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A NON-COMMUTATIVE GENERALIZATION OF MV-ALGEBRAS

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1. Introduction

It is well-known that MV-algebras (as algebras of type $(\oplus, \odot, \neg, 0, 1)$ of signature $\langle 2, 2, 1, 0, 0\rangle$) have been introduced and studied by C. C. Chang in [4] and [5] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. Moreover, Chang proved in [5] that every linearly ordered MV-algebra is isomorphic to an MV-algebra in the form $\Gamma(G, u)$, where G is a commutative linearly ordered (additive) group, $0 \leqslant u \in G$ is a strong order unit in G, $\Gamma(G, u) = [0, u] = \{x \in G;\ 0 \leqslant x \leqslant u\}$, and the operations on $\Gamma(G, u)$ are as follows: $1 = u$, $x \oplus y = \min(x + y, u)$, $\neg x = u - x$, $x \odot y = \max(x + y - u, 0) = \neg(\neg x \oplus \neg y)$ for any $x, y \in \Gamma(G, u)$. D. Mundici generalized this result in [11] to arbitrary MV-algebras. Namely, let, in general, G be a commutative lattice ordered group (l-group) and $0 \leqslant u \in G$. If $x \oplus y = (x + y) \wedge u$, $\neg x = u - x$, $x \odot y = \neg(\neg x \oplus \neg y)$ for any $x, y \in [0, u]$, then $\Gamma(G, u) = ([0, u], \oplus, \odot, \neg, 0, u)$ is an MV-algebra. Mundici proved that every MV-algebra is isomorphic to $\Gamma(G, u)$ for some commutative l-group G and some strong unit $u \in G$, and moreover, that the category of MV-algebras is equivalent to the category of commutative l-groups with strong units. Mundici further proved in [12] that MV-algebras are also categorically equivalent to bounded commutative BCK-algebras. The author in [13] and [14] showed that one can obtain MV-algebras as special cases of dually residuated l-monoids (DRl-monoids) because MV-algebras are equivalent to DRl-monoids of a class of bounded DRl-monoids.

In the paper we define a generalization of MV-algebras as algebras $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$ of signature $\langle 2, 2, 1, 1, 0, 0\rangle$ in which the binary operations $\oplus$ and $\odot$ in general need not be commutative. (MV-algebras introduced by Chang are then called *commutative.*) In the first part of the paper some properties of non-commutative MV-algebras are described. In the second part it is proved that (non-commutative) MV-algebras are in a one-to-one correspondence with some bounded non-commutative DRl-monoids, and in the last section it is shown that every interval $[0,u]$ of any (non-commutative) l-group can be viewed as an MV-algebra and that every linearly ordered MV-algebra is isomorphic to an analogous interval of some linearly ordered loop.

(Note that the class of commutative l-groups is the smallest non-trivial variety of l-groups. Therefore, from this point of view, intervals of non-commutative l-groups represent a very essential generalization of Chang's MV-algebras.)

We use the terminology and results of [2], [7] and [9] concerning the theory of lattice ordered groups and of [1] concerning the theory of loops.

2. Properties of non-commutative MV-algebras

An algebra $A=(A,\oplus,\odot,\neg,\sim,0,1)$ of signature $\langle 2,2,1,1,0,0\rangle$ is called a *(non-commutative) MV-algebra* if for any $x,y,z,u\in A$:

(1) $x\oplus(y\oplus z)=(x\oplus y)\oplus z$; (1′) $x\odot(y\odot z)=(x\odot y)\odot z$
(2) $x\oplus\sim x=1=\neg x\oplus x$; (2′) $x\odot\sim x=0=\neg x\odot x$
(3) $x\oplus 1=1=1\oplus x$; (3′) $x\odot 0=0=0\odot x$
(4) $x\oplus 0=x=0\oplus x$; (4′) $x\odot 1=x=1\odot x$
(5) $\neg(x\oplus y)=\neg x\odot\neg y$; (5′) $\neg(x\odot y)=\neg x\oplus\neg y$
(6) $\sim(x\oplus y)=\sim x\odot\sim y$; (6′) $\sim(x\odot y)=\sim x\oplus\sim y$
(7) $\neg\sim x=x=\sim\neg x$
(8) $\neg 0=1=\sim 0$
(9) $y\oplus(x\odot\sim y)=(\neg y\odot x)\oplus y$; (9′) $y\odot(x\oplus\sim y)=(\neg y\oplus x)\odot y$

Denote $x\vee y=y\oplus(x\odot\sim y)$, $x\wedge y=y\odot(x\oplus\sim y)$. Then

(10) $x\vee y=y\vee x$; (10′) $x\wedge y=y\wedge x$
(11) $x\vee(y\vee z)=(x\vee y)\vee z$; (11′) $x\wedge(y\wedge z)=(x\wedge y)\wedge z$
(12a) $x\oplus(y\wedge z)\oplus u=(x\oplus y\oplus u)\wedge(x\oplus z\oplus u)$
(12b) $x\oplus(y\vee z)\oplus u=(x\oplus y\oplus u)\vee(x\oplus z\oplus u)$
(12′a) $x\odot(y\wedge z)\odot u=(x\odot y\odot u)\wedge(x\odot z\odot u)$
(12′b) $x\odot(y\vee z)\odot u=(x\odot y\odot u)\vee(x\odot z\odot u)$

Note. It is obvious that the above system of axioms is not the simplest one, but we use it in this form to have the possibility comparing the introduced notion with that of a commutative MV-algebra. We can see that A can be considered as a commutative (Chang's) MV-algebra if and only if both operations $\oplus$ and $\odot$ are commutative and the unary operations $\neg$ and $\sim$ coincide. (Note that conditions (12b) and (12′a) are not directly required but can be derived for Chang's MV-algebras. See e.g. [8], Theorem 3.1.) A simple axiomatic system for commutative MV-algebras is used e.g. in [6].

Theorem 1. *If A is an MV-algebra, $x, y \in A$, then*

(a) $\neg 1 = 0 = \sim 1$;

(b) $x \odot y = \neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$;

(c) $x \vee 0 = x = x \wedge 1$, $x \wedge 0 = 0$, $x \vee 1 = 1$;

(d) $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg(x \wedge y) = \neg x \vee \neg y$,
$\sim(x \vee y) = \sim x \wedge \sim y$, $\sim(x \wedge y) = \sim x \vee \sim y$;

(e) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.

Proof. (a) It follows from (7) and (8).

(b) From (5), (6) and (7).

(c) From (4′) and (8).

(d)
$$\neg(x \vee y) = \neg(y \oplus (x \odot \sim y)) = \neg y \odot \neg(x \odot \sim y)$$
$$= \neg y \odot (\neg x \oplus \neg\sim y) = \neg y \odot (\neg x \oplus \sim\neg y) = \neg x \wedge \neg y.$$

The remaining equalities are analogous.

