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A NON-COMMUTATIVE THEORY OF INTEGRATION FOR A SEMI-FINITE AW^*-ALGEBRA AND A PROBLEM OF FELDMAN

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1. Introduction. In [5], I.Kaplansky introduced a class of C^*-algebras called AW^*-algebras. For these, while being algebraically defined, much of the Murray-von Neumann structure theory for von Neumann algebras, in particular, the lattice structure theory of the set of projections can be developed. Dixmier showed that this class of AW^*-algebras is exactly broader than that of von Neumann algebras [1]. Therefore, it is an interesting problem for us to investigate the difference between AW^*-algebras and von Neumann algebras. From this point of view, we shall extend Feldman's result on "Embedding of AW^*-algebras" to semi-finite AW^*-algebras, that is, we shall show that a semi-finite AW^*-algebra with a separating set of states which are completely additive on projections (c. a. states) has a faithful representation as a semi-finite von Neumann algebra on some Hilbert space (Theorem 5. 2). He showed that a finite AW^*-algebra which possesses a separating set of c. a. states admits a faithful representation as a von Neumann algebra [3].

In the previous paper [7], we constructed the algebra C of "measurable operators" for a semi-finite AW^*-algebra M in algebraic fashion and studied the structure of C. Throughout this paper, we always assume M to be a semi-finite AW^*-algebra with a separating set $\mathfrak{S}$ of c. a. states and C to be the algebra of "measurable operators" for it.

The contents of this paper are as follows. Section 2 is preliminary. We review briefly the definitions and elementary properties of M which will be used later. In section 3, along the same lines with [10], we shall prove the existence theorem of a dimension function (Theorem 3. 2) for M and introduce the notion of convergence nearly everywhere of sequences in C. Section 4 concerns with the existence of a faithful semi-finite numerical trace τ on M and the non-commutative integration theory with respect to τ. We shall show that the set $\mathfrak{H}_\tau$ of square τ-integrable elements in C is a Hilbert space under a suitable norm (Theorem 4. 7). Section 5 is the main part of this paper and is devoted to prove the theorem: M can be represented faithfully as a semi-finite von Neumann algebra (Theorem 5. 2). As a corollary, we give the alternative proof of Theorem 2 in [6], more precisely, an AW^*-algebra of type I whose center is a W^*-algebra admits a faithful representation as a von Neumann algebra of type I.

2. Definitions and preliminary results. By an AW^*-algebra we mean a C^*-algebra M with the following two conditions:

(a) In the set of projections, any collection of orthogonal projections has a least upper bound.

(b) Any maximal commutative self-adjoint subalgebra is generated by its projections.

Let M_{sa}, M^+, M_p, M_{pi} and M_u be the set of all self-adjoint elements, positive elements, projections, partial isometries and unitary elements in M, respectively. Denote the two-sided ideal generated algebraically by all finite projections in M by $\mathfrak{M}$, then $\mathfrak{M}$ contains only finite projections. If every non-zero projection in M contains a non-zero finite projection in M, then M is called semi-finite. For informations about AW^*-algebras, especially the lattice structure theory of projections, and the algebras of "measurable operators" for them, see [3], [6] and [7].

Let N be a W^*-algebra, namely a C^*-algebra with a dual structure as a Banach space, N_* be the predual of N, that is, the Banach space of all bounded normal functionals on N. Then N can be represented faithfully as a von Neumann algebra ([2]) on some Hilbert space [9] and in particular, N becomes an AW^*-algebra. For other informations about W^*-algebras, see [9].

Now we prove the fundamental results which will be used later.

LEMMA 2.1. *For e in M_p, let $z(e)$, $c(e)$ and $s(e)$ be the least central projection $\geqq e$,* $\mathrm{Sup}\{ueu^*, u \in M_u\}$ *and* $\mathrm{Sup}\{f; f\sim e, f \in M_p\}$, *respectively* [5, Corollary 3], *then $z(e)=c(e)=s(e)$. We say that $z(e)$ is the central carrier of e.*

PROOF. First, we shall prove $c(e) \in Z$ (the center of M). For $v \in M_u$, we have $vc(e)v^* = v(\mathrm{Sup}\{ueu^*, u \in M_u\})v^* \geqq \mathrm{Sup}\{vue(vu)^*, u \in M_u\} = c(e)$. Therefore, $vc(e)v^* \geqq c(e)$ and $v^*c(e)v \geqq c(e)$, that is, $vc(e)v^* = c(e)$. Since every element of M can be written as a finite linear combination of elements in M_u, it follows that $c(e) \in Z$. Thus $c(e) \geqq z(e) \geqq e$. On the other hand, $z(e) = uz(e)u^* \geqq ueu^*$ for all $u \in M_u$ and $z(e) \geqq c(e)$, which implies $z(e) = c(e)$. By the definition of $s(e)$ and $c(e)$, $s(e) \geqq c(e) = z(e)$, while $f \sim e$ implies $f \leqq z(e)$. Consequently $s(e) = c(e) = z(e)$. The lemma follows.

REMARK. By [5, Corollary 1], the right annihilator of eM is $M(1-z(e))$.

LEMMA 2. 2. *For* $e \in M_p$ *and* $z \in Z_p, z(ez) = zz(e)$.

PROOF. From the above remark, the right annihilator of $zeM = M(1-z(ze))$. It is easily seen that $M(1-zz(e)) \subset$ right annihilator of zeM. Hence it suffices to show the converse assertion. If $x \in$ right annihilator of zeM, then $zx \in$ right annihilator of $eM(=M(1-z(e)))$. $zxz(e)=0$, that is, $x \in M(1-zz(e))$. Therefore we have $z(ze) = zz(e)$.

LEMMA 2. 3. *For any* e *in* M_p, *the center of the* AW^**-algebra* eMe ([5, Theorem 2. 4]) *is* Ze. *Moreover*, $Ze \cong Zz(e)$.

PROOF. Let $(eMe)^\natural$ be the center of eMe, then it is clear that $(eMe)^\natural \supseteqq Ze$. Hence from this fact and the spectral decomposition theorem, it is sufficient to show that $(eMe)^\natural_p \subseteqq (Ze)_p$. If $g \in (eMe)^\natural_p$, then $g \leqq e$ and $g \leqq ez(g)$. Noting that the right annihilator of $gM = M(1-z(g))$, if we put $k = ez(g)-g$, $gMk = gMek$ $= geMek = eMegk = 0$. Therefore $k \in M(1-z(g))$, which implies $k=0$, that is, $g = ez(g) \in (Ze)_p$. For $x \in Zz(e)$, put $\psi(x) = xe$, then ψ is a $*$-homomorphism of $Zz(e)$ onto Ze. We shall show that ψ is one to one. In fact, if $xe=0$ for some $x \in Zz(e)$, then $ebx = 0$ for all $b \in M$, that is, $x \in$ right annihilator of $eM(=M(1-z(e)))$. Therefore we have $z(e)x = 0$. This completes the proof of Lemma 2. 3.

LEMMA 2. 4. *Let* N *be an arbitrary* AW^**-algebra and let* p *and* q *be projections in* N *such that* $p \sim q$. *Then, there exist orthogonal families of projections* $\{p_i\}_{i=1}^n$ *and* $\{q_i\}_{i=1}^n$ *in* N *such that* $p = \sum_{i=1}^n p_i$, $q = \sum_{i=1}^n q_i$ *and for each* i, p_i *is unitarily equivalent to* q_i *in* N.

PROOF. The proof is the same as that of [10, Remark 1. 1]. But for the sake of completeness, we sketch them. Since, by [5, Theorem 4. 2], there exists a central projection e in N such that Ne is finite, $N(1-e)=0$ or properly infinite and $N = Ne \oplus N(1-e)$, we have only to consider two cases: (a) p and q are finite, and (b) p and q are properly infinite. For the first case, the assertion is clear from [5, Theorem 5. 7]. Therefore it suffices to show this for case (b). Suppose that p and q are properly infinite, then, there is a central projection z in N such that $(1-q)z \precsim (1-p)z$ and $(1-q)(1-z) \succsim (1-p)(1-z)$[5, Theorem 5. 6]. Hence we may suppose $1-p \succsim 1-q$ without loss of generality. By the similar reason, we may assume that: either (a) $1-q$ is finite or (b) $1-q$ is properly infinite; either (α) $1-p \precsim p$ or (β) $p \precsim 1-p$. Since p and q are properly infinite, by [5, Lemma 4. 5], there exist projections p_1 and q_1 in N such that $p_1 \leqq p$, $p-p_1 \sim p_1 \sim p$, $q_1 \leqq q$ and $q-q_1 \sim q_1 \sim q$. If (α) holds, then $1-p_1 = p-p_1$

$+1-p\precsim p-p_1+p_1=p\sim p_1$ and it follows $1-p_1\precsim p_1$. On the other hand since $1-p_1\geqq p-p_1\sim p_1$, we have $1-p_1\sim p_1$. Next we show that $1-q_1\sim q_1$. In fact, $1-q_1=(q-q_1)+1-q\precsim q-q_1+q_1=q\sim q_1$ and $1-q_1\geqq q-q_1\sim q_1$, which implies that $1-q_1\sim q_1$. Therefore $1-p_1\sim 1-q_1$, $p_1\sim q_1$ and p_1 and q_1 are unitarily equivalent in N. By symmetry, $p-p_1$ and $q-q_1$ are unitarily equivalent. In case $(a\beta)$, $1=1-q+q\precsim q-q_1+q_1=q$ and $1\sim p$, thus $1-p\precsim p$ and we can arrive at case (a). If $(b\beta)$ holds, we may suppose without loss of generality that either $q\precsim 1-q$ or $1-q\precsim q$. If $1-q\precsim q$, then $1-q\precsim q\sim q-q_1$ and $1=1-q+q\precsim q-q_1+q_1=q$. Case (α) reappears. If $q\precsim 1-q$, then noting that there exists a projection q_1' in N such that $q_1'\leqq 1-q$ and $1-q\sim q_1'\sim 1-q-q_1'$, we have $1=1-q+q\precsim 1-q-q_1'+q_1'=1-q, 1-q\sim 1$ and $1-p\sim 1$, which implies that p and q are unitarily equivalent in N. This completes the proof.

Let $\mathfrak{S}$ be a fixed separating set of states on M which are completely additive on projections, and $\tilde{\mathfrak{S}}$ be the set of finite linear combinations of elements in $\{a^*\omega a, \omega\in\tilde{\mathfrak{S}}, a\in M\}$, where $(a^*\omega a)(x)=\omega(axa^*)$ for all $x\in M$. For any positive number ε and any positive integer n, put $V_{\varepsilon,n}(\omega_1,\omega_2,\cdots,\omega_n)(0)=\{a;|\omega_i(a)|<\varepsilon, i=1,2,\cdots,n, \omega_1,\omega_2,\cdots,\omega_n\in\tilde{\mathfrak{S}}\}$, and we define the $\sigma(\mathfrak{S})$-topology of M by assigning sets of the form $V_{\varepsilon,n}(\omega_1,\omega_2,\cdots,\omega_n)(0)$ to be its neighborhood system of 0. Since $\tilde{\mathfrak{S}}$ is a separating set of continuous linear functionals on M, this topology is the separated locally convex topology defined by the family of semi-norms $q_\omega(x)=|\omega(x)|\,\omega\in\mathfrak{S}$. Then we have the following

LEMMA 2.5. *Let* $\{e_\alpha\}_{\alpha\in A}$ *be an orthogonal set of projections in* M *such that* $e=\mathrm{Sup}[\sum\{e_\alpha,\alpha\in I\}; A\supset I\in\mathcal{F}]$ *where* $\mathcal{F}$ *is the family of all finite subsets of* A, *then* $\sum\{e_\alpha,\alpha\in I\}\uparrow e\,(I\in\mathcal{F})$ *in the* $\sigma(\mathfrak{S})$*-topology.*

PROOF. By [3, Lemma 3], $\omega(\sum\{e_\alpha,\alpha\in I\})\rightarrow\omega(e)$ for all $\omega\in\tilde{\mathfrak{S}}$. Hence the assertion is clear from the definition of the $\sigma(\mathfrak{S})$-topology.

LEMMA 2.6. *Any abelian AW*-subalgebra* N, *especially, the center* Z, *of* M *is a W*-algebra and the* $\sigma(\mathfrak{S})$*-topology restricted to this subalgebra is equivalent to the* σ*-topology on bounded spheres.*

PROOF. For any increasing net $\{a_\alpha\}_{\alpha\in A}$ of positive elements in N with supremum a and $\omega\in\mathfrak{S}$, by [4, Lemma 2.2], we have $\omega(\mathrm{Sup}\{a_\alpha,\alpha\in A\})=\mathrm{Sup}\{\omega(a_\alpha),\alpha\in A\}$. Therefore N has a separating set of normal positive measures and the first half part of the result follows from [1, Theorem 1]. Since the $\sigma(\mathfrak{S})$-topology is weaker than the σ-topology and the unit sphere of N is σ-compact, while the $\sigma(\mathfrak{S})$-topology is separated, it is equivalent to the σ-topology on bounded spheres of N.

THEOREM 2.1 ([4], [11]). *Let N be a finite AW^*-algebra with a separating set $\mathfrak{S}'$ of c.a. states. Then N has a center-valued trace $\Phi(\cdot)$ in the sense of* [4, Definition 1] *with the following continuity property: If $\{a_\alpha\}$ is an increasing net of positive elements in N such that $a_\alpha \uparrow a$ in the $\sigma(\mathfrak{S}')$-topology for some a in N, then $\operatorname{Sup}_\alpha \Phi(a_\alpha) = \Phi(a)$ in Z (the center of N).*

We sketch the proof after the fashion of [9, Chap. II, 4]. By [5, Theorem 6] and [6, Lemma 18], N is a direct sum of an AW^*-algebra of type II_1 and homogeneous AW^*-algebras. Therefore, it is no loss of generality to suppose that either N is homogeneous or of type II_1. In the former case, there is a finite family $\{e_i\}_{i=1}^n$ of abelian projections in N such that $1=\sum_{i=1}^n e_i$, $e_i \sim e_1$ for all i and $e_ie_j=0$ if $i \neq j$. Let v_{1i} be a partial isometry in N such that $v_{1i}^*v_{1i}=e_1$ and $v_{1i}v_{1i}^*=e_i$ for each i. Put $v_{11}=e_1$. Since $z(e_1)=1$, by Lemma 2.3, e_1Ne_1 is $*$-isomorphic to Z. Let ψ be the $*$-isomorphism of Z onto e_1Ne_1 defined in the proof of Lemma 2.3 and ϕ be its inverse map. Then if we put $\Phi(x) = (1/n)\sum_{i=1}^n \phi(e_1v_{1i}^*xv_{1i}e_1)$, it is easy to show that Φ is the center-valued trace in the sense of Definition 1 in [4]. For any directed increasing net $\{x_\alpha\}$ of positive elements in N such that $x_\alpha \uparrow x$ in the $\sigma(\mathfrak{S}')$-topology, by the definition of the $\sigma(\mathfrak{S}')$-topology, $e_1v_{1i}^*x_\alpha v_{1i}e_1 \to e_1v_{1i}^*xv_{1i}e_1$ for the $\sigma(\mathfrak{S}')$-topology. By Lemma 2.6 and the σ-continuity of ϕ, it follows that $\Phi(x_\alpha) \to \Phi(x)$ for the σ-topology on Z. Now we assume that N is an AW^*-algebra of type II_1. Before going into the proof, we need following definitions and lemmas.

DEFINITION 2.1. We say that a projection e in N is fundamental if there exist a central projection z and a set of orthogonal equivalent projections $\{e_1, e_2, \cdots, e_{2^n}\}$ in N such that $e=e_1$ and $\sum_{i=1}^{2^n} e_i = z$.

First, note that z and n depend only on e. In fact, let z_1 and $\{e_1', e_2', \cdots, e_{2^m}'\}$ be another such family for e. Suppose $m<n$, then $z_1=\sum_{i=1}^{2^m} e_i' \sim \sum_{i=1}^{2^m} e_i \lneqq \sum_{i=1}^{2^n} e_i = z$ and $z_1 = z(e) = z$ by Lemma 2.1, which contradicts the finiteness of N. Hence, we can define unambiguously a center-valued operation Φ on all fundamental projections as $\Phi(e) = 2^{-n}z$. Then we have

LEMMA 2.7. *For any pair of fundamental projections p_1 and p_2, $\Phi(p_1) = \Phi(p_2)$ if and only if $p_1 \sim p_2$.*

PROOF. Since "if" part is clear, it is sufficient to prove that $\Phi(p_1)=\Phi(p_2)$

implies $p_1 \sim p_2$. By [5, Theorem 5.6], there exists a central projection z such that $p_1 z \precsim p_2 z$ and $p_1(1-z) \succsim p_2(1-z)$, hence we may assume $p_1 \precnsim p_2$ without loss of generality. Thus there is a set of orthogonal projections $\{f_1', f_2', \cdots, f_{2^n}'\}$ such that $e_i \sim f_i' \lneqq f_i$, where $e_i \sim p_1, f_i \sim p_2, e_i e_j = 0 (i \neq j), f_i f_j = 0 (i \neq j), \sum_{i=1}^{2^n} e_i = z(p_1)$ and $\sum_{i=1}^{2^n} f_i = z(p_2)$. Thus it follows that $2^n \cdot \Phi(p_1) = \sum_{i=1}^{2^n} e_i \sim \sum_{i=1}^{2^n} f_i' \lneqq \sum_{i=1}^{2^n} f_i = 2^n \cdot \Phi(p_2)$ and this is a contradiction. The lemma follows.

LEMMA 2.8. *Let e_1 and e_2 be two fundamental projections such that $\Phi(e_1) = 2^{-n} z$ and $\Phi(e_2) = 2^{-m} z$. If $m \geqq n$, then there exists a set of orthogonal equivalent projections $\{p_i\,;\ i = 1, 2, \cdots, 2^{m-n}\}$ such that $e_1 = \sum_{i=1}^{2^{m-n}} p_i$, and $p_i \sim e_2$ for each i. Moreover, let $e_1, e_2, \cdots, e_s, f_1, f_2, \cdots, f_t$ and $\bar{f}$ be fundamental projections such that $\{e_i\}_{i=1}^{s}$ and $\{f_j\}_{j=1}^{t}$ are orthogonal. Putting $e = \sum_{i=1}^{s} e_i$ and $f = \sum_{j=1}^{t} f_j$, if $f \leqq e$ and $\Phi(\bar{f}) + \sum_{j=1}^{t} \Phi(f_j) \leqq \sum_{i=1}^{s} \Phi(e_i)$, then there exists a projection f_{t+1} such that $f_{t+1} \sim \bar{f}$ and $f_{t+1} \leqq e - f$.*

PROOF. Let $\{q_1, q_2, \cdots, q_{2^m}\}$ be a set of orthogonal equivalent projections such that $e_2 = q_1$ and $z = \sum_{i=1}^{2^m} q_i$, then putting $e_3 = \sum_{i=1}^{2^{m-n}} q_i$, e_3 is a fundamental projection such that $z(e_3) = z = z(e_2)$ and it follows that $\Phi(e_3) = 2^{-n} z(e_2) = \Phi(e_1)$, which implies $e_1 \sim e_3$. Hence the first half part of the lemma follows. Now we shall prove the second assertion of the lemma. If we decompose N suitably, we may assume that $e_1, e_2, \cdots, e_s, f_1, f_2, \cdots, f_t$ and $\bar{f}$ have the same central carrier. Thus we can write $\Phi(e_i) = 2^{-n_i} z_0$, $\Phi(f_j) = 2^{-m_j} z_0$ and $\Phi(\bar{f}) = 2^{-m} z_0$ for some central projection z_0. Denote the largest number of $\{n_i, m_j, m\}$ by r. By [5, Lemma 4.12], there exists a fundamental projection p in N such that $\Phi(p) = 2^{-r} z_0$. Then by the above arguments, $e_i = \sum_{k=1}^{2r-n_i} (e_i)^k$, $f_j = \sum_{k=1}^{2r-m_j} (f_j)^k$ and $\bar{f} = \sum_{k=1}^{2r-m} \bar{f}^k$, $\{(e_i)^k\}$ (resp. $\{(f_j)^k\}$, $\{\bar{f}^k\}$) is a set of orthogonal equivalent projections such that $(e_i)^k \sim p$, $(f_j)^k \sim p$ and $\bar{f}^k \sim p$. Since, by our assumption $2^{r-m} + \sum_{j=1}^{t} 2^{r-m_j} \leqq \sum_{i=1}^{s} 2^{r-n_i}$, the assertion follows from [5, Theorem 5.7].

