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A NON-LIFTABLE CALABI-YAU THREEFOLD IN CHARACTERISTIC 3

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Abstract. We show the existence of a Calabi-Yau threefold in characteristic 3 with its third Betti number zero. This example admits no lifting to characteristic zero and hence indicates that a theorem by Deligne that any $K3$ surface in positive characteristic has a lifting to characteristic zero cannot be generalized straightforward to the case of Calabi-Yau threefolds.

0. Introduction. Calabi-Yau threefolds as complex manifolds have been studied by a number of algebraic geometers as well as physicists, and a great deal of advancement has been achieved in the theory. On the other hand, $K3$ surfaces in positive characteristics have also been studied intensively through the seventies and eighties. It is the purpose of our study to see to what extent we can understand Calabi-Yau threefolds in positive characteristics with the help of these two theories. In this paper, we observe several results with strong emphasis on specific phenomena of Calabi-Yau threefolds in positive characteristics which are known at this stage.

One of the interesting problems of Calabi-Yau threefolds in characteristic p is whether they have liftings to characteristic zero or not. For $K3$ surfaces it was proved by Deligne [2] that any $K3$ surface lifts projectively to characteristic zero.

We consider, in this paper, quotient varieties of $\boldsymbol{P}^3$ by p-closed rational vector fields, and obtain a Calabi-Yau threefold X with its third Betti number zero. Then it is seen that this X admits no lifting to characteristic zero, which illustrates a clear difference from the case of $K3$ surfaces.

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1. Preliminaries. We consider a smooth projective variety X defined over an algebraically closed field k of characteristic $p > 0$.

DEFINITION 1.1. A smooth projective threefold X is said to be a Calabi-Yau threefold if $K_X \cong \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$.

One of the most important properties of Calabi-Yau threefolds in positive characteristics is that the invariant $\mathrm{ht}(X)$, called the height of X, can be defined.

DEFINITION 1.2 (Artin-Mazur [1]). Let X be a Calabi-Yau threefold and $\Phi^3(X/k, \boldsymbol{G}_m)$ be the Artin-Mazur formal group associated to X. Then we define the height $\mathrm{ht}(X)$ associated to X to be the height of $\Phi^3(X/k, \boldsymbol{G}_m)$.

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Let $W\mathcal{O}_X$ denote the sheaf of Witt vectors over X introduced by Serre [14]. Then the above definition is equivalent to the following:

$$\mathrm{ht}(X) = \begin{cases} \dim_K H^3(X, W\mathcal{O}_X) \otimes_W K & \text{if } H^3(X, W\mathcal{O}_X) \otimes_W K \neq 0, \\ \infty & \text{if } H^3(X, W\mathcal{O}_X) \otimes_W K = 0, \end{cases}$$

where K is the quotient field of the ring of Witt vectors $W(k)$. In particular, we say that X is a supersingular Calabi-Yau if $\mathrm{ht}(X) = \infty$, after the case of $K3$ surfaces.

As is the case with $K3$ surfaces, this invariant is expected to be closely related to the specific phenomena of positive characteristics. It is known that the following property, which is well-known for $K3$ surfaces, continues to hold for Calabi-Yau threefolds.

THEOREM 1.3. *If a Calabi-Yau threefold X is uniruled, then X is supersingular.*

The proof of this theorem is based on the following observation (cf. [5]). Let k be a field of characteristic $p > 0$ and $f : Y \to X$ be a generically finite surjective morphism of smooth complete varieties over k. If $H^j(W\mathcal{O}_Y) \otimes_W K = 0$, then $H^j(W\mathcal{O}_X) \otimes_W K = 0$. In particular, if $H^j(\mathcal{O}_Y) = 0$, then $H^j(W\mathcal{O}_X) \otimes_W K = 0$.

