A non-Gaussian Ornstein-Uhlenbeck process for electricity spot price modeling and derivatives pricing

Thilo Meyer-Brandis
Center of Mathematics for Applications / University of Oslo

Based on joint work with

F.E. Benth, *CMA / University of Oslo*
J. Kallsen, *TU Munich*

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Electricity markets

Electricity exchanges organize trade in:

- ▶ Hourly spot electricity, next-day delivery
- ▶ Financial forward/futures contracts
- ▶ European options on forwards

In particular, from a mathematical finance point of view:

- ▶ Spot electricity: non-storability of electricity renders market highly incomplete, underlying not tradable
- ▶ Forwards: Delivery of electricity over period of weeks/months/quarters of year rather than fixed delivery

Electricity price modelling

Two categories of approaches for electricity price modelling:

1. Direct modelling of futures prices
   - Transfer of concepts from interest rate theory (HJM approach) (Clewlow & Strickland 1999, Manoliu & Tompaidis 2002, Benth & Koekebakker 2005)
   - Advantage: complete market and risk neutral pricing machinery available
   - Problem: no inference about spot prices possible (arbitrage relations not valid)

Electricity price modelling

2. Spot price modelling

- Various structured OTC products depend on spot evolution $\rightarrow$ spot price model required
- Use spot price model to derive prices of futures (and other derivatives)
- breakdown of spot-futures relationship $\rightarrow$ identification of market price of risk (pricing measure) necessary to derive futures prices

In this work we want to introduce a model of the second category, i.e. a new electricity spot price model.

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Essential features of spot prices

- Daily NordPool system price 01.1997-01.2001
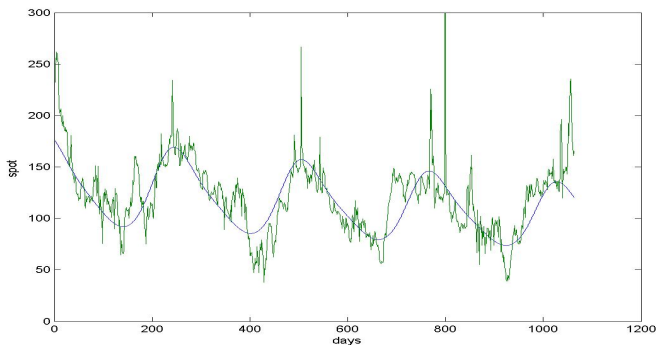
Essential features of spot prices

Stylized features of electricity spot prices are:

- ▶ mean reversion
- ▶ seasonality
  - ▶ yearly price cycle (in examples above winter has higher prices than summer)
  - ▶ weekly seasonality
  - ▶ intra-daily cycles
- ▶ intrinsic feature of sharp spikes followed by sharp drops (leptokurtic returns)
- ▶ level dependent volatility

Modelling requirements of spot dynamics

A spot price model should

- ▶ reflect statistics and path properties of historical data
- ▶ reflect physical conditions and constraints
- ▶ but also allow for sufficient analytical tractability:
  - ▶ risk evaluation
  - ▶ forward/futures price dynamics
  - ▶ option pricing

In particular, analytical pricing of forwards and futures is very desirable.

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Common spot price models

Most common reduced form spot price models are of exponential Ornstein-Uhlenbeck type

- ▶ guarantees positive prices
- ▶ enhances robustness of calibration procedure

However

- ▶ Is the exponential structure the right transformation for electricity prices?
  - ▶ exponential structure originates from population growth modelling (in finance compound interest modelling)
- ▶ Most importantly, no manageable analytic expressions for corresponding forward/futures contracts!

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An arithmetic model

Benth, Kallsen, M.-B. (Appl. Math. Fin. 2006):

We propose to model the spot price as a *sum of non-Gaussian OU-processes*:

$$S(t) = \Lambda(t) + \sum_{i=1}^{n} Y_i(t)$$

where

$$dY_i(t) = -\lambda_i Y_i(t)\, dt + \sigma_i(t)\, dL_i(t)$$

- $L_i(t)$ are independent *increasing* time inhomogeneous pure jump Lévy processes (additive processes).
- We suppose a Lévy measure of $L_i(t)$ of the form

$$\nu_i(dt, dz) = \rho_i(t)\, dt\, \nu_i(dz)$$

where $\rho_i(t)$ controls seasonal variation of jump intensity.