LEMMA 2.9. *If $\{e_\alpha\}_{\alpha \in A}$ is a set of orthogonal fundamental projections and if e is a fundamental projection such that $e = \sum \{e_\alpha, \alpha \in A\}$, then $\Phi(e) = \sum \{\Phi(e_\alpha), \alpha \in A\}$ in Z, that is, Φ is completely additive on fundamental projections.*

PROOF. By Lemma 2. 8, we can easily show that $\sum\{\Phi(e_\alpha), \alpha\in A\}\leqq\Phi(e)$. Conversely let ψ be a functional in $\mathfrak{S}'$, put $z_0=1-\mathrm{Sup}\{z\,;\,\psi(z)=0,\ z\in Z_p\}$. Then, by the complete additivity of ψ and Schwarz' inequality, we can easily show that ψ is faithful on Zz_0. Noting that $\sum\{\Phi(e_\alpha\cdot z_0), \alpha\in A\}\leqq\Phi(ez_0)$, by [4, Lemma 2. 2], $\sum\{\psi(\Phi(e_\alpha\cdot z_0), \alpha\in A\}\leqq\psi(\Phi(ez_0))$; therefore without an exception of countable set $\{\psi(\Phi(e_iz_0)\,;\,i=1,2,3,\cdots\}$, $\psi(\Phi(e_\alpha\cdot z_0))=0$, that is, $e_\alpha\cdot z_0=0$. Thus we may reduce the problem to the case $\sum_{i=1}^{\infty}\Phi(e_i)+2^{-n}1<\Phi(e)$ for some positive integer n. By [5, Lemma 4. 12], there is a fundamental projection e_0 such that $\Phi(e_0)=2^{-n}1$ and $e_0\leqq e$. Hence by Lemma 2. 8, we can take a sequence $\{e_1', e_2', \cdots\}$ of orthogonal equivalent projections in $(e-e_0)N(e-e_0)$ with $e_1'\sim e_1$. Thus it follows that $e=\sum_{i=1}^{\infty}e_i\sim\sum_{i=1}^{\infty}e_i\ \leqq e-e_0\lneqq e$, which contradicts the finiteness of N. Therefore $\sum\{\Phi(e_\alpha)\ \alpha\in A\}=\Phi(e)$. The lemma follows.

LEMMA 2. 10. *Every non-zero projection e in N contains a non-zero fundamental projection in N.*

PROOF. Let $\{e_i\}_{i\in I}$ be a maximal family of orthogonal equivalent projections such that $e_i\sim e$ for each $i\in I$. By the finiteness of N, I is a finite set, say $\{1, 2, \cdots, n\}$. By [5, Theorem 5. 6], there exists a non-zero central projection z such that $(1-\sum_{i=1}^{n}e_i)z\precsim e_1z$. Thus we can write $z=\sum_{i=1}^{n}e_iz+e_{n+1}$, where e_{n+1} is a projection such that $e_{n+1}z\prec ze_n\sim ze_1$ and $e_{n+1}e_iz=0$ for $i=1,2,\cdots,n$. By [5, Lemma 4. 12], we can find a family $\{f_1, f_2, \cdots, f_{2^m}\}$ of orthogonal equivalent projections such that $\sum_{i=1}^{2^m}f_i=z$ and $n+1<2^m$. Again by [5, Theorem 5. 6], there is a central projection z_1 such that $e_1z_1\prec f_1z_1$ and $e_1(1-z_1)\succ f_1(1-z_1)$. Now we shall show $e_1z_1=0$. In fact, if $e_1z_1\neq 0$, then $zz_1=\sum_{i=1}^{n}e_iz_1z+e_{n+1}zz_1\prec\sum_{i=1}^{n+1}f_iz_1$ $\lneqq\sum_{i=1}^{2^m}f_iz_1=zz_1$. Thus this is a contradiction because of the finiteness of N. Therefore, it follows that $e_1\succ f_1(1-z_1)$. If $f_1(1-z_1)=0$, then $f_1\leqq z_1$, $z(f_1)=z\leqq z_1$, $e_1\leqq 1-z_1\leqq 1-z$ and $ez=0$. While by the above argument if $e_1z=0$, then $1-\sum_{i=1}^{n}e_i\succsim e_1$ and this contradicts the maximality of $\{e_i\}_{i=1}^{n}$. Thus this is a contradiction. It follows that $f_1(1-z_1)$ is a non-zero fundamental projection such that $e_1\succsim f_1(1-z_1)$. This completes the proof.

Now let N_p^f be the set of all fundamental projections of N and put $\bar{\varphi}(e)=\varphi(\Phi(e))$ for $e\in N_p^f$ and $\varphi\in Z_*$, $\varphi\geqq 0$ and $\|\varphi\|=1$. For any positive number

ε, let $\{z_\alpha\}$ and $\{\psi_\alpha\}$ be a maximal family of non-zero orthogonal central projections and positive functionals in $\widetilde{\mathfrak{S}}'z_\alpha(=\{\psi z_\alpha, \psi\in\widetilde{\mathfrak{S}}'\})$ respectively, satisfying the following: $1/(1+\varepsilon)\psi_\alpha(f)\leqq\bar{\varphi}(f)<(1+\varepsilon)^2\psi_\alpha(f)$ for $f\in Nz_\alpha\cap N_p^f$ and for all α. From now on, we shall show $\sum z_\alpha=1$. In fact, if $0<1-\sum z_\alpha$, then there is a positive functional ψ_1 in $\widetilde{\mathfrak{S}}$ such that $\psi_1\left(1-\sum z_\alpha\right)=\bar{\varphi}\left(1-\sum z_\alpha\right)$. Then,

LEMMA 2.11. *There exist a positive number θ and a non-zero projection e_0 in $N\left(e_0\leqq 1-\sum z_\alpha\right)$ such that $\theta\psi_1(p)\leqq\bar{p}(p)\leqq\theta(1+\varepsilon)\psi_1(p)$ for all fundamental projections $p\leqq e_0$.*

PROOF. If the contrary holds, then for all $f\in N\left(1-\sum z_\alpha\right)\cap N_p^f$, there is a non-zero fundamental projection such that $0\lneqq f_1\leqq f$ and $\psi_1(f_1)\gneqq\bar{\varphi}(f_1)$. By Lemma 2.10, there exists a set of orthogonal fundamental projections $\{f_\beta\}$ such that $\psi_1(f_\beta)>\bar{\varphi}(f_\beta)$ and $\sum f_\beta=1-\sum z_\alpha$. Since Φ is completely additive, $\psi_1\left(1-\sum z_\alpha\right)=\sum_\beta\psi_1(f_\beta)>\sum_\beta\bar{\varphi}(f_\beta)=\bar{\varphi}\left(1-\sum_\alpha z_\alpha\right)$ and hence this is a contradiction. Therefore there exists a projection $e_1\in N_p^f\cap N\left(1-\sum_\alpha z_\alpha\right)$ such that $\psi_1(e_1)\neq 0$ and for all $p\in N_p^f$ with $p\leqq e_1$, $\psi_1(p)\leqq\bar{p}(p)$. Let θ be the least upper bound of the numbers η such that $\eta\cdot\psi_1(e_1)\leqq\bar{\varphi}(e_1)$, then $\theta\cdot(1+\varepsilon)\cdot\psi_1(e_1)\geqq\bar{\varphi}(e_1)$ and thus $\theta\psi_1(e_1)\leqq\bar{\varphi}(e_1)\leqq\theta(1+\varepsilon)\psi_1(e_1)$. By the same argument as above we can prove the lemma.

Thus we have $\theta\psi_1(u^*eu)\leqq\bar{\varphi}(u^*eu)=\bar{\varphi}(e)\leqq\theta(1+\varepsilon)\psi_1(e)$ for all $u\in(e_1Ne_1)_u$ and $e\leqq e_0$ with $e\in N_p^f$. Therefore, $\psi_1(u^*eu)\leqq(1+\varepsilon)\psi_1(e)$ for all $e\in N_p^f$ with $e\leqq e_1$ and $u\in(e_1Ne_1)_u$. By virtue of the complete additivity of ψ_1 and the spectral decomposition theorem, it follows that $\psi_1(u^*au)\leqq(1+\varepsilon)\cdot\psi_1(a)$ for all $a\in(e_1Ne_1)$, $a\geqq 0$. Now by the finiteness of e_1Ne_1 and the polar decomposition theorem ([11, Lemma 2.1]), we can easily show $\psi_1(a^*a)\leqq(1+\varepsilon)\cdot\psi_1(aa^*)$ for all $a\in e_1Ne_1$. Let $\{q_1, q_2, \cdots, q_{2^n}\}$ be a family of orthogonal equivalent projections such that $e_1=q_1$ and $\sum_{i=1}^{2^n}q_i=z$, $z\in Z_p$ and w_i $(i=1,2,\cdots,2^n)$ be the partial isometry such that $w_i^*w_i=q_1=e_1$ and $w_iw_i^*=q_i$. If we define $\psi(a)=\sum_{i=1}^{2^n}\psi_1(w_i^*aw_i)$ for $a\in Nz$, then $\psi\in\widetilde{\mathfrak{S}}'z$ $(\psi\geqq 0)$ and putting $a_{ij}=w_j^*aw_i(\in e_1Ne_1)$, we have for $a\in Nz$

$$\psi(a^*a)=\sum_{ij}\psi_1(w_i^*a^*w_jw_j^*aw_i)=\sum_{ij}\psi_1(a_{ij}^*a_{ij})$$

and

$$\psi(aa^*) = \sum_{ij}\psi_1(w_i{}^*aw_jw_j{}^*a^*w_i) = \sum_{ij}\psi_1(a_{ji}a_{ji}{}^*)$$
$$\leqq \sum_{ij}(1+\varepsilon)\cdot\psi_1(a_{ji}{}^*a_{ji}).$$

Therefore we have $\psi(a^*a)\leqq(1+\varepsilon)\cdot\psi(aa^*)$ for all $a\in Nz$. Let f be a fundamental projection in Nz such that $\Phi(f)=2^{-m}z_0(z_0\in Z_p)$, then by [5, Lemma 4.12], there is a family $\{f_1, f_2, \cdots, f_{2^n}\}$ of orthogonal equivalent projections such that $f=\sum_{i=1}^{2^n} f_i$. Noting that f_i is fundamental for each i, $\Phi(f_i)=2^{-n-m}z_0\leqq 2^{-n}z=\Phi(e_1)$, which implies $f_i\precsim e_1$, that is, there is a projection g_i in N with $g_i\leqq e_1$ and $f_i\sim g_i$. Therefore $\bar{\varphi}(f)=\sum_{i=1}^{2^n}(f_i)=\sum_{i=1}^{2^n}\bar{\varphi}(g_i)\leqq\theta\cdot(1+\varepsilon)\cdot\sum_{i=1}^{2^n}\psi(g_i)\leqq\theta(1+\varepsilon)^2\sum_{i=1}^{2^n}\psi(f_i)=\theta(1+\varepsilon)^2\cdot\psi(f)$. Similarly we get $\bar{\varphi}(f)=\sum_{i=1}^{2^n}\bar{\varphi}(f_i)\geqq\theta\sum_{i=1}^{2^n}\psi(g_i)>\theta/(1+\varepsilon)\cdot\sum_{i=1}^{2^n}\psi(f_i)=\theta/(1+\varepsilon)\cdot\psi(f)$. Therefore it follows that

$$\theta/(1+\varepsilon)\cdot\psi(f)\leqq\bar{\varphi}(f)\leqq\theta(1+\varepsilon)^2\cdot\psi(f)\ \text{ for } f\in Nz\cap N_p^f.$$

Putting $\psi'=\theta\psi(\in\widetilde{\mathfrak{S}'z})$, $\{z, z_\alpha\}$ and $\{\psi, \psi_\alpha\}$ has the same property as $\{z_\alpha\}$ and $\{\psi_\alpha\}$, which contradicts the maximality. Thus $\sum z_\alpha=1$.

Now we define a functional on N_p^f as follows: $\psi_\varepsilon(e)=\sum_\alpha\psi_\alpha(ez_\alpha)$ for $e\in N_p^f$. Then, $(1/(1+\varepsilon))\psi_\varepsilon(e)=\sum_\alpha(1/(1+\varepsilon))\psi_\alpha(ez_\alpha)\leqq\sum_\alpha\bar{\varphi}(ez_\alpha)=\bar{\varphi}\left(\sum_\alpha ez_\alpha\right)=\bar{\varphi}(e)$ $\leqq\sum_\alpha(1+\varepsilon)^2\psi_\alpha(ez_\alpha)=(1+\varepsilon)^2\psi_\varepsilon(e)$ and hence ψ_ε is bounded on N_p^f. Therefore if we put $\widetilde{\psi_\varepsilon}(a)=\sum_\alpha\psi_\alpha(az_\alpha)$ for $a\in N$, then $\widetilde{\psi_\varepsilon}$ is a bounded positive linear functional on N which is completely additive on projections. Since $\widetilde{\psi_\varepsilon}(1)$ $\leqq(1+\varepsilon)\bar{\varphi}(1)$, $\{\widetilde{\psi_{1/n}}, n=1,2,\cdots\}$ is a bounded set in N^* and thus it is $\sigma(N^*,N)$-relatively compact. Let $\widetilde{\psi_0}$ be an accumulation point of $\{\widetilde{\psi_{1/n}}, n=1,2,\cdots\}$, then $1/(1+1/n)\widetilde{\psi_{1/n}}(e)\leqq\bar{\varphi}(e)\leqq(1+1/n)^2\widetilde{\psi_{1/n}}(e)$ for each $e\in N_p^f$ implies that $\widetilde{\psi_0}(e)=\bar{\varphi}(e)$ for all $e\in N_p^f$. On the other hand, $\widetilde{\psi_0}$ is completely additive on projections. In fact, for any orthogonal family of projections $\{e_\alpha\}_{\alpha\in A}$ with $e=\sum\{e_\alpha, \alpha\in A\}$, we can write by Lemma 2.10, $e_\alpha=\sum\{e_{\alpha,\beta_\alpha}, \beta_\alpha\in A_\alpha\}$, $1-e$ $=\sum\{f_\beta, \beta\in B\}$ where $\{e_{\alpha,\beta_\alpha}\}$ and $\{f_\beta\}$ are families of orthogonal fundamental projections in N. Since $\sum_\alpha\sum_{\beta_\alpha}e_{\alpha,\beta_\alpha}+\sum_\beta f_\beta=1$, $\sum_\alpha\sum_{\beta_\alpha}\bar{\varphi}(e_{\alpha,\beta_\alpha})+\sum_\beta\bar{\varphi}(f_\beta)=1$, which implies $\sum_\alpha\sum_{\beta_\alpha}\widetilde{\psi_0}(e_{\alpha,\beta_\alpha})+\sum_\beta\widetilde{\psi_0}(f_\beta)=1$. Moreover $\sum_\beta\widetilde{\psi_0}(f_\beta)\leqq\widetilde{\psi_0}\left(\sum_\beta f_\beta\right)$

$=\widetilde{\psi}_0(1-e)$, which implies $\sum_\alpha\sum_{\beta\alpha}\widetilde{\psi}_0(e_{\alpha,\beta\alpha})\geqq\widetilde{\psi}_0(e)$. On the other hand $\sum_\alpha\sum_{\beta\alpha}\widetilde{\psi}_0(e_{\alpha,\beta\alpha})$ $\leqq\widetilde{\psi}_0\Big(\sum_\alpha\sum_{\beta\alpha}e_{\alpha,\beta\alpha}\Big)=\widetilde{\psi}_0(e)$, therefore it follows $\sum_\alpha\sum_{\beta\alpha}\widetilde{\psi}_0(e_{\alpha,\beta\alpha})=\widetilde{\psi}_0(e)$. Since $\sum_{\beta\alpha}\widetilde{\psi}_0(e_{\alpha,\beta\alpha})\leqq\widetilde{\psi}_0\Big(\sum_{\beta\alpha}e_{\alpha,\beta\alpha}\Big)=\widetilde{\psi}_0(e_\alpha)$, $\widetilde{\psi}_0(e)=\sum_\alpha\widetilde{\psi}_0(e_\alpha)$, which is the desired property. Now let $\{e_\alpha\}_{\alpha\in A}$ and $\{f_\beta\}_{\beta\in B}$ be two families of orthogonal fundamental projections such that $\sum_\alpha e_\alpha=\sum_\beta f_\beta$, then $\varphi\Big(\sum_{\alpha\in J}\Phi(e_\alpha)\Big)=\sum_{\alpha\in J}\bar{\varphi}(e_\alpha)=\sum_{\alpha\in J}\psi_0(e_\alpha)$ $=\widetilde{\psi}_0\Big(\sum_{\alpha\in J}e_\alpha\Big)\leqq\widetilde{\psi}_0\Big(\sum_\alpha e_\alpha\Big)\leqq\widetilde{\psi}_0(1)=\varphi(1)$ for all $\varphi\in\mathfrak{S}'$ and for all finite subsets J of A and $\varphi\Big(\sum_\alpha\Phi(e_\alpha)\Big)=\varphi\Big(\sum_\beta\Phi(f_\beta)\Big)$ for all $\varphi\in Z_*$, $\varphi\geqq0$ and $\|\varphi\|=1$. Therefore $\sum_\alpha\Phi(e_\alpha)$ and $\sum_\beta\Phi(f_\beta)$ exists in Z and moreover it follows that $\sum_\alpha\Phi(e_\alpha)$ $=\sum_\beta\Phi(f_\beta)$. Since for any projection e in N, there is by Lemma 2. 10, a set $\{e_\alpha\}_{\alpha\in A}$ of orthogonal fundamental projections such that $e=\sum e_\alpha$, then put $\Phi(e)$ $=\sum_\alpha\Phi(e_\alpha)$. By the above arguments, $\Phi(e)$ can be therefore unambiguously defined. Thus Φ is extensible on all projections of N; suppose that $\sum_{i=1}^n\alpha_ip_i=0$ ($p_i\in N_p$ α_i is a complex number for each i), then $\varphi\Big(\sum_{i=1}^n\alpha_i\cdot\Phi(p_i)\Big)=\sum_{i=1}^n\alpha_i\cdot\widetilde{\psi}_0(p_i)=\widetilde{\psi}_0\Big(\sum_{i=1}^n\alpha_ip_i\Big)$ $=0$ for all $\varphi\in\mathfrak{S}'$, which implies $\sum_{i=1}^n\alpha_i\Phi(p_i)=0$; on the other hand since elements of the form $\sum_{i=1}^n\alpha_ip_i$ is uniformly dense in N and $\Big\|\Phi(\sum_{i=1}^n\alpha_ip_i)\Big\|\leqq4\Big\|\sum_{i=1}^n\alpha_ip_i\Big\|$, then Φ has a unique continuous extension (denote it by the same symbol) on N. It is easy to see that this unique extension satisfies all the properties mentioned in Definition 1 in [4]. Now let $\{a_\mu\}$ be an increasing net of positive elements in N such that $a_\mu\uparrow a$ in the $\sigma(\mathfrak{S}')$-topology for some a in N, then $\psi(a-a_\mu)\geqq0$ and $\psi(a-a_\mu)\to0$ for all $\psi\in\widetilde{\mathfrak{S}'}$ with $\psi\geqq0$. Thus we have $\psi((a-a_\mu)z_\alpha)\to0$ for all such ψ and α, therefore $\psi_\alpha((a-a_\mu)z_\alpha)\to0$ for all α. On the other hand, by Lemma 2. 10 and the spectral theorem it follows that

$$(1/(1+\varepsilon))\cdot\psi_\alpha(a)\leqq\varphi(\Phi(a))\leqq(1+\varepsilon)^2\cdot\psi_\alpha(a)$$

for all $a\in Nz_\alpha$, $a\geqq0$. Hence $\varphi(\Phi(a-a_\mu)z_\alpha))\leqq(1+\varepsilon)^2\psi_\alpha((a-a_\mu)z_\alpha)\to0,\varphi(\Phi(a-a_\mu)z_\alpha)$ $\to0$ for all α and $\varphi\in Z_*$ with $\varphi\geqq0$. From this fact, Φ satisfies all requirements. This completes the proof of Theorem 2. 1.

Now let M be the semi-finite AW^*-algebra in the preceding paragraph of this section and Z be the center of M. Since Z can be represented as the algebra of the complex-valued continuous functions on a hyperstone space Ω, we identify them. Let $\boldsymbol{Z}$ be the set of all continuous $[0,+\infty]$-valued functions on Ω. By our convensions we recall here that $0\cdot+\infty=0$. If $f, f'\in\boldsymbol{Z}$ and λ is a non-negative number, then $f+f'\in\boldsymbol{Z}$ and $\lambda\cdot f\in\boldsymbol{Z}$. Since $\omega\to f(\omega)f'(\omega)(\omega\in\Omega)$ is a lower semi-continuous function, thus it coincides, except on a non-dense set, with a unique continuous function ff'. Then we have:

LEMMA 2.12. *If $\{a_\alpha\}$ is an increasing directed set of elements in $\boldsymbol{Z}$, for any positive element b in Z,* $\operatorname{Sup}\{ba_\alpha, \alpha\}=b\operatorname{Sup}\{a_\alpha,\alpha\}$.

PROOF. Let ψ be a faithful normal pseudo-measure on Ω ([1, Definition 2 and Proposition 7]), then since $\operatorname{Sup}_\alpha\{ba_\alpha\}\leqq b\cdot\operatorname{Sup}_\alpha\{a_\alpha\}$, it is sufficient to show that $\psi(b\cdot\operatorname{Sup}\{a_\alpha,\alpha\})=\psi(\operatorname{Sup}\{ba_\alpha\ \alpha\})$. By [1, Proposition 7], we have only to show that $\mu(b\operatorname{Sup}\{a_\alpha,\alpha\})=\mu(\operatorname{Sup}\{ba_\alpha,\alpha\})$ for each positive normal measure μ on Ω. Since the functions $\omega\ (\omega\in\Omega)\to b(\omega)(\operatorname{Sup}_\alpha\{a_\alpha(\omega)\})$ and $\omega\ (\omega\in\Omega)\to(b\operatorname{Sup}_\alpha\{a_\alpha\})(\omega)$ coincide except on a μ-null set, we have

$$\begin{aligned}\mu(b\operatorname{Sup}_\alpha\{a_\alpha\}) &= \int b(\omega)\operatorname{Sup}_\alpha\{a_\alpha(\omega)\}\,d\mu(\omega)\\ &= \int\operatorname{Sup}_\alpha\{b(\omega)a_\alpha(\omega)\}\,d\mu(\omega)\\ &= \operatorname{Sup}_\alpha\left\{\int b(\omega)a_\alpha(\omega)d\mu(\omega)\right\}\\ &= \operatorname{Sup}_\alpha\left\{\int (ba_\alpha)(\omega)d\mu(\omega)\right\}\quad\text{(by [1, Proposition 1])}\\ &= \int\operatorname{Sup}_\alpha\{(ba_\alpha)(\omega)\}\,d\mu(\omega)\\ &= \mu(\operatorname{Sup}(ba_\alpha))\quad\text{(by the same reason as above).}\end{aligned}$$

This completes the proof of Lemma 2.12.