REMARK 1.4 (The Hodge Symmetry). For a smooth projective complex n-fold X, it is well-known that the following equalities, known as the Hodge symmetry, hold:

$$\dim_C H^j(\Omega_X^i) = \dim_C H^i(\Omega_X^j), \quad 0 \le i, j \le n.$$

In characteristic $p > 0$, the Hodge symmetry does not hold in general. However, Rudakov-Shafarevich proved in [10] that these equalities hold for $K3$ surfaces, by showing the non-existence of non-zero vector fields. For a Calabi-Yau threefold X, we have the equality $\Omega_X^2 \cong T_X$ and the Serre duality. So we see that the Hodge symmetry would follow if one could prove the vanishing $H^0(\Omega_X^1) = H^0(\Omega_X^2) = 0$, which is one of the main questions about Calabi-Yau threefolds in positive characteristics.

We use the following notation in this paper:

NOTATION 1.5.

X	: a smooth projective threefold defined over an algebraically closed field k of characteristic $p > 0$.
$b_i(X)$	: the l-adic Betti number of X given by $\dim_{Q_l} H^i_{\text{ét}}(X, Q_l)$ $(l \neq p)$, which is also equal to $\mathrm{rank}_W H^i_{\text{crys}}(X/W)$.
$b_i^{\mathrm{DR}}(X)$	: the de Rham Betti number of X, which is given by $\dim_k H^i_{\mathrm{DR}}(X)$. If τ_i denotes the number of generators of the torsion part of $H^i_{\text{crys}}(X/W)$, then $b_i^{\mathrm{DR}}(X) = b_i(X) + \tau_i + \tau_{i+1}$ holds.
$e(X)$	: the Euler number of X defined by $e(X) = \sum_{i=0}^{6}(-1)^i b_i(X)$.
$X \to X^{(-1)}$	: the relative Frobenius morphism of X.
δ	: a rational vector field on X which is p-closed, i.e., $\delta^p = \alpha\delta$ for some $\alpha \in k(X)$.

(δ)	: the divisor on X associated to a p-closed rational vector field δ, which is given as follows: Locally δ is expressed as $\delta = \alpha(A\partial/\partial x + B\partial/\partial y + C\partial/\partial z)$, where x, y, z are local coordinates and A, B, C are regular functions without common factors. Then the divisor (α) is given in each affine open set, and can be glued together to form a divisor (δ) on X.
$\mathrm{Sing}\,\delta$	: the set of singular points of a p-closed rational vector field δ. This is given locally by $\{A = B = C = 0\}$ under the expression of δ as above.
$\mathcal{L} \subset T_X$	: the 1-foliation induced by a p-closed rational vector field δ, i.e., a saturated invertible subsheaf of the tangent bundle T_X which is locally generated by δ.
$\boldsymbol{P}^n$	: the n-dimensional projective space defined over k. When considering a different base field, for example $\boldsymbol{F}_p$, we indicate it as $\boldsymbol{P}^n_{\boldsymbol{F}_p}$.

For a Calabi-Yau threefold X, we have $\chi(\mathcal{O}_X) = 0$ and $e(X) = -2\chi(\Omega^1_X)$ by the Riemann-Roch theorem.

We call a morphism $f : X \to S$ a fibration if S is normal and $f_*\mathcal{O}_X = \mathcal{O}_S$. We say that X has a projective lifting to characteristic zero if there exists a smooth projective morphism

$$\mathfrak{X} \to \mathrm{Spec}\, R$$

over a discrete valuation ring R such that the closed fiber is isomorphic to X, and the quotient field of R is of characteristic zero.

2. Construction. In this section, we investigate a Calabi-Yau threefold obtained as the quotient of $\boldsymbol{P}^3$ by a p-closed rational vector field. Our method of constructing quotient varieties by rational vector fields was introduced by Rudakov-Shafarevich, and has been used in various works. We refer the reader to [3] and [11].