(e)
$$x \wedge (x \vee y) = x \wedge ((\neg x \odot y) \oplus x) = x \odot (((\neg x \odot y) \oplus x) \oplus \sim x)$$
$$= x \odot ((\neg x \odot y) \oplus 1) = x \odot 1 = x$$

Analogously the second equality. □

We obtain the following theorem as a consequence of (10), (10′), (11), (11′), and (c) and (e) from Theorem 1.

Theorem 2. *If A is an MV-algebra then $(A, \vee, \wedge)$ is a bounded lattice with the least element 0 and the greatest element 1.*

Hence we can consider the order relation $\leqslant$ on A induced by the lattice $(A, \vee, \wedge)$, i.e.

$$x \leqslant y \Longleftrightarrow x \vee y = y \Longleftrightarrow x \wedge y = x.$$

Theorem 3. *For any $x, y, z, u \in A$ we have*

(a)
$$x \leqslant y \Longrightarrow u \oplus x \oplus z \leqslant u \oplus y \oplus z,$$
$$u \odot x \odot z \leqslant u \odot y \odot z$$

(i.e. $(A, \oplus, \leqslant)$ and $(A, \odot, \leqslant)$ are ordered monoids);

(b) $x \odot y \leqslant x \wedge y \leqslant x \vee y \leqslant x \oplus y$;

(c) $x \leqslant y \Longleftrightarrow \neg y \leqslant \neg x \Longleftrightarrow \sim y \leqslant \sim x$.

Proof. (a) If $x \leqslant y$ then $u \oplus (x \wedge y) \oplus z = u \oplus x \oplus z$, hence by (12a) $(u \oplus x \oplus z) \wedge (u \oplus y \oplus z) = u \oplus x \oplus z$, thus $u \oplus x \oplus z \leqslant u \oplus y \oplus z$.

Using (12) and (12′) we also obtain the remaining implications.

(b) Since $y \leqslant 1$, by (a) and (4′) we get $x \odot y \leqslant x \odot 1 = x$. Similarly $x \odot y \leqslant y$, and so $x \odot y \leqslant x \wedge y$.

The equality $x \vee y \leqslant x \oplus y$ is dual.

(c) Let $x \leqslant y$. Then $\neg(x \vee y) = \neg y$, hence by Theorem 1(d) we have $\neg x \wedge \neg y = \neg y$, and thus $\neg y \leqslant \neg x$.

Suppose that $\neg y \leqslant \neg x$. Then $\sim(\neg y \wedge \neg x) = \sim\neg y$, hence by Theorem 1(d), $\sim\neg y \vee \sim\neg x = \sim\neg y$, that means $y \vee x = y$, therefore $x \leqslant y$.

The second equivalence is analogous. □

Theorem 4. *For any $x, y \in A$ we have*

(a) $x \oplus y = 0 \Longrightarrow x = 0 = y$;

(b) $x \odot y = 1 \Longrightarrow x = 1 = y$.

Proof. (a) Let $x \oplus y = 0$. Then by Theorem 1(c) and conditions (4) and (12a),

$$0 = x \wedge 0 = x \wedge (x \oplus y) = (x \oplus 0) \wedge (x \oplus y) = x \oplus 0 = x.$$

Hence $0 = x \oplus y = 0 \oplus y = y$.

(b) Dually. □

Theorem 5. *For any $x, y \in A$ the following conditions are equivalent:*

(a) $x \leqslant y$;

(b) $y \oplus \sim x = 1$;

(c) $\neg x \oplus y = 1$;

(d) $x \odot \sim y = 0$;

(e) $\neg y \odot x = 0$.

Proof. (a)⇔(b): Let $x \leqslant y$. Then by (2) and Theorem 3, $1 = x \oplus \sim x \leqslant y \oplus \sim x$, hence $y \oplus \sim x = 1$.

Conversely, if $y \oplus \sim x = 1$, then by (4′) we get $x \wedge y = x \odot (y \oplus \sim x) = x \odot 1 = x$, thus $x \leqslant y$.

(a)⇔(c): Analogously.

(a)⇔(d), (a)⇔(e): Dually. □

Theorem 6. a) *If $x \oplus z = y \oplus z$ ($z \oplus x = z \oplus y$) and $x, y \leqslant \neg z$ (x, $y \leqslant \sim z$, respectively), then $x = y$.*

b) *If $x \odot z = y \odot z$ ($z \odot x = z \odot y$) and $\neg z \leqslant x$, y ($\sim z \leqslant x, y$, respectively), then $x = y$.*

P r o o f. a) Let $x \oplus z = y \oplus z$ and x, $y \leqslant \neg z$. Then by Theorem 5, (9′) and (10′),

$$x = 1 \odot x = (\neg x \oplus \neg z) \odot x = \neg z \odot (x \oplus \sim\neg z)$$
$$= \neg z \odot (x \oplus z) = \neg z \odot (y \oplus z) = y.$$

Analogously the second implication.

b) Dually. □

Theorem 7. *For any $x, y \in A$ the following conditions are equivalent:*

(a) $x \oplus y = y$;

(b) $x \odot y = x$;

(c) $\sim x \vee y = 1$;

(d) $x \wedge \neg y = 0$.

P r o o f. (a)⇔(d): Let $x \oplus y = y$. Then

$$x \wedge \neg y = \neg y \odot (x \oplus \sim\neg y) = \neg y \odot y = 0.$$

Conversely, let $x \wedge \neg y = 0$. Then

$$y = 0 \oplus y = (x \wedge \neg y) \oplus y = (x \oplus y) \wedge (\neg y \oplus y)$$
$$= (x \oplus y) \wedge 1 = x \oplus y.$$

(b)⇔(c): If $x \odot y = x$ then

$$\sim x \vee y = (\neg\sim x \odot y) \oplus \sim x = x \oplus \sim x = 1.$$

Conversely, if $\sim x \vee y = 1$ then

$$x = x \odot 1 = x \odot (\sim x \vee y) = (x \odot \sim x) \vee (x \odot y)$$
$$= 0 \vee (x \odot y) = x \odot y.$$

(c)⇔(d): If $\sim x \vee y = 1$ then $\neg(\sim x \vee y) = \neg 1$, hence $x \wedge \neg y = 0$.
Conversely, let $x \wedge \neg y = 0$. Then $\sim(x \wedge \neg y) = \sim 0 = 1$, thus $\neg x \vee y = 1$. □

The proof of the following theorem is analogous.