- $\sigma_i(t)$ controls seasonal variation of jump sizes
- λ_i different level of mean reversion
- $\Lambda(t)$ deterministic seasonality function

$\rightarrow$ The model guarantees positive prices because the $L_i(t)$'s are increasing.

$\rightarrow$ Upward jumps are followed by downward drops whose sharpness is controlled by the corresponding λ_i.

$\rightarrow$ The model allows for analytical pricing of corresponding forward and fututres contracts.

Pricing of forward/futures contracts

- Let $F(t; T_1, T_2)$ be time t forward price of a contract which delivers electricity at a rate $S(t)/T_2 - T_1$ during the settlement period $[T_1, T_2]$:

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u)\, du\,.$$

- Forward price defined so that time t value is zero, given information about the spot price up to time t:

$$F(t; T_1, T_2) = \mathbb{E}_Q \left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u)\, du \,|\, \mathcal{F}_t \right],$$

where Q is a pricing measure to be determined.

Proposition:

$$F(t; T_1, T_2) = F(0; T_1, T_2) +$$
$$\sum_{i=1}^{n} \frac{1}{\lambda_i(T_2 - T_1)} \int_0^t \sigma_i(s)(e^{-\lambda_i(T_1 - s)} - e^{-\lambda_i(T_2 - s)}) d\bar{L}_i(s) \,,$$

where

$$F(0; T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \{\Lambda(u) +$$
$$+ \sum_{i=1}^{n} \left(y_i e^{-\lambda_i u} + \int_0^u \int_{\mathbb{R}^+} \sigma_i(s) e^{-\lambda_i(u-s)} z \hat{\nu}_i(dz, ds) \right) \right\} du$$

and $\bar{L}_i(t)$ is the compensated jump process with compensating measure $\hat{\nu}_i(dz, ds)$ under Q.

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Pricing of forward/futures contracts

Or, expressed in terms of the components Y_i:

Proposition:

$$F(t; T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \Lambda(u) du + \Theta(t, T_1, T_2)$$
$$+ \sum_{i=1}^{n} \frac{e^{-\lambda_i(T_1-t)} - e^{-\lambda_i(T_2-t)}}{\lambda_i(T_2 - T_1)} Y_i(t),$$

where

$$\Theta(t, T_1, T_2) = \sum_{i=1}^{n} \int_{T_1}^{T_2} \int_{t}^{u} \int_{\mathbb{R}^+} \sigma_i(s) e^{-\lambda_i(u-s)} z \, \hat{\nu}_i(dz, ds) \, du$$

and $\hat{\nu}_i(dz, ds)$ is the Lévy measure of $L_i(t)$ under the pricing measure Q.

Pricing of options on forward/futures contracts

▶ Some notations:

$$\Sigma_i(t, T_1, T_2) = \frac{\sigma_i(t)}{\lambda_i(T_2 - T_1)} \left(e^{-\lambda_i(T_1 - t)} - e^{-\lambda_i(T_2 - t)} \right).$$

$$\widetilde{\psi}^i_{t,T}(\theta) := \ln \mathbb{E}_Q \left[\exp(i \int_t^T \theta(s) dL_i(s)) \right]$$
$$= \int_t^T \int_0^\infty \left\{ e^{i\theta(s)z} - 1 \right\} \widehat{\nu}_i(dz, ds)$$

▶ Let $g \in L^1(\mathbb{R})$ be payoff of an option written on $F(T; T_1, T_2)$, $T \leq T_1$. Then the price is given by

$$p(t; T; T_1, T_2) = e^{-r(T-t)} \mathbb{E}_Q \left[g(F(T; T_1, T_2)) \, | \, \mathcal{F}_t \right].$$

Propositions:

If $g(F(T, T_1, T_2)) \in L^1(Q)$, then we have that

$$p(t; T; T_1, T_2) = e^{-r(T-t)} \left(g \star \Phi_{t,T}\right)\left(F(t; T_1, T_2)\right)$$

where the function $\Phi_{t,T}$ is defined via its Fourier transform

$$\widehat{\Phi}_{t,T}(y) = \exp\left(\sum_{i=1}^{n} \widetilde{\psi}_{t,T}^{i}(y\Sigma_i(\cdot, T_1, T_2))\right),$$

and $\star$ is the convolution product.