PROPOSITION 2.1. i) *Let* $\boldsymbol{A}^3 := \mathrm{Spec}\, k[x, y, z] \subset \boldsymbol{P}^3$ *be an affine open set. The derivation*

$$\delta := (x^p - x)\frac{\partial}{\partial x} + (y^p - y)\frac{\partial}{\partial y} + (x^p - z)\frac{\partial}{\partial z}$$

determines a p-closed rational vector field on $\boldsymbol{P}^3$ *with* $p^3 + p^2 + p + 1$ *isolated singular points* $\mathrm{Sing}\,\delta$. *Each singular point of* δ *can be resolved by one point blowing-up.*

ii) *Let* $\pi : S \to \boldsymbol{P}^3$ *be the blowing-ups at* $p^3 + p^2 + p + 1$ *singular points* $\mathrm{Sing}\,\delta$. *Then the smooth rational vector field on S, which we denote by* $\pi^*\delta$, *induces a smooth projective threefold X as its quotient*:

$$\text{(2-A)} \qquad \begin{array}{ccccc} S & \xrightarrow{\ g\ } & X & \xrightarrow{\ \tilde{g}\ } & S^{(-1)} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow \\ \boldsymbol{P}^3 & \xrightarrow{\ g_0\ } & V & \xrightarrow{\ \tilde{g}_0\ } & \boldsymbol{P}^{3(-1)} \end{array},$$

where g (resp. g_0) is the finite and flat (resp. finite) morphism of degree p which is induced by $\pi^*\delta$ (resp. δ). $\tilde{\pi}$ is a naturally induced birational morphism. In particular, we have

$$g^*K_X \cong \pi^*\mathcal{O}_{\boldsymbol{P}^3}((p-1)^2-4) \otimes \mathcal{O}_S\left((3-p)\sum_{i=1}^{p^3+p^2+p+1} E_i\right),$$

where $\{E_i\}$ are the exceptional divisors of π.

THEOREM 2.2. *Suppose $p = 3$. Then the birational morphism $\tilde{\pi} : X \to V$ in* (2-A) *is a crepant resolution. The smooth projective threefold X satisfies the following properties*:

i) *X is a Calabi-Yau threefold.*
ii) *X is unirational, therefore supersingular.*
iii) $\pi_1^{\mathrm{alg}}(X) = \{1\}$.
iv) $b_2(X) = 41, b_3(X) = 0$.
v) $H^0(\Omega_X^1) = H^0(T_X) = 0$, *therefore the Hodge symmetry holds.*
vi) *X has quasi-elliptic fibrations.*

COROLLARY 2.3. *The Calabi-Yau threefold X in $p = 3$ obtained above does not admit a projective lifting to characteristic zero.*

PROOF. Suppose that the Calabi-Yau threefold X in question has a projective lifting to characteristic zero:

$$\mathfrak{X} \to \operatorname{Spec} R,$$

over a discrete valuation ring R (cf. Section 1). Let $\mathfrak{X}_{\bar{\eta}}$ be its geometric generic fiber. Then we have, by the Hodge theory, $b_3(\mathfrak{X}_{\bar{\eta}}) = \dim H^3_{\mathrm{DR}}(\mathfrak{X}_{\bar{\eta}}) = \sum_{i+j=3} h^j(\Omega^i_{\mathfrak{X}_{\bar{\eta}}})$. However, from the fact that the Betti numbers and the arithmetic genus are invariant under deformation, we deduce that $b_3(\mathfrak{X}_{\bar{\eta}}) = 0$ and $h^3(\mathcal{O}_{\mathfrak{X}_{\bar{\eta}}}) = 1$. But this is absurd. □

Before proceeding to the proof of the theorem, we first introduce the following notation.

NOTATION 2.4.

$\mathcal{L} \hookrightarrow T_S$ stands for the smooth 1-foliation on S, which is locally generated by $\pi^*\delta$. We denote a general hyperplane of $\boldsymbol{P}^n$ by $\mathcal{O}_{\boldsymbol{P}^n}(1)$. The hyperplanes in $\boldsymbol{P}^3_{\boldsymbol{F}_p}$ are denoted by $\{F_i \mid i = 1, \dots, p^3+p^2+p+1\}$. The base change $F_i \times_{\operatorname{Spec}\boldsymbol{F}_p} \operatorname{Spec} k$ is also denoted by the same F_i.