3. Existence of a dimension function for M and the convergence "nearly everywhere" of sequences of elements in C. In this section we shall construct a dimension function (Definition 3.1) on M_p and using this dimension function we introduce the notions of the convergence "nearly everywhere" of sequences in C and study some of its topological properties. The methods which we use are similar to those of I. E. Segal [10].

THEOREM 3.1. *Let M be a semi-finite AW^*-algebra with a separating set $\mathfrak{S}$ of states which are completely additive on projections (c. a. states), Z be its center and $\boldsymbol{Z}$ be the set of all $[0, +\infty]$-valued continuous functions on Ω (the spectrum of Z). Then there is an operation Φ from M^+ (the positive part of M) to $\boldsymbol{Z}$ having the following properties:*

(1) $\Phi(h_1+h_2) = \Phi(h_1)+\Phi(h_2)$ $h_1, h_2 \in M^+$;

(2) $\Phi(\lambda h) = \lambda\cdot\Phi(h)$ *if* λ *is a positive number and* $h \in M^+$;

(3) $\Phi(st) = t\cdot\Phi(s)$ $s \in M^+, t \in Z^+$;

(4) $\Phi(uau^{-1}) = \Phi(a)$ *if* $a \in M^+$ *and* $u \in M_u$;

(5) *for any* $a \in M^+$ *with* $\Phi(a)=0$, $a=0$;

(6) *for every directed increasing net* $\{a_\mu\}$ *in* M^+ *such that* $a_\mu \uparrow a$ *in the* $\sigma(\mathfrak{S})$*-topology for some* a *in* M, $\Phi(a_\mu) \uparrow \Phi(a)$ *in* $\boldsymbol{Z}$;

(7) *for every non-zero* a *in* M^+, *there exists a non-zero* $b \in M^+$ *majorized by* a *such that* $\Phi(b) \in Z^+$.

PROOF. The semi-finiteness of M implies that there is a non-zero finite projection p. Let $\{p_\alpha, \alpha \in A\}$ be a maximal family of orthogonal equivalent projections such that $p \sim p_\alpha$ for each α, then by [5, Theorem 5.6], there exists a non-zero contral projection z as follows; $p_0 = \left(1-\sum_{\alpha\in A} p_\alpha\right)z \prec pz \neq 0$; thus $z = p_0 + \left(\sum_{\alpha\in A} p_\alpha\right)z$ by Lemma 2.12; let v_α be a partial isometry in M giving the equivalence $zp \sim zp_\alpha$, v_0 a partial isometry such that $v_0{}^*v_0 \leqq zp$ and $v_0v_0{}^* = p_0$. Let $zp\mathfrak{S}zp = \{zp\psi zp, \psi \in \mathfrak{S}\}$, then $zp\mathfrak{S}zp$ is a separating set of c. a. states (by [3, Lemma 3]) on the finite AW^*-algebra $zpMzp$. Therefore by Theorem 2.1, there is a Zzp-valued (note that by Lemma 2.3, the center of $zpMzp$ is Zzp) operation Φp on $zpMzp$. Since by Lemma 2.3, the map $\psi(a) = apz$ of $Zzz(p)$ onto Zzp is a $*$-isomorphism, then we define a $*$-isomorphism $\phi(a)$ on Zzp as $\phi(a) = \psi^{-1}(zpazp)$ and a new linear operation Φz on $(Mz)^+$ to $\boldsymbol{Z}$ as follows:

$$\Phi z(h) = \sum_{\beta\in A\cup\{0\}}{}' \phi(\Phi p((v_\beta)^* h v_\beta)), \text{ for } h \in (Mz)^+,$$

where $\sum'_{\alpha\in A\cup\{0\}} a_\alpha = \mathrm{Sup}\left\{\sum_{\alpha\in B} a_\alpha, B\subset A\cup\{0\}, \text{finite set}\right\}$, $a_\alpha \in \boldsymbol{Z}$. Noting that $\sum_{\beta\in A\cup\{0\}} v_\beta(v_\beta)^* = z$, for $h\in(Mz)^+$ and $u\in(Mz)_u$,

$$\Phi z(uhu^*) = \sum_{\beta\in A\cup\{0\}}' \phi(\Phi p((v_\beta)^* uhu^* v_\beta)).$$

Since $(v_\beta)^* uh^{1/2}zh^{1/2}u^* v_\beta \in zpMzp$ and $z=\sum_{\beta\in A\cup\{0\}} v_\beta(v_\beta)^*$ in the $\sigma(\mathfrak{S})$-topology, we get that $(v_\beta)^* uh^{1/2}zh^{1/2}u^* v_\beta = \sum_{\gamma\in A\cup\{0\}} (v_\beta)^* uh^{1/2} v_\gamma(v_\gamma)^* h^{1/2}u^* v_\beta$ in the $\sigma(\mathfrak{S})$-topology and therefore by Theorem 2. 1, it follows that

$$\sum_{\gamma\in A\cup\{0\}}' \phi(\Phi p((v_\beta)^* uh^{1/2}v_\gamma(v_\gamma)^* h^{1/2}u^* v_\beta)) = \phi(\Phi p((v_\beta)^* uhzu^* v_\beta))$$

and

$$\begin{aligned}\Phi z(uhu^*) &= \sum_{\beta\in A\cup\{0\}}' \left\{\sum_{\gamma\in A\cup\{0\}}' \phi(\Phi p((v_\beta)^* uh^{1/2}v_\gamma(v_\gamma)^* h^{1/2}u^* v_\beta))\right\} \\ &= \sum_{\beta\in A\cup\{0\}}' \left\{\sum_{\gamma\in A\cup\{0\}}' \phi(\Phi p((v_\gamma)^* h^{1/2}u^* v_\beta(v_\beta)^* uh^{1/2}v_\gamma))\right\} \\ &= \sum_{\gamma\in A\cup\{0\}}' \left\{\sum_{\beta\in A\cup\{0\}}' \phi(\Phi p((v_\gamma)^* h^{1/2}u^* v_\beta(v_\beta)^* uh^{1/2}v_\gamma))\right\}\end{aligned}$$

(by Fubini's Theorem)

$$\begin{aligned}&= \sum_{\gamma\in A\cup\{0\}}' \phi(\Phi p((v_\gamma)^* hv_\gamma)) \\ &= \Phi z(h).\end{aligned}$$

Next let h be in $(Mz)^+$ and a be in $(Zz)^+$, then by Lemma 2. 3, we have

$$\begin{aligned}\phi(\Phi p((v_\beta)^* hav_\beta)) &= \phi(\Phi p((v_\beta)^* hv_\beta zpa)) \\ &= \phi(zpa)\cdot\phi(\Phi p((v_\beta)^* hv_\beta)) \\ &= a\cdot\phi(\Phi p((v_\beta)^* hv_\beta)).\end{aligned}$$

Since $\phi(\Phi p((v_\beta)^* hv_\beta))\geqq 0$, then by Lemma 2. 12, it follows that $\Phi z(ha)=a\cdot\Phi z(h)$. To prove the faithfulness of Φz we argue as follows. Let h be in $(Mz)^+$ such that $\Phi z(h)=0$, then since $\Phi z(h)=\sum_{\beta\in A\cup\{0\}} \phi(\Phi p((v_\beta)^* hv_\beta))\geqq \phi(\Phi p((v_\beta)^* hv_\beta))\geqq 0$ for all $\beta \in A \cup \{0\}$, we get that $(v_\beta)^* h v_\beta = 0$ for all $\beta \in A \cup \{0\}$, that is, $hp_\beta z = 0$ for all $\beta \in A \cup \{0\}$, which implies by [5, Lemma 2.2], $h = 0$. Suppose that $\{a_\mu\}$ is a directed increasing net of positive elements in Mz such that $a_\mu \uparrow a$ for some $a \in Mz$ in the $\sigma(\mathfrak{S})$-topology. We shall prove $\Phi z(a_\mu) \uparrow \Phi z(a)$ in $\boldsymbol{Z}$. In fact, since $(v_\beta)^* a_\mu v_\beta \uparrow (v_\beta)^* a v_\beta$ in the $\sigma(zp\mathfrak{S}zp)$-topology for each $\beta \in \{0\} \cup A$, we have by Theorem 2.1, $\phi(\Phi p((v_\beta)^* a_\mu v_\beta)) \uparrow \phi(\Phi p((v_\beta)^* a v_\beta))$ for each $\beta \in \{0\} \cup A$. Therefore, it follows that $\sum'_{\beta \in \{0\} \cup A} \phi(\Phi p((v_\beta)^* a_\mu v_\beta)) \rightarrow \sum'_{\beta \in \{0\} \cup A} \phi(\Phi p((v_\beta)^* a v_\beta))$, that is, $\Phi z(a_\mu) \uparrow \Phi z(a)$. To prove the semi-finiteness of Φz, we have only to show that for any e in $(Mz)_p$, there is a projection f in Mz majorized by e such that $\Phi z(f) \in Z^+$. Since there exists a non-zero projection f in Mz such that $f \leqq e$ and $f \precsim zp$. By the definition of Φz and Lemma 2.4, we have $\Phi z(f) \leqq \Phi z(pz) = \Phi z(p_\alpha z)$ for any $\alpha \in A$ and $\Phi z(f) \in Z^+$. Therefore Φz satisfies the conditions (1)–(7) in Theorem 3.1. Let $\{z_\alpha\}$ and $\{\Phi z_\alpha\}$ be a maximal family of non-zero orthogonal central projections in M and linear operations on $(Mz_\alpha)^+$ to $\boldsymbol{Z}$ respectively satisfying the conditions (1)–(7) in Theorem 3.1, then from the above arguments $\sum_\alpha z_\alpha = 1$. Define a new linear operation Φ on M^+ to $\boldsymbol{Z}$ as follows:

$$\Phi(h) = \sum'_{z_\alpha \in \{z_\alpha\}} \Phi z_\alpha(z_\alpha h) \quad h \in M^+,$$

then by the same reason as above discussions we can easily show that Φ satisfies the desired properties. This completes the proof of Theorem 3.1.

Now let $\mathfrak{P}$ be the set $\{s \in M^+, \Phi(s) \in Z^+\}$ and $\mathfrak{M}'$ be the set $\{b\,;\, b \in M, b^*b \in \mathfrak{P}\}$, then we can easily see that $\mathfrak{M}' \cdot \mathfrak{M}'$(denote it by $\mathfrak{N}$) is the two-sided ideal such that $\mathfrak{N}^+$ (the positive part of $\mathfrak{N}) = \mathfrak{P}$. Since every element of $\mathfrak{N}$ is a linear combination of elements of $\mathfrak{N}^+$, by the properties of Φ there is a linear operation $\dot{\Phi}$ on $\mathfrak{N}$ to Z which coincides with Φ on $\mathfrak{N}^+$. If $a \in \mathfrak{N}$ and if $u \in M_u$, then $\dot{\Phi}(u^*au) = \dot{\Phi}(a)$; therefore $\dot{\Phi}(au) = \dot{\Phi}(uauu^*) = \dot{\Phi}(ua)$; since every element of M is a linear combination of unitary elements, we have $\dot{\Phi}(ab) = \dot{\Phi}(ba)$ for $a \in \mathfrak{N}$ and $b \in M$. Let $\{t_\mu\}$ be a uniformly bounded increasing net of positive elements in $\mathfrak{N}$ such that $t_\mu \uparrow t$ for the $\sigma(\mathfrak{S})$-topology for some $t \in M$. If $\{\Phi(t_\mu)\}$ is uniformly bounded, then $t \in \mathfrak{N}$ and $\dot{\Phi}(t) = \mathrm{Sup}\{\dot{\Phi}(t_\mu)\,;\, \mu\}$. In fact, by the property of $\Phi, \Phi(t_\mu) \uparrow \Phi(t)$ in $\boldsymbol{Z}$ and $\mathrm{Sup}\{\Phi(t_\mu), \mu\} = \Phi(t)$ in $\boldsymbol{Z}$. On the other hand $0 \leqq \dot{\Phi}(t_\mu) \leqq k1$ for all μ which implies that $\Phi(t) \in Z^+$, that is, $t \in \mathfrak{N}$ and $\dot{\Phi}(t) = \Phi(t)$. Next we shall show that every non-negative element of M is the least upper bound of a set of non-negative elements in $\mathfrak{N}$. In fact, from the argument used in the proof of the above theorem, there is an increasing mutually commuting net of projections $\{f_\beta\}_{\beta \in B}$ in $\mathfrak{N}$ such that $\mathrm{Sup}\{f_\beta, \beta \in B\} = 1$. For every non-negative element a in M, $a^{1/2}f_\beta a^{1/2} \to a$ for the $\sigma(\mathfrak{S})$-topology and $a^{1/2}f_\beta a^{1/2} \in \mathfrak{N}^+$. Thus we have

THEOREM 3.2. *In Theorem* 3.1, *let* $\mathfrak{P}$ *be the set* $\{s \in M, s \geqq 0, \Phi(s) \in Z\}$, *then* $\mathfrak{P}$ *is the positive part of a two-sided ideal* $\mathfrak{N}$ *and there exists a unique linear operation* $\dot{\Phi}$ *on* $\mathfrak{N}$ *to* Z *which coincides with* Φ *on* $\mathfrak{P}$; *moreover this linear operation satisfies the following properties*:

(1) *If* $t \in \mathfrak{N}$ *with* $t \geqq 0$, *then* $\dot{\Phi}(t) \geqq 0$ *and* $\dot{\Phi}(t) = 0$ *only if* $t = 0$;

(2) $\dot{\Phi}(st) = \dot{\Phi}(ts)$ *if* $s \in M$ *and* $t \in \mathfrak{N}$;

(3) $\dot{\Phi}(st) = s \cdot \dot{\Phi}(t)$ *if* $s \in Z$ *and* $t \in \mathfrak{N}$;

(4) *let* $\{t_\mu\}$ *be a directed increasing net of positive elements in* $\mathfrak{N}$ *such that* $t_\mu \to t$ *in the* $\sigma(\mathfrak{S})$-*topology for some positive element* t *in* M *and if* $\{\dot{\Phi}(t_\mu)\}$ *is uniformly bounded, then* $t \in \mathfrak{N}$ *and* $\dot{\Phi}(t) = \mathrm{Sup}\{\dot{\Phi}(t_\mu), \mu\}$;

(5) *every non-negative element in* M *is the supremum of a set of non-negative elements in* $\mathfrak{N}$.

LEMMA 3.1. *Let* N *be a finite AW*-algebra with a separating set of c.a. states and if the center of* N *is* σ-*finite, then* N *is also* σ-*finite.*

PROOF. By Theorem 2.1, N has a center-valued trace $\Phi'(\cdot)$ in the sense of [4, Definition 1]. On the other hand the σ-finiteness of the center of N implies there is a faithful positive normal measure μ on the center. Now let $\{e_\alpha\}$ be a set of orthogonal projections in N with $e = \sum_\alpha e_\alpha (e \in N_p)$, then $\mu(\Phi'(e)) = \sum_\alpha \mu(\Phi'(e_\alpha))$. This implies that all but countably many must vanish and the faithfulness of μ and Φ yields the desired property. The lemma follows.

Now we are in the position to prove the existence theorem of a dimension function.

THEOREM 3.3. *In* M, *we can define a dimension function* $D(e)$ *with values in* $\boldsymbol{Z}$ *for all projections* $e \in M$, *in such a way that*

(1) $D(e)(\omega) < \infty$ *except on a non-dense set if and only if* $e \in \mathfrak{M}$;

(2) *if* $p, q \in M_p$ *and* $pq = 0$, *then* $D(p+q) = D(p) + D(q)$;

(3) *for any indexed chain of projections* $\{e_\lambda;\ \lambda\in\Lambda\}$ *in* M, $D(\bigvee_{\lambda\in\Lambda} e_\lambda) = \operatorname{Sup}\{D(e_\lambda), \lambda\in\Lambda\}$;

(4) *if* u *is in* M_{pi}, *then* $D(u^*u)=D(uu^*)$;

(5) *for* $e\in Z_p$ *and* $p\in M_p$, $D(e)\neq 0$ *and* $D(ep)=eD(p)$.

PROOF. First note that if $\{e_\mu\}$ is any indexed set of mutually orthogonal projections in Z such that $\sum_\mu e_\mu=1$, then in order to prove the existence of a dimension function for M, we have only to show that Me_μ all admit dimension functions. Since 1 is the sum of orthogonal central projections which are σ-finite with respect to the center, it suffices to consider the case in which the center Z is σ-finite. Now let p be a projection in $\mathfrak{M}$, then pMp is a finite AW^*-algebra with a separating set $p\mathfrak{S}p(=\{p\varphi p, \varphi\in\mathfrak{S}\})$ of c. a. states whose center Zp is σ-finite and by Lemma 3.1, pMp is also σ-finite. Therefore, by Theorem 3.2 (5), there is a sequence of mutually orthogonal projections $\{p_n\}_{n=1}^\infty$ in $\mathfrak{N}$ such that $p=\sum_{n=1}^\infty p_n$. Now write $D(p)=\sum_{n=1}^{\infty}{}' \dot\Phi(p_n)$. In order to show that $D(p)$ is well defined, suppose $p=\sum_{n=1}^\infty p_n'$ with $p_n'\in\mathfrak{N}$ and $p_n'p_m'=0$ if $n\neq m$. By symmetry, we have only to show that $\sum_{n=1}^{\infty}{}'\dot\Phi(p_n')\geqq\sum_{n=1}^{n}\dot\Phi(p_n')$ for all n. Let $e_n'=\sum_{i=1}^n p_i'$ and $e_n=\sum_{i=1}^n p_i$, then $e_me_n'e_m\leqq e_mpe_m=e_m$ and $\dot\Phi(e_me_n'e_m)\leqq\dot\Phi(e_m)$. Since $\dot\Phi(e_me_n'e_m)=\dot\Phi(e_me_n')=\dot\Phi(e_n'e_me_n')$ and $e_n'e_me_n'\uparrow e_n'pe_n'$ as $m\to\infty$ in the $\sigma(\mathfrak{S})$-topology, we get $\dot\Phi(e_me_n'e_m)\uparrow\dot\Phi(e_n')$ for the s-topology in Z. Hence we have $\dot\Phi(e_n')\leqq\sum_{n=1}^\infty\dot\Phi(p_n)$. Thus the definition is unambiguous. Next, we shall show that $D(p)(\omega)<\infty$ on a dense open set. Let $\Omega_0=\{\omega;\ \omega\in\Omega, D(p)(\omega)=\infty\}$. If $(\Omega_0)^i\neq\emptyset$, denoting the central projection corresponding to the clopen set $(\Omega_0)^i$ by e and considering the situation on Me, we have only to see that it is impossible that $D(p)(\omega)=\infty$ on Ω. On the other hand $\left\{\omega;\ \sum_{n=1}^\infty\dot\Phi(p_n)(\omega)<\infty\right\}=\left\{\omega;\ \sum_{n=1}^\infty\dot\Phi(p_n)(\omega)\neq D(p)(\omega)\right\}$ is a set of first category and hence, by [1, p. 10, Corollary], it is non-dense. Note that the closure of a non-dense set is also non-dense, and we can easily show that $\left\{\omega;\ \sum_{n=1}^\infty\dot\Phi(p_n)(\omega)=\infty\right\}$ contains a non empty clopen set. Thus by the same reason as above, it suffices to show the statement that $\sum_{n=1}^\infty\dot\Phi(p_n)(\omega)=\infty$ for all $\omega\in\Omega$ is false. In fact, since Ω is compact, by Dini's theorem, $\sum_{n=1}^m\dot\Phi(p_n)(\omega)\uparrow\infty$ uniformly as $m\uparrow\infty$. Now, since $\dot{\Phi}(p_1)$ is a bounded function, there is a positive interger n_1 such that

$$\dot{\Phi}(p_1) \leqq \sum_{i=2}^{n_1} \dot{\Phi}(p_i) \quad \text{for all} \quad \omega \in \Omega;$$

similarly, $\dot{\Phi}(p_2) \leqq \sum_{i=n_1+1}^{n_2} \dot{\Phi}(p_i)$ for all $\omega \in \Omega$ and a suitable integer $n_2 > n_1$ and by mathematical induction, we can choose a strictly increasing sequence $\{n_i\}$ of positive integers such that

$$\dot{\Phi}(p_i) \leqq \sum_{j=n_i+1}^{n_{i+1}} \dot{\Phi}(p_j) \text{ for all } \omega \in \Omega.$$