Theorem 8. *For any $x, y \in A$ the following conditions are equivalent:*

(a) $x \oplus y = x$;

(b) $x \odot y = y$;

(c) $x \vee \neg y = 1$;
(d) $\sim x \wedge y = 0$.

As a consequence we get

Theorem 9. *For any $x \in A$ the following conditions are equivalent:*
(a) $x \oplus x = x$;
(b) $x \odot x = x$;
(c) $\sim x \vee x = 1$;
(d) $x \vee \neg x = 1$;
(e) $x \wedge \neg x = 0$;
(f) $\sim x \wedge x = 0$;
(g) $\neg x \oplus \neg x = \neg x$;
(h) $\neg x \odot \neg x = \neg x$;
(i) $\sim x \oplus \sim x = \sim x$;
(j) $\sim x \odot \sim x = \sim x$.

Let us now consider the set of all elements in A having the properties from the preceding theorem.

Theorem 10. *The set $B = \{x \in A; x \oplus x = x\}$ is a sublattice of the lattice $(A, \vee, \wedge)$ containing 0 and 1 such that if $x \in B$ then also $\neg x$ and $\sim x$ belong to B and both are complements of x in B.*

Proof. Let $x, y \in B$. Then by ($12'$b) and Theorem 3(b),

$$\begin{aligned}(x \vee y) \odot (x \vee y) &= (x \odot x) \vee (x \odot y) \vee (y \odot x) \vee (y \odot y)\\ &= x \vee y \vee (x \odot y) \vee (y \odot x) = x \vee y,\end{aligned}$$

hence $x \vee y \in B$. Similarly we get $x \wedge y \in B$.

By Theorem 9, $\neg x$, $\sim x \in B$, by (4) and ($4'$), 0, $1 \in B$, and thus by Theorem 9, $\neg x$ and $\sim x$ are complements of x in B. □

Now let us denote by $\mathcal{MV}$ the class of all (non-commutative) MV-algebras. From the definition it is obvious that $\mathcal{MV}$ is a variety of algebras of type $(\oplus, \odot, \neg, \sim, 0, 1)$. Recall that a variety of algebras is called *arithmetical* if it is congruence distributive and permutable. It is well known (e.g. [3], Theorem II.12.5) that by Pixley's theorem a variety $\mathcal{V}$ is arithmetical if and only if there is a term $w(x, y, z)$ of given type such that the identities

$$w(y, y, x) = w(x, y, x) = w(x, y, y) = x$$

are satisfied in $\mathcal{V}$.

Theorem 11. *The variety $\mathcal{MV}$ of MV-algebras is arithmetical.*

P r o o f. Set

$$w(x, y, z) = ((\neg y \oplus z) \odot x) \vee ((\neg y \oplus x) \odot z) \vee (x \wedge z).$$

It is obvious that $w(x, y, z)$ is indeed a term in the language $(\oplus, \odot, \neg, \sim, 0, 1)$ because the operations $\vee$ and $\wedge$ only express abbreviations of the corresponding terms of the type of MV-algebras.

Let A be an MV-algebra and $x, y, z \in A$. Then

$$w(y, y, x) = ((\neg y \oplus x) \odot y) \vee ((\neg y \oplus y) \odot x) \vee (y \wedge x),$$

and so (9′), (2) and (4′) of the definition of an MV-algebra yield

$$w(y, y, x) = (x \wedge y) \vee x \vee (x \wedge y) = x.$$

Further,

$$w(x, y, x) = ((\neg y \oplus x) \odot x) \vee ((\neg y \oplus x) \odot x) \vee x,$$

and because $(\neg y \oplus x) \odot x \leqslant x$ by Theorem 3, we get

$$w(x, y, x) = x.$$

Finally,

$$\begin{aligned} w(x, y, y) &= ((\neg y \oplus y) \odot x) \vee ((\neg y \oplus x) \odot y) \vee (x \wedge y) \\ &= x \vee (x \wedge y) \vee (x \wedge y) = x. \end{aligned}$$

□

3. Connections between MV-algebras and dually residuated lattice ordered semigroups

Definition. An algebra $A = (A, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of signature $\langle 2, 0, 2, 2, 2, 2 \rangle$ is called a *dually residuated* (*non-commutative*) *lattice ordered monoid* (a *DRl-monoid*) if:

(1) $(A, +, 0)$ is a (non-commutative) monoid;
(2) $(A, \vee, \wedge)$ is a lattice;

(3) $(A,+,0,\vee,\wedge)$ is a lattice ordered monoid, i.e. for any $x,y,z,v\in A$,

$$x+(y\vee z)+v=(x+y+v)\vee(x+z+v),$$
$$x+(y\wedge z)+v=(x+y+v)\wedge(x+z+v);$$

(4) if $\leqslant$ denotes the order on A induced by the lattice $(A,\vee,\wedge)$ then for any $x,y\in A$,
$x\rightharpoonup y$ is the least element $s\in A$ such that $s+y\geqslant x$,
$x\leftharpoondown y$ is the least element $r\in A$ such that $y+r\geqslant x$;

(5) A satisfies the identities
(a) $((x\rightharpoonup y)\vee 0)+y\leqslant x\vee y$, $y+((x\leftharpoondown y)\vee 0)\leqslant x\vee y$,
(b) $x\rightharpoonup x\geqslant 0$, $x\leftharpoondown x\geqslant 0$.

Remark. Condition (4) is equivalent to the following identities:
(4a) $(x\rightharpoonup y)+y\geqslant x$, $y+(x\leftharpoondown y)\geqslant x$;
(4b) $(x\rightharpoonup y)\leqslant((x\vee z)\rightharpoonup y)$, $(x\leftharpoondown y)\leqslant((x\vee z)\leftharpoondown y)$;
(4c) $((x+y)\rightharpoonup y)\leqslant x$, $((y+x)\leftharpoondown y)\leqslant x$.

Therefore DRl-monoids form a variety of algebras of signature $\langle 2,0,2,2,2,2\rangle$.

Commutative DRl-monoids have been introduced and studied by K. L. N. Swamy in [15], [16] and [17] as common generalizations of commutative lattice ordered groups and Brouwerian (and so also Boolean) algebras. (For commutative DRl-monoids, the operations $\rightharpoonup$ and $\leftharpoondown$ coincide and the common result $x\rightharpoonup y=x\leftharpoondown y$ is denoted by $x-y$ for any elements x and y. Then commutative DRl-monoids are regarded as algebras in the language $(+,0,\vee,\wedge,-)$ of signature $\langle 2,0,2,2,2\rangle$.) Connections between commutative MV-algebras and DRl-monoids were described by the author in [13] and [14]. The properties of non-commutative DRl-monoids were studied by T. Kovář in [10].

Now we will deal with connections between MV-algebras and DRl-monoids in non-commutative cases.

