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A Nodal Basis for C^1 Piecewise Polynomials of Degree $n \geq 5$

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Abstract. A basis for the space of C^1 piecewise polynomials in two variables of degree $n \geqslant 5$ is constructed. The basis is parametrized by "nodal variables," namely, the values and derivatives of the basis functions at a discrete set of points.

Let Π be a bounded domain in $\mathbf{R}^2$ such that $\partial\Pi$ consists of a finite number of nonintersecting polygonal arcs (by a polygonal arc, we mean a continuous curve consisting of a finite number of line segments), and let T be a rectilinear triangulation of Π (i.e., a triangulation with straight edges). Denote by $S_n = S_n(\Pi, T)$ the subspace of $C^1(\Pi)$ consisting of functions whose restriction to each triangle is a polynomial of degree $\leqslant n$ (the space of C^1 *piecewise polynomials of degree n*). Strang [3] conjectured a formula for the dimension of S_n (in terms of the number of triangles, edges, and vertices in T), and we mention later some results of our study [2] of the dimension of S_n. But the main purpose of this note is to describe the structure of S_n completely for $n \geqslant 5$ by exhibiting a *nodal* basis for S_n. This basis makes S_n convenient to use for the approximation of solutions of differential equations via the finite element method (see, e.g., [1], [4]). By a nodal basis, we mean that the functions in S_n are determined by their values and derivatives at points in Π (these are called *nodal values*). The nodal values describing S_n are

(1) the value and x and y derivatives at each vertex in T,

(2) the value at each of $n - 5$ distinct points in the interior of each edge,

(3) the (edge) normal derivative at each of $n - 4$ distinct points in the interior of each edge (the points may coincide with those of (2)), and

(4) the value at $\frac{1}{2}(n - 4)(n - 5)$ distinct points in the interior of each triangle, chosen so that if a polynomial of degree $n - 6$ vanishes at all the points, it vanishes identically.

The remaining nodal values are more complicated to describe, as they involve second derivatives at the vertices. Suppose e_1 is an edge in T, v is one of its vertices and $f \in S_n$. By the "second e_1 derivative of f at v", we mean $\partial_{e_1}\partial_{e_1}(f|\tau)(v)$, where ∂_{e_1} is differentiation in the e_1 direction away from v and τ is a triangle with e_1 as an edge. There are two such triangles, and the value $\partial_{e_1}\partial_{e_1}(f|\tau)(v)$ is independent of the choice for $f \in S_n$. Similarly, if e_1 and e_2 are two edges of a triangle τ in T, we define the "e_1, e_2 cross derivative of f at v" to be $\partial_{e_1}\partial_{e_2}(f|\tau)(v)$ (here $v = e_1 \cap e_2$). Referring to Fig. 1, we have

$$\partial_{e_1}\partial_{e_2}(f|\tau)(v) = \partial_{e_1}\partial_{e_2}(f|\tau_0)(v) = \partial_{e_1}\partial_{e_2}(f|\tau_1)(v)$$

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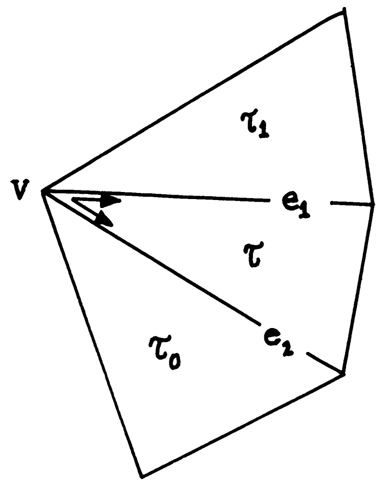

FIGURE 1

for $f \in S_n$. Returning to our list of nodal values, we add

(5) one cross derivative at each vertex (i.e., for each vertex, choose adjacent edges e_1, e_2 emanating from it, and put in the e_1, e_2 cross derivative at that vertex), and

(6) at each vertex, the second edge derivative for all the edges meeting there, with one exception: if the vertex is an interior vertex (i.e., not on $\partial\Pi$), we omit one of the second edge derivatives, where the omitted edge is chosen so that its two adjacent edges are not collinear (if there are no such edges, no omission is made).