By Theorem 3.2 and [5, Theorem 5.6], we get that $p_i \precsim \sum_{j=n_i+1}^{n_{i+1}} p_j$ for each i. Since $\{p_i\}_{i=1}^{\infty}$ and $\left\{\sum_{j=n_i+1}^{n_{i+1}} p_j\right\}_{i=1}^{\infty}$ are families of orthogonal projections, by [5, Theorem 5.5], $p \precsim p - p_1$ and the finiteness of p implies $p_1 = 0$. The arbitrariness in the ordering of the p_i now shows that all the p_i are zero, so that $D(p) = 0$ and this is a contradiction. Thus the set $\{\omega; D(p)(\omega) < \infty\}$ is a dense open set. Next for any properly infinite projection p with the central carrier $z(p)$, $D(p)(\omega)$ is defined as $\infty \cdot z(p)(\omega)$. For an arbitrary projection p in M with finite and properly infinite parts p_1 and p_2 respectively, $D(p)$ is defined as $D(p_1) + D(p_2)$. Since the assertion (1) is clear from the definition, it remains to show that $D(\cdot)$ satisfies conditions (2)−(5). With regard to (2) and (4), it is easily shown from the definition of $D(\cdot)$. In order to prove (5) we have only to consider the cases in which p is either finite or properly infinite. If $p \in \mathfrak{M}$ and $e \in Z_p$, then there is a sequence $\{p_i\}_{i=1}^{\infty}$ of mutually orthogonal projections in $\mathfrak{N}$ such that $p = \sum_{i=1}^{\infty} p_i$. Then $ep = \sum_{i=1}^{\infty} ep_i$ and $D(ep) = \sum_{i=1}^{\infty} D(ep_i) = \sum_{i=1}^{\infty} eD(p_i) = eD(p)$ by Lemma 2.12. For the case where p is properly infinite the assertion is clear from Lemma 2.2. Now we are in the position to prove (3). Making use of (5), we may assume either $\operatorname{Sup}\{D(e_\lambda)(\omega), \lambda\} < \infty$ on a dense open set or it is infinite on an open dense set. Set $e = \vee \{e_\lambda, \lambda \in \Lambda\}$ and $D(e)(\omega) \geqq \operatorname{Sup}\{D(e_\lambda)(\omega); \lambda \in \Lambda\}$. Thus we have only to prove the statement for the first case. First we note that the non-negative continuous functions on $\Omega (\in \boldsymbol{Z})$ are order-isomorphic to the continuous functions with values in $[0, \pi/2]$, via the transformation $f \rightarrow$ arctangent f. Thus it suffices to show (3) in case where $\{D(e_\lambda)\} \subset Z^+$ and are uniformly bounded. Next we observe that the supremum of any collection of the elements in Z^+ is the supremum of some subcollection which is at most countable. Since Z is σ-finite, by [1, Proposition 7], there is a faithful positive measure μ on Ω such that,

$$\begin{aligned}\mu(\operatorname{Sup}\{D(e_\lambda), \lambda \in \Lambda\}) &= \operatorname{Sup}\{\mu(D(e_\lambda)), \lambda \in \Lambda\} \\ &= \operatorname{Sup}\{\mu(D(e_{\lambda_i}))\,, \text{ for some } \lambda_1, \lambda_2, \cdots \in \Lambda\} \\ &= \mu(\operatorname{Sup}\{D(e_{\lambda_i}), i = 1, 2, \cdots\})\,,\end{aligned}$$

that is, $[\operatorname{Sup}\{D(e_\lambda), \lambda \in \Lambda\}](\omega) = [\operatorname{Sup}\{D(e_{\lambda_i}), i\}](\omega)$ except on a non-dense set. By [1], it follows that $\operatorname{Sup}\{D(e_\lambda), \lambda \in \Lambda\} = \operatorname{Sup}\{D(e_{\lambda_i}), i = 1, 2, 3, \cdots\}$. Since $\{e_\lambda\}_{\lambda \in \Lambda}$ is an indexed chain, putting $e' = \bigvee_{i=1}^{\infty} e_{\lambda_i} (\leqq e)$, for any $\lambda \in \Lambda$, $e_\lambda \leqq e'$ or $e_\lambda \geqq e$. If $e_\lambda \geqq e'$, then $D(e_\lambda) \geqq D(e') \geqq \operatorname{Sup}\{D(e_{\lambda_i}), i = 1, 2, 3, \cdots\} = \operatorname{Sup}\{D(e_\lambda), \lambda\} \geqq D(e_\lambda)$ and so $D(e_\lambda) = D(e')$, that is, $e', e_\lambda \in \mathfrak{M}$ and $e_\lambda \sim e'$ or $e_\lambda = e'$, which implies that $e' = e$. Therefore, we have only to show (3) in case where $e = \bigvee_{i=1}^{\infty} e_i, e_i \in \mathfrak{M}$. Let $s_i = e_{i+1} - e_i (i = 1, 2, 3, \cdots)$ and $s_i = \sum_{j=1}^{\infty} s_{ij}$ where the s_{ij} are mutually orthogonal projections in $\mathfrak{N}$. Now by the definition of $D(\cdot), D(s_i) = \sum_{j=1}^{\infty} D(s_{ij})$ and by the same reason, $D(e) = \sum_{i,j=1}^{\infty} D(s_{ij}) = \sum_{i=1}^{\infty} D(s_i) = \operatorname{Sup}\{D(e_i), i = 1, 2, 3, \cdots\}$. This completes the proof of Theorem 3. 3.

In the previous paper [7], we defined a "measurable operator" for a semi-finite AW^*-algebra in algebraic fashion and studied the structure of the $*$-algebra C of "measurable operators". Now we are in the position to introduce the notion of "the convergence nearly everywhere of sequences in C".

DEFINITION 3. 2. We say that a sequence $\{x(n)\}_{n=1}^{\infty}$ of C converges nearly everywhere (or converges n. e.) to an element x in C if for any positive ε, there exist a positive integer $n_0(\varepsilon)$ and an SDD$\{e_n(\varepsilon)\}$ such that

$$(x(n) - x)[e_n(\varepsilon), 1] \in \bar{M} \quad \text{for all} \quad n \geqq n_0(\varepsilon)$$

and

$$\|(x(n) - x)[e_n(\varepsilon), 1]\|_\infty < \varepsilon \quad \text{for all} \quad n \geqq n_0(\varepsilon)\,,$$

where we write $\|x\|_\infty = \|x\|$ for $x = [x, 1]$.

REMARK. We must note that a limit nearly everywhere is unique. In fact, suppose that $x(n) \to x$ n. e. and $x(n) \to x'$ n. e. ($x(n)$, x and $x' \in C, n = 1, 2, \cdots$) then for any positive number ε, there exist a positive integer $n_0(\varepsilon)$ and SDD's $\{e_n(\varepsilon)\}$ and $\{e_n(\varepsilon)'\}$ such that $(x(n) - x)[e_n(\varepsilon), 1] \in \bar{M}$, $(x(n) - x')[e_n(\varepsilon)', 1] \in \bar{M}$ for all $n \geqq n_0(\varepsilon)$, $\|(x(n)-x)[e_n(\varepsilon), 1]\|_\infty < \varepsilon$ and $\|(x(n)-x')[e_n(\varepsilon)', 1]\|_\infty < \varepsilon$ for all $n \geqq n_0(\varepsilon)$. Let $f_n(\varepsilon) = e_n(\varepsilon) \wedge e_n(\varepsilon)'$, then by [7, Lemma 3.1], $\{f_n(\varepsilon)\}$ is also an SDD. Moreover, $(x-x')[f_n(\varepsilon), 1] \in \bar{M}$ and $\|(x-x')[f_n(\varepsilon), 1]\|_\infty < 2\varepsilon$ for all $n \geqq n_0(\varepsilon)$. Write $x-x' = [x_n, e_n]$ and by [7, Definition 3.3], we have $(x-x')[e_n \wedge f_n(\varepsilon), 1] = [x_n(e_n \wedge f_n(\varepsilon)), 1]$ for each n. Thus by [7, The Remark following Theorem 3.1], it follows that for all $n \geqq n_0(\varepsilon)$,

$$\|x_n(e_n \wedge f_n(\varepsilon))\| < 2\varepsilon$$

and for $1 \leqq n \leqq n_0(\varepsilon)$, since $x_n(e_n \wedge f_n(\varepsilon)) = x_{n_0}(e_{n_0} \wedge f_{n_0}(\varepsilon))(e_n \wedge f_n(\varepsilon))$,

$$\|x_n(e_n \wedge f_n(\varepsilon))\| < 2\varepsilon.$$

Therefore by [7, Theorem 5.3], $x-x' \in \bar{M}$ and $\|x-x'\|_\infty < 2\varepsilon$, that is, $x = x'$. Hence a limit n. e. is unique.

THEOREM 3.4. *If $\{x(n)\}_{n=1}^\infty$ and $\{y(n)\}_{n=1}^\infty$ are sequences of elements in C converging n. e. to x and y in C, respectively.* Then $\{x(n)+y(n)\}_{n=1}^\infty$ *converges to $x+y$ nearly everywhere.*

PROOF. For every positive number ε, there exist a positive integer $n_0(\varepsilon)$ and SDD's $\{e_n(\varepsilon)\}$, $\{f_n(\varepsilon)\}$ such that $(x(n)-x)[e_n(\varepsilon), 1]$, $(y(n)-y)[f_n(\varepsilon), 1] \in \bar{M}$, $\|(x(n)-x)[e_n(\varepsilon), 1]\|_\infty < \varepsilon$ and $\|(y(n)-y)[f_n(\varepsilon), 1]\|_\infty < \varepsilon$ for all $n \geqq n_0(\varepsilon)$. By the same reason as above, $\{e_n(\varepsilon) \wedge f_n(\varepsilon)\}$ is an SDD. It is plain that $(x(n)+y(n)-x-y)[e_n(\varepsilon) \wedge f_n(\varepsilon), 1] \in \bar{M}$ and $\|(x(n)+y(n)-x-y)[e_n(\varepsilon) \wedge f_n(\varepsilon), 1]\|_\infty < 2\varepsilon$ for all $n \geqq n_0(\varepsilon)$. This completes the proof.

LEMMA 3.2. *For any* SDD $\{e_n\}$ *and $x \in C$, $\{x^{-1}[e_n]\}$ (where $x^{-1}[e_n]$ is the largest projection in M right annihilating $(1-[e_n, 1])x$) is also an* SDD.

PROOF. Using [7, Theorem 6.3 and 6.4], we can prove the assertion by the same way as that used in [7, Lemma 3.1].

THEOREM 3.5. *Let $\{x(n)\}_{n=1}^\infty$ be a sequence of elements in C which converges n. e. to x in C. Suppose that there is a central projection e which is σ-finite with respect to the center such that $x(n)[1-e, 1] = 0$ for all n. Then there exists a strictly increasing subsequence $\{n_i\}$ of positive integers such that $\{x(n_i)^*\}_{i=1}^\infty$ converges n. e. to x^*.*

PROOF. First of all, we suppose $e = 1$, that is, Z is σ-finite. By the above theorem, we may assume $x=0$ without loss of generality. Since Z is σ-finite, there is a faithful positive normal measure μ on Ω (the spectrum of Z). Write $x(n)=[x_m(n), e_m(n)]$. First we note that the choice of $\{e_m(n)\}$ is independent on the index n. In fact, since $1-e_m(n)\downarrow 0$ $(m\to\infty)$ and $1-e_m(n)\in\mathfrak{M}$ it follows that $D(1-e_m(n))(\omega)\downarrow 0$ μ-a. e.$(m\to\infty)$. By Egoroff's theorem, there are a family $\{\Omega(m,n)\}_{n,m=1}^{\infty}$ of clopen subsets of Ω and a sequence $\{i(m,n)\}$ of positive integers such that

$$\mu(\Omega-\Omega(m,n))<(1/m)2^{-n}$$

and

$$D(1-e_{i(m,n)}(n))(\omega)<1/2^m \quad \omega\in\Omega(m,n)$$

for each pair of positive integers m and n. Moreover, $\Omega(m,n)\uparrow(m\uparrow)$ and $i(m,n)\uparrow\infty$ as m and $n\uparrow\infty$. Write $\Omega(m)=\left(\bigcap_{n=1}^{\infty}\Omega(m,n)\right)^i$, $\Omega(m)$ is a clopen set and by [1, Corollary of Proposition 6], we get

$$\begin{aligned}\mu(\Omega-\Omega(m)) &= \mu\left(\Omega-\bigcap_{n=1}^{\infty}\Omega(m,n)\right)\\ &<\sum_{n=1}^{\infty}\mu(\Omega-\Omega(m,n))\\ &<(1/m)\sum_{n=1}^{\infty}2^{-n}=1/m\,.\end{aligned}$$

Write $\Omega_0=\bigcup_{m=1}^{\infty}\Omega(m)$ and noting that $\Omega(m)\uparrow(m\uparrow)$, we have $\mu(\Omega-\overline{\Omega_0})=\mu(\Omega-\Omega_0)<\mu(\Omega-\Omega(m))<1/m$ for all m. The faithfulness of μ implies that $\Omega-\overline{\Omega_0}=\emptyset$, that is, Ω_0 is dense in Ω. If $\omega\in\Omega_0$, then $\omega\in\Omega(m_0)$ for some positive integer m_0. Since $\Omega(m)\uparrow(m\uparrow)$, it follows that $\omega\in\Omega(m)$ for all $m\geqq m_0$ and that

$$D(1-e_{i(m,m)}(m))(\omega)<1/2^m \text{ for all } \omega\in\Omega(m) \text{ and } m\geqq m_0\,.$$

Therefore,

$$\sum_{m=1}^{\infty}D(1-e_{i(m,m)}(m))(\omega)<\infty \text{ for all } \omega\in\Omega_0\,.$$

Putting $f_k=\bigwedge_{m=1}^{\infty}e_{ki(m,m)}(m)$, $f_k\in M_p$ and $f_k\uparrow(k\uparrow)$. Since $1-f_k=\bigvee_{m=1}^{\infty}(1-e_{ki(m,m)}(m))$, we get that $D(1-f_k)(\omega)\leqq\sum_{m=1}^{\infty}D(1-e_{ki(m,m)}(m))(\omega)<\infty$ on a dense open set Ω_0. Thus $1-f_k\in\mathfrak{M}_p$ and $D(1-e_{ki(m,m)}(m))\downarrow 0$ implies that $D(1-f_k)\downarrow 0$, that is, $1-f_k\downarrow 0$. Hence $\{f_k\}$ is an SDD. For any pair of positive integers j and k with $j\geqq k$, $x_{ki(m,m)}(m)f_k=x_{ki(m,m)}(m)e_{ki(m,m)}(m)f_k=x_{ji(m,m)}(m)e_{ki(m,m)}(m)f_k$ $=x_{ji(m,m)}f_k$ and similarly $(x_{ki(m,m)}(m))^*f_k=(x_{ji(m,m)}(m))^*f_k$. Therefore $\{x_{ki(m,m)}(m), f_k\}$ is an EMO and since $\{e_k(m)\}$ implements the equivalence of $\{x_k(m),e_k(m)\}$ and $\{x_{ki(m,m)}(m),f_k\}$, $x(m)=[x_{ki(m,m)}(m),f_k]$, which is the desired property. On the other hand, by the assumption, for any positive number ε, there are a positive integer $n_0(\varepsilon)$ and an SDD$\{e_n(\varepsilon)\}$ such that $x(n)[e_n(\varepsilon),1]\in\bar{M}$ and $\|x(n)[e_n(\varepsilon),1]\|_\infty$ $<\varepsilon$ for all $n\geqq n_0(\varepsilon)$. Write $f(n,k,\varepsilon)=f_k\wedge((x(n)^*)^{-1}[e_n(\varepsilon)])\wedge e_n(\varepsilon), D(1-f(n,k,\varepsilon))$ $\to 0$ as n and $k\to\infty$ on a dense set. Therefore by the same arguments as above, we can take a subsequence $\{n_i(\varepsilon)\}$ with $n_i(\varepsilon)\geqq n_0(1/i)$ for each i such that

$$\sum_{i=1}^{\infty}D(1-f(n_i(\varepsilon)\ n_i(\varepsilon),\varepsilon))(\omega)<\infty \text{ on a dense open set.}$$

Take $g_k(\varepsilon)=\bigwedge_{i=1}^{\infty}f(kn_i(\varepsilon)\ kn_i(\varepsilon),\varepsilon)$, then $\{g_k(\varepsilon)\}$ is an SDD. Since $D(1-g_k(1/n))$ $\downarrow 0$ as $k\to\infty$, again by the same arguments as above, there is a subsequence $\{k(n)\}$ of positive integers such that $\sum_{n=1}^{\infty}D(1-g_{k(n)}(1/n))(\omega)<\infty$ on a dense open set. Write $g_j=\bigwedge_{n=1}^{\infty}g_{jk(n)}(1/n)$ and $k_j=jn_j(1/j)k(j)$, noting that $\{g_j\}$ is an SDD it follows that

$$\begin{aligned}
x(k_j)^*[g_j,1] &= x(jn_j(1/j)k(j))^*[g_j,1]\\
&= x(jn_j(1/j)k(j))^*[g_{jk(j)}(1/j),1][g_j,1]\\
&= x(jn_j(1/j)k(j))^*[f(jk(j)n_j(1/j),jk(j)n_j(1/j),1/j),1][g_j,1]\\
&= x(jn_j(1/j)k(j))^*(((x(jk(j)n_j(1/j))^*)^{-1}[e_{jk(j)n_j(1/j)}(1/j),1][g_j,1]\\
&= (x(jn_j(1/j)k(j))[e_{jk(j)n_j(1/j)}(1/j),1])^*[g_j,1]\,.
\end{aligned}$$

Since $jn_j(1/j)k(j)\geqq n_j(1/j)\geqq n_0(1/j)$, we have $x(k_j)^*[g_j,1]\in\bar{M}$ and $\|x(k_j)^*[g_j,1]\|_\infty$ $<1/j$ for each j. Thus we get the result for the case $e=1$. To prove the assertion for the general case we argue as follows. First note that $x[1-e,1]=0$. In fact, for every positive number ε, writing $x=[x_n,e_n]$, there are a positive integer $n_0(\varepsilon)$ and an SDD $\{e_n(\varepsilon)\}$ such that

$$\|x_n(1-e)(e_n(\varepsilon)\wedge e_n)\|<\varepsilon \quad \text{for all} \quad n\geqq n_0(\varepsilon).$$

This implies that $x[1-e,1]=0$ and $x(n), x \in C[e,1] (\cong C(Me)$ by [7, Theorem 3.3]). Therefore by the above arguments, there exists a subsequence $\{n_i\}(n_i \uparrow \infty)$ of integers such that $x(n_i)^* \to x^*$ n.e. $(i \to \infty)$ in $C(Me)$, that is, for every positive number ε, we can take a positive integer $n_0(\varepsilon)$ and an SDD $\{e_n(\varepsilon)\}$ in Me such that for each $i \geqq n_0(\varepsilon)$,

$$(x(n_i)^* - x^*)[e_i(\varepsilon), 1] \in \overline{Me}$$

and

$$\|(x(n_i)^* - x^*)[e_i(\varepsilon), 1]\|_\infty < \varepsilon .$$

Now put $e_n(\varepsilon)' = e_n(\varepsilon) + 1 - e$ and $\{e_n(\varepsilon)'\}$ is an SDD in M and $(x(n_i)^* - x^*)[e_i(\varepsilon), 1] = (x(n_i)^* - x^*)[e_i(\varepsilon)', 1]$. This completes the proof.

THEOREM 3.6. *Let $\{x(n)\}_{n=1}^\infty$ be a sequence of elements in C which converges nearly everywhere to x in C and e be a central projection such that $x(n)[1-e,1]=0$ for each n and that it is σ-finite with respect to the center. Then for any y in C, there exist subsequences $\{n_i\}$ and $\{m_i\}$ of positive integers such that $x(n_i)y \to xy$ $(i \to \infty)$ and $yx(m_i) \to yx$ $(i \to \infty)$ nearly everywhere.*

PROOF. By the same reason as that used in the proof of the above theorem, we may assume $e=1$ without loss of generality. Now let $y=[y_n, f_n]$ then for every positive number ε, we can take a positive integer $n_0(j,\varepsilon)$ and an SDD $\{e_n(j,\varepsilon)\}$ such that

$$(x(n) - x)[e_n(j, \varepsilon), 1] \in \bar{M}$$

and

$$\|(x(n) - x)[e_n(j, \varepsilon), 1]\|_\infty < \varepsilon / \|y_j f_j\| \quad \text{for each} \quad n \geqq n_0(j, \varepsilon)$$

and for any positive integer j. For $\varepsilon = 1/j$, we denote $e_n(j, 1/j)$ by $e_n(j)$. Taking $f(i,n,j) = y^{-1}[e_n(j)] \wedge f_i$, $D(1-f(n,n,j)) \to 0$ as $n \to \infty$ on a dense set for each j. Therefore, by the same argument used in the proof of Theorem 3.5, there is a subsequence $\{n(j)\}(n(j) \geqq n_0(j, 1/j))$ of positive integers such that

$$\sum_{j=1}^{\infty} D(1 - f(n(j), n(j), j))(\omega) < \infty \quad \text{on a dense open set.}$$

Write $g_k = \bigwedge_{j=1}^{\infty} f(kn(j), kn(j), j)$,

$$D(1-g_k) \leqq \sum_{j=1}^{\infty} D(1-f(kn(j), kn(j)\ j)),$$

which implies that $D(1-g_k)(\omega) \downarrow 0$ except on a non-dense set and $\{g_k\}$ is an SDD. For each positive integer j_k which satisfies $j_k > kn(1)$, it follows that

$$\begin{aligned}(x(kn(j_k))-x)y[g_k, 1] &= (x(kn(j_k))-x)y[f(kn(j_k), kn(j_k), j_k), 1][g_k, 1] \\ &= (x(kn(j_k))-x)[e_{kn(j_k)}(j_k), 1]y[g_k, 1] \\ &= (x(kn(j_k))-x)[e_{kn(j_k)}(j_k), 1]y\ [f_{kn(1)}, 1][g_k, 1] \\ &= (x(kn(j_k))-x)[e_{kn(j_k)}(j_k), 1][y_{kn(1)}f_{kn(1)}, 1][g_k, 1].\end{aligned}$$

Since $kn(j_k) \geqq n_0(j_k, 1/j_k)$, we have that

$$(x(kn(j_k))-x)[e_{kn(j_k)}(j_k), 1] \in \bar{M}$$

and

$$\|(x(kn(j_k))-x)[e_{kn(j_k)}(j_k), 1]\|_\infty < 1/j_k\|y_{j_k}f_{j_k}\|.$$

On the other hand, $j_k > kn(1)$, which implies that $(x(kn(j_k))-x)y[g_k, 1] \in \bar{M}$ and $\|(x(kn(j_k))-x)y[g_k, 1]\|_\infty < 1/j_k$ for all k. The first half part of the result follows. By Theorem 3. 5 and the above result we can choose a subsequence $\{m_i\}$ of positive integers such that $yx(m_i) \to yx$ n. e. $(i \to \infty)$. This completes the proof of Theorem 3. 6.