Theorem 12. *Let $A=(A,+,0,\vee,\wedge,\rightharpoonup,\leftharpoondown)$ be a DRl-monoid with a greatest element 1 which satisfies the conditions*
(i) $\forall x\in A$; $1\leftharpoondown(1\rightharpoonup x)=x=1\rightharpoonup(1\leftharpoondown x)$;
(ii) $\forall x,y\in A$; $1\rightharpoonup((1\leftharpoondown x)+(1\leftharpoondown y))=1\leftharpoondown((1\rightharpoonup x)+(1\rightharpoonup y))$.
Set

$$\neg x=1\rightharpoonup x,\quad {\sim}x=1\leftharpoondown x,\quad x\cdot y=\neg({\sim}x+{\sim}y)$$

for any $x,y\in A$.

Then $(A,+,\cdot,\neg,\sim,0,1)$ is an MV-algebra.

P r o o f. By [10], Theorem 1.2.3, if a DRl-monoid A is bounded above, then it is bounded below too, and moreover, 0 is the least element in A.

We will verify the axioms of an MV-algebra for A. Since $(A, +, 0)$ is a monoid, (1) and (4) are satisfied.

(2) By [10], Lemma 1.7, in any DRl-monoid we have

$$((x \rightharpoonup y) \vee 0) + y = x \vee y = y + ((x \leftharpoondown y) \vee 0).$$

Hence in our case we get for any $x \in A$

$$x + {\sim}x = x + (1 \leftharpoondown x) = 1 + (x \leftharpoondown 1) \geqslant 1 + 0 = 1.$$

Analogously,

$$\neg x + x = (1 \rightharpoonup x) + x = (x \rightharpoonup 1) + 1 \geqslant 0 + 1 = 1.$$

(3) $$x + 1 \geqslant 0 + 1 = 1, \quad 1 + x \geqslant 1 + 0 = 1.$$

(7) By condition (i) we get

$$\neg{\sim}x = 1 \rightharpoonup (1 \leftharpoondown x) = x, \quad {\sim}\neg x = 1 \leftharpoondown (1 \rightharpoonup x) = x.$$

(5) By (7),

$$\neg x \cdot \neg y = \neg({\sim}\neg x + {\sim}\neg y) = \neg(x + y).$$

(6) By condition (ii) and property (7) we have

$${\sim}x \cdot {\sim}y = {\sim}(\neg{\sim}x + \neg{\sim}y) = {\sim}(x + y).$$

(8) $\neg 0 = 1 \rightharpoonup 0$ is the smallest of the elements $r \in A$ such that $0 + r \geqslant 1$, hence $\neg 0 = 1$. Similarly ${\sim}0 = 1$.

(1′) $$\begin{aligned} x \cdot (y \cdot z) &= x \cdot (\neg({\sim}y + {\sim}z)) \\ &= \neg({\sim}x + {\sim}\neg({\sim}y + {\sim}z)) = \neg({\sim}x + ({\sim}y + {\sim}z)), \\ (x \cdot y) \cdot z &= \neg({\sim}\neg({\sim}x + {\sim}y) + {\sim}z) = \neg(({\sim}x + {\sim}y) + {\sim}z), \end{aligned}$$

and thus by (1), condition (1′) is satisfied.

(2′) By (7) and (2) we have

$$x \cdot {\sim}x = {\sim}(\neg x + \neg{\sim}x) = {\sim}(\neg x + x) = {\sim}1 = 0.$$

Similarly $\neg x \cdot x = 0$.

(3′) By (3),

$$x \cdot 0 = \neg({\sim}x + {\sim}0) = \neg({\sim}x + 1) = \neg 1 = 0,$$

and similarly also $0 \cdot x = 0$.

$$(4') \qquad x \cdot 1 = \neg({\sim}x + {\sim}1) = \neg({\sim}x + 0) = \neg{\sim}x = x.$$

Similarly $1 \cdot x = x$.

(5′) By condition (ii),

$$\neg(x \cdot y) = \neg{\sim}(\neg x + \neg y) = \neg x + \neg y.$$

$$(6') \qquad {\sim}(x \cdot y) = {\sim}\neg({\sim}x + {\sim}y) = {\sim}x + {\sim}y.$$

Put now $x \preceq_1 y \iff y + (x \cdot {\sim}y) = y$ for any $x, y \in A$. Let $x \preceq_1 y$. Then by the definition of the operation $\leftharpoondown$ we have $x \cdot {\sim}y \geqslant x \leftharpoondown y$, and hence $y + (x \leftharpoondown y) = y$, that means, by [10], Lemma 1.1.7, $x \vee y = y$, and therefore $x \leqslant y$.

Conversely, let $x \leqslant y$. Then the equality $(1 \rightharpoonup x) + x = 1$ implies $(1 \rightharpoonup x) + y = 1$. Hence

$$\begin{aligned} y + (x \cdot {\sim}y) &= y + {\sim}(\neg x + y) = y + (1 \leftharpoondown ((1 \rightharpoonup x) + y)) \\ &= y + (1 \leftharpoondown 1) = y + 0 = y, \end{aligned}$$

that means $x \preceq_1 y$.

Therefore the relation $\preceq_1$ coincides with the relation $\leqslant$ of the DRl-monoid A.

Similarly, put $x \preceq_2 y \iff (\neg y \cdot x) + y = y$ for any $x, y \in A$. Analogously we can prove (using the definition of the operation $\rightharpoonup$ and [10], Lemma 1.1.7) that the relation $\preceq_2$ also coincides with $\leqslant$.

Hence $x \vee y = y + (x \cdot {\sim}y) = (\neg y \cdot x) + y$ is the lattice join and $x \wedge y = y \cdot (x + {\sim}y) = (\neg y + x) \cdot y$ is the lattice meet also in the MV-algebra constructed. (So at the same time we have verified the validity of conditions (9) and (9′).) Therefore (10), (10′), (11) and (11′) are satisfied.

(12) It follows from the fact that $(A, +, 0, \leqslant)$ is an l-monoid.