An interior vertex for which the adjacent edges of each edge are collinear is called *singular*; the star of such a vertex is simply a convex quadrilateral with the diagonals drawn in.

Let us count the number D of nodal values listed above. Let T be the number of triangles, E (resp. E_0) be the number of edges (resp. interior edges), and V (resp. V_0, σ) be the number of vertices (resp. interior vertices, singular vertices) in $\mathcal{T}$. Then

$$D = 3V + (n-5)E + (n-4)E + \tfrac{1}{2}(n-4)(n-5)T + V + (2E - V_0 + \sigma).$$

Combining and using the obvious relations

$$E - E_0 = V - V_0 \quad \text{and} \quad 3T = E + E_0,$$

we obtain the formula

$$\text{(I)} \qquad D = \tfrac{1}{2}(n+1)(n+2)T - (2n+1)E_0 + 3V_0 + \sigma.$$

This is the formula for the dimension of S_n conjectured by Strang [3], except that singular vertices were not explicitly mentioned. Let us number the nodal values 1, 2, $\ldots$, D. Then we have the following:

THEOREM. *There is a basis* $\{\varphi_j\}_{j=1}^D$ *of* $S_n(\Pi, T)$ $(n \geq 5)$ *where each* φ_j *has the* j*th nodal value equal to* 1 *and all the other nodal values zero.*

Remark. It is important for applications to know how large the support of each basis function is. If the jth node is one of type 1, 5, or 6, involving the value or a derivative at a vertex, then φ_j is supported in the star of that vertex. For the edge nodes (2 or 3), the support of the associated basis function is the union of the two triangles sharing that edge. The basis function for a 4 node is supported in the relevant triangle.

Proof of the Theorem. The nodal values give us a map from S_n to $\mathbf{R}^D$, and we must show that it is an isomorphism. We begin by showing that it is injective. Suppose that $f \in S_n$ has all nodal values zero. Then we shall show that $f \equiv 0$. Looking locally at one triangle for the moment, a polynomial of degree n is completely determined by its value, first and second derivatives at the three vertices plus the relevant quantities at edge and interior nodes as in 2, 3, and 4. Thus we need to show that, in addition to all nodal values being zero, all remaining second derivatives at vertices are zero. We determine these by looking at the star of a vertex in T. Referring to Fig. 2, suppose we know $\partial_{e_1}\partial_{e_2}(f|\tau_1)(v)$

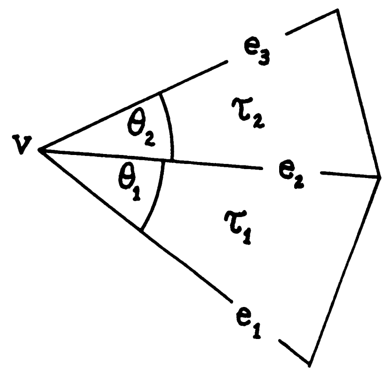

FIGURE 2

and $\partial_{e_2}\partial_{e_2}(f|\tau_1)(v)$. Then as remarked above, we already know $\partial_{e_1}\partial_{e_2}(f|\tau_2)(v)$ and $\partial_{e_2}\partial_{e_2}(f|\tau_2)(v)$ (they are the same). We wish to calculate $\partial_{e_2}\partial_{e_3}(f|\tau_2)(v)$. The answer is derived by writing ∂_{e_3} in terms of ∂_{e_1} and ∂_{e_2}, namely

$$\partial_{e_3} = -\frac{\sin\theta_2}{\sin\theta_1}\partial_{e_1} + \left(\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 + \cos\theta_2\right)\partial_{e_2}.$$

Applying ∂_{e_2}, we find

$$\text{(II)}\quad \begin{aligned}\partial_{e_2}\partial_{e_3}(f|\tau_2)(v) &= -\frac{\sin\theta_2}{\sin\theta_1}\partial_{e_1}\partial_{e_2}(f|\tau_1)(v)\\ &\quad + \left(\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 + \cos\theta_2\right)\partial_{e_2}\partial_{e_2}(f|\tau_1)(v).\end{aligned}$$