4. A non-commutative theory of integration for a faithful semi-finite trace of M. First we show the existence of a faithful semi-finite trace on M, that is,

THEOREM 4. 1. *There exists a* $[0, \infty]$*-valued function* τ *on* M^+ *having the following properties*:

(1) *If* $a, b \in M^+$, *then* $\tau(a+b) = \tau(a) + \tau(b)$;

(2) *if* $a \in M^+$ *and* λ *is a positive number* $\tau(\lambda a) = \lambda \cdot \tau(a)$

(*we recall here* $0 \cdot +\infty = 0$ *by our conventions*);

(3) *if* $a \in M^+$ *and* $u \in M_u$, $\tau(u^*au) = \tau(a)$;

(4) $\tau(a) = 0$ $(a \in M^+)$ *implies* $a = 0$;

(5) *for any non-zero a in* M^+, *there is a non-zero b in* M^+ *majorized by a such that* $\tau(b)<\infty$;

(6) *let* $\{a_\alpha\}$ *be a directed increasing net of positive elements in M such that* $a_\alpha \uparrow a$ *in the* $\sigma(\mathfrak{S})$*-topology for some* $a\in M$, *then* $\tau(a_\alpha)\uparrow \tau(a)$.

REMARK. We call such a function a faithful semi-finite trace on M^+. A gauge space Γ is a pair $\{M,\tau\}$ composed of the AW^*-algebra M and a trace τ.

PROOF. By [1, Proposition 7(*a*)], there is a faithful normal semi-finite pseudo measure μ on Ω. Now we define $\tau(a)=\mu(\Phi(a))$ for $a\in M^+$, then it is plain that $\tau(\cdot)$ meets all requirements. This completes the proof.

Then by the same arguments used in the proof of Theorem 3.2, there are a two-sided ideal $\mathcal{E}$ whose positive part is $\{a\,;\, a\in M^+,\ \tau(a)<\infty\}$ and a linear non-negative functional $\dot\tau$ on $\mathcal{E}$ coincides with τ on $\{a\,;\, a\in M^+, \tau(a)<\infty\}$ with the following properties:

(a) $\dot\tau(xy)=\dot\tau(yx)$ if x or $y\in\mathcal{E}$, x and $y\in M$,

(b) $\dot\tau(u^*xu)=\dot\tau(x)$ if $x\in\mathcal{E}$ and $u\in M_u$.

Let $\mathcal{F}$ be the set $\{a\,;\, a\in M, \tau(LP(a))<\infty\}$, then $\mathcal{F}$ is a two-sided ideal contained in $\mathcal{E}$ such that $\mathcal{E}_p=\mathcal{F}_p$.

Now we define

DEFINITION 4.1. For $t\in\mathcal{F}$, we define $\|t\|_1=\operatorname{Sup}\{|\tau(st)|\,;\, s\in M,\ \|s\|\leqq 1\}$.

The function $t\to\|t\|_1$ deserves the name "norm", that is, it satisfies the following properties;

(i) $\|t\|_1\geqq 0$ for $t\in\mathcal{F}$ and $\|t\|_1=0$ if and only if $t=0$,

(ii) $\|s+t\|_1\leqq\|s\|_1+\|t\|_1$ if $t,s\in\mathcal{F}$,

(iii) $\|\alpha t\|_1=|\alpha|\,\|t\|_1$ where α is a complex number and $t\in\mathcal{F}$,

(iv) $\|t\|_1=\|t^*\|_1=\dot\tau(|t|)$ where $t=u|t|$ is the polar decomposition of t,

(v) if $v\in M$ and $t\in\mathcal{F}$, then $\|vt\|_1\leqq\|v\|\,\|t\|_1$ and $\|tv\|_1\leqq\|v\|\,\|t\|_1$.

In fact, the first half part of the statement (v) is clear from the definition of $\|t\|_1$. On the other hand, $\|tv\|_1=\text{Sup}\{|\dot{\tau}(stv)|, \|s\|\leqq 1, s\in M\}=\text{Sup}\{|\dot{\tau}(vst)|, \|s\| \leqq 1, s\in M\}\leqq \|v\|\|t\|_1$. Let $t=u|t|$ be the polar decomposition of $t(t\in \mathcal{F})$, then $u^*t=|t|$ and $|t|\in \mathcal{F}$. Therefore $\|u^*t\|_1\leqq \|t\|_1$ and $\||t|\|_1\leqq \|t\|_1$, which implies $\|t\|_1 =\||t|\|_1$. Since $\dot{\tau}$ is non-negative, by Schwarz' inequality, we have $|\dot{\tau}(s|t|)|^2 =|\tau(|t|^{1/2}s|t|^{1/2})|^2\leqq \tau(|t|)\cdot\tau(|t|^{1/2}s^*s|t|^{1/2})=\tau(|t|)\cdot\dot{\tau}(s^*s|t|)$. $\|s^*s\|\leqq 1$ implies that $|\dot{\tau}(s|t|)|^2\leqq \tau(|t|)\||t|\|_1$ and that $\||t|\|_1^2\leqq \tau(|t|)\||t|\|_1$. Therefore $\|t\|_1 =\||t|\|_1=\tau(|t|)$. Now $\|t^*\|_1=\||t|u^*\|_1\leqq \||t|\|_1=\|t\|_1$ and by symmetry it follows that $\|t\|_1=\|t^*\|_1$. If $\|t\|_1=0$, then $\dot{\tau}(|t|)=\tau(|t|)=0$. The faithfulness of τ implies $t=0$. It is easy to verify the remainder of the above assertions.

Now we are in the position to introduce the class of integrable elements in $\mathcal{C}$ via

DEFINITION 4.2. An element x in $\mathcal{C}$ is integrable if there exists a sequence $\{x(n)\}_{n=1}^{\infty}$ in $\mathcal{F}$ such that $[x(n), 1]\to x$(n. e.) and $\|x(n)-x(m)\|_1\to 0$ as n and $m\to\infty$. The integral of x, in symbol $\tilde{\tau}(x)$, is defined by $\tilde{\tau}(x)=\lim_{n\to\infty}\dot{\tau}(x(n))$. The set of all integrable elements in $\mathcal{C}$ is denoted by $L^1(\Gamma)$.

REMARK. Note first that the value $\tilde{\tau}(x)$ of the integral of x in fact exists and is finite and that it is uniquly determined by any particular such sequences. Since $|\dot{\tau}(x(n))-\dot{\tau}(x(m))|=|\dot{\tau}(x(n)-x(m))|\leqq\|x(n)-x(m)\|_1\to 0$ as n $m\to\infty$, lim $\dot{\tau}(x(n))$ exists and is finite. To prove the second statement, we argue as follows. Let $\{x(n)\}$ and $\{x(n)'\}$ be two sequences in $\mathcal{F}$ which converge n. e. to x in $\mathcal{C}$ and are L^1-Cauchy, that is, $\|x(n)-x(m)\|_1\to 0$ as $n, m\to\infty$ and $\|x(n)'-x(m)'\|_1\to 0$ as $n, m\to\infty$. Since $\{x(n)-x(n)'\}$ converges to 0 n. e. and is L^1-Cauchy, we have only to show the following statement: If $\{x(n)\}_{n=1}^{\infty}$ is an L^1-Cauchy sequence in $\mathcal{F}$ which converges to 0 n. e., then $\dot{\tau}(x(n))\to 0(n\to\infty)$. For every positive number δ, there is a positive integer $n_1(\delta)$ such that $|\dot{\tau}(x(n))-\dot{\tau}(x(n_1(\delta)))|<\delta$ for all $n\geqq n_1(\delta)$. Since $x(n_1(\delta))\in\mathcal{F}$, then $RP(x(n_1(\delta)))\in\mathcal{F}_p$. Therefore, $|\dot{\tau}(x(n)(1-RP(x(n_1(\delta)))))|<|\dot{\tau}(x(n) -x(n_1(\delta)))|+|\dot{\tau}((x(n_1(\delta))-x(n))RP(x(n_1(\delta))))|+|\dot{\tau}(x(n_1(\delta))(1-RP(x(n_1(\delta)))))|<2\delta$ for all $n\geqq n_1(\delta)$. Thus it suffices to show that $\dot{\tau}(x(n)p)\to 0$ as $n\to\infty$ for all $p\in\mathcal{F}_p$. In fact, for every positive number ε, there are a positive integer $n_2(\varepsilon)$ and an SDD$\{e_n(\varepsilon)\}$ such that $\|x(n)e_n(\varepsilon)\|<\varepsilon$ for any $n\geqq n_2(\varepsilon)$.

$$\begin{aligned} px(n) &= px(n)e_n(\varepsilon)+px(n)(1-e_n(\varepsilon)) \\ &= px(n)e_n(\varepsilon)+px(n)(1-e_n(\varepsilon))+p(x(n)-x(n_1))(1-e_n(\varepsilon)). \end{aligned}$$

Therefore,

$$|\dot{\tau}(p(x(n)-x(n_1))(1-e_n(\varepsilon)))|\leqq\|x(n)-x(n_1)\|_1<\delta \quad (n\geqq n_1(\delta)),$$

and

$$|\dot{\tau}(px(n_1)(1-e_n(\varepsilon)))| = |\dot{\tau}((1-e_n(\varepsilon))px(n_1))|$$
$$< \dot{\tau}(x(n_1)^*x(n_1))^{1/2}\cdot\dot{\tau}((1-e_n(\varepsilon))p(1-e_n(\varepsilon)))^{1/2}$$
$$= \dot{\tau}(x(n_1)^*x(n_1))^{1/2}\dot{\tau}(p(1-e_n(\varepsilon))p)^{1/2}.$$

Since $p(1-e_n(\varepsilon))p \downarrow 0$ as $n\to\infty$ in the $\sigma(\mathfrak{S})$-topology, $\dot{\tau}(p(1-e_n(\varepsilon))p)\to 0$. Therefore taking ε as $\varepsilon\|p\|_1<\delta$, we have $\dot{\tau}(p(1-e_n(\varepsilon))p)\leqq\delta^2(\dot{\tau}(|x(n_1)|^2)^{-1}$ for all $n\geqq n_3(\delta)$ for some positive integer $n_3(\delta)$. Combining the above estimations, it follows that $|\dot{\tau}(px(n))|\leqq 3\delta$ for all $n\geqq\max.(n_1,n_2,n_3)$. Thus $\tilde{\tau}$ is unambiguously defined. Morever by Theorem 3. 4 and the above results $\tilde{\tau}$ is linear on $L^1(\Gamma)$. Secondly we note that if $x\in\mathcal{E}$, then $[x,1]$ is integrable and its integral is equal to $\dot{\tau}(x)$. To prove this assertion, we argue as follows. We may suppose $x\geqq 0$ without loss of generality. Let u be the Cayley transform of x and $\{u\}''=C(\Omega)$ where Ω is the spectrum of $\{u\}''$. Then, noting that $[x,1]=i(1+[u,1])(1-[u,1])^{-1}$ in C, we have $[x,1](1-[u,1])=i(1+[u,1])$ and therefore $1+u\in\mathcal{E}$. Let $\Gamma_n=\{\gamma;$ $|u(\gamma)+1|>2/((r_n)^2+1)\}^-$ where $\{r_n\}$ is an increasing sequence of positive numbers such that $r_n>\|x\|$ and $r_n\uparrow\infty(n\to\infty)$, and f_n be the projection in $\{u\}''$ corresponding to the clopen set Γ_n of Ω. Then the function $\omega\in\Gamma_n\to(1+u(\omega))^{-1}$ is continuous. Therefore if we set $w_n(\omega)=(1+u(\omega))^{-1}(1-u(\omega))f_n$, $w_n\in M$, xw_n $=f_n$ and hence $f_n\in\mathcal{F}$. Write $x_n=f_nx$, $x_n\in\mathcal{F}$ and if $n>m$, then x_n-x_m $=x(f_n-f_m)\geqq 0$. $\|x_n-x_m\|_1=\dot{\tau}(xf_n)-\dot{\tau}(xf_m)$. The fact that $\dot{\tau}(xf_n)\uparrow(n\to\infty)$ and $\dot{\tau}(xf_n)\leqq\dot{\tau}(x)<\infty$ implies that $\{x_n\}_{n=1}^{\infty}$ is L^1-Cauchy. Noting that x_n-x $=x(f_n-1)$, $\|x_n-x\|<4r_n((r_n)^2+2)^{1/2}((r_n)^2+1)^{-2}\to 0$ as $n\to\infty$ and therefore $\dot{\tau}(x_n)$ $\to\dot{\tau}(x)(n\to\infty)$. This is the desired property.

Next we show

THEOREM 4. 2. *For any* $s\in L^1(\Gamma)$, $[s,1]t$, $t[s,1]$ *and* $t^*\in L^1(\Gamma)$. *Moreover,* $\tilde{\tau}([s,1]t)=\tilde{\tau}(t[s,1])$ *and* $\overline{\tilde{\tau}(t^*)}=\tilde{\tau}(t)$ (*where* $\bar{\alpha}$ *is the complex conjugate of a complex number* α).

PROOF. First note that we may assume that Z is σ-finite without loss of generality. In fact $t\in L^1(\Gamma)$, there is an L^1-Cauchy sequence $\{x(n)\}_{n=1}^{\infty}$ of elements in $\mathcal{F}$ which converges to t nearly everywhere. Let $e=\bigvee_{n=1}^{\infty}LP(x(n))\vee RP(x(n))$ and $z(e)$ be its central carrier. If $\{g_\lambda;\lambda\in\Lambda\}$ be a mutually orthogonal family of projections in Z such that $z(e)=\sum\{g_\lambda,\lambda\in\Lambda\}$ then $\tau(z(e)(LP(x(n))\vee RP(x(n))))$ $=\sum_{\lambda\in\Lambda}\tau(g_\lambda(LP(x(n))\vee RP(x(n))))$ and $\tau(z(e)(LP(x(n))\vee RP(x(n))))<\infty$ implies that the family of non-zero $g_\lambda(LP(x(n))\vee RP(x(n)))$ is at most countable for each n. Therefore we have that the set of non-zero $z(e)g_\lambda$ is at most countable. The assertion follows. Thus $(1-z(e))x(n)=0$ for all n implies $[1-z(e),1]t=0$.

Hence by Theorems 3.5 and 3.6, $\{x(n_i)s\}$ converges to $t[s,1]$ n.e., $\{sx(n_i)\}$ converges to $[s,1]t$ n.e. and $\{x(n_i)^*\}$ converges to t^* n.e. for some subsequence $\{n_i\}$ of positive integers. Moreover the sequences $\{x(n_i)s\}$, $\{sx(n_i)\}$, $\{x(n_i)^*\}$ are L^1-Cauchy, which implies $[s,1]t$, $t[s,1]$ and $t^* \in L^1(\Gamma)$ and by the definition of integral $\tilde{\tau}$ and Theorem 4.1, it follows that $\tilde{\tau}([s,1]t)=\tilde{\tau}(t[s,1])$ and $\tilde{\tau}(t^*)=\overline{\tilde{\tau}(t)}$. This completes the proof of the theorem.

THEOREM 4.3. *If p ($\in M_p$) is integrable, then $p \in \mathcal{F}$ and $\tilde{\tau}([p,1])=\tau(p)$.*

PROOF. First we suppose that 1 is integrable. We wish to show that M is a finite algebra. If $1 \sim q$, $q \in M_p$ and $q \lneqq 1$, then the semi-finiteness of τ implies that there is a non-zero projection g in $\mathcal{F}$ such that $g \leqq 1-q (q \neq 1)$. $1-g \geqq q \sim 1$ implies $1-g \sim 1$ and thus $1-g$ is also integrable. Since $\tilde{\tau}(1)=\tilde{\tau}([1-g,1])$, it follows that $\tilde{\tau}(1)=\tilde{\tau}([1-g,1])+\dot{\tau}(g)$ and $\tau(g)=0$. That is, $g=0$. This is a contradiction and the above statement is proved. Next we show that 1 can be represented as a sum of orthogonal central τ-finite projections. In fact, let p be any non-zero τ-finite projection, then $D(p) \neq 0$ and there are a non-zero central projection r and a positive integer n such that

$$2^{-n-1}r \leqq D(p)r \leqq 2^{-n}r .$$

Then there exists a set $\{p_i\}_{i=1}^{2^n}$ of orthogonal projections in M such that $p_1, p_2, \cdots, p_{2^n} \leqq r$, $p_i \sim pr$ for each i and $r-\sum_{j=1}^{2^n} p_j \precsim \sum_{j=1}^{2^n} p_j$ (see for example [8, Lemma 5.1]). $\dot{\tau}\left(\sum_{j=1}^{2^n} p_j\right)=2^n \cdot \dot{\tau}(pr) \leqq 2^n \cdot \dot{\tau}(p) < \infty$ and $\dot{\tau}\left(r-\sum_{j=1}^{2^n} p_j\right) < \infty$, which implies that $\tau(r)<\infty$. The finiteness of τ shows the above assertion. Since 1 is integrable, by the first paragraph of the proof of the above theorem, Z is σ-finite. Therefore there is an increasing sequence $\{p_r\}_{r=1}^{\infty}$ of τ-finite central projections such that $p_r \uparrow 1 (r \rightarrow \infty)$. By the assumption, there exists an L^1-Cauchy sequence $\{t(n)\}_{n=1}^{\infty}$ of elements in $\mathcal{F}$ such that $[t(n),1] \rightarrow 1$ nearly everywhere. Now let us consider the set $\{t(n)p_k, n=1,2,3,\cdots\}$, then it is L^1-Cauchy and converges to p_k n.e. for each k. Since $\lim_{n\to\infty} \dot{\tau}(t(n)p_k)=\tilde{\tau}([p_k,1])=\dot{\tau}(p_k)=\tau(p_k)$ for each k, there is a subsequence $\{n(k)\}$ of positive integers such that

$$\lim_{k\to\infty} |\dot{\tau}(t(n(k))p_k)-\tau(p_k)| = 0 .$$

On the other hand, $\tau(p_k) \uparrow \tau(1)$, therefore, in order to prove that $1 \in \mathcal{F}$ and $\tilde{\tau}(1)=\dot{\tau}(1)$, we have only to show that $\{t(n(k)p_k\}$ is L^1-Cauchy and converges to 1 n.e.. For every positive number ε, there is a positive integer $k_0(\varepsilon)$ such that for all k_1 and $k_2 \geqq k_0(\varepsilon)(k_1 \geqq k_2)$,

$$\|t(n(k_1)) - t(n(k_2))\|_1 < \varepsilon$$

and

$$\begin{aligned}
&\|t(n(k_1))p_{k_1} - t(n(k_2))p_{k_2}\|_1 \\
&< \|t(n(k_1))p_{k_1} - t(n(k_0))p_{k_1}\|_1 + \|t(n(k_0))(p_{k_1} - p_{k_2})\|_1 + \|(t(n(k_2)) - t(n(k_0)))p_{k_2}\|_1 \\
&< 2\varepsilon + \|t(n(k_0))(p_{k_1} - p_{k_2})\|_1 \\
&= 2\varepsilon + \dot{\tau}((p_{k_1} - p_{k_2})|t(n(k_0))|) \\
&= 2\varepsilon + \dot{\tau}(|t(n(k_0))|^{1/2}(p_{k_1} - p_{k_2})|t(n(k_0))|^{1/2}),
\end{aligned}$$

which implies that $\|t(n(k_1))p_{k_1} - t(n(k_2))p_{k_2}\|_1 \to 0 (k_1, k_2 \to \infty)$. Since $t(n(k)) \to 1$ n. e., for every positive number δ, there are a positive integer $n_0(\delta)$ and an SDD $\{e_n(\delta)\}$ such that $\|t(n(k)) - 1)e_k(\delta)\| < \delta$ for all $k > n_0(\delta)$. Write $f_k(\delta) = e_{n(k)}(\delta)p_k$, $\{f_k(\delta)\}$ is an SDD and $(t(n(k))p_k - 1)f_k(\delta) = t(n(k))p_k e_{n(k)}(\delta) - p_k e_{n(k)}(\delta) = (t(n(k)) - 1)p_k e_{n(k)}(\delta)$, which implies the desired result. Thus $\tilde{\tau}(1) = \lim_{k\to\infty} \dot{\tau}(t(n(k))p_k) = \lim_{k\to\infty} \tau(p_k) = \tau(1) < \infty$. For the general case, for an integrable projection p, $[p, 1]C[p, 1] \cong C(pMp)$, and set $\tau_p(a) = \tau(a)$ for $a \in (pMp)^+$, by the above arguments, we have $p \in \mathcal{F}(pMp)$. Since $\mathcal{F}(pMp) = p\mathcal{F}p$, then $p \in \mathcal{F}$ and by the definition of integral $\tilde{\tau}$, it follows that $\tau_p(1) = \tau(p) = \tilde{\tau}_p(1) = \tilde{\tau}([p, 1])$ and the result follows.

