A nilpotent-generated semigroup associated with a semigroup of full transformations

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Synopsis

Let X be a set with infinite regular cardinality $\mathbf{m}$ and let $\mathcal{T}(X)$ be the semigroup of all self-maps of X. The semigroup $Q_{\mathbf{m}}$ of 'balanced' elements of $\mathcal{T}(X)$ plays an important role in the study by Howie [**3, 5, 6**] of idempotent-generated subsemigroups of $\mathcal{T}(X)$, as does the subset $S_{\mathbf{m}}$ of 'stable' elements, which is a subsemigroup of $Q_{\mathbf{m}}$ if and only if $\mathbf{m}$ is a regular cardinal. The principal factor $P_{\mathbf{m}}$ of $Q_{\mathbf{m}}$, corresponding to the maximum $\mathcal{J}$-class $J_{\mathbf{m}}$, contains $S_{\mathbf{m}}$ and has been shown in [**7**] to have a number of interesting properties.

Let N_2 be the set of all nilpotent elements of index 2 in $P_{\mathbf{m}}$. Then the subsemigroup $\langle N_2 \rangle$ of $P_{\mathbf{m}}$ generated by N_2 consists exactly of the elements in $P_{\mathbf{m}} \backslash S_{\mathbf{m}}$. Moreover $P_{\mathbf{m}} \backslash S_{\mathbf{m}}$ has 2-nilpotent-depth 3, in the sense that $N_2 \cup N_2^2 \subset P_{\mathbf{m}} \backslash S_{\mathbf{m}} = N_2 \cup N_2^2 \cup N_2^3$.

1. Introduction and background

Let X be a set with infinite cardinality $\mathbf{m}$ and let $\mathcal{T}(X)$ be the full transformation semigroup of X. (For this and other undefined semigroup notions see [**4**].) In his study of idempotent-generated subsemigroups of $\mathcal{T}(X)$, Howie [**3, 5**] made use of the following subsets associated with an element α of $\mathcal{T}(X)$:

$$S(\alpha) = \{x \in X : x\alpha \neq x\}, \quad Z(\alpha) = X \backslash X\alpha,$$
$$C(\alpha) = \bigcup \{y\alpha^{-1} : y \in X\alpha \text{ and } |y\alpha^{-1}| \geqq 2\}.$$

The cardinal numbers $|S(\alpha)|$, $|Z(\alpha)|$ and $|C(\alpha)|$ are called, respectively, the *shift*, the *defect* and the *collapse* of α, and the subsemigroup

$$Q_{\mathbf{m}} = \{\alpha \in T(X) : |S(\alpha)| = |Z(\alpha)| = |C(\alpha)| = \mathbf{m}\}$$

of *balanced* elements of weight $\mathbf{m}$ is an important part of the subsemigroup of $\mathcal{T}(X)$ generated by the idempotents.

The principal factor $P_{\mathbf{m}}$ associated with the maximum $\mathcal{J}$-class $J_{\mathbf{m}}$ in $Q_{\mathbf{m}}$ is the Rees quotient $Q_{\mathbf{m}}/I_{\mathbf{m}}$, where $I_{\mathbf{m}} = \{\alpha \in Q_{\mathbf{m}} : |X\alpha| < \mathbf{m}\}$. It is usually convenient to think of it as $J_{\mathbf{m}} \cup \{0\}$, where $J_{\mathbf{m}} = \{\alpha \in Q_{\mathbf{m}} : |X\alpha| = \mathbf{m}\}$, and where the product of two elements of $J_{\mathbf{m}}$ is taken to be zero if it falls in $I_{\mathbf{m}}$.

The importance of cardinals in semigroup theory was demonstrated by Preston

* Research supported by *Instituto Nacional de Investigação Científica*, Portugal.

