha > 0,\ \beta > 0,\ \lambda > 0.$$

▷ Weighted exponential (WE) distribution [12] with density function

$$g(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1-e^{-\alpha\lambda x}), \quad x>0, \alpha>0,\ \lambda>0.$$

▷ Weighted Lindley (WL) distribution [8] with density function

$$g(x) = \frac{\theta^{c+1}}{(\theta+c)\Gamma(c)} x^{c-1}(1+x)e^{-\theta x}, \quad x>0,\ c>0,\ \theta>0.$$

▷ New weighted Lindley (NWL) distribution [3] with density function

$$g(x) = \frac{\lambda^2(1+\alpha)^2}{\alpha\lambda(1+\alpha)+\alpha(2+\alpha)}(1+x)(1-e^{-\lambda\alpha x})e^{-\lambda x}, \quad x>0,\ \alpha>0,\ \lambda>0.$$

The first data set shows the breaking stress of carbon fibers of 50 mm in length (see [28]) listed in Table 7. The second data set is given in Table 8 and represents the strength of 1.5 cm glass fibers measures at the National Physical Laboratory in England (see [36]).

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11	3.56
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90	1.57
2.67	2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85
1.25	4.38	1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03
1.89	2.88	2.82	2.05	3.65	3.75	2.43	2.95	2.97	3.39	2.96
2.35	2.55	2.59	2.03	1.61	2.12	3.15	1.08	2.56	1.80	2.53

Table 7. Breaking stress of carbon fibers data.

0.55	0.74	0.77	0.81	0.84	0.93	1.04	1.11	1.13	1.24	1.25
1.27	1.28	1.29	1.30	1.36	1.39	1.42	1.48	1.48	1.49	1.50
1.50	1.51	1.52	1.53	1.54	1.55	1.55	1.58	1.59	1.60	1.61
1.61	1.61	1.61	1.62	1.62	1.63	1.64	1.66	1.66	1.66	1.67
1.68	1.68	1.69	2.00	2.01	2.24					

Table 8. Strength of 1.5 cm glass fibers data.

For each data set, we estimate the unknown parameters of each distribution by the maximum-likelihood method, and with those estimates, we obtain the values of Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC) and Hannan-Quinn Information Criterion (HQIC). Further, we get the Kolmogorov-Smirnov (K-S) statistic with its corresponding p-value, Cramer-von Mises (W^*) and Anderson-Darling (A^*) goodness-of-fit statistics. The results obtained are presented in Tables 9–12. As we can see, the smallest values of the AIC, BIC, CAIC, HQIC, K-S, W^*, and A^* and the largest value of the p-value are obtained for the WGo distribution. Therefore, we conclude that the WGo distribution provides the best fit among the compared distributions. This conclusion is confirmed again by Figures 5 and 6, where the estimated densities and estimated survival functions for the considered distributions of both data sets are plotted based on the density and the survival function of each distribution and replacing the parameters with their MLEs given in Tables 9 and 11.

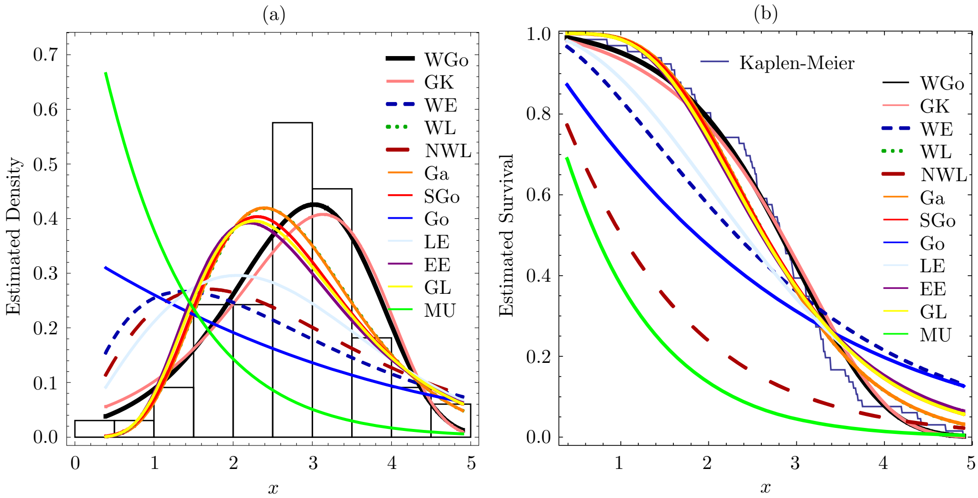

Figure 5. Estimated densities and survival functions for the considered distributions for breaking stress of carbon fibers data.