(12′) First verify that ${\sim}(x \vee y) = {\sim}x \wedge {\sim}y$ and $\neg(x \wedge y) = \neg x \vee \neg y$.

$$\begin{aligned} &{\sim}(x \vee y) = {\sim}((\neg y \cdot x) + y) = {\sim}(\neg y \cdot x) \cdot {\sim}y \\ &= (y + {\sim}x) \cdot {\sim}y = (\neg{\sim}y + {\sim}x) \cdot {\sim}y = {\sim}x \wedge {\sim}y, \\ &\neg(x \wedge y) = \neg(y \cdot (x + {\sim}y)) = \neg y + \neg(x + {\sim}y) \\ &= \neg y + (\neg x \cdot y) = \neg y + (\neg x \cdot {\sim}\neg y) = \neg x \vee \neg y. \end{aligned}$$

Now we get

$$\begin{aligned}x\cdot(y\vee z)&=\neg(\sim x+\sim(y\vee z))=\neg(\sim x+(\sim y\wedge\sim z))\\&=\neg((\sim x+\sim y)\wedge(\sim x+\sim z))=\neg(\sim(x\cdot y)\wedge\sim(x\cdot z))\\&=\neg\sim(x\cdot y)\vee\neg\sim(x\cdot z)=(x\cdot y)\vee(x\cdot z).\end{aligned}$$

The equality $(y\vee z)\cdot x=(y\cdot x)\vee(z\cdot x)$ can be proved analogously, hence (12′b) is proved. Dually (12′a). □

In the next theorem we will show that also conversely, any MV-algebra induces some DRl-monoid.

Theorem 13. *Let $A=(A,\oplus,\odot,\neg,\sim,0,1)$ be an MV-algebra. For any $x,y\in A$ set*

$$x\rightharpoonup y=\neg y\odot x,\quad x\leftharpoondown y=x\odot\sim y.$$

If $x\vee y$ $(x\wedge y)$ denotes the supremum (the infimum) of elements $x,y\in A$ in the order induced on A, then $(A,\oplus,0,\vee,\wedge,\rightharpoonup,\leftharpoondown)$ is a bounded DRl-monoid with the greatest element 1 *satisfying conditions* (i) *and* (ii) *of Theorem* 12.

P r o o f. By the definition of an MV-algebra and by Theorem 2 we get that $y\oplus(x\odot\sim y)=x\vee y\geqslant x$. Let $v\in A$ be such that $y\oplus v\geqslant x$. Then by Theorem 5, $x\odot\sim y\leqslant v$, and hence $x\odot\sim y$ is the least of the elements $v\in A$ with the property $y\oplus v\geqslant x$, thus $x\odot\sim y=x\leftharpoondown y$.

Similarly $(\neg y\odot x)\oplus y=x\vee y\geqslant x$, and if $w\oplus y\geqslant x$ for $w\in A$, then by Theorem 5 we get $\neg y\odot x\leqslant w$, that means $\neg y\odot x=x\rightharpoonup y$.

We will show the validity of conditions (5) of the definition of a DRl-monoid.

(a) Let $x,y\in A$. Then (9) of the definition of an MV-algebra yields

$$\begin{aligned}(x\rightharpoonup y)\oplus y&=(\neg y\odot x)\oplus y=x\vee y,\\ y\oplus(x\leftharpoondown y)&=y\oplus(x\odot\sim y)=x\vee y.\end{aligned}$$

(b) For any $x\in A$ we get by (2) and (2′) of the definition of an MV-algebra:

$$\begin{aligned}x\rightharpoonup x&=\neg x\odot x=0,\\ x\leftharpoondown x&=x\odot\sim x=0.\end{aligned}$$

Hence $(A,\oplus,0,\vee,\wedge,\rightharpoonup,\leftharpoondown)$ is a DRl-monoid and 1 is its greatest element.

We will verify conditions (i) and (ii).

(i) $$\begin{aligned}1\leftharpoondown(1\rightharpoonup x)&=1\leftharpoondown(\neg x\odot 1)=1\leftharpoondown\neg x=1\odot\sim\neg x=x,\\ 1\rightharpoonup(1\leftharpoondown x)&=1\rightharpoonup(1\odot\sim x)=1\rightharpoonup\sim x=\neg\sim x\odot 1=x.\end{aligned}$$

(ii)
$$1 \rightharpoonup ((1 \leftharpoondown x) \oplus (1 \leftharpoondown y)) = 1 \rightharpoonup ((1 \odot {\sim} x) \oplus (1 \odot {\sim} y))$$
$$= 1 \rightharpoonup ({\sim} x \oplus {\sim} y) = \neg({\sim} x \oplus {\sim} y) \odot 1 = x \odot y,$$
$$1 \leftharpoondown ((1 \rightharpoonup x) \oplus (1 \rightharpoonup y)) = 1 \leftharpoondown ((\neg x \odot 1) \oplus (\neg y \odot 1))$$
$$= 1 \leftharpoondown (\neg x \oplus \neg y) = 1 \odot {\sim}(\neg x \oplus \neg y) = x \odot y.$$

□

Remark. Let $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$ be an MV-algebra and let $A_1 = (A, \oplus, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ be the DRl-monoid generated by A by the method of Theorem 13, that means, in A_1, $x \rightharpoonup y = \neg y \odot x$, $x \leftharpoondown y = x \odot {\sim} y$ for any $x, y \in A$. Then in the MV-algebra $A_2 = (A, \oplus, \cdot, \dashv, \approx, 0, 1)$ induced by A_1 by the method of Theorem 12, we have in A_2 for any $x \in A$,

$$\dashv x = 1 \rightharpoonup x = \neg x \odot 1 = \neg x,$$
$$\approx x = 1 \leftharpoondown x = 1 \cdot {\sim} x = {\sim} x,$$

and hence by Theorem 1(b) also

$$x \cdot y = x \odot y$$

for any $x, y \in A$. Therefore the MV-algebras A and A_2 coincide and thus every MV-algebra is induced by a DRl-monoid.

Let $\mathcal{C}_\infty$ and $\mathcal{C}_\in$ be classes of algebras of given types. Recall that two classes $\mathcal{C}_\infty$ and $\mathcal{C}_\in$ of algebras are *equivalent* if there exists a one-to-one correspondence $\mathcal{F}$ between $\mathcal{C}_\infty$ and $\mathcal{C}_\in$ such that for any $A \in \mathcal{C}_\infty$, A and $\mathcal{F}(\mathcal{A})$ have the same underlying set, and for each $A, B \in \mathcal{C}_\infty$ and each mapping f of A into B, f is a $\mathcal{C}_\infty$-homomorphism of the algebra A into the algebra B if and only if f is a $\mathcal{C}_\in$-homomorphism of $\mathcal{F}(\mathcal{A})$ into $\mathcal{F}(\mathcal{B})$. It is obvious that the algebraic categories corresponding to the equivalent classes $\mathcal{C}_\infty$ and $\mathcal{C}_\in$ are isomorphic.

Let us denote by $\mathcal{MV}$ the class of MV-algebras (which is a variety of algebras of signature $\langle 2, 2, 1, 1, 0, 0 \rangle$). For the class of bounded DRl-monoids we will consider the greatest element 1 as a nullary operation and so we will extend the type of such DRl-monoids to $(+, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown, 1)$ of signature $\langle 2, 0, 2, 2, 2, 2, 0 \rangle$. Now, let us denote by $\mathcal{DR}l_{1(i)(ii)}$ the class of DRl-monoids with 1 satisfying conditions (i) and (ii) viewed as a variety of the above type. Then we get the following theorem.