[9] when he showed that the set

$$M = \{\alpha \in \mathscr{T}(X): |X\alpha| = \mathbf{m} \text{ and } (\forall y \in X\alpha)\, |y\alpha^{-1}| < \mathbf{m}\}$$

is a subsemigroup of $\mathscr{T}(X)$ if and only if $\mathbf{m}$ is a *regular* cardinal, i.e. if and only if $|\Lambda| < \mathbf{m}$ and $\mathbf{m}_\lambda < \mathbf{m}$ for all λ in Λ together imply that

$$\sum_{\lambda \in \Lambda} \mathbf{m}_\lambda < \mathbf{m}.$$

The same equivalence holds good for the set $S_\mathbf{m} = M \cap Q_\mathbf{m}$ considered in [6] by Howie, who also showed that, in the case where $\mathbf{m}$ is regular, $S_\mathbf{m}$ is a regular, bisimple, idempotent-generated semigroup.

In an analogous manner Marques [7] showed that $P_\mathbf{m}$ is a regular, 0-bisimple, idempotent-generated semigroup.

The semigroup $P_\mathbf{m}$ has a zero element and is easily seen to contain nilpotent elements. If $\mathbf{m}$ is regular then (see Theorem 2.17) the subsemigroup $\langle N \rangle$ generated by the nilpotent elements is $P_\mathbf{m} \backslash S_\mathbf{m}$. In fact one requires only the set

$$N_2 = \{\alpha \in P_\mathbf{m}: \alpha^2 = 0,\ \alpha \in 0\}$$

of nilpotents of index 2, and

$$N_2 \cup N_2^2 \subset P_\mathbf{m} \backslash S_\mathbf{m} = N_2 \cup N_2^2 \cup N_2^3.$$

2. Regular cardinals

Throughout this section X will be a set with infinite cardinal $\mathbf{m}$, and $\mathbf{m}$ will be regular. Hence, as remarked in the last section, the set

$$S_\mathbf{m} = \{\alpha \in Q_\mathbf{m}: |X\alpha| = \mathbf{m} \text{ and } (\forall y \in X\alpha)\, |y\alpha^{-1}| < \mathbf{m}\} \tag{2.1}$$

is a subsemigroup of $P_\mathbf{m} = J_\mathbf{m} \cup \{0\}$, where

$$J_\mathbf{m} = \{\alpha \in Q_\mathbf{m}: |X\alpha| = \mathbf{m}\}. \tag{2.2}$$

We also have the following proposition:

PROPOSITION 2.1. *The set $P_\mathbf{m} \backslash S_\mathbf{m}$ is a regular subsemigroup of $P_\mathbf{m}$.*

Proof. Let $\alpha \in P_\mathbf{m} \backslash S_\mathbf{m}$. Then the equivalence relation $\ker\alpha (= \{(x, y): x\alpha = y\alpha\})$ on X has at least one class $y\alpha^{-1}$ $(y \in X\alpha)$ with cardinality $\mathbf{m}$. For all β in $P_\mathbf{m}$ we have $\ker(\alpha\beta) \supseteq \ker\alpha$ and so certainly $\ker(\alpha\beta)$ has at least one class of cardinality $\mathbf{m}$. Thus $\alpha\beta$ is in $P_\mathbf{m} \backslash S_\mathbf{m}$. Notice that we have actually shown that $P_\mathbf{m} \backslash S_\mathbf{m}$ is a right ideal of $P_\mathbf{m}$. It is not hard to see that the left ideal property does not hold.

To see that the semigroup is regular, consider the $\mathbf{m}$ sets $y\alpha^{-1}$, where $y \in X\alpha$, and choose an element x_y in each $y\alpha^{-1}$. Define $\xi \in \mathscr{T}(X)$ by the rule that

$$\begin{aligned} y\xi &= x_y \quad (y \in X\alpha), \\ x\xi &= x_0, \quad (x \in Z(\alpha)), \end{aligned}$$

where x_0 is some arbitrarily chosen fixed element of X. Then $\alpha\xi\alpha = \alpha$. Moreover,

$$S(\xi) \supseteq Z(\alpha)\backslash\{x_0\}, \quad C(\xi) \supseteq Z(\alpha), \quad X\xi \supseteq \{x_y: y \in X\alpha\},$$

from which it follows that $|S(\xi)| = |C(\xi)| = |X\xi| = \mathbf{m}$. Also, at least one of the sets $y\alpha^{-1}$ is of cardinality $\mathbf{m}$; hence $Z(\xi)$ contains $y\alpha^{-1}\backslash\{x_y\}$ and so $|Z(\xi)| = \mathbf{m}$. Finally, $x_0\xi^{-1} = Z(\alpha)$, of cardinality $\mathbf{m}$, and we conclude that $\xi \in P_{\mathbf{m}}\backslash S_{\mathbf{m}}$.