$\bar{F}_i$ is the strict transform of F_i by $\pi : S \to \boldsymbol{P}^3$ in (2-A). In particular, $\pi\,|_{\bar{F}_i} : \bar{F}_i \to F_i$ corresponds to blowing-ups at $\boldsymbol{F}_p$-rational points of $F_i \cong \boldsymbol{P}^2_{\boldsymbol{F}_p}$.

PROOF OF PROPOSITION 2.1. i) Suppose that the local coordinates are given by

$$U := \operatorname{Spec} k[x, y, z], \quad U_1 := \operatorname{Spec} k[x_1, y_1, z_1] \subset \boldsymbol{P}^3 := \operatorname{Proj} k[X_0, X_1, X_2, X_3],$$

where $(X_0, X_1, X_2, X_3) = (1, x, y, z) = (x_1, 1, y_1, z_1)$. In U_1, the derivation δ is expressed as:

$$\delta = \frac{1}{x_1^{p-1}}\left[(x_1^p - x_1)\frac{\partial}{\partial x_1} + (y_1^p - y_1)\frac{\partial}{\partial y_1} + (z_1^p - z_1)\frac{\partial}{\partial z_1}\right].$$

It can be observed that δ has a pole of degree $p-1$ at $x_1 = 0$, and the singular points $\operatorname{Sing}\delta$ correspond to the $\boldsymbol{F}_p$-rational points of $\boldsymbol{P}^3_{\boldsymbol{F}_p} := \operatorname{Proj}\boldsymbol{F}_p[X_0, X_1, X_2, X_3]$.

Consider the blowing-up at the origin: $x = s$, $y = st$, $z = su$. Then we have $\partial/\partial x = \partial/\partial s - (t/s)\partial/\partial t - (u/s)\partial/\partial u$, $\partial/\partial y = (1/s)\partial/\partial t$, $\partial/\partial z = (1/s)\partial/\partial u$ and

$$\pi^*\delta = s\left[(s^{p-1}-1)\frac{\partial}{\partial s} + s^{p-2}(t^p - t)\frac{\partial}{\partial t} + s^{p-2}(u^p - u)\frac{\partial}{\partial u}\right].$$

We see that $\pi^*\delta$ vanishes along an exceptional divisor $E_1 := \{s = 0\}$ with degree one, that is, the equality of divisors $(\pi^*\delta) = \pi^*(\delta) + E_1$ holds. Moreover, $\pi^*\delta$ has no singular points lying on E_1, so the singularity at the origin is resolved. Other singular points in $\operatorname{Sing}\delta$ can also be resolved in the same way.

ii) The first assertion follows from the result (i). For the second, we use the canonical bundle formula (cf. [11]):

$$g^*K_X \sim K_S - (p-1)(\pi^*\delta),$$

where $(\pi^*\delta) \sim -(p-1)\pi^*\mathcal{O}_{\boldsymbol{P}^3}(1) + \sum_{i=1}^{p^3+p^2+p+1} E_i$. □

REMARK 2.5. Let $q \in \operatorname{Sing}\delta$ be a singular point of δ in Proposition 2.1. Then the complete local ring of the singular point $g_0(q) \in \operatorname{Sing} V$ is given as:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{\boldsymbol{P}^3,q} & \hookleftarrow & \hat{\mathcal{O}}_{V,g_0(q)} \\ \| & & \| \\ k[[x,y,z]] & \hookleftarrow & k[[x^i y^j z^k \mid i+j+k \equiv 0 \bmod p,\ 0 \le i,j,k < p]] \end{array}$$

In particular, this is a toric singularity of type $(1/p)(1,1,1)$, and there exists a crepant resolution if $p = 3$.