- Numerical pricing by fast Fourier transform techniques.
- Not available for exponential models in this explicit form.
- Exponential damping for payoffs not in $L^1(\mathbb{R})$ (see e.g. Carr & Madan 1999.

Case study: simulation of the NordPool spot
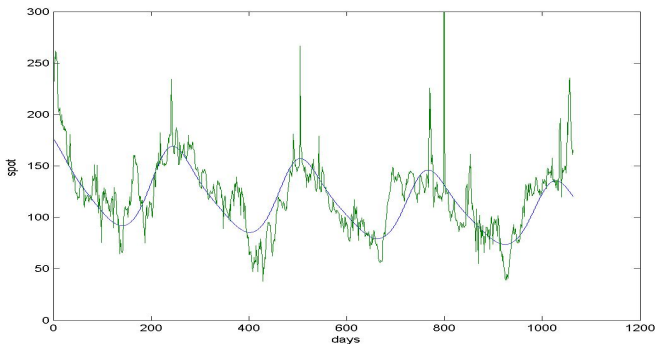
We want to fit the model

$$S(t) = \Lambda(t) + \sum_{i=1}^{n} Y_i(t)$$

$$dY_i(t) = -\lambda_i Y_i(t)\, dt + \sigma_i(t)\, dL_i(t)$$

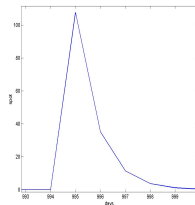# Case study: simulation of the NordPool spot

In order to fit the model to the time series of daily Nordpool spot price given above, we proceed in four steps:

1. Identification of the first OU-process $Y_1(t)$ modelling the seaonal spikes.

2. We remove the spikes from the spot series and fit a deterministic seasonal mean $\Lambda(t)$ of cosines to the remaining time series.

3. We remove the seasonal mean and fit a sum $\sum_{i=2}^{n} Y_i(t)$ of *stationary* OU-processes to the remaining time series.
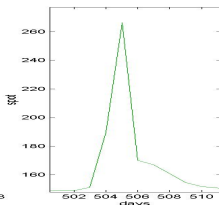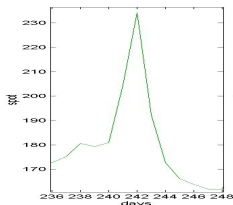
4. Simulation of a sample path.

Case study: simulation of the NordPool spot

1. Identification of $Y_1(t)$ modelling the seaonal spikes:

Simulated spike:

Zoom into the 3 biggest spikes of 1998, 1999, 2000:
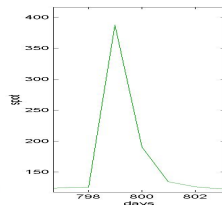
Case study: simulation of the NordPool spot

- For an estimated mean reversion $\hat{\lambda}_1 = 1.12$ (2/3 decay after one day), find the OU-path $\hat{Y}_1(t)$ with jump times $\hat{\tau}_1 \leq ... \leq \hat{\tau}_{\hat{r}}$ and corresponding path values $\hat{\mu}_1, ..., \hat{\mu}_{\hat{r}}$ that minimize

$$\min_{\substack{1 \leq \tau_1 \leq ... \leq \tau_r \leq N \\ \mu_1, ..., \mu_r \\ r \in \{1, ..., N\}}} \left\{ \gamma \cdot r + \sum_{i=1}^{r+1} \sum_{t=\tau_{i-1}}^{\tau_i - 1} \left(spot(t) - \mu_{\tau_{i-1}} e^{-\hat{\lambda}(t - \tau_{i-1})} \right)^2 \right\}$$

- γ represents penalization of jumps
- Using dynamic programming, an adoption of an algorithm from (Winkler, Liebscher 00) yields an exact algorithm to solve the above min-problem.