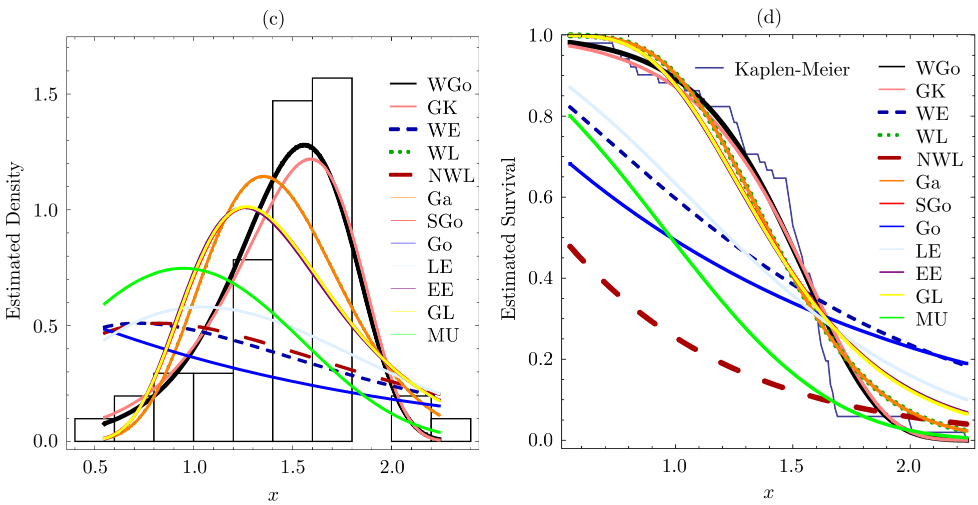

Figure 6. Estimated densities and survival functions for the considered distributions for strength of 1.5 cm glass of fibers data.

Consequently, the variance-covariance matrices of the MLEs of the WGo model for breaking stress and glass fibers data sets are

$$\begin{pmatrix} 0.00706446 & -0.00514303 \\ -0.00514303 & 0.00406333 \end{pmatrix}$$

and

$$\begin{pmatrix} 0.0572936 & -0.00421193 \\ -0.00421193 & 0.000329407 \end{pmatrix},$$

respectively. Note that the diagonal entries of these matrices are the variances of the MLEs of the WGo parameters λ and σ of each of the data sets while the values -0.00514303 and -0.00421193 represent the covariances between the MLEs of λ and σ for the first and second data sets, respectively. Also, the 90% and 95% confidence intervals for the WGo parameters λ and σ are given in Table 13 for both the data sets.

Table 14 gives some descriptive statistics for both the data sets and it is noted that the two data sets have negative skewness and positive kurtosis. Meanwhile, some of the corresponding theoretical measures of the WGo distribution of both the data sets are summarized in Table 15, where the parameters of the WGo are replaced by their corresponding MLEs for each data set using Tables 9 and 11. From Tables 14 and 15 it can be concluded that the considered measures of the WGo distribution are close to the sample measures given by Table 14 for both the data sets.

Distributions	Estimates			AIC	BIC	CAIC	HQIC
$\mathrm{Mu}(\alpha)$	0.05676	—	—	364.986	367.175	365.851	365.048
$\mathrm{GL}(\alpha,\gamma)$	1.2460	7.0410	—	191.594	195.973	191.784	193.324
$\mathrm{EE}(\alpha,\lambda)$	9.19917	1.00755	—	194.745	199.124	194.935	196.475
$\mathrm{LE}(\alpha,\beta)$	$1.562\cdot 10^{-19}$	0.23815	—	200.417	204.796	202.147	200.607
$\mathrm{Go}(\lambda,\sigma)$	0.08047	4.27515	—	257.638	262.017	257.828	259.368
$\mathrm{SGo}(\alpha,\beta)$	11.4174	1.09324	—	189.253	193.632	189.444	190.984
$\mathrm{Ga}(\alpha,\beta)$	7.48803	0.368528	—	186.335	190.714	186.526	188.066
$\mathrm{GK}(\alpha,\beta,\lambda)$	0.037281	1.07068	$5.113\cdot 10^{-9}$	182.177	188.746	182.564	184.772
$\mathrm{WE}(\alpha,\beta)$	$4.907\cdot 10^{-7}$	0.724757	—	228.008	232.387	228.198	229.738
$\mathrm{WL}(c,\theta)$	6.99868	2.79513	—	186.049	190.238	186.049	187.589
$\mathrm{NWL}(\alpha,\beta)$	0.0000152	0.96886	—	217.452	221.831	217.642	219.182
$\mathrm{WGo}(\lambda,\sigma)$	0.75910	0.20431	—	177.023	181.402	177.213	178.753

Table 9. The MLEs of the parameters for some models fitted to the breaking stress of carbon fibers data and the values of AIC, BIC, CAIC, and HQIC statistics.

Distributions	K-S	p-value	$-\log L$	W^*	A^*
MU	0.712623	0.0000	181.493	14.228	90.6943
GL	0.146994	0.115416	93.797	0.330919	1.8593
EE	0.15495	0.0840311	95.372	0.376008	2.1188
LE	0.226512	0.0022894	98.2084	1.01888	5.20603
Go	0.361949	$6.177 \cdot 10^{-8}$	126.819	3.01643	14.485
SGo	0.136481	0.17097	92.6266	0.289118	1.65346
Ga	0.13284	0.194487	91.1675	0.248153	1.32674
GK	0.111974	0.379499	88.0884	0.14077	0.95967
WE	0.250166	0.0005168	112.004	1.54251	8.09264
WL	0.131825	0.201544	90.9291	0.24376	1.30117
NWL	0.227033	0.0022191	106.726	1.21146	6.51955
WGo	0.097974	0.550746	86.5114	0.090301	0.61579

Table 10. The values of K-S, p-value, $-\log L$, W^* and A^* statistics for some models fitted to the breaking stress of carbon fibers data.