LEMMA 2.6. *Let $\pi_0 : \bar{F} \to \boldsymbol{P}^2 (\cong \boldsymbol{P}^2_{\boldsymbol{F}_p} \times \operatorname{Spec} k)$ be the birational morphism obtained by blowing up the $\boldsymbol{F}_p$-rational points in $\boldsymbol{P}^2_{\boldsymbol{F}_p}$. Then we have*

$$H^0\left(\pi_0^*\mathcal{O}_{\boldsymbol{P}^2}((p-1)p) \otimes \mathcal{O}_{\bar{F}}\left(-p\sum_{j=1}^{p^2+p+1} e_j\right)\right) = 0,$$

where $e_1, \dots, e_{p^2+p+1}$ are the exceptional curves for π_0.

PROOF. Consider the lines $\{l_i \mid i = 1, \dots, p^2+p+1\}$ in $\boldsymbol{P}^2_{\boldsymbol{F}_p}$. We denote the strict transform of $l_i \times_{\operatorname{Spec}\boldsymbol{F}_p} \operatorname{Spec} k$ for $\pi_0 : \bar{F} \to \boldsymbol{P}^2$ by the same l_i. Then it can be expressed as $l_k \sim \pi_0^*\mathcal{O}_{\boldsymbol{P}^2}(1) - \sum_{l=1}^{p+1} e_{j_l}$.

Suppose that there exists an effective divisor $\bar{D} \in H^0(\pi_0^*\mathcal{O}_{\boldsymbol{P}^2}((p-1)p) \otimes \mathcal{O}_{\bar{F}}(-p\sum_{j=1}^{p^2+p+1} e_j))$. Then we have $(\bar{D}.l_k) = -2p < 0$. This implies that $\bar{D}$ has $\sum_{k=1}^{p^2+p+1} l_k$ as its component. On the other hand, we have the intersection number $(\bar{D} - \sum_{k=1}^{p^2+p+1} l_k . \pi_0^*\mathcal{O}_{\boldsymbol{P}^2}(1)) = -2p-1 < 0$, which contradicts the fact that $\pi_0^*\mathcal{O}_{\boldsymbol{P}^2}(1)$ is nef. Thus we have the desired assertion. □

PROOF OF THEOREM 2.2. If $p = 3$, we have $g^* K_X \sim 0$ by Proposition 2.1 (ii), that is, K_X is numerically equivalent to zero. Here, we show that X is indeed a Calabi-Yau threefold.

First, we prove $H^0(\mathcal{L}^{-p}) = 0$, where $\mathcal{L} \cong \pi^* \mathcal{O}_{\boldsymbol{P}^3}(-(p-1)) \otimes \mathcal{O}(\sum_{i=1}^{p^3+p^2+p+1} E_i)$. Suppose there exists an effective divisor $D \in H^0(\mathcal{L}^{-p})$. Then consider the exact sequence

$$0 \to H^0(\mathcal{O}_S(D - \bar{F}_i)) \to H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_{\bar{F}_i}(D\,|_{\bar{F}_i})),$$

where the last term vanishes because of Lemma 2.6. This implies that $D - \sum_{i=1}^{p^3+p^2+p+1} \bar{F}_i$ is an effective divisor. On the other hand, we have

$$\left(D - \sum_{i=1}^{p^3+p^2+p+1} \bar{F}_i . (\pi^* \mathcal{O}_{\boldsymbol{P}^3}(1))^2 \right) < 0 .$$

But this is absurd. Thus, we have $H^0(\mathcal{L}^{-p}) = 0$.