REMARK. For any $t \in L^1(\Gamma)$, we define $\|t\|_1 = \mathrm{Sup}\{|\tau([s, 1]t)|, s \in M, \|s\| \leqq 1\}$, Then the function $t \to \|t\|_1 (t \in L^1(\Gamma))$ satisfies actually the properties of a norm;

(a) $0 \leqq \|t\|_1 < \infty$ for $t \in L^1(\Gamma)$ and $\|t\|_1 = 0$ if and only if $t = 0$,

(b) $\|s + t\|_1 \leqq \|s\|_1 + \|t\|_1$ if $s, t \in L^1(\Gamma)$,

(c) $\|\alpha \cdot t\|_1 = |\alpha| \|t\|_1$ if $t \in L^1(\Gamma)$ and α is a complex number,

(d) $\|t\|_1 = \|t^*\|_1$,

(e) if $s \in M$, then $\|[s, 1]t\|_1 \leqq \|s\| \|t\|_1$ and $\|t[s, 1]\|_1 \leqq \|s\| \|t\|_1$,

In fact, if $\|t\|_1 = 0$, then $\tilde{\tau}([s, 1]t) = 0$ for all $s \in M$. Let $t = [w, 1]|t| (w \in M_{pi})$ be the polar decomposition of t (see [7, Theorem 6. 3]), $|t| = [w^*, 1]t$ and $\tilde{\tau}(|t|[s, 1]) = 0$ for all $s \in M$. Let $[u, 1]$ be the Cayley transform of t^*t and $|t| = [t_n, e_n]$ where $t_n, e_n \in \{u\}''$, $t_n e_n = t_n$ and $0 \leqq t_n \uparrow$ for all n, then by the same reason as that used in the proof of [7, Theorem 6. 3] we can choose for each $n, m = 1, 2, 3, \cdots$, positive elements $c_m^n (\in \{u\}'')$ and projecticns $e_m^n (\in \{u\}'')$ satisfying [7, Theorem 6. 3 (1)-(5)]. Moreover, $|t|[c_m^n, 1] = [e_m^n, 1]$ and $|t| \in L^1(\Gamma)$ implies $[e_m^n, 1] \in L^1(\Gamma)$. Therefore by Theorem 4. 3, $e_m^n \in \mathcal{F}_p$ and $0 = \tilde{\tau}(|t|[c_m^n\ 1]) = \tilde{\tau}([e_m^n, 1]) = \tau(e_m^n)$ for all

n and m. Since $e_m^n \uparrow RP(t)$, $\tau(RP(t))=0$, that is, $t=0$. The remainder of the above statements are verified by the same method as that used in the paragraph following Definition 4. 1.

THEOREM 4. 4. *The integral of a non-negative integrable element of C is non-negative.*

PROOF. Let t be an integrable non-negative element in C and $[u, 1]$ be its Cayley transform. Write $t=[t_n, e_n]$, where $0 \leqq t_n \uparrow$, $t_n e_n = t_n$ and $t_n, e_n \in \{u\}''$. Note that by [7, Theorem 5. 2], e_n is the projection corresponding to the clopen subset $\{\omega ; |u(\omega)-1| > 2((r_n)^2+1)^{-1/2}\}^-(=\Omega_n)$ of the spectrum Ω of $\{u\}''$ where $\{r_n\}$ is a strictly increasing sequence of positive numbers satisfying $r_n > \|x_n\|$ with $t=[x_n, g_n]$ and $r_n \uparrow \infty(n\to\infty)$. Let $\Gamma_n=\{\gamma ; |u(\gamma)+1| > 2((r_n)^2+1)^{-1}\}^-$, since the function $\gamma(\in\Gamma_n)\to(1+u(\gamma))^{-1}$ is continuous, setting f_n is the projection in $\{u\}''$ corresponding to the clopen set Γ_n and $w_n(\gamma)=(1+u(\gamma))^{-1}(1-u(\gamma))f_n e_n$, then $w_n \in M$ and $t[w_n, 1]=[w_n, 1]t=[e_n f_n, 1]$. $t \in L^1(\Gamma)$ implies that $e_n f_n \in \mathcal{F}_p$ by Theorem 4. 3. Let $t_n' = t_n f_n$, then $t_n' \geqq 0$ and $t_n' \in \mathcal{F}$ for each n. First, we show that $\{t_n'\}$ is L^1-Cauchy. In fact, since for each pair of positive integers m and n with $m<n, t_n'-t_m' = t_n f_n - t_m f_m = t_n f_n - t_n e_m f_m = t_n f_n(1-e_m f_m) \geqq 0$, it follows that $\|t_n'-t_m'\|_1 = \dot\tau(|t_n'-t_m'|) = |\dot\tau(t_n'-t_m')| = \dot\tau(t_n')-\dot\tau(t_m')$. On the other hand, $\tau(t_n')\uparrow(n\uparrow)$ and $|\dot\tau(t_n')| = |\tilde\tau(t[f_n e_n, 1])| \leqq \|t\|_1 < \infty$, which implies $\lim_{n\to\infty}\dot\tau(t_n')$ exists and is finite. Therefore $\{t_n'\}$ is L^1-Cauchy. Next we show that $\{[t_n', 1]\}$ converges to t n. e.. In fact, $[t_n', 1]-t = t[e_n f_n, 1]-t = t([e_n f_n - 1, 1])$, therefore $([t_n', 1]-t)[e_n, 1] = [t_n(e_n f_n - 1), 1]$. On the other hand if $\omega \in \Omega_n$, then $|(1-u(\omega))^{-1}| \leqq (1/2)((r_n)^2+1)^{1/2}$ and $t_n(\omega) = (1+u(\omega))(1-u(\omega))^{-1}$. Since $|1+u(\omega)| < 2/((r_n)^2+1)$ for $\omega \in (\Gamma_n)^c$, it follows that $\|t_n(1-f_n)\| \leqq 1/((r_n)^2+1)^{1/2}$ for all n. Thus $\{[t_n', 1]\}_{n=1}^{\infty}$ converges nearly everywhere to t. Hence $\tilde\tau(t) = \lim_{n\to\infty}\dot\tau(t_n') \geqq 0$. This completes the proof.

REMARK. If $\omega \in (\Gamma_n)^c$, then $|1-u(\omega)| \leqq 2((r_n)^2+1)^{-1/2} < 2$ and $1+u(\omega) \neq 0$. Let $y_n(\omega) = (u(\omega)+1)^{-1}(1-e_n)$, then $y_n \in M$ and $(1+u)y_n = y_n(1+u) = 1-e_n$. On the other hand $1+[u, 1] \in L^1(\Gamma)$, which implies $1-e_n \in \mathcal{F}$ for all n by Theorem 4.3.

COROLLARY. *Let $t \in L^1(\Gamma)$ with $t \geqq 0$, then there is an L^1-Cauchy sequence $\{t(n)\}_{n=1}^{\infty}$ of positive elements in $\mathcal{F}$ such that $t \geqq [t(n), 1]$ and $[t(n), 1] \to t(n\to\infty)$ nearly everywhere.*

PROOF. The assertion is clear from the proof of the above theorem.

THEOREM 4. 5. *Let $t \in C$ with $t \geqq 0$, $\tilde\tau(t) = \mathrm{Sup}\{\dot\tau(s), s \in \mathcal{F}, 0 \leqq [s, 1] \leqq t\}$,* *in the sense that if either side exists and is finite, then the other side exists and has the same value.*

PROOF. Suppose $t \in L^1(\Gamma)$ with $t \geqq 0$ and $s \in \mathcal{F}$ with $0 \leqq [s, 1] \leqq t$, then $\tilde{\tau}(t-[s,1]) \geqq 0$, which implies $\tilde{\tau}(t) \geqq \dot{\tau}(s)$. Therefore by the above corollary, $\tilde{\tau}(t) = \operatorname{Sup}\{\dot{\tau}(s), s \in \mathcal{F}\ 0 \leqq [s,1] \leqq t\}$. Conversely, let u be the Cayley transform of t, $\{e_n\}$ and $\{f_n\}$ be the families of projections used in the proof of Theorem 4. 4. Then $t[e_n, 1] - t[e_n, 1][f_m, 1] = [t_n, 1] - [t_n, 1][f_m, 1] = [t_n, 1][1 - f_m, 1] \geqq 0$. It is plain that for $\omega \in \Gamma_m$, $|1-u(\omega)| \leqq 2r_m((r_m)^2+2)^{1/2}((r_m)^2+1)^{-1}$ and for $\omega \in \Omega_n \cap \Gamma_m$ $|(1-u(\omega))^{-1}| \geqq ((r_m)^2+1)/2r_m((r_m)^2+2)^{1/2}$. Hence $t \geqq 1/r_m((r_m)^2+2)^{1/2}[f_m, 1]$ for each m. Note that $\tau(f_m) = \operatorname{Sup}\{\tau(p), p \in \mathcal{F}_p, p \leqq f_m\}$, $\tau(f_m) < \infty$, that is, $f_m \in \mathcal{F}$. Let $[t_n', 1] = t[e_n, 1][f_n, 1]$, then $t_n' \in \mathcal{F}$ and $0 \leqq [t_n', 1] \leqq t$. For each pair of integers m and n, with $m > n$, $\dot{\tau}(|t_m' - t_n'|) = \dot{\tau}(t_m' - t_n')$, therefore it follows that $\{t_n'\}$ is an L^1-Cauchy sequence. Since the clopen subset of Ω corresponding to the projection $1-f_n$ is contained in $\{\omega;\ |1+u(\omega)| \leqq 2/((r_n)^2+1)\}$ and $|1-u(\omega)|^{-1} \leqq ((r_n)^2+1)/2$ for $\omega \in \Omega_n$, making use of the functional representation we have

$$\|(t-[t_n', 1])[e_n, 1]\|_\infty < 1/((r_n)^2+1)^{1/2} \text{ for each } n.$$

This implies that $[t_n', 1] \to t$ n. e. and $t \in L^1(\Gamma)$. This completes the proof of Theorem 4. 5.

Theorem 4. 5 naturally leads us to the following

DEFINITION 4. 3. For $t \in C$ with $t \geqq 0$, we define $\tilde{\tilde{\tau}}(t) = \operatorname{Sup}\{\dot{\tau}(s), s \in \mathcal{F}, 0 \leqq [s, 1] \leqq t\}$.

Thus by the above theorem, if $t \in L^1(\Gamma)(t \geqq 0)$, $\tilde{\tau}(t) = \tilde{\tilde{\tau}}(t)$. Moreover, if s, $t \in C^+$, then it is plain that $\tilde{\tilde{\tau}}(s+t) = \tilde{\tilde{\tau}}(s) + \tilde{\tilde{\tau}}(t)$.

COROLLARY. *For any t in C, $\tilde{\tilde{\tau}}(t^*t) = \tilde{\tilde{\tau}}(tt^*)$.*

PROOF. Let $t = [v, 1]t$ be the polar decomposition of t ([7, Theorem 6. 3]), then $t^*t = |t|^2$ and $tt^* = [v, 1]|t|^2[v^*, 1] = [v, 1]t^*t[v^*, 1]$. The assertion is clear from Definition 4. 3.

THEOREM 4. 6. *For any non-negative element t in C, the following two statements are equivalent.*

(1) $t \in L^1(\Gamma)$.

(2) $[t_n, 1] \in L^1(\Gamma)$ *and* $\mathrm{Sup}\{\tilde{\tau}([t_n, 1]), n=1,2 \cdots\} < \infty$ *for some* $t = [t_n, e_n]$ $(t_n \geqq 0)$ *in Theorem* 5.2 *in* [7], *and in this case,* $\tilde{\tau}(t) = \mathrm{Sup}\{\tilde{\tau}([t_n, 1]), n = 1, 2, \cdots\}$.

PROOF. Let $t \in L^1(\Gamma)$ with $t \geqq 0$ and $t = [t_n, e_n]$ by the representation in Theorem 5.2 of [7], then $t \geqq [t_n, 1] \geqq 0$ and $\tilde{\tilde{\tau}}([t_n, 1]) < \infty$ implies $[t_n, 1] \in L^1(\Gamma)$ and $\mathrm{Sup}\{\tilde{\tau}([t_n, 1]), n = 1, 2, \cdots\} \leqq \tilde{\tau}(t) < \infty$. On the other hand, $\tilde{\tau}(t - [t_n, 1]) = \tilde{\tau}(t) - \tilde{\tau}([t_n, 1]) = \tilde{\tau}(t[1 - e_n, 1]) \to 0$ by Proposition 4.4. Therefore (1) implies (2). Conversely, suppose (2), then let f_n be the projection used in the proof of the above theorem, $f_n e_n \in \mathcal{F}_p$. Therefore write $t_n' = t_n f_n$, $\{t_n'\}_{n=1}^{\infty}$ is an L^1-Cauchy sequence which converges to t n.e.. Hence $t \in L^1(\Gamma)$ and the statement (1) follows.

The rest of our discussions in this section is devoted to the space $L^2(\Gamma)$ defined as follows.

DEFINITION 4.4. Let $L^2(\Gamma)(= \mathfrak{H}_\tau)$ be the set $\{t\,;\, t \in C, t^*t = |t|^2 \in L^1(\Gamma)\}$.

Then,

PROPOSITION 4.1. *If* $s, t \in \mathfrak{H}_\tau$, then $s^*t \in L^1(\Gamma)$ and $|\tilde{\tau}(s^*t)|^2 \leqq \tilde{\tau}(s^*s) \cdot \tilde{\tau}(t^*t)$.

PROOF. Suppose s and t are self-adjoint, then we have $(s-t)(s-t)^* \geqq 0$, $(s+t)(s+t)^* \geqq 0$, which implies $s^2 + t^2 \geqq ts + st \geqq -(s^2 + t^2)$ and $ts + st \in L^1(\Gamma)$. On the other hand, $(s+it)(s+it)^* \geqq 0$, $(s-it)(s-it)^* \geqq 0$, which shows $s^2 + t^2 \geqq i(st - ts) \geqq -(s^2 + t^2)$ and $st - ts \in L^1(\Gamma)$. Therefore, st and $ts \in L^1(\Gamma)$. For the general case, let $s = s_1 + is_2$ and $t = t_1 + it_2$ with s_1, s_2, t_1 and $t_2 \in C_{sa}$, then by the above argument, it follows that $s^*s \in L^1(\Gamma)$. Now for any pair of complex numbers λ and μ, $(\lambda x + \mu y)^*(\lambda x + \mu y) \geqq 0$, that is, $\lambda\bar{\lambda} x^*x + \bar{\lambda}\mu x^*y + \lambda\bar{\mu} y^*x + \bar{\mu}\mu y^*y \geqq 0$, therefore, it follows that $\bar{\lambda}\lambda\tilde{\tau}(x^*x) + \bar{\lambda}\mu\tilde{\tau}(x^*y) + \lambda\bar{\mu}\tilde{\tau}(y^*x) + \bar{\mu}\mu\tilde{\tau}(y^*y) \geqq 0$ and by the same way as that in the proof of Schwarz' inequality that $|\tilde{\tau}(x^*y)|^2 \leqq \tilde{\tau}(x^*x) \cdot \tilde{\tau}(y^*y)$. This completes the proof.

PROPOSITION 4.2. *For any* $t \in C$, *we define* $\|t\|_2 = (\tilde{\tilde{\tau}}(|t|^2))^{1/2}$. *Then* $\|t\|_2 = \mathrm{Sup}\{\|ts\|_1, \|s\|_2 \leqq 1, ts \in L^1(\Gamma)\}$. *Moreover* $L^2(\Gamma) = \{t \in C, \|t\|_2 < \infty\}$ *is a prehilbert space with respect to the norm* $\|\ \|_2$.

PROOF. First of all we note that it suffices to show the above statement for t with $t \geqq 0$. In fact, for $t \in C$, let $t = [w, 1]|t|$ be the polar decomposition of t. Suppose there exists a sequence $\{r_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ with $\|r_n\|_2 \leqq 1$ such that

$|t^*|[r_n, 1] \in L^1(\Gamma)$ for each n and $\| |t^*|[r_n, 1]\|_1 \to \| |t^*| \|_2 = \|t^*\|_2 = \|t\|_2$. Write $s_n = w^* r_n (\in \mathcal{F})$, it follows that $|t^*|[r_n, 1] = [w, 1]|t|[w^*, 1][r_n, 1] = [w, 1]|t|[s_n, 1] = t[s_n, 1] \in L^1(\Gamma)$ for each n and $\| |t^*|[r_n, 1]\|_1 = \|t[s_n, 1]\|_1 \to \|t\|_2 (n \to \infty)$. Note that $\|s_n\|_2 = (\tilde{\tau}([s_n^* s_n, 1]))^{1/2} = (\dot{\tau}(s_n^* s_n))^{1/2} = (\dot{\tau}(r_n^* w w^* r_n))^{1/2} \leqq \dot{\tau}(r_n^* r_n)^{1/2} = \|r_n\|_2 \leqq 1$. Therefore the above statement follows. Making use of the same notations as that used in the proof of the above theorem, let $t = [t_n, e_n]$ and $t_n' = t_n f_n$, then in order to prove the theorem, we have only to consider two cases: (a)$\|t_n'\|_2 < \infty$ for all n, and (b) there is an n_0 such that $\|t_n'\|_2 = \infty$. Suppose that (a) holds, since $[t_n', 1] \in L^2(\Gamma)$ and $t_n'^2 \geqq (1/r_n^2((r_n)^2+2)) f_n e_n$ for each positive integer n, it follows that $[f_n e_n, 1] \in L^1(\Gamma)$ and therefore $e_n f_n \in \mathcal{F}$. Thus $t_n' \in \mathcal{F}$ follows. Let $s_n = (1/\|t_n'\|_2) t_n'$ (we may assume $\|t_n'\|_2 \neq 0$ without loss of generality), then $s_n \in \mathcal{F}$ and $t[s_n, 1] = [t_n s_n, 1] \in L^1(\Gamma)$. It is plain that $\|s_n\|_2 = 1$ and $\|t[s_n, 1]\|_1 = \|t_n s_n\|_1 = (1/\|t_n'\|_2)\|t_n'\|_2^2 = \|t_n'\|_2 = (\tilde{\tau}([t_n', 1]))^{1/2}$. On the other hand $t_n'^2 \uparrow$ and $[(t_n')^2, 1] \leqq t^2$, it follows that $\tilde{\tau}(t_n'^2)^{1/2} \to \tilde{\tilde{\tau}}(t^2)^{1/2} = \|t\|_2^2 < \infty$ as $n \to \infty$. The result follows. Next if we suppose that (b) holds, then for all positive integers $n \geqq n_0$, there is a projection g_n in $\mathcal{F}$ such that $f_n e_n \geqq g_n$ and $\|g_n\|_2 = \tau(g_n)^{1/2} \geqq r_n((r_n)^2+2)^{1/2} n$. Write $s_n = (1/\|g_n\|_2) g_n (\in \mathcal{F})$, then $\|s_n\|_2 = 1$ and $t[s_n, 1] = [t_n' s_n, 1] \in \bar{\mathcal{F}} \subset L^1(\Gamma)$. Since $s_n t_n' s_n \geqq (1/r_n((r_n)^2 + 2)^{1/2})(s_n)^2 e_n f_n = (1/r_n((r_n)^2 + 2)^{1/2})(s_n)^2 = (1/r_n((r_n)^2 + 2)^{1/2}\|g_n\|_2^2) g_n \geqq 0$, therefore

$$\|t[s_n, 1]\|_1 \|s_n\| \geqq 1/r_n((r_n)^2 + 2)^{1/2} \text{ for each } n.$$

Note that $\|s_n\| = 1/\|g_n\|_2$, $\|t[s_n, 1]\|_1 \geqq n$ for all n and $\|t[s_n, 1]\|_1 \to \|t\|_2$ as $n \to \infty$. Therefore the first half part of the statement of Proposition 4.2 follows. The second part of the assertion is clear from the triangular inequality and Proposition 4.1. Thus $L^2(\Gamma)$ is a normed linear space with the property that $\|t^*\|_2 = \|t\|_2$ for all $t \in L^2(\Gamma)$. This completes the proof.