Distributions	Estimates			AIC	BIC	CAIC	HQIC
Mu(α)	0.882517	—	—	86.8127	88.7445	86.8943	87.5509
GL(α,γ)	3.08433	24.9005	—	51.2492	55.1128	51.4992	52.7256
EE(α,λ)	29.6639	2.6983	—	52.3535	56.2171	52.6035	53.8299
LE(α,β)	$2.15 \cdot 10^{-32}$	0.91598	—	80.8089	84.6726	81.0589	82.2853
Go(λ,σ)	0.077567	8.77178	—	138.053	141.917	138.303	139.53
SGo(α,β)	31.6607	2.74601	—	51.1369	55.005	51.386	52.6133
Ga(α,β)	16.2198	0.088889	—	41.8238	45.6874	42.0738	43.3002
GK(α,β,λ)	0.017439	3.296	$3.582 \cdot 10^{-10}$	36.2874	42.0829	36.798	38.502
WE(α,λ)	$6.282 \cdot 10^{-6}$	1.38718	—	107.093	110.957	107.343	108.57
WL(c,θ)	15.8894	11.4243	—	41.7278	45.5914	41.9778	43.2042
NWL(α,λ)	$2.947 \cdot 10^{-8}$	1.75647	—	100.713	104.576	100.963	102.189
WGo(λ,σ)	2.42682	0.044765	—	31.9246	35.7883	32.1746	33.401

Table 11. The MLEs of the parameters for some models fitted to the strength 1.5 cm glass fibers data and the values of AIC, BIC, CAIC, and HQIC statistics.

7. Concluding remarks

A weighted version of the Gompertz distribution is introduced by making use of a new weighted function. This version represents a mixture of classical Gompertz

Distribution	K-S	p-value	$-\log L$	W^*	A^*
MU	0.509217	$6.523 \cdot 10^{-12}$	42.4063	5.2078	42.4063
GL	0.221037	0.013701	23.6246	0.670228	3.5518
EE	0.222876	0.0126063	24.1767	0.688759	3.64489
LE	0.329031	0.000032	38.4045	1.81703	9.02137
Go	0.411066	$6.542 \cdot 10^{-8}$	23.5685	3.18956	15.2271
SGo	0.217577	0.0159946	18.9119	0.659491	3.51528
Ga	0.221848	0.0132077	18.9119	0.533687	2.84555
GK	0.191839	0.0468544	15.1437	0.308231	1.77064
WE	0.250166	0.0000192	51.5466	2.07798	10.5031
WL	0.221575	0.0133722	18.8639	0.532304	2.83704
NWL	0.320534	0.0000562	48.3563	1.89421	9.68732
WGo	0.178637	0.077163	13.9623	0.26702	1.54241

Table 12. The values of K-S, p-value, $-\log L$, W^* and A^* statistics for some models fitted to the strength 1.5 cm glass fibers data.

CI	λ	σ
90%	[0.651392, 0.866822]	[0.122624, 0.286007]
95%	[0.620856, 0.897357]	[0.0994652, 0.309165]
CI	λ	σ
90%	[2.12007, 2.73357]	[0.0215358, 0.068055]
95%	[2.03311, 2.82053]	[0.014942, 0.0746488]

Table 13. Confidence intervals for the breaking stress and glass fibers data, respectively.

Data	Mean	Median	SD	Skewness	Kurtosis	MD-mean	MD-median	Shannon Entropy
Breaking stress data	2.75955	2.835	0.891455	−0.13046	0.17421	0.683223	0.678939	4.021619
Strength of 1.5 cm data	1.4417	1.52	0.3268	−0.63760	0.727899	0.244383	0.228824	3.595379

MD:= Mean deviation

Table 14. Descriptive statistics of both data sets.

and second upper record value of Gompertz densities, gives a new version of the two-parameter Lindley distribution using a certain transformation, and can be also regarded as a member of the log-Lindley-X family. Properties, estimation of the

Data	Mean	Median	SD	MD-mean	MD-median	Shannon Entropy
Breaking stress data	2.74354	2.82545	0.94329	0.759559	0.756782	4.53298
Strength of 1.5 cm data	1.43556	1.47933	0.34316	0.268458	0.26612	5.55912

Table 15. Some measures of the WGo distribution for both data sets.

parameters and inference of this version are obtained. Flexibility of the model with respect to other models is illustrated by fitting two real data sets and using some goodness-of-fit statistics. Some of the obtained theoretical measures of the model have been computed for the data and compared with the corresponding descriptive measures. Some issues for future research may be considering different estimation methods of the unknown parameters of the model and a studying the corresponding step-stress model.

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Authors' address: *Hassan S. Bakouch*, *Ahmed M. T. Abd El-Bar*, Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt, e-mail: hnbakouch@yahoo.com, ahmed_stat2006@yahoo.com.