Secondly, we show that $H^1(\mathcal{O}_X) = 0$ is derived from $H^0(\mathcal{L}^{-p}) = 0$. Consider the smooth 1-foliation $\mathcal{L} \hookrightarrow T_S$ locally generated by $\pi^*\delta$, and let $\Omega_S \to \mathcal{L}^{-1}$ be its dual. Consider the composition map with the universal derivation d.

$$\mathcal{O}_S \xrightarrow{d} \Omega_S \to \mathcal{L}^{-1} .$$

This composition map is the one which sends $s \in \mathcal{O}_S$ to $\delta(s) \in \mathcal{L}^{-1}$. Taking the direct images by the quotient morphism $g : S \to X$, we have the following diagram:

$$\begin{array}{ccccccc} 0 & \to & \mathcal{O}_X & \to & g_*\mathcal{O}_S & \to & g_*\mathcal{O}_S/\mathcal{O}_X \to 0 \\ & & \| & & \| & & \cap \\ 0 & \to & \mathcal{O}_X & \to & g_*\mathcal{O}_S & \to & g_*\mathcal{L}^{-1} . \end{array}$$

Here, these two rows are exact by definition. Then the assertion verified above $H^0(\mathcal{L}^{-p}) = 0$ indicates that the first term in the following exact sequence vanishes:

$$H^0(g_*\mathcal{O}_S/\mathcal{O}_X) \to H^1(\mathcal{O}_X) \to H^1(g_*\mathcal{O}_S) .$$

Indeed, the last term also vanishes, since g is a finite morphism and S is a smooth rational threefold. Thus we obtain the desired assertion $H^1(\mathcal{O}_X) = 0$.

Thirdly, we prove $H^2(\mathcal{O}_X) = 0$ and $K_X \cong \mathcal{O}_X$. By the Riemann-Roch formula, we have $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X) = 0$. Then by the Serre duality and the facts: $h^0(\mathcal{O}_X) = 1$, $h^1(\mathcal{O}_X) = 0$, we have the following inequality:

$$1 \leq 1 + h^2(\mathcal{O}_X) = h^3(\mathcal{O}_X) = h^0(K_X) .$$

Here, we see that the last term is at most one, because K_X is numerically trivial in $p = 3$, from which the assertions $H^2(\mathcal{O}_X) = 0$ and $K_X \cong \mathcal{O}_X$ follow. Thus X is a Calabi-Yau threefold.

The assertions ii), iii) follow from the construction, iv) follows from the equalities $b_i(S) = b_i(X)$ for $i = 0, \ldots, 6$, since the quotient morphism g in (2-A) is finite and purely inseparable. The quasi-elliptic fibrations in vi) are induced from the projection $\tilde{\boldsymbol{P}}^3 \to \boldsymbol{P}^2$, where $\tilde{\boldsymbol{P}}^3$ is a one point blowing-up of $\boldsymbol{P}^3$. So there remains to prove v).

Let $\mathcal{M} := T_{X/S^{(-1)}} \hookrightarrow T_X$ be the smooth 1-foliation of rank two on X, which corresponds to the purely inseparable finite morphism $\tilde{g} : X \to S^{(-1)}$ of degree p^2. Then we have the following exact sequences:

$$0 \to g^*\mathcal{M}^{-1} \to \Omega_S \to \mathcal{L}^{-1} \to 0\,,$$
$$0 \to \tilde{g}^*\mathcal{L}^{-1} \to \Omega_X \to \mathcal{M}^{-1} \to 0\,.$$

Then look at the long exact sequence:

$$0 \to H^0(\tilde{g}^*\mathcal{L}^{-1}) \to H^0(\Omega_X) \to H^0(\mathcal{M}^{-1}) \to \cdots\,.$$

Here we have $H^0(\mathcal{M}^{-1}) = 0$ because of the inclusion $H^0(g^*\mathcal{M}^{-1}) \hookrightarrow H^0(\Omega_S) = 0$. Moreover, $H^0(\tilde{g}^*\mathcal{L}^{-1}) = 0$ holds, since we have

$$H^0(\tilde{g}^*\mathcal{L}^{-1}) \hookrightarrow H^0(g_*(g^*\tilde{g}^*\mathcal{L}^{-1})) = H^0(\mathcal{L}^{-p})$$

and we already know that the last term vanishes. Thus we have $H^0(\Omega_X) = 0$.