We know that $S_{\mathbf{m}}$ is a subsemigroup of $P_{\mathbf{m}}$ not containing the zero element of $P_{\mathbf{m}}$. It follows that no element α of $S_{\mathbf{m}}$ can be nilpotent, since $\alpha^k \in S_{\mathbf{m}}$ for every k and so cannot be zero. Thus N, the set of nilpotents in $P_{\mathbf{m}}$, is contained in $P_{\mathbf{m}}\backslash S_{\mathbf{m}}$. and it now follows from Proposition 2.1 that $\langle N\rangle$, the smallest subsemigroup of $P_{\mathbf{m}}$ containing N, is also contained in $P_{\mathbf{m}}\backslash S_{\mathbf{m}}$; that is

$$\langle N\rangle \subseteq P_{\mathbf{m}}\backslash S_{\mathbf{m}}. \tag{2.3}$$

We now aim to show that $P_{\mathbf{m}}\backslash S_{\mathbf{m}} \subseteq \langle N_2\rangle$ $(\subseteq \langle N\rangle)$, from which our first main result will follow. First, we give an alternative characterisation of the set

$$N_2 = \{\eta \in Q_{\mathbf{m}} \colon |X\eta| = \mathbf{m} \quad \text{and} \quad |X\eta^2| < \mathbf{m}\}.$$

PROPOSITION 2.2. *Let α be a non-zero element of $P_{\mathbf{m}}\backslash S_{\mathbf{m}}$, and let*

$$A = \cup\{y\alpha^{-1} \colon y \in X\alpha,\ |y\alpha^{-1}| < \mathbf{m}\}.$$

Then $\alpha \in N_2$ if and only if (i) $|X\alpha \cap A| < \mathbf{m}$ *and* (ii) $|\{y \in X\alpha \colon |y\alpha^{-1}| = \mathbf{m}$ *and* $X\alpha \cap y\alpha^{-1} \neq \varnothing\}| < \mathbf{m}$.

Proof. Let

$$B = \bigcup\{y\alpha^{-1} \colon y \in X\alpha,\ |y\alpha^{-1}| = \mathbf{m}\}. \tag{2.4}$$

Since X is the disjoint union of A and B, it follows that

$$X\alpha = (X\alpha \cap A) \cup (X\alpha \cap B),\ X\alpha^2 = (X\alpha \cap A)\alpha \cup (X\alpha \cap B)\alpha. \tag{2.5}$$

Suppose now that α satisfies (i) and (ii).
From (i) it follows that $|(X\alpha \cap A)\alpha| < \mathbf{m}$. Also

$$\begin{aligned} y \in (X\alpha \cap B)\alpha &\Rightarrow y\alpha^{-1} \cap X\alpha \cap B \neq \varnothing \\ &\Rightarrow |y\alpha^{-1}| = \mathbf{m} \quad \text{and} \quad y\alpha^{-1} \cap X\alpha \neq \varnothing, \end{aligned} \tag{2.6}$$

and so from (ii) it follows that $|(X\alpha \cap B)\alpha| < \mathbf{m}$. Hence $|X\alpha^2| < \mathbf{m}$ and so $\alpha \in N_2$ as required.

Conversely, let $\alpha \in N_2$. By (2.5) it follows that both $|(X\alpha \cap A)\alpha|$ and $|(X\alpha \cap B)\alpha|$ are less than $\mathbf{m}$. Now

$$\begin{aligned} y \in (X\alpha \cap A)\alpha &\Rightarrow y\alpha^{-1} \cap X\alpha \cap A \neq \varnothing \\ &\Rightarrow |y\alpha^{-1}| < \mathbf{m}. \end{aligned}$$

Hence by the regularity of $\mathbf{m}$ the set

$$X\alpha \cap A \subseteq \bigcup\{y\alpha^{-1} \colon y \in (X\alpha \cap A)\alpha\}$$

has cardinality less than $\mathbf{m}$. To show (ii) we simply observe that the sequence of implications (2.6) is reversible.