Theorem 14. *The classes $\mathcal{MV}$ and $\mathcal{DR}l_{1(i)(ii)}$ are equivalent.*

Proof. If $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$ is an MV-algebra, denote by $\mathcal{F}(A) = (A, \oplus, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown, 1)$ the induced DRl-monoid. By the preceding remark it is clear that $\mathcal{F}$ is a one-to-one correspondence between $\mathcal{MV}$ and $\mathcal{DR}l_{1(i)(ii)}$. Moreover, if $f\colon A \to B$ is an MV-homomorphism then (since the operations $\vee, \wedge, \rightharpoonup$ and $\leftharpoondown$ are defined by MV-terms) f is also an DRl_1-homomorphism, and vice versa. $\square$

Corollary 15. *The categories $\mathcal{MV}$ and $\mathcal{DR}l_{1(i)(ii)}$ are isomorphic.*

Let $\mathcal{CMV}$ denote the class of commutative MV-algebras and $\mathcal{CDR}l_{1(i)}$ the class of commutative DRl-monoids with the greatest element 1 satisfying condition (i) which is now in the form $1-(1-x)=x$. (Note that condition (ii) is trivially satisfied for commutative DRl-monoids.)

Thus as a consequence we have (see also [14]):

Theorem 16. *The classes $\mathcal{CMV}$ and $\mathcal{CDR}l_{1(i)}$ are equivalent (and hence the corresponding equational categories are isomorphic).*

Moreover, by [10], Theorem 1.1.23, the lattice $(A, \vee, \wedge)$ of any DRl-monoid A is distributive. Hence we have:

Theorem 17. *If $A = (A, \oplus, \odot, \neg, \sim, 0, 1)$ is any MV-algebra, then the lattice $(A, \vee, \wedge)$ is distributive.*

At the same time we obtain as an immediate consequence of Theorem 10:

Corollary 18. *For every MV-algebra A the set $B = \{x \in A;\ x \oplus x = x\}$ of its additive idempotents is a Boolean algebra in which the complement x' of arbitrary element $x \in B$ fulfils $x' = \neg x = \sim x$.*

4. Intervals of lattice ordered groups and loops

Let G be a commutative l-group, $0 \leqslant u \in G$ and $A = [0, u] = \{x \in G;\ 0 \leqslant x \leqslant u\}$. Set $x \oplus y = (x+y) \wedge u$, $\neg x = u - x$ and $x \odot y = \neg(\neg x \oplus \neg y)$ for any $x, y \in A$. Then $\Gamma(G, u) = (A, \oplus, \odot, \neg, 0, u)$ is a commutative MV-algebra. C. C. Chang in [5] proved that any linearly ordered commutative MV-algebra is isomorphic to $\Gamma(G, u)$ for a commutative linearly ordered group G and a strong order unit u in G. D. Mundici in [11] generalized this result to arbitrary commutative MV-algebras. Namely, he showed that if A is any commutative MV-algebra then there are a commutative l-group G and a strong unit u in G such that A is isomorphic to $\Gamma(G, u)$.

In this section we will examine analogous intervals of non-commutative l-groups.

Theorem 19. *Let $G = (G, +, 0, -(\cdot), \vee, \wedge)$ be a (non-commutative) lattice ordered group, $0 \leqslant u \in G$ and $A = [0, u]$. For any $x, y \in A$ set*

$$x \oplus y = (x + y) \wedge u, \quad x \rightharpoonup y = (x - y) \vee 0, \quad x \leftharpoondown y = (-y + x) \vee 0.$$

Then $(A, \oplus, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a bounded DRl-monoid (with the greatest element u) in which any $x, y \in A$ satisfy

(i) $$u \leftharpoondown (u \rightharpoonup x) = x = u \rightharpoonup (u \leftharpoondown x);$$

(ii) $$u \rightharpoonup ((u \leftharpoondown x) \oplus (u \leftharpoondown y)) = u \leftharpoondown ((u \rightharpoonup x) \oplus (u \rightharpoonup y)).$$

Proof. We will verify conditions (1)–(5) of a DRl-monoid.

Condition (1):

$$x \oplus (y \oplus z) = (x + y + z) \wedge u = (x \oplus y) \oplus z,$$

hence $(A, \oplus, 0)$ is a monoid.

Condition (2), (3): $(A, \vee, \wedge)$ is a sublattice of the lattice $(G, \vee, \wedge)$ and e.g.

$$\begin{aligned} x \oplus (y \wedge z) = (x + y) \wedge (x + z) \wedge u &= ((x + y) \wedge u) \wedge ((x + z) \wedge u) \\ &= (x \oplus y) \wedge (x \oplus z), \end{aligned}$$

hence $(A, \oplus, 0, \vee, \wedge)$ is an l-monoid.

Condition (4):

$$\begin{aligned} y \oplus (x \leftharpoondown y) = y \oplus ((-y + x) \vee 0) &= (y \oplus (-y + x)) \vee y \\ &= (x \wedge u) \vee y = x \vee y, \end{aligned}$$

thus $y \oplus (x \leftharpoondown y) \geqslant x$.

Let $r \in A$, $y \oplus r \geqslant x$. Then $(y + r) \wedge u \geqslant x$, hence $y + r \geqslant x$, i.e. $r \geqslant -y + x$. Moreover, $r \geqslant 0$, therefore $r \geqslant (-y + x) \vee 0 = x \leftharpoondown y$. Thus $x \leftharpoondown y$ satisfies condition (4). Similarly for $x \rightharpoonup y$. (Moreover, $(x \rightharpoonup y) \oplus y = x \vee y$, as well.)

Condition (5a): Since 0 is the least element in A, we have

$$\begin{aligned} ((x \rightharpoonup y) \vee 0) \oplus y &= (x \rightharpoonup y) \oplus y = x \vee y, \\ y \oplus ((x \leftharpoondown y) \vee 0) &= y \oplus (x \leftharpoondown y) = x \vee y. \end{aligned}$$

Condition (5b): Evidently $x \rightharpoonup x = 0 = x \leftharpoondown x$.

It remains to show the validity of conditions (i) and (ii).