The assertion $H^0(T_X) = 0$ follows from Proposition 2.7 mentioned below. Thus we complete the proof of Theorem 2.2. □

PROPOSITION 2.7. *Consider the p-closed rational vector field on $\boldsymbol{P}^3$ given by*

$$\delta = (G_1^p - x)\frac{\partial}{\partial x} + (G_2^p - y)\frac{\partial}{\partial y} + (G_3^p - z)\frac{\partial}{\partial z}$$

with $G_1, G_2, G_3 \in k[x, y, z]$. Let $g_0 : \boldsymbol{P}^3 \to V$ be its quotient and suppose that the resolution of singularities $\tilde{\pi} : X \to V$ such that $X \setminus \tilde{\pi}^{-1}(\mathrm{Sing}\, V) \cong V \setminus \mathrm{Sing}\, V$ exists. Suppose further that $\{1, G_1, G_2, G_3\} \cup \{G_iG_j \,|\, i, j \in \{1, 2, 3\}\}$ in $k[x, y, z]$ are k-linearly independent and $\delta \notin H^0(T_{\boldsymbol{P}^3})$. Then we have $H^0(T_X) = 0$.

PROOF. For the proof, we consider the purely inseparable morphisms which factor the Frobenius morphism:

$$\boldsymbol{P}^3 \xrightarrow{g_0} V \xrightarrow{\tilde{g}_0} \boldsymbol{P}^{3(-1)}\,.$$

Then there exist 1-foliations $\mathcal{L}_0 := T_{\boldsymbol{P}^3/V} \subset T_{\boldsymbol{P}^3}$ and $\mathcal{M}_0 := T_{V/\boldsymbol{P}^{3(-1)}} \subset T_V$ which correspond to g_0 and $\tilde{g}_0$, respectively. Consider the exact sequence:

$$0 \to \mathcal{L}_0 \to T_{\boldsymbol{P}^3} \to T_{\boldsymbol{P}^3}/\mathcal{L}_0 \to 0\,.$$

We also have an exact sequence on $V_0 := V \setminus \mathrm{Sing}\, V$:

$$0 \to \mathcal{M}_0 \to T_V \to \tilde{g}_0^*\mathcal{L}_0 \to 0\,,$$

and $T_{\boldsymbol{P}^3}/\mathcal{L}_0 \cong g_0^*\mathcal{M}_0$ holds on $g_0^{-1}(V_0)$ (cf. [3]). So the following long exact sequences are induced:

$$0 \to H^0(\mathcal{L}_0) \to H^0(T_{\boldsymbol{P}^3}) \to H^0(T_{\boldsymbol{P}^3}/\mathcal{L}_0) \to 0\,,$$
$$0 \to H^0(V_0, \mathcal{M}_0) \to H^0(V_0, T_V) \to H^0(V_0, \tilde{g}_0^*\mathcal{L}_0)\,.$$

Here $H^0(\mathcal{L}_0)=0$ holds from the hypothesis $\delta\notin H^0(T_{\boldsymbol{P}^3})$. Then $H^0(V_0,\tilde{g}_0^*\mathcal{L}_0)=0$ also follows. By computation of local cohomologies, we have $H^0(\boldsymbol{P}^3,T_{\boldsymbol{P}^3}/\mathcal{L}_0)\cong H^0(g_0^{-1}(V_0), T_{\boldsymbol{P}^3}/\mathcal{L}_0)$. So, we obtain the inclusion $H^0(V_0,T_V)\hookrightarrow H^0(T_{\boldsymbol{P}^3})$.