Notice that from this proof it follows that for every α in N_2

$$|X\alpha \cap B| = \mathbf{m}. \tag{2.7}$$

Also,

$$(X\alpha \cap B)\alpha = \{y \in X\alpha \colon |y\alpha^{-1}| = \mathbf{m} \quad \text{and} \quad X\alpha \cap y\alpha^{-1} \neq \varnothing\}.$$

Notice now that for each non-zero α in $P_{\mathbf{m}}\backslash S_{\mathbf{m}}$ the set $Y(\alpha)$ defined by

$$Y(\alpha)=\{y\in X\alpha\colon |y\alpha^{-1}|=\mathbf{m}\} \tag{2.8}$$

is non-empty.

PROPOSITION 2.3. *Let α be a non-zero element of $P_{\mathbf{m}}\backslash S_{\mathbf{m}}$. If there exists y in $Y(\alpha)$ such that $|X\alpha\cap y\alpha^{-1}|<\mathbf{m}$ then $\alpha\in N_2^2$.*

Proof. Suppose that $|X\alpha\cap y_0\alpha^{-1}|<\mathbf{m}$, where $y_0\in Y(\alpha)$. Let θ be a bijection from $X\alpha$ onto $y_0\alpha^{-1}$ and define $\eta_1=\alpha\theta$. Then $\ker\eta_1=\ker\alpha$ and $X\eta_1=y_0\alpha^{-1}$, from which it follows that $\eta_1\in P_{\mathbf{m}}\backslash S_{\mathbf{m}}$. Also $X\eta_1^2=\{y_0\theta\}$ and so certainly $\eta_1\in N_2$.

Next, if $x\in X\eta_1=y_0\alpha^{-1}$, we define

$$x\eta_2=x\theta^{-1}(\in X\alpha).$$

Otherwise we define $x\eta_2=a$, where a is some arbitrarily chosen element in $X\alpha$. Then $\eta_2\in P_{\mathbf{m}}\backslash S_{\mathbf{m}}$. Also

$$\begin{aligned}X\eta_2^2&=(X\alpha)\eta_2\\&=(X\alpha\cap y_0\alpha^{-1})\eta_2\cup(X\alpha\backslash y_0\alpha^{-1})\eta_2\\&=(X\alpha\cap y_0\alpha^{-1})\eta_2\cup\{a\}\end{aligned}$$

and so $|X\eta_2^2|<\mathbf{m}$. Thus $\eta_2\in N_2$.

It is now clear that $\eta_1\eta_2=\alpha$, and so $\alpha\in N_2^2$ as required.

The converse of Proposition 2.3 is not true, as the next proposition makes clear.

PROPOSITION 2.4. *Let α be a non-zero element of $P_{\mathbf{m}}\backslash S_{\mathbf{m}}$ and suppose that $|X\alpha\cap y\alpha^{-1}|=\mathbf{m}$ for all y in $Y(\alpha)$. Then $\alpha\in N_2^2$ if and only if there exists z in $Y(\alpha)$ for which $|z\alpha^{-1}\backslash X\alpha|=\mathbf{m}$.*

Proof. Suppose first that there is an element z_0 in $Y(\alpha)$ such that $|z_0\alpha^{-1}\backslash X\alpha|=\mathbf{m}$. We may write $z_0\alpha^{-1}\backslash X\alpha$ as a disjoint union $A\cup B$, where $|A|=|B|=\mathbf{m}$. Let θ be a bijection from $X\alpha$ onto A, and define $\eta_i=\alpha\theta$. Then $\eta_i\in P_{\mathbf{m}}\backslash S_{\mathbf{m}}$, $|X\eta_1|=|A|=\mathbf{m}$, $X\eta_1^2=\{z_0\theta\}$; hence $\eta_1\in N_2$.

Then by analogy with Proposition 2.3 we define $\eta_2\colon X\to X$ by

$$x\eta_2=\begin{cases}x\theta^{-1} & (x\in A)\\ t & (x\notin A)\end{cases}$$

where t is an arbitrarily chosen element of $X\alpha$. Then $\eta_2\in P_{\mathbf{m}}\backslash S_{\mathbf{m}}$ and $X\eta_2^2=\{t\}$. Hence $\eta_2\in N_2$, and it is clear that $\eta_1\eta_2=\alpha$.