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Case study: simulation of the NordPool spot

▶ We assume the first OU-component $Y_1(t)$ given through

	λ	$\sigma(\mathbf{t})$	$\nu(\mathbf{dz})$	$\rho(\mathbf{t})$
Y_1	1.12	1	Exp(180)	$0.07 \cdot \left(\frac{2}{\left| \sin(\frac{\pi(t-6)}{261}) \right| + 1} - 1 \right)$

Case study: simulation of the NordPool spot

2. We fit a deterministic seasonal mean $\Lambda(t)$ of cosines to the time series $spot(t) - \hat{Y}_1(t)$.

3. We de-seasonalize the spot price process by removing seasonal spikes and deterministic mean level:

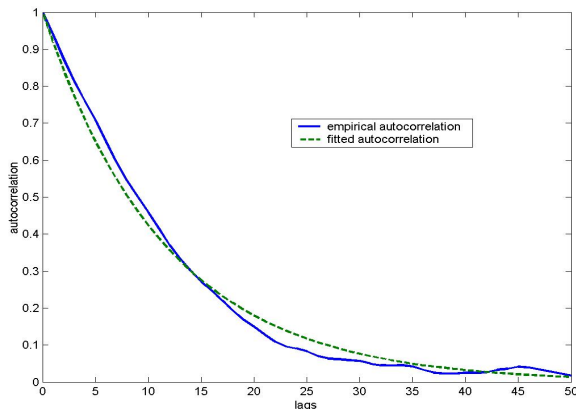$$despot(t) = spot(t) - \Lambda(t) - \hat{Y}_1(t),$$

and calibrate a sum of *stationary* OU-processes to the de-seasonalized spot price:

$$X(t) := \sum_{i=2}^{n} Y_i(t) \sim despot(t),$$

where $dY_i(t) = -\lambda_i Y_i(t)\, dt + dL_i(t)$ with now L_i increasing Lévy processes (no variation over time in controls).
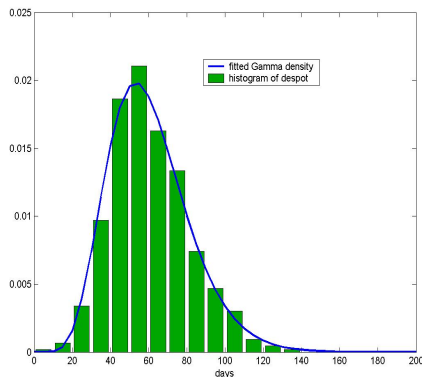
▶ Already one component $X(t) = Y_2(t)$ with $\hat{\lambda}_2 = 0.0846$ is sufficient to optimally fit the empirical autocorrelation structure:
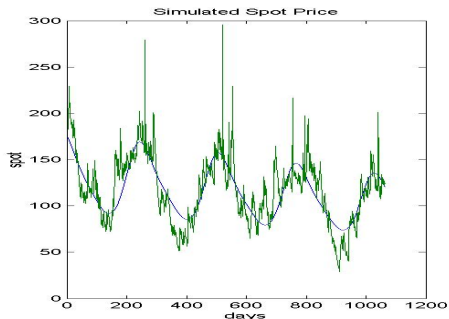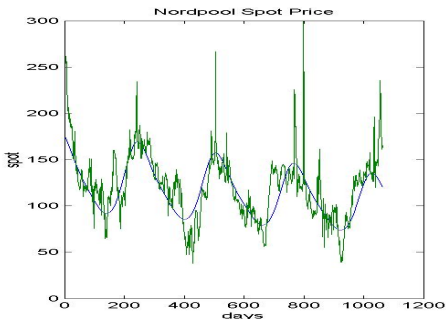
Case study: simulation of the NordPool spot

▶ We assume $Y_2 \sim \text{Gamma}(\nu, \alpha)$ and estimate $\nu = 8.055$, $\alpha = 0.132$ through performing maximum likelihood on *despot*:

4. Simulation of a complete path of the estimated process
$S(t) = \Lambda(t) + Y_1(t) + Y_2(t)$:

▶ Empirical moments of NordPool spot price versus simulated moments (averaged over 3000 simulation paths):

	Mean	Std. Dev.	Skewness	Kurtosis
Empirical	121.2387	35.9166	0.8516	6.7061
Simulated	120.6239	36.4370	0.8276	6.4120

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Conclusion

- Most common spot models are of geometric type and become unfeasible for further analysis of derivatives pricing.

- We propose an arithmetic model that is simple enough to yield analytical forward prices. Option pricing by fast Fourier transform techniques.

- The arithmetic model describes well both path properties and statistics of electricity spot prices.

- Future work includes the calibration of the market price of risk and the study of futures prices induced by the model.

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Coordinates

- e-mail: meyerbr@math.uio.no
- tel: 0047 22857787
- www.cma.uio.no