Condition (i):

$$u \leftharpoondown (u \rightharpoonup x) = -(u - x) + u = x, u \rightharpoonup (u \leftharpoondown x) = u - (-x + u) = x.$$

Condition (ii):

$$\begin{aligned} u \rightharpoonup ((u \leftharpoondown x) \oplus (u \leftharpoondown y)) &= u \rightharpoonup ((-x + u) \oplus (-y + u)) \\ &= u - ((-x + u - y + u) \wedge u) = (y - u + x) \vee 0, \\ u \leftharpoondown ((u \rightharpoonup x) \oplus (u \rightharpoonup y)) &= u \leftharpoondown ((u - x) \oplus (u - y)) \\ &= -((u - x + u - y) \wedge u) + u = (y - u + x) \vee 0. \end{aligned}$$

□

As an immediate consequence of Theorems 12 and 19 we obtain

Theorem 20. *Let $G = (G, +, 0, -(\cdot), \vee, \wedge)$ be a (non-commutative) l-group, $0 \leqslant u \in G$ and $A = [0, u]$. If we set*

$$x \oplus y = (x + y) \wedge u, \quad x \odot y = (y - u + x) \vee 0,$$
$$\neg x = u - x, \quad \sim x = -x + u$$

for any $x, y \in A$ then $\Gamma(G, u) = (A, \oplus, \odot, \neg, \sim, 0, u)$ is an MV-algebra.

Remark. a) By Remark after the proof of Theorem 12, for arbitrary $0 \leqslant u \in G$, the set B_u of additive idempotents of $\Gamma(G, u)$ is a Boolean algebra. Evidently $x \in B_u$ if and only if $x \wedge (u - x) = x \wedge (-x + u) = 0$.

b) If for $0 < u \in G$ the interval $[0, u]$ is a chain (i.e. u is a basic element in the l-group G), then $B_u = \{0, u\}$. Hence, if G is a linearly ordered group, then for each $0 < u \in G$, B_u is a two-element Boolean algebra.

c) Let an l-group G be the direct sum of linearly ordered groups $G_1, \ldots, G_n$ and let $0 < u = (u_1, \ldots, u_n) \in G$. If $u_i \neq 0$ for each $i = 1, \ldots, n$, then B_u is a finite Boolean algebra having 2^n elements. Hence for any finite Boolean algebra B there are an l-group G and $0 \leqslant u \in G$ such that $B \cong B_u$.

d) Let G be an l-group and $0 < u \in G$. Let us suppose that u is a singular element in G, i.e. for any $s, t \in [0, u]$, $s + t = u$ implies $s \wedge t = 0$. Since for every $x \in [0, u]$, $x + (-x + u) = u = (u - x) + x$, we have $x \wedge \sim x = 0 = \neg x \wedge x$, and so $x \in B_u$. Hence in this case $B_u = A = [0, u]$, therefore A is a Boolean algebra and $x' = \neg x = \sim x$ for any $x \in A$.

Recall that any commutative MV-algebra is isomorphic to $\Gamma(G, u)$ for an appropriate commutative lattice ordered group G and $0 \leqslant u \in G$ (where moreover u can be a strong unit of G). This fact has been proved for linearly ordered MV-algebras and linearly ordered groups by C. C. Chang in [5], and it has been generalized to arbitrary MV-algebras and commutative l-groups by D. Mundici in [11] using the possibility of expressing any commutative MV-algebra as a subdirect sum of linearly ordered MV-algebras.

We will show that for non-commutative MV-algebras the similar construction does not lead in general to a group, but, nevertheless, we will show that any linearly ordered MV-algebra can be represented as an interval of a linearly ordered loop.

Definition. a) Let $(G,+,0,/,\backslash)$ be a loop and let $\leqslant$ be an order on G. Then $G=(G,+,0,/,\backslash,\leqslant)$ is called an *ordered loop* if

$$\forall x,y,z,v \in G;\ x \leqslant y \Rightarrow v+x+z \leqslant v+y+z.$$

b) If G is an ordered loop and $(G,\leqslant)$ is a lattice with the lattice operations $\vee$ and $\wedge$, then $G=(G,+,0,/,\backslash,\vee,\wedge)$ is called a *lattice ordered loop* if for any $x,y,z,v \in G$,

$$v+(x\vee y)+z=(v+x+z)\vee(v+y+z),$$
$$v+(x\wedge y)+z=(v+x+z)\wedge(v+y+z).$$

(Recall that for any $x,y\in G$, y/x $(y\setminus x)$ denotes the unique solution v (w) of the equation $x+v=y$ ($w+x=y$, respectively).)

Lemma 21. *If G is an ordered loop and $x\in G$, then*

$$0\leqslant x \Leftrightarrow 0/x\leqslant 0 \Leftrightarrow 0\setminus x\leqslant 0.$$

Proof. If $0\leqslant x$ then $0/x\leqslant x+(0/x)=0$, and if $0/x\leqslant 0$ then $0=x+(0/x)\leqslant x$. Similarly $0\leqslant x$ if and only if $0\setminus x\leqslant 0$. □

Lemma 22. *Let G be a lattice ordered loop, $0\leqslant u\in G$, and $A=[0,u]$. For any $x,y\in A$ set*

$$x\oplus y=(x+y)\wedge u,\ \neg x=(0\setminus x)+u,\ \sim\! x=u+(0/x),$$
$$x\odot y=\sim\!(\neg x\oplus\neg y).$$

Then $\oplus$ and $\odot$ are binary and $\neg$ and $\sim$ are unary operations on A. (Denote $\Gamma(G,u)=(A,\oplus,\odot,\neg,\sim,0,u)$.)

P r o o f. Let $x, y \in A$. Obviously $x \oplus y \in A$. By Lemma 21, $0 \setminus x \leqslant 0$, hence $(0 \setminus x) + u \leqslant u$. Further, $0 = (0 \setminus x) + x \leqslant (0 \setminus x) + u$, thus $(0 \setminus x) + u \in A$. Analogously $u + (0/x) \in A$.