Thus, knowing one cross derivative plus all but one second edge derivative at a vertex, we can determine all the cross derivatives (work clockwise and counter-clockwise from the known cross derivative, stopping upon arrival at the unknown second edge derivative or a boundary edge). All that is left is to determine the unknown second edge derivative at a nonsingular interior vertex. We invert (II) to obtain

$$\text{(III)} \qquad \partial_{e_2}\partial_{e_2}(f|\tau_1)(v) = \frac{\csc\theta_1\partial_{e_1}\partial_{e_2}(f|\tau_1)(v) + \csc\theta_2\partial_{e_2}\partial_{e_3}(f|\tau_2)(v)}{\cot\theta_1 + \cot\theta_2}.$$

Our requirement that e_1 and e_3 not be collinear means that the denominator is nonzero, so the remaining second edge derivative is determined. When all of the nodal values for $f \in S_n$ are zero, the above argument shows that all second derivatives at each vertex are zero. Thus $f \equiv 0$. Also, when the nodal values for f are zero locally, $f \equiv 0$ in the relevant triangles, proving the Remark about support of the basis functions.

We now show that the nodal value map $S_n \longrightarrow \mathbf{R}^D$ is onto. Given a set of nodal values, we first construct a piecewise polynomial triangle by triangle (not necessarily C^1) by using the formulae above to define the remaining second derivatives at the vertices of each triangle. What we must check is that a function f so constructed is C^1. Suppose τ_1 and τ_2 are two triangles in T sharing an edge e. f is C^1 iff $f|_{\tau_1} - f|_{\tau_2}$ vanishes to second order on e for all such τ_i. The one-variable polynomial $(f|_{\tau_1} - f|_{\tau_2})|_e$ vanishes at the endpoints of e to third order plus at $n - 5$ other points, hence it is zero. Denoting by $\partial_{e^\perp}$ the normal derivative to e, we have $(\partial_{e^\perp}f|_{\tau_1} - \partial_{e^\perp}f|_{\tau_2})|_e$ equal to zero at the endpoints plus at $n - 4$ other points. In addition, we must show that

$$\text{(IV)} \qquad \partial_e(\partial_{e^\perp}f|_{\tau_1} - \partial_{e^\perp}f|_{\tau_2}) = 0$$

at the endpoints of e. By our construction, we have $\partial_e\partial_{e'}f|_{\tau_1} - \partial_e\partial_{e'}f|_{\tau_2} = 0$ there, where e' is an edge adjacent to e. Since also $\partial_e\partial_e f|_{\tau_1} - \partial_e\partial_e f|_{\tau_2} = 0$, we recover (IV) by writing $\partial_{e^\perp}$ in terms of ∂_e and $\partial_{e'}$. Since $(\partial_{e^\perp}f|_{\tau_1} - \partial_{e^\perp}f|_{\tau_2})|_e$ is a polynomial of degree $n - 1$, it must be $\equiv 0$. Thus f is C^1, and the theorem is complete.

Remark. If the number of edges meeting at a nonsingular interior vertex is odd, it is possible to replace the cross derivative at that vertex with the remaining second edge derivative, and still have a nodal basis. (Eq. (II) allows us to compute all cross derivatives: starting at τ_1, work around v until returning to τ_1, thus obtaining an equation for the e_1, e_2 cross derivative in terms of all second edge derivatives at v.) In either case, Eqs. (II) and (III) (via the arguments in the proof above) allow us to determine explicitly the nodal basis functions $\{\varphi_j\}$ for computational purposes.

Further Results. In [2], we prove that (I) gives the dimension of $S_n(\Pi, T)$ for $n \geqslant 4$, and we construct a nodal basis for $n = 4$, but it is defined in a global fashion rather than a local one, as was done here. For $n = 3, 2$,* we verify (I) under increasingly more stringent requirements on T. For $n = 2$, we construct an example where (I) does not give the correct dimension.

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*The cases $n = 0, 1$ are trivial.

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