To prove the converse implication, we need the following simple lemma concerning regular semigroups with zero:

LEMMA 2.5. *Let S be a regular semigroup with zero. Let $a=zb$, where $a,b\neq 0$ and z is a nilpotent of index 2. Then there exists a nilpotent z_1 of index 2 in S such that $a=z_1b$ and $z_1\mathcal{R}a$.*

Proof. Let $z_1=aa'z$, where a' is an inverse of a. Then

$$z_1b=aa'zb=aa'a=a,$$

and it is clear that $z_1 \mathcal{R} a$. Also

$$z_1^2 = aa'zaa'z = aa'z^2ba'z = 0.$$

We now return to the proof of Proposition 2.4. Let $\alpha = \eta_1\eta_2 \in N_2^2$, with $|X\alpha \cap y\alpha^{-1}| = \mathbf{m}$ for all y in $Y(\alpha)$. By the lemma we may assume that $\ker \eta_1 = \ker \alpha$. Now assume by way of contradiction that $|z\alpha^{-1} \backslash X\alpha| < \mathbf{m}$ for every z in $Y(\alpha)$.

Consider the set

$$D = \{y \in X\eta_1 \colon |y\eta_1^{-1}| = \mathbf{m} \text{ and } X\eta_1 \cap y\eta_1^{-1} \neq \varnothing\}.$$

By Proposition 2.2 this set has cardinality less than $\mathbf{m}$. On the other hand, in the notation of (2.4),

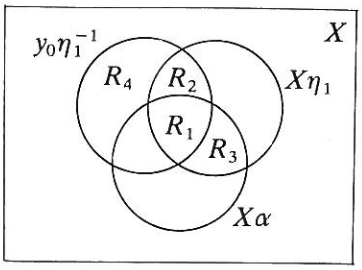
$$\bigcup\{X\eta_1 \cap y\eta_1^{-1} \colon y \in D\} = X\eta_1 \cap B$$

and so, by (2.7), has cardinality $\mathbf{m}$. Hence, by the regularity of $\mathbf{m}$,

$$|X\eta_1 \cap y_0\eta_1^{-1}| = \mathbf{m}$$

for some y_0 in D.

We illustrate the situation in the Venn diagram below:



We have just established that $|R_1 \cup R_2| = \mathbf{m}$ and we are assuming that $R_2 \cup R_4$ $(= (y_0\eta_2)\alpha^{-1} \backslash X\alpha)$ has cardinality less than $\mathbf{m}$. Hence $|R_2| < \mathbf{m}$. Hence certainly $|R_1 \cup R_3| = \mathbf{m}$; that is,

$$|X\alpha \cap X\eta_1| = \mathbf{m}.$$

Now, from the fact that $\ker \eta_1 = \ker \alpha$ it follows that η_2 must map $X\eta_1$ in a one–one fashion. Hence

$$|(X\alpha \cap X\eta_1)\eta_2| = \mathbf{m}.$$

But $X\eta_2 \supseteq X\alpha$ and so $X\eta_2^2 \supseteq X\alpha\eta_2 \supseteq (X\alpha \cap X\alpha \cap X\eta_1)\eta_2$. It follows that $|X\eta_2^2| = \mathbf{m}$, contrary to the assumption that $\eta_2 \in N_2$.

This completes the proof of Proposition 2.4.

Propositions 2.3 and 2.4 give a characterisation of those elements of $P_{\mathbf{m}} \backslash S_{\mathbf{m}}$ that can be expressed as products of 2 nilpotents of index 2. The remaining elements can be expressed as products of length 3:

PROPOSITION 2.6. *Let α be a non-zero element in $P_{\mathbf{m}} \backslash S_{\mathbf{m}}$, and suppose that $|X\alpha \cap y\alpha^{-1}| = \mathbf{m}$ and $|y\alpha^{-1} \backslash X\alpha| < \mathbf{m}$ for all y in $Y(\alpha)$. Then there exist η_1, η_2, η_3 in N_2 such that $\alpha = \eta_1\eta_2\eta_3$.*

Proof. Choose y_0 in $Y(\alpha)$ and express $y_0\alpha^{-1}$ as a disjoint union $A \cup B$, where $|A| = |B| = \mathbf{m}$. Let $\theta: X\alpha \to A$ be a bijection, and define $\eta_1: X \to X$ by $\eta_1 = \alpha\theta$. Then $\eta_1 \in N_2$.