Now, we show that there exists no element $\theta\in H^0(T_{\boldsymbol{P}^3})$ such that the restriction $\theta\big|_{k(V)}$ determines a derivation of $k(V)$. Take a basis of $H^0(T_{\boldsymbol{P}^3})$:

$$\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x},\ y\frac{\partial}{\partial x},\ z\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial y},\ z\frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ x\frac{\partial}{\partial z},\ y\frac{\partial}{\partial z},\ z\frac{\partial}{\partial z},$$
$$x\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right),\quad y\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right),\quad z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right).$$

The function field of V is given by $k(V)=k(x^p,y^p,z^p,w_1,w_2)$, where $w_1:=(G_1^p-x)(G_2^p-y)^{p-1}$ and $w_2:=(G_2^p-y)(G_3^p-z)^{p-1}$.

So, it suffices to show that there exists no element $\theta\in H^0(T_{\boldsymbol{P}^3})$ such that $(\delta(\theta w_1), \delta(\theta w_2))=0$ in $k(\boldsymbol{P}^3)\oplus k(\boldsymbol{P}^3)$. This is equivalent to the following elements in $k(\boldsymbol{P}^3)\oplus k(\boldsymbol{P}^3)$ being k-linearly independent:

$$\left(\delta\left(\frac{\partial}{\partial x}w_1\right),\delta\left(\frac{\partial}{\partial x}w_2\right)\right),\ \left(\delta\left(x\frac{\partial}{\partial x}w_1\right),\delta\left(x\frac{\partial}{\partial x}w_2\right)\right),\ \left(\delta\left(y\frac{\partial}{\partial x}w_1\right),\delta\left(y\frac{\partial}{\partial x}w_2\right)\right),$$
$$\left(\delta\left(z\frac{\partial}{\partial x}w_1\right),\delta\left(z\frac{\partial}{\partial x}w_2\right)\right),\ \left(\delta\left(\frac{\partial}{\partial y}w_1\right),\delta\left(\frac{\partial}{\partial y}w_2\right)\right),\ \left(\delta\left(x\frac{\partial}{\partial y}w_1\right),\delta\left(x\frac{\partial}{\partial y}w_2\right)\right),$$
$$\left(\delta\left(y\frac{\partial}{\partial y}w_1\right),\delta\left(y\frac{\partial}{\partial y}w_2\right)\right),\ \left(\delta\left(z\frac{\partial}{\partial y}w_1\right),\delta\left(z\frac{\partial}{\partial y}w_2\right)\right),\ \left(\delta\left(\frac{\partial}{\partial z}w_1\right),\delta\left(\frac{\partial}{\partial z}w_2\right)\right),$$
$$\left(\delta\left(x\frac{\partial}{\partial z}w_1\right),\delta\left(x\frac{\partial}{\partial z}w_2\right)\right),\ \left(\delta\left(y\frac{\partial}{\partial z}w_1\right),\delta\left(y\frac{\partial}{\partial z}w_2\right)\right),\ \left(\delta\left(z\frac{\partial}{\partial z}w_1\right),\delta\left(z\frac{\partial}{\partial z}w_2\right)\right),$$
$$\left(\delta\left(x\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)w_1\right),\delta\left(x\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)w_2\right)\right),$$
$$\left(\delta\left(y\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)w_1\right),\delta\left(y\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)w_2\right)\right),$$
$$\left(\delta\left(z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)w_1\right),\delta\left(z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)w_2\right)\right).$$

This is, indeed, the case under the assumption of Proposition 2.7. Then the desired assertion follows from the inclusion:

$$H^0(X,T_X)\hookrightarrow H^0(\pi^{-1}(V_0),T_X)\cong H^0(V_0,T_V)=0.$$

This completes the proof of Proposition 2.7. □

REMARKS 2.8. i) The smooth quotient threefold X obtained in Proposition 2.1 in other characteristics is classified as a rational threefold if $p=2$, and as a threefold of general type (i.e., the Kodaira dimension $\kappa(X)=3$) if $p\geq 5$.

ii) It is not known if the existence of Calabi-Yau threefolds with the third Betti number zero is a phenomenon specific to characteristic three or not. It follows that such Calabi-Yau threefolds are supersingular.

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