PROPOSITION 4.3. *$\bar{\mathcal{F}}(=\{[x, 1], x \in \mathcal{F}\})$ is norm-dense in $L^2(\Gamma)$ and $L^1(\Gamma)$, respectively. Moreover $\tilde{\tau}(xy) = \tilde{\tau}(yx)$ for each pair of elements x and y in $L^2(\Gamma)$.*

PROOF. Let $t \in L^2(\Gamma)$ and $t = [w, 1]|t|$ be the polar decomposition of t. First we show that it is sufficient to prove the statement for t with $t \geqq 0$. In fact, suppose for $t \in L^2(\Gamma)$ (resp. $\in L^1(\Gamma)$), there is a sequence $\{s_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\|[s_n, 1] - |t|\|_2 \to 0$ ($\|[s_n, 1] - |t|\|_1 \to 0$ resp.). Observe that by Proposition 4.2, for any $y \in M$, $t[y, 1]$ and $[y, 1]t \in L^2(\Gamma)$, $\|[y, 1]t\|_2 \leqq \|y\| \|t\|_2$ and $\|t[y, 1]\|_2 \leqq \|y\| \|t\|_2$, it follows that $[w, 1][s_n, 1](\in \mathcal{F}) \to t$ in $L^2(\Gamma)$(resp. in $L^1(\Gamma)$)$(n \to \infty)$ strongly. Let t be a non-negative element in $L^2(\Gamma)$ (resp. in $L^1(\Gamma)$) and u be the Cayley transform of t, then we can write $t = [t_n, e_n]$, $t_n, e_n \in \{u\}''$, $t_n \geqq 0$, $t_n \uparrow$ and $t_n e_n = t_n$ by Theorem 5.2 in [7]. Let $\{t_n'\}$ and $\{f_n\}$ be the sequences of positive elements and projections in $\{u\}''$ respectively, used in the proof of Theorem 4. 6. Observe that $t_n' \in \mathcal{F}$ and $t^2-[t_n'^2, 1]=[(t_m)^2(1-e_nf_n), e_m]=(t-[t_n', 1])^2$, it follows that $\|t-[t_n', 1]\|_2^2=\tilde{\tau}(t^2[1-e_nf_n, 1])$ (resp. $\|t-[t_n', 1]\|_1=\tilde{\tau}(t[1-e_nf_n, 1])$). Therefore by Proposition 4. 4, $\|t-[t_n', 1]\|_2\to 0$ (resp. $\|t-[t_n', 1]\|_1\to 0$) as $n\to\infty$. By the above argument and the properties of τ, it is easy to show that $\tilde{\tau}(xy)=\tilde{\tau}(yx)$ for all x and y in $L^2(\Gamma)$. This completes the proof.

To prove the completeness of the spaces $L^2(\Gamma)$ and $L^1(\Gamma)$, we need the following Proposition.

PROPOSITION 4. 4. *For $t\in C(t\geqq 0)$, let $\varphi(x)=\tilde{\tilde{\tau}}(t^{1/2}xt^{1/2})(x\in M^+)$(we call this functional φ an extended indefinite integral of t), then φ is completely additive on projections.*

PROOF. It is plain that φ is finitely additive on projections. Let $\{e_\mu\}$ be an indexed family of mutually orthogonal projections in M such that $e=\sum_\mu e_\mu$ for some projection e in M, then $\varphi(e)\geqq\sum_\mu\varphi(e_\mu)$. Therefore if uncountable many of the $\varphi(e_\mu)$ are positive, then $\varphi(e)\leqq\sum_\mu\varphi(e_\mu)$. Thus we have only to prove the statement for the case $\mu=i=1, 2, \cdots$. We show that $\varphi(e)\leqq\sum_{i=1}^\infty\varphi(e_i)$. Let s be in $\mathcal{F}$ such that $0\leqq[s, 1]\leqq[e, 1]t[e, 1]$, then $se=s$ and $s=\lim\sum_{i=1}^n s^{1/2}e_is^{1/2}$ in the $\sigma(\mathfrak{S})$-topology $=\lim\sum_{i=1}^n s^{1/2}e_is^{1/2}RP(s)$ in the $\sigma(\mathfrak{S})$-topology. Thus by the property of τ, we have

$$\dot{\tau}(s)=\lim_n\sum_{i=1}^n\dot{\tau}(s^{1/2}e_is^{1/2})$$

$$=\lim_n\sum_{i=1}^n\dot{\tau}(e_ise_i)$$

$$\leqq\lim_n\sum_{i=1}^n\tilde{\tilde{\tau}}([e_i, 1]t[e_i, 1])\quad([e_ise_i, 1]\leqq[e_i, 1]t[e_i, 1])$$

$$=\lim_n\sum_{i=1}^n\varphi(e_i)$$

$$=\sum_{i=1}^\infty\varphi(e_i)\,.$$

This completes the proof of Proposition 4. 4.

Now we are in the position to prove

THEOREM 4.7. *$L^1(\Gamma)$ (resp. $L^2(\Gamma)$) is a Banach space with respect to the norm $\| \ \|_1$ (resp. $\| \ \|_2$). In particular, $L^2(\Gamma)$ is a Hilbert space (denote it by $\mathfrak{H}_\tau$).*

PROOF. Let $\{s(n)\}$ be a $\| \ \|_p$ ($p=1,2$)-Cauchy sequence in $L^p(\Gamma)(p=1,2)$. We show that there is an element s in $L^p(\Gamma)$ such that $\|s(n)-s\|_p(p=1,2) \to 0(n\to\infty)$. Since $\bar{\mathcal{F}}$ is uniformly dense in $L^p(\Gamma)(p=1,2)$ and $\|t\|_p=\|t^*\|_p$ for all $t\in L^p(\Gamma)(p=1,2)$, we may assume that $\{s(n)\}\subset\mathcal{F}_{sa}$ (we write $s(n)=[s_n,1]$) without loss of generality. Moreover, we can suppose $\|[s_n,1]-[s_{n+1},1]\|_p<1/4^{np}(p=1,2)$. Now by the spectral theorem, there is a sequence $\{e_n\}_{n=1}^\infty$ of projections in M such that

$$\|(s_n-s_{n+1})e_n\|\leqq 2^{-n} \quad \text{for all } n$$

and

$$(s_n-s_{n+1})^2\geqq 2^{-2n}(1-e_n) \quad \text{for all } n.$$

Since $|s_n-s_{n+1}|\geqq 2^{-n}(1-e_n)$ and $s_n-s_{n+1}\in\mathcal{F}$, it follows that $1-e_n\in\mathcal{F}$ for each n. Observe that $1/4^n\geqq 1/4^{np}>\|[s_n,1]-[s_{n+1},1]\|_p\geqq 2^{-n}(\dot\tau(1-e_n))^{1/p}$ for $p=1$ and 2, write $g_n=\bigvee_{k=n}^{\infty}(1-e_k)$, $g_n\downarrow$ and $\dot\tau(g_n)\leqq\sum_{k=n}^{\infty}\dot\tau(1-e_k)<\sum_{k=n}^{\infty}1/4^k$. Therefore $\dot\tau(g_n)\downarrow 0$ as $n\to\infty$, that is, $g_n\downarrow 0$. Note that $e_n\geqq 1-g_n\geqq 1-g_k$ if $n\geqq k$, we have

$$\begin{aligned}\|(s_n-s_{n+1})(1-g_k)\| &= \|(s_n-s_{n+1})e_n(1-g_k)\|\\ &\leqq \|(s_n-s_{n+1})e_n\|\\ &\leqq 2^{-n} \quad \text{for all } n\geqq k.\end{aligned}$$

Let us consider the sequence $\{s_n(1-g_k)\}_{n\geqq k}$, then putting $a_n^{(k)}=(1-g_k)s_n(1-g_k)+(1-g_k)s_ng_k+g_ks_n(1-g_k)$ for each pair of positive integers n and k, it follows that $\{a_n^{(k)}\}\subset M_{sa}$ and

$$\begin{aligned}\|a_n^{(k)}-a_{n+1}^{(k)}\| &\leqq \|(1-g_k)(s_n-s_{n+1})(1-g_k)\|+\|(1-g_k)(s_n-s_{n+1})g_k\|\\ &\quad+\|g_k(s_n-s_{n+1})(1-g_k)\|\\ &\leqq 3\|(s_n-s_{n+1})(1-g_k)\|\leqq 3\cdot 2^{-n} \quad \text{for all } n\geqq k.\end{aligned}$$

Hence $\{a_n^{(k)}\}$ is a uniformly Cauchy sequence for each k, so that there exists for each positive integer k, a self-adjoint element $s^{(k)}$ in M such that for k, $a_n^{(k)} \to s^{(k)} (n \to \infty)$ uniformly. Observe that $s_n(1-g_k)=a_n^{(k)}(1-g_k) \to s^{(k)}(1-g_k)(n \to \infty)$ uniformly, it follows that $s_n(1-g_{k_2})=s_n(1-g_{k_1})(1-g_{k_2})$ if $k_1 \geqq k_2$ implies that $s^{(k_2)})(1-g_{k_2})=s^{(k_1)}(1-g_{k_2})$. Therefore note that $g_k \in \mathfrak{F}_p \subset \mathfrak{M}_p$, $\{s^{(k)}, 1-g_k\}$ is an EMO. Write $t=[s^{(k)}, 1-g_k]$, $t \in C_{sa}$. Since $(s_k - s^{(k)})(1-g_k)=s_k(1-g_k)-s^{(k)}(1-g_k)$, $\|([s_k, 1]-t)[1-g_k, 1]\|_\infty \leqq 2^{-k+1}$ for all k, so that $[s_n, 1] \to t$ n. e. $(n \to \infty)$. Now in case of $L^1(\Gamma)$, t is integrable by the above arguments. Let $s \in M$ with $\|s\| \leqq 1$, then $\|s_n s - s_m s\|_1 \leqq \|s_n - s_m\|_1$, which implies $\{s_n s\}$ is an L^1-Cauchy sequence and $[s_{n_i} s, 1] \to t[s, 1]$ n. e. $(i \to \infty)$ for some strictly increasing subsequence $\{n_i\}$ of positive integers. Let $[s_n, 1]-t=[w_n 1] | [s_n, 1]-t |$ be the polar decomposition of $[s_n, 1]-t$, then we have

$$\begin{aligned}\|[s_n 1]-t\|_1 &= \tilde{\tau}(|[s_n, 1]-t|) \\ &= \tilde{\tau}(|[s_n, 1]-t|[g_k, 1]) + \tilde{\tau}(|[s_n, 1]-t|(1-[g_k, 1])) \\ &= \tilde{\tau}(|[s_n, 1]-t|[g_k, 1]) + \tilde{\tau}([1-g_k, 1][w_n^*, 1]([s_n, 1] \\ &\quad - t)(1-[g_k, 1]))\end{aligned}$$

and since for any projection p in $\mathfrak{F}$ with $p \leqq 1-g_k$,

$$t[p, 1] = t[1-g_k, 1][p, 1] = [s^{(k)}(1-g_k), 1][p, 1] = [s^{(k)}p, 1],$$

it follows that for every $p \leqq 1-g_k$ with $p \in M_p$,

$$\begin{aligned}\tilde{\tau}([p, 1]|[s_n, 1]-t|[p, 1]) &= \tilde{\tau}([p, 1][w_n^*, 1]([s_n, 1]-t)[p, 1]) \\ &\leqq \lim_{m \to \infty} \tilde{\tau}([p, 1][w_n^*, 1][(s_n - s_m)p, 1]) \\ &\leqq \lim_{m \to \infty} \operatorname{Sup} \|s_n - s_m\|_1 .\end{aligned}$$

Now the complete additivity of the indefinite integral shows that $\|[s_n, 1]-t\|_1 \leqq \lim_{k \to \infty} \operatorname{Sup} \|s_n - s_k\|_1$, which implies the desired result. In case of $L^2(\Gamma)$, let $s \in \mathfrak{F}$ with $\|s\|_2 \leqq 1$, then $[s_{n_i} s, 1] \to t[s, 1]$ n. e. $(i \to \infty)$ for some strictly increasing subsequence $\{n_i\}$ of positive integers and $\{s_{n_i} s\}$ is L^1-Cauchy. Therefore $t[s, 1] \in L^1(\Gamma)$. Now the completeness of $L^1(\Gamma)$ implies there is an integrable element r such that $\|[s_{n_i} s, 1]-r\|_1 \to 0 (i \to \infty)$. By the same argument as that in case of $L^1(\Gamma)$, we can take a strictly increasing subsequence $\{m_i\}$ of positive integers such that $[s_{m_i} s, 1] \to r (i \to \infty)$ n. e.. Therefore $r=t[s, 1]$ and $\|t[s, 1]\|_1 = \|r\|_1 = \lim_{n \to \infty} \|s_n s\|_1 \leqq \lim_{n \to \infty} \operatorname{Sup} \|s_n\|_2 \|s\|_2 < \infty$ by Proposition 4. 1. Since $\overline{\mathfrak{F}}$ is strongly dense in $L^2(\Gamma)$,

$t\in L^2(\Gamma)$ and $\|t\|_2\leqq\lim\operatorname{Sup}\|s_n\|_2$ by Proposition 4.2. Next we show that $\|[s_n, 1]-t\|_2\to 0(n\to\infty)$. In fact, $\{s_n-s_m\}_{m=1}^{\infty}$ is an L^1-Cauchy sequence which converges n. e. to $[s_n, 1]-t(m\to\infty)$. By the same reason as above, it follows that $\|[s_n, 1]-t\|_2\leqq\lim_{m\to\infty}\operatorname{Sup}\|s_n-s_m\|_2$. The above statement follows and $L^2(\Gamma)=\mathfrak{H}_\tau$ is a Hilbert space. This completes the proof of Theorem 4.7.

5. Representation of *M*. Let $B(\mathfrak{H}_\tau)$ be the algebra of all bounded linear operators on $\mathfrak{H}_\tau$. For any $x\in M$, we define $\pi_1(x)a=[x, 1]a(a\in\mathfrak{H}_\tau)$ (resp. $\pi_r(x)a=a[x, 1]$), then we have $\|\pi_1(x)a\|_2\leqq\|x\|\,\|a\|_2$ and $\|\pi_r(x)a\|_2\leqq\|x\|\,\|a\|_2$. Therefore, it is easy to show that $\pi_1(x)$ and $\pi_r(x)\in B(\mathfrak{H}_\tau)$ for all $x\in M$. Moreover, it is immediate that

$$\pi_1(\lambda x+\mu y)=\lambda\pi_1(x)+\mu\pi_1(y),\ \pi_1(xy)=\pi_1(x)\pi_1(y),$$
$$\pi_1(x)^*=\pi_1(x^*),\ \pi_r(\lambda x+\mu y)=\lambda\pi_r(x)+\mu\pi_r(y),$$
$$\pi_r(xy)=\pi_r(y)\cdot\pi_r(x),\ \pi_r(x)^*=\pi_r(x^*)$$

for x, y in M and complex numbers λ, μ. On the other hand if $\pi_1(x)=0$ (resp. $\pi_r(x)=0$), then $[x, 1]a=0$ $(a[x, 1]=0)$ for all a in $\mathfrak{H}_\tau$. Since τ is semi-finite, there is an orthogonal set $\{e_\alpha\}$ of projections in $\mathcal{F}$ such that $\sum_\alpha e_\alpha=1$. Therefore $\overline{\mathcal{F}}\subset\mathfrak{H}_r$ implies that $xe_\alpha=0$ $(e_\alpha x=0)$ for all α. Hence by [5, Lemma 2.2], $x=0$. Therefore $\pi_1(\cdot)$(resp. $\pi_r(\cdot)$) is a $*$-isomorphism (resp. $*$-antiisomorphism) of M into $B(\mathfrak{H}_\tau)$. Let $\{g_i\}_{i\in I}$ be a set of mutually orthogonal projections of M with $e=\sum_{i\in I}g_i$, then

$$\begin{aligned}\|\pi_1(e)a-\sum_{i\in J}\pi_1(g_i)a\|_2^2&=\left\|\left[e-\sum_{i\in J}g_i, 1\right]a\right\|_2^2\\&=\tilde{\tau}\left(a^*\left[e-\sum_{i\in J}g_i, 1\right]a\right)\\&=\tilde{\tau}\left(aa^*\left[e-\sum_{i\in J}g_i, 1\right]\right)\end{aligned}$$

for any finite subset J of I. By Proposition 4.4, $\sum_{i\in J}\pi_1(g_i)\to\pi_1(e)$ strongly. Similarly, $\sum_{i\in J}\pi_r(g_i)\to\pi_r(e)$ strongly. Therefore $\pi_1(M)$ and $\pi_r(M)$ are AW^*-subalgebras of $B(\mathfrak{H}_\tau)$ in the sense of [6, 3. Definition].

THEOREM 5.1. *Let **M**(resp. **N**) be the weak closure of $\pi_1(M)$(resp. $\pi_r(M)$). Then **M** and **N** are von Neumann algebras such that **M**=**N**′ where **N**′ is the* *commutant of* $\boldsymbol{N}$ *in the sense of* [2, 1. 1].

To prove the above theorem, we need the following definition.

DEFINITION 5. 1. An element a in $\mathfrak{H}_\tau$ is said left bounded (resp. right bounded) if there exists an operator $\pi_1(a)$ (resp. $\pi_r(a)$) of $B(\mathfrak{H}_\tau)$ such that $\pi_1(a)b = \pi_r(b)a$ (resp. $\pi_r(a)b = \pi_1(b)a$) for all b in $\bar{M}\cap\mathfrak{H}_\tau$.

Now we sketch the proof after [2, Chapter 1, Section 5]. Let $[\boldsymbol{M}\mathfrak{H}_\tau]$ be the closed subspace of $\mathfrak{H}_\tau$ generated by the $T\xi$ $(T\in\boldsymbol{M}, \xi\in\mathfrak{H}_\tau)$, then $\bar{\mathcal{F}}$ is dense in $\mathfrak{H}_\tau$ and $\pi_1(M)\bar{\mathcal{F}}=\bar{\mathcal{F}}$, which implies $[\boldsymbol{M}\mathfrak{H}_\tau]=\mathfrak{H}_\tau$. Similarly, $[\boldsymbol{N}\mathfrak{H}_\tau]=\mathfrak{H}_\tau$. Therefore by [2, Chapter 1, Section 3, Corollary of Theorem 2], $\boldsymbol{M}$ and $\boldsymbol{N}$ are von Neumann algebras. By the definition of π_1 and π_r, it is plain that $\boldsymbol{M}\subset\boldsymbol{N}'$. Thus we have only to show the converse assertion. First we shall prove that if a is left bounded then for $T\in\boldsymbol{N}'$, Ta is also left bounded and $T\pi_1(a)=\pi_1(Ta)$. Let $\mathfrak{M}_1=\{\pi_1(a)$, a is left bounded$\}$, then $\mathfrak{M}_1$ is a left ideal of $\boldsymbol{N}'$. In fact, let a be a left bounded element of $\mathfrak{H}_\tau$, b and c be in $\bar{M}\cap\mathfrak{H}_\tau$, then

$$\begin{aligned}\pi_1(a)\cdot\pi_r(b)c &= \pi_1(a)cb = \pi_r(cb)a = \pi_r(b)\cdot\pi_r(c)a\\ &= \pi_r(b)\cdot\pi_1(a)c\,.\end{aligned}$$

Since $\bar{\mathcal{F}}\subset\bar{M}\cap\mathfrak{H}_\tau$ and $\bar{\mathcal{F}}$ is dense in $\mathfrak{H}_\tau$, $\pi_1(a)\in\boldsymbol{N}'$. If $T\in\boldsymbol{N}'$, then $T\pi_1(a)b = T\pi_r(b)a=\pi_r(b)Ta$ for all $b\in\bar{M}\cap\mathfrak{H}_\tau$. Therefore Ta is left bounded and $\pi_1(Ta) = T\pi_1(a)$. Similarly we have that if a is right bounded and if $T\in\boldsymbol{M}'$, then Ta is right bounded, $T\pi_r(a)=\pi_r(Ta)$ and that let $\mathfrak{M}_2=\{\pi_r(a)$, a is right bounded$\}$, then $\mathfrak{M}_2$ is a left ideal of $\boldsymbol{M}$. Let $\mathfrak{M}_3=\mathfrak{M}_1\cap\mathfrak{M}_1^*$ and $\mathfrak{M}_4=\mathfrak{M}_2\cap\mathfrak{M}_2^*$, then $\mathfrak{M}_3''\subset\boldsymbol{N}'$ and $\mathfrak{M}_4'\subset\boldsymbol{M}'$. Let T be in $\boldsymbol{N}'$ and T_1 be in $\mathfrak{M}_3'$, then for each pair of elements a and b in $\bar{M}\cap\mathfrak{H}_\tau$, $\pi_1(b)^*T\pi_1(a)\in\mathfrak{M}_3$. Therefore $T_1\pi_1(b)^*T\pi_1(a) =\pi_1(b)^*T\pi_1(a)T_1$. The semi-finiteness of τ implies that there is an increasing net $\{e_\alpha\}$ of projections in $\mathcal{F}$ such that $\pi_1(e_\alpha)\uparrow 1$ for the σ-topology. Since $\bar{\mathcal{F}}\subset\bar{M}\cap\mathfrak{H}_\tau$, $T_1T=TT_1$, that is, $T\in\mathfrak{M}_3'$. Therefore $\boldsymbol{N}=\mathfrak{M}_3'$ and similarly $\boldsymbol{M}'=\mathfrak{M}_4''$. Next we show that $\mathfrak{M}_3\subset\mathfrak{M}_4$. Let $\pi_1(a)\in\mathfrak{M}_3$ and $\pi_r(b)\in\mathfrak{M}_4$ then by the definition of $\mathfrak{M}_3$ and $\mathfrak{M}_4$, $\pi_1(a)^*=\pi_1(c)$ for some left bounded element c and $\pi_r(b)^*=\pi_r(d)$ for some right bounded element d. Observe that the inner product $(\ ,\ )_\tau$ of $\mathfrak{H}_\tau$ is defined by $(x,y)_\tau=\tilde{\tau}(y^*x)(x,y\in\mathfrak{H}_\tau)$, it follows that for each pair of elements x and y in $\mathfrak{H}_\tau\cap\bar{M}$, $(a,xy)_\tau=(c^*,xy)_\tau$, therefore $a=c^*$. Since $\bar{\mathcal{F}}\subset\bar{M}\cap\mathfrak{H}_\tau\subset\mathfrak{H}_\tau$, $\bar{M}\cap\mathfrak{H}_\tau$ is uniformly dense in $\mathfrak{H}_\tau$. Thus there exists a sequence $\{x_n\}$ in $\bar{M}\cap\mathfrak{H}_\tau$ such that $\|x_n-a\|_2\to 0(n\to\infty)$. $\|x_n-a\|_2=\|x_n^*-a^*\|_2=\|x_n^*-c\|_2\to 0(n\to\infty)$. Similarly there is a sequence $\{y_n\}$ in $\bar{M}\cap\mathfrak{H}_\tau$ such that $y_n^*\to b$ and $y_n\to d$ uniformly $(n\to\infty)$. Therefore we have

$$\begin{aligned}(\pi_1(a)\cdot\pi_r(b)x, y)_\tau &= (\pi_r(b)x, \pi_1(c)y)_\tau \\ &= (\pi_1(x)b, \pi_r(y)c)_\tau \\ &= \lim_{n\to\infty}(\pi_1(x)y_n^*, \pi_r(y)x_n^*)_\tau \\ &= \lim_{n\to\infty}(xy_n^*, x_n^*y)_\tau \\ &= \lim_{n\to\infty}\tilde{\tau}((x_n^*y)^*(xy_n^*)) \\ &= \lim_{n\to\infty}\tilde{\tau}(y^*x_nxy_n^*) \\ &= \lim_{n\to\infty}\tilde{\tau}((yy_n)^*x_nx) \\ &= \lim_{n\to\infty}(x_nx, yy_n)_\tau \\ &= \lim_{n\to\infty}(\pi_r(x)x_n, \pi_1(y)y_n)_\tau \\ &= (\pi_r(x)a, \pi_1(y)d)_\tau \\ &= (\pi_1(a)x, \pi_r(d)y)_\tau \\ &= (\pi_r(b)\cdot\pi_1(a)x, y)_\tau,\end{aligned}$$

which implies that $\pi_1(a)\cdot\pi_r(b)=\pi_r(b)\cdot\pi_1(a)$ and therefore $\boldsymbol{N}'=\mathfrak{M}_3''\subset\mathfrak{M}_4''=\boldsymbol{M}'=\boldsymbol{M}$. Thus we have $\boldsymbol{M}=\boldsymbol{N}$. This completes the proof.