Next, let

$$P = Z(\alpha)\backslash y_0\alpha^{-1}, \quad Q = Z(\alpha) \cap y_0\alpha^{-1} \quad (= y_0\alpha^{-1}\backslash X\alpha);$$

then $Z(\alpha) = P \cup Q$, $|P| = \mathbf{m}$, $|Q| < \mathbf{m}$. Define η_2 as a bijection from $X\eta_1$ onto P, and complete the definition by specifying that

$$(X\backslash X\eta_1)\eta_2 = a,$$

where a is arbitrarily chosen in P. Then $\eta_2 \in N_2$. In fact $X\eta_2^2 = P\eta_2 = \{a\}$, since

$$P \subseteq X\backslash y_0\alpha^{-1} \subseteq X\backslash A = X\backslash X\eta_1.$$

Finally, define η_3 for each x in $X\eta_1\eta_2$ by

$$x\eta_3 = x\eta_2^{-1}\theta^{-1}$$

and complete the definition by putting

$$(X\backslash X\eta_1\eta_2)\eta_3 = b,$$

where b is chosen arbitrarily in $X\alpha$. Then

$$X\eta_3 = X\alpha \subseteq X\backslash P = X\backslash X\eta_1\eta_2$$

and so $X\eta_3^2 = \{b\}$. Since it is clear that $\eta_3 \in P_{\mathbf{m}}\backslash S_{\mathbf{m}}$, we thus have $\eta_3 \in N_2$. It is easily seen that $\eta_1\eta_2\eta_3 = \alpha$.

We have now shown that every non-zero element of $P_{\mathbf{m}}\backslash S_{\mathbf{m}}$ belongs to $\langle N_2 \rangle$. It is evident that $0 = \eta^2 \in N_2^2 \subseteq \langle N_2 \rangle$, where η is any element of N_2. Thus we have the following result:

THEOREM 2.7. *Let X be a set with infinite cardinality $\mathbf{m}$, where $\mathbf{m}$ is regular, and let $P_{\mathbf{m}}$, $S_{\mathbf{m}}$ be defined by equations (2.1) and (2.2). Let N be the set of nilpotents in $P_{\mathbf{m}}$, and let N_2 be the set of nilpotents of index 2. Then $\langle N \rangle$, the subsemigroup of $P_{\mathbf{m}}$ generated by N, is equal to $\tilde{T}$. Moreover,*

$$N_2 \cup N_2^2 \subset P_{\mathbf{m}}\backslash S_{\mathbf{m}} = N_2 \cup N_2^2 \cup N_2^3.$$

Remark. It is not hard to see that the restriction of regularity is necessary. If $\mathbf{m}$ is irregular, then the subset $S_{\mathbf{m}}$ (no longer a semigroup) contains nilpotent elements. To see this, write $X = A \cup B$, where $|A| = |B| = \mathbf{m}$ and $A \cap B = \varnothing$, and suppose that

$$A = \bigcup\{A_\lambda : \lambda \in \lambda\},$$

where $|\Lambda| < \mathbf{m}$ and $|A_\lambda| < \mathbf{m}$ for all λ. Let $\theta: \Lambda \cup B \to A$ be a bijection and define $\eta: X \to X$ by

$$x\eta = \begin{cases} \lambda\theta & \text{if } x \in A_\lambda \quad (\lambda \in \Lambda) \\ x\theta & \text{if } x \in B \end{cases}.$$

Then the $(\ker \eta)$-classes are A_λ $(\lambda \in \Lambda)$ and $\{b\}$ $(b \in B)$ and so $\eta \in S_{\mathbf{m}}$. However $X\eta^2 = \Lambda\theta$, of cardinality less than $\mathbf{m}$, and so η is a nilpotent of index 2 in $P_{\mathbf{m}}$.

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4