Finally, $x \odot y = \sim(\neg x \oplus \neg y) = \big(0 \setminus ((u + (0 \setminus x)) \oplus (u + (0 \setminus y)))\big) + u$, and because $u + (0 \setminus x)$, $u + (0 \setminus y) \in A$, we have $x \odot y \in A$. $\square$

Theorem 23. *Let A be a bounded linearly ordered DRl-monoid satisfying* (i) *and* (ii). *Then there exist a linearly ordered loop G and $0 \leqslant u \in G$ such that $\Gamma(G, u)$ is an MV-algebra and A is isomorphic to $\Gamma(G, u)$.*

P r o o f. Let A be a linearly ordered non-commutative DRl-monoid with a greatest element 1 satisfying conditions (i) and (ii) and let $\mathbb{Z}$ be the additive group of integers linearly ordered by the natural order. Denote by $B = \mathbb{Z} \vec{\times} A$ the cartesian product of $\mathbb{Z}$ and A ordered by the lexicographic order. (That means, $(m, x) < (n, y)$ if and only if $m < n$ or $m \leqslant n$ and $x < y$.) We will define a binary operation $\oplus$ on B as follows: If $(m, x), (n, y) \in B$ then

$$(m, x) \oplus (n, y) = (m + n, x + y), \quad \text{if } x + y < 1,$$
$$(m, x) \oplus (n, y) = \big(m + n + 1, \quad 1 \rightharpoonup ((1 \leftharpoondown x) + (1 \leftharpoondown y))\big), \quad \text{if } x + y = 1.$$

Moreover, for any $m \in \mathbb{Z}$ put

$$(m, 1) = (m + 1, 0).$$

a) Let us show that for any $(m, x), (n, y) \in B$ there exists a unique $(s, w) \in B$ such that $(m, x) \oplus (s, w) = (n, y)$.

aα) Suppose that $x \leqslant y$.

aαI) Let $y \neq 1$. Then

$$(m, x) \oplus (n - m, \, y \leftharpoondown x) = (n, y).$$

Show that $(s, w) = (n - m, y \leftharpoondown x)$ is a unique solution of the above equation. Let $(p, z) \in B$ be such that also $(m, x) \oplus (p, z) = (n, y)$.

aαI1) Let $x + z < 1$. Then $m + p = n$, $x + z = y$, hence $p = n - m$, $z \geqslant y \leftharpoondown x$, and $x + (y \leftharpoondown x) = x + z$. We have $y \leftharpoondown x \leqslant 1 \leftharpoondown x$, and moreover, $x + z = y < 1$ and $x + (1 \leftharpoondown x) = 1$ imply $z < 1 \leftharpoondown x$. Therefore in the induced MV-algebra we get $x + (y \leftharpoondown x) = x + z$ and $z, y \leftharpoondown x \leqslant \sim x$, hence, by Theorem 6, $z = y \leftharpoondown x$.

aαI2) Let $x + z = 1$. Then $m + p + 1 = n$, thus $p = n - m - 1$ and $1 \rightharpoonup ((1 \leftharpoondown x) + (1 \leftharpoondown z)) = y$. At the same time, $(1 \leftharpoondown x) + (1 \leftharpoondown z) \geqslant 1 \leftharpoondown x$, hence $1 \rightharpoonup ((1 \leftharpoondown x) + (1 \leftharpoondown z)) \leqslant 1 \rightharpoonup (1 \leftharpoondown x) = x$, and so $y \leqslant x$. That means $x = y$.

Hence by [10], Lemma 1.1.12, $(1 \rightharpoonup (1 \leftharpoondown z)) \rightharpoonup (1 \leftharpoondown x) = x$, and therefore by (i), $z \rightharpoonup (1 \leftharpoondown x) = x$.

Moreover, from $x + 1 = 1$ we get $z \geqslant 1 \leftharpoondown x$, hence by [10], Lemma 1.1.12, $(z \rightharpoonup (1 \leftharpoondown x)) + (1 \leftharpoondown x) = z$. Thus $x + (1 \leftharpoondown x) = z$, that means $z = 1$.

Therefore $(p, z) = (n - m - 1, 1) = (n - m, 0)$. (At the same time, in this case, $(n - m, y \leftharpoondown x) = (n - m, x \leftharpoondown x) = (n - m, 0)$.)

aαII) Let $y = 1$. Then

$$(m, x) \oplus (n - m, 1 \leftharpoondown x) = (n, 1).$$

Let $(m, x) \oplus (p, z) = (n, 1)$. If $x + z < 1$, then by the definition of $\oplus$, $x + z = 1$, a contradiction. Hence $x + z = 1$, i.e. $z \geqslant 1 \leftharpoondown x$. It is obvious that $1 \rightharpoonup ((1 \leftharpoondown x) + (1 \leftharpoondown z))$ is equal either to 1 or to 0.

If $1 \rightharpoonup ((1 \leftharpoondown x) + (1 \leftharpoondown z)) = 1$, then by Theorem 4 we have $x = z = 1$. Then $(m, 1) \oplus (p, 1) = (n, 1)$, thus $m + p + 1 = n$, i.e. $p = n - m - 1$. Hence $(p, 1) = (n - m - 1, 1) = (n - m, 0) = (n - m, 1 \leftharpoondown 1)$.

Let $1 \rightharpoonup ((1 \leftharpoondown x) + (1 \leftharpoondown z)) = 0$. Then $x \odot z = 0$, hence $\neg{\sim}x \odot z = 0$, and therefore by Theorem 5, $z \leqslant {\sim}x = 1 \leftharpoondown x$, i.e. $z = 1 \leftharpoondown x$.

aβ) Suppose $x > y$.

Let $(m, x) \oplus (p, z) = (n, y)$. We have $x + z > y$, thus $x + z = y$ cannot hold. Hence $x + 1 = 1$, so $z \geqslant 1 \leftharpoondown x$ and $1 \rightharpoonup ((1 \leftharpoondown x) + (1 \leftharpoondown z)) = y$. Therefore $x \odot z = x \odot ((1 \leftharpoondown x) + y)$, and since $z \geqslant 1 \leftharpoondown x$ and $(1 \leftharpoondown x) + y \geqslant 1 \leftharpoondown x$, we have, by Theorem 6, $z = (1 \leftharpoondown x) + y$.

b) Analogously, for any $(m, x), (n, y) \in B$ there is in B a unique solution (r, v) of the equation $(r, v) \oplus (m, x) = (n, y)$.

Therefore B is a quasigroup. Obviously, $(0, 0)$ is a zero element in B, hence B is a loop.

Now it is evident that the MV-algebra corresponding to A is isomorphic to $\Gamma(B, (0, 1))$. □

Remark. If A is a commutative MV-algebra, then the loop B from the proof of Theorem 23 is a (linearly ordered) group. But in general, for a non-commutative case, B is not a group. Namely, for any $(m, x) \in B$ we have

$$(m, x) \oplus (-m - 1, {\sim}x) = (m - m - 1 + 1, x \odot {\sim}x) = (0, 0),$$
$$(-m - 1, \neg x) \oplus (m, x) = (-m - 1 + m + 1, \neg x \odot x) = (0, 0),$$

that means $(-m - 1, {\sim}x)$ is a right and $(-m - 1, \neg x)$ is a left opposite element of (m, x) in the loop B. But in general, ${\sim}x$ is not equal to $\neg x$, therefore $(B, \oplus)$ need not be a semigroup, and so B need not be a group.

Problem. The question whether any linearly ordered (non-commutative) MV-algebra is isomorphic to $\Gamma(G, u)$ for a linearly ordered group G and an element $0 \leqslant u \in G$ remains open.

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