REMARK. $\boldsymbol{M}=(\pi_r(M))'$ and $\boldsymbol{N}=(\pi_1(M))'$.

Now we are in the position to state

THEOREM 5.2. *$\pi_1(M)=\boldsymbol{M}$, that is, M is a semi-finite W^*-algebra.*

In order to prove the theorem, we need some lemmas.

LEMMA 5.1. *Let a be in $\mathfrak{F}$ and $\xi=[a,1](\xi\in\mathfrak{H}_\tau)$. Denote the orthogonal projection on the subspace $[\boldsymbol{M}'\xi]$ generated by $T\xi(T\in\boldsymbol{M}')$ by $P_{[\boldsymbol{M}'\xi]}$. Then $P_{[\boldsymbol{M}\xi]}\in\pi_1(M)$.*

PROOF. Since for any $b\in\mathfrak{F}$, $\pi_r(b)\xi=\pi_r(b)[a,1]=[a,1][b,1]=[ab,1]=\pi_1(a)[b,1]$, so that $\pi_r(b)\xi\in\mathrm{Range}(\pi_1(a))$ for all $b\in\mathfrak{F}$. Note that $\pi_r(\mathfrak{F})$ is σ-dense in $\pi_r(M)$. In fact, the semi-finiteness of τ implies that there is a set

$\{e_i\}_{i\in I}$ of orthogonal projections in $\mathcal{F}$ such that $\sum_{i\in I} e_i = 1$. For each finite subset J of I, $\|\pi_r\left(x\left(1-\sum_{i\in J} e_i\right)y\|_2^2 = \left(\pi_r\left(x\left(1-\sum_{i\in J} e_i\right)\right)y, \pi_r\left(x\left(1-\sum_{i\in J} e_i\right)\right)y\right)_\tau$ $= \left(y\left[x\left(1-\sum_{i\in J} e_i\right), 1\right], y\left[x\left(1-\sum_{i\in J} e_i\right), 1\right]\right)_\tau = \tilde{\tau}\left(y[x, 1]\left[1-\sum_{i\in J} e_i, 1\right][x^*, 1]y^*\right)$ for $y\in \mathfrak{H}_\tau$ and $x\in\mathcal{F}$. Therefore by Proposition 4. 4, the above statement follows. By the above argument and Theorem 5. 1, it follows that $[\boldsymbol{M}'\xi]\subset\overline{\mathrm{Range}(\pi_1(a))}$ (the strong closure of Range($\pi_1(a)$)). On the other hand, $\pi_1(a)[b, 1]=\pi_r(b)\xi\in[\boldsymbol{M}'\xi]$ for all $b\in\mathcal{F}$. The strong density of $\mathcal{F}$ in $\mathfrak{H}_\tau$ implies that $\overline{\mathrm{Range}(\pi_1(a))}\subset[\boldsymbol{M}'\xi]$, or $[\boldsymbol{M}'\xi]=\overline{\mathrm{Range}(\pi_1(a))}$. Next we show that $P_{[M'\xi]}=P_{\overline{\mathrm{Range}(\pi_1(a))}}=LP_{B(\mathfrak{H}_\tau)}(\pi_1(a))$ where $LP_{B(\mathfrak{H}_\tau)}(\pi_1(a))$is the left projection of $\pi_1(a)$ in the AW^*-algebra $B(\mathfrak{H}_\tau)$. By the definition of $LP_{B(\mathfrak{H}_\tau)}(\pi_1(a))$, it is plain that $P_{\overline{\mathrm{Range}(\pi_1(a))}}\geqq LP_{B(\mathfrak{H}_\tau)}(\pi_1(a))$. Let $\int_0^\infty \lambda dE_\lambda=\pi_1(aa^*)$ be the spectral decomposition of $\pi_1(aa^*)$, then $LP_{B(\mathfrak{H}_\tau)}(\pi_1(a))\pi_1(aa^*)$ $=\pi_1(aa^*)$ implies $LP_{B(\mathfrak{H}_\tau)}(\pi_1(a))\int_{1/n}^\infty dE_\lambda=\int_{1/n}^\infty dE_\lambda$ for each positive integer n. Therefore $LP_{B(\mathfrak{H}_\tau)}(\pi_1(a))\geqq P_{\overline{\mathrm{Range}(\pi_1(a))}}$, which is the desired property. Since $\pi_1(M)$ is an AW^*-subalgebra of $B(\mathfrak{H}_\tau)$, by [6, Lemma 2], $P_{[M'\xi]}=LP_{B(\mathfrak{H}_\tau)}(\pi_1(a))\in\pi_1(M)$. This completes the proof.

LEMMA 5. 2. *For any* ξ *in* $\mathfrak{H}_\tau$, *let* $\xi=[u, 1]|\xi|$ *be the polar decomposition of* $\xi(u\in M_{pi})$, *then* $P_{[M'\xi]}\sim P_{[M'|\xi|]}$ *and* $P_{[M'\xi]}=\pi_1(u)P_{[M'|\xi|]}\pi_1(u)^*$. *Therefore if* $P_{[M'|\xi|]}\in\pi_1(M)$, *then* $P_{[M'\xi]}\in\pi_1(M)$.

PROOF. First observe that $\xi=\pi_1(u)|\xi|$ and $|\xi|=\pi_1(u^*)\xi$. For any $b\in M$, $\pi_r(b)\xi=\pi_r(b)\pi_1(u)|\xi|=\pi_1(u)\pi_r(b)|\xi|$, Therefore $\pi_r(M)\xi=\pi_1(u)\pi_r(M)|\xi|$. If $\eta\in[\boldsymbol{M}'\xi]$, then there is a sequence $\{A_n\}_{n=1}^\infty\subset\boldsymbol{M}$ such that $A_n\xi\to\eta$ in $\mathfrak{H}_\tau$. Thus $A_n|\xi|=A_n\pi_1(u^*)\pi_1(u)|\xi|=A_n\pi_1(u^*)\xi=\pi_1(u^*)A_n\xi\to\pi_1(u^*)\eta(n\to\infty)$ in $\mathfrak{H}_\tau$, which implies that $\pi_1(u^*)\eta\in[\boldsymbol{M}'|\xi|]$ and $\pi_1(u)\pi_1(u^*)\eta\in\pi_1(u)[\boldsymbol{M}'|\xi|]$. On the other hand, since $\pi_1(u)\pi_1(u^*)A_n\xi=A_n\pi_1(u)\pi_1(u^*)\xi=A_n\xi$ and $\pi_1(u)\pi_1(u^*)A_n\xi\to\pi_1(u)\pi_1(u^*)\eta$ $(n\to\infty)$ in $\mathfrak{H}_\tau$, it follows that $\pi_1(u)\pi_1(u^*)\eta=\eta$ and $\eta\in\pi_1(u)[\boldsymbol{M}'|\xi|]$, that is, $[\boldsymbol{M}'\xi]\subset\pi_1(u)[\boldsymbol{M}'|\xi|]$. Conversely, since $\pi_r(M)\xi=\pi_1(u)\pi_r(M)|\xi|$, we have $[\boldsymbol{M}'\xi]$ $\supset\pi_1(u)[\boldsymbol{M}'|\xi|]$, that is, $[\boldsymbol{M}'\xi]=\pi_1(u)[\boldsymbol{M}'|\xi|]$. Next we show $\pi_1(RP(\xi))[\boldsymbol{M}'|\xi|]$ $=[\boldsymbol{M}'|\xi|]$([7, Theorem 6. 4]). In fact, for every b in M, $\pi_1(RP(\xi))\pi_r(b)|\xi|$ $=\pi_r(b)\pi_1(RP(\xi))|\xi|=\pi_r(b)[u^*u, 1]|\xi|=\pi_r(b)|\xi|$ by [7, Theorem 6. 3]. Therefore $\pi_1(RP(\xi))[\boldsymbol{M}'|\xi|]\subset[\boldsymbol{M}'|\xi|]$. On the other hand if $\eta\in\overline{\pi_1(RP(\xi))[\boldsymbol{M}'|\xi|]}$, then observe that $\pi_1(RP(\xi))\eta=\eta$ and $\pi_1(RP(\xi))(\boldsymbol{M}'|\xi|)$ is dense in $\pi_1(RP(\xi))[\boldsymbol{M}'|\xi|]$, then there is a sequence $\{A_n\}$ in $\boldsymbol{M}$ such that $\pi_1(RP(\xi))A_n|\xi|\to\eta(n\to\infty)$. $A_n|\xi|=A_n\pi_1(RP(\xi))|\xi|=\pi_1(RP(\xi))A_n|\xi|\to\eta(n\to\infty)$, so that $\eta=\pi_1(RP(\xi))\eta$

$\in \pi_1(RP(\xi))[\boldsymbol{M}'|\xi|]$. Therefore $\pi_1(RP(\xi))[\boldsymbol{M}'|\xi|]$ is closed linear subspace of $\mathfrak{H}_\tau$. It follows, from the fact that $\pi_1(RP(\xi))\cdot\pi_r(M)|\xi| = \pi_r(M)|\xi|$, that $\pi_1(RP(\xi[\boldsymbol{M}((' |\xi|]$ $= [\boldsymbol{M}'|\xi|]$. From the above arguments, an easy calculation shows that $\pi_1(\boldsymbol{u})P_{[M'|\xi|]}\pi_1(u^*)$ is a projection and $\pi_1(u)P_{[M'|\xi|]}\pi_1(u^*) \leqq P_{[M'\xi]}$. On the other hand, for all $\eta \in \mathfrak{H}_\tau$, $P_{[M'\xi]}\eta = \pi_1(u)P_{[M'|\xi|]}\zeta$ for some $\zeta \in \mathfrak{H}_\tau$, $P_{[M'|\xi|]}\zeta = P_{[M'|\xi|]}\pi_1(u^*)\pi_1(u)P_{[M'|\xi|]}\zeta$ $= P_{[M'|\xi|]}\pi_1(u^*)P_{[M'\xi]}\eta$, which implies that $P_{[M'\xi]}\eta = \pi_1(u)P_{[M'|\xi|]}\pi_1(u^*)P_{[M'\xi]}\eta$ and $P_{[M'\xi]} = \pi_1(u)P_{[M'|\xi|]}\pi_1(u)^*$. Therefore the proof is completed.

LEMMA 5. 3. *For every* $\xi \in \mathfrak{H}_\tau$, $P_{[M'\xi]} \in \pi_1(M)$.

PROOF. By Lemma 5. 2, we may assume $\xi \geqq 0$ without loss of generality. Let $[u, 1]$ be the Cayley transform of ξ. Write $\xi = [t_n, e_n]$, $t_n, e_n \in \{u\}''$ $0 \leqq t_n \uparrow$ and $t_n e_n = t_n$. Choosing a family $\{t_n', f_n$ where $t_n \geqq 0$, f_n is a projection$\}$ used in the proof of Proposition 4. 3, then $0 \leqq t_n' \uparrow$, $[t_n', 1] \leqq \xi$ and $t_n' \in \mathcal{F}$ for each n. Moreover write $\xi_n = [t_n', 1]$, $\|\xi_n - \xi\|_2 \to 0 (n \to \infty)$. By Lemma 5. 1 $P_{[M'\xi_n]} \in \pi_1(M)$ for each n. First we show that $P_{[M'\xi_n]} \uparrow$ and $P_{[M'\xi_n]} \leqq P_{[M'\xi]}$. In fact, $\xi_n = [t_n, 1]$ $= [e_n f_n t_{n+1} e_{n+1} f_{n+1}, 1] = \xi_{n+1}[e_n f_n, 1] = \pi_r(e_n f_n)\xi_{n+1}$, which implies $[\boldsymbol{M}'\xi_n] \subset [\boldsymbol{M}'\xi_{n+1}]$. Similarly, $\xi_n = [t_n', 1] = \pi_r(e_n f_n)\xi$ and $[\boldsymbol{M}'\xi_n] \subset [\boldsymbol{M}'\xi]$. Next we show $\bigvee_{n=1}^{\infty} P_{[M'\xi_n]}$ $= P_{[M'\xi]}$ in $\boldsymbol{M}$. If there is a non-zero projection Q in $\boldsymbol{M}$ such that $P_{[M'\xi]} - \bigvee_{n=1}^{\infty} P_{[M'\xi_n]}$ $\geqq Q$, then $QP_{[M'\xi_n]} = 0$. Therefore $Q\xi_n = 0$ for each n, so that $Q\xi = 0$, that is, $Q[\boldsymbol{M}'\xi] = 0$. Hence this is a contradiction. Thus $P_{[M'\xi_n]} \uparrow P_{[M'\xi]}$ weakly $(n \to \infty)$ in $\boldsymbol{M}$. By [3, Lemma 2], $P_{[M\xi]} \in \pi_1(M)$. This completes the proof.

PROOF OF THEOREM 5. 2. It is sufficient to show that $\pi_1(M)_p = \boldsymbol{M}_p$ (see for example [3, Lemma 1]). Suppose then $P \in \boldsymbol{M}_p$. For any $\xi \in P\mathfrak{H}_\tau$, $\pi_r(b)P\xi$ $= P\pi_r(b)\xi \in P\mathfrak{H}_\tau$ for all $b \in M$, $[\boldsymbol{M}\xi] \subset P\mathfrak{H}_\tau$, that is, $P_{[M'\xi]} \leqq P$. Let $\{P_{[M'\xi]}\}$ be a maximal family of orthogonal cyclic projections majorized by P, then $P = \sum_{\xi} P_{[M'\xi]}$ in $\boldsymbol{M}$. By Lemma 5. 3, $P_{[M'\xi]} \in \pi_1(M)$, so that by [3, Lemma 2], $P \in \pi_1(M)$. This completes the proof.

Making use of this theorem, we show the following

THEOREM 5. 3([6]). *Let M be an AW^*-algebra of type* I *whose center Z is a W^*-algebra, then M is a W^*-algebra of the same type.*

PROOF. It is sufficient to show that M has a separating set of c. a. states. Since Z is a W^*-algebra and by [5, Lemma 4. 8] M can be represented as a direct sum of homogeneous AW^*-algebras and we may assume that M is a homogeneous AW^*-algebra whose center Z is σ-finite without loss of generality. By the structure theorem, there are an abelian projection e_0 with $z(e_0)=1$ and a family of orthogonal projections $\{e_i\}_{i\in I}$ containing e_0 such that $e_i \sim e_0 (i\in I)$ and $\sum_{i\in I} e_i = 1$. By Lemma 2.3, $e_0 M e_0 = Z e_0 \cong Z$. Let ϕ be an inverse map of the map $a(\in Z) \to a e_0 (\in Z e_0)$ and $\Phi(a) = \sum_\alpha \phi((v_\alpha)^* a v_\alpha)$ in $\boldsymbol{Z}$ for $a \in M^+$ where v_α is a partial isometry such that $(v_\alpha)^* v_\alpha = e_0$, $(v_\alpha)(v_\alpha)^* = e_\alpha (\alpha \in I)$. Then we have

(1) $\Phi(\lambda a + \mu b) = \lambda\Phi(a) + \mu\Phi(b)$ if $a, b \in M^+$ and λ and μ are complex numbers,

(2) $\Phi(as) = s\Phi(a)$ for $s \in Z^+$ and $a \in M^+$,

(3) if $u \in M_u$ and $a \in M^+$, then $\Phi(uau^*) = \Phi(a)$,

(4) if $\Phi(a) = 0$ with $a \in M^+$, then $a = 0$,

(5) let $\{f_\beta\}$ be a mutually orthogonal projections in M with $f = \sum_\beta f_\beta$ then $\Phi(f) = \sum_\beta{}' \Phi(f_\beta)$ in $\boldsymbol{Z}$,

(6) for every a in M^+, there is a non-zero b in M^+ with $\Phi(b) \in Z^+$.

In fact, the assertions (1) and (4) are clear from the definition of Φ. First we show the statement (2). Since $(v_\alpha)^* a s v_\alpha = s(v_\alpha)^* a v_\alpha$, it follows that $\phi((v_\alpha)^* a s v_\alpha) = \phi(s(v_\alpha)^* a v_\alpha) = \phi(s e_0 e_0 (v_\alpha)^* a v_\alpha) = \phi(s e_0)\cdot\phi((v_\alpha)^* a v_\alpha) = s\phi((v_\alpha)^* a v_\alpha)$. Hence by Lemma 2.12, $\Phi(as) = s\Phi(a)$. To prove the statement (3), we argue as follows. Since for $u \in M_u$ and $a \in M^+$, $uau^* = ua^{1/2}a^{1/2}u^*$ and $(v_\alpha)^* uau^* v_\alpha = (v_\alpha)^* ua^{1/2}a^{1/2}u^* v_\alpha$, by [6, Lemma 7], it follows that $(v_\alpha)^* uau^* v_\alpha \in Ze_0$ and $(v_\alpha)^* uau^* v_\alpha = \sum_\beta (v_\alpha)^* ua^{1/2} e_\beta a^{1/2} u^* v_\alpha$ in Ze_0 for each α. Therefore by the same way as that used in Theorem 3.1, $\Phi(uau^*) = \Phi(a)$. To prove the statement (5), let $\{f_\beta\}$ be a mutually orthogonal projections in M with $f = \sum_\beta f_\beta$. Again by [6, Lemma 7], $(v_\alpha)^* f v_\alpha \in Ze_0$ and $(v_\alpha)^* f v_\alpha = \sum_\beta (v_\alpha)^* f_\beta v_\alpha$ in Ze_0. Thus by the same reason as above, the statement (5) follows. Now we show the last assertion (6). Let a be a non-zero element in M^+, then there are a positive number α and a non-zero projection p in M such that $a \geqq \alpha p$. Then we can easily choose a non-zero abelian projection f in M such that $f \leqq p$ and $f \precsim e_0$. By lemma 2.4, $\Phi(f) \leqq \Phi(e_0)$. Write $b = \alpha f$, b satisfies all requirements.

Next let $\mathfrak{P} = \{s \in M^+, \Phi(s) \in Z^+\}$, then $\mathfrak{P}$ is the positive part of a two-sided ideal $\mathfrak{N}$. By the same way as that used in the proof of Theorem 3.2, there is a unique linear operation $\dot{\Phi}$ on $\mathfrak{N}$ to Z which coincides with Φ on $\mathfrak{P}$ satisfying (a), (b), (c) and (e) in Theorem 3.2. Moreover this operation satisfies: (d′)

if $\{f_\beta\}$ be a set of mutually orthogonal projections in M, then $\sum_\beta \Phi(f_\beta) = \Phi\left(\sum_\beta f_\beta\right)$. Let μ be a faithful positive normal measure on Z, then set $\sigma a(x) = \mu(\Phi(ax))$ for $a \in \mathfrak{P}$ and $x \in M$ and we have by [6, Lemma 7] $\sigma a(f) = \sum_\beta \sigma a(f_\beta)$. An easy computation shows that $\{\sigma a, a \in \mathfrak{P}\}$ is a separating set of positive c.a. functionals on M. This completes the proof of Theorem 5.3.

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