TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 257, Number 2, February 1980

A NONCOMMUTATIVE GENERALIZATION AND q-ANALOG OF THE LAGRANGE INVERSION FORMULA[1]

BY

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ABSTRACT. The Lagrange inversion formula is generalized to formal power series in noncommutative variables. A q-analog is obtained by applying a linear operator to the noncommutative formula before substituting commuting variables.

1. Introduction. Given a (formal) power series $g(t)$, we may define implicitly a series $f(z)$ by

$$f(z) = zg[f(z)]. \tag{1.1}$$

Lagrange's inversion formula [6, pp. 148–151] expresses the coefficients of powers of f in terms of coefficients of powers of g:

$$\langle z^n\rangle[f(z)]^k = k\langle t^{n-k}\rangle[g(t)]^n/n. \tag{1.2}$$

(Here $\langle z^n\rangle[f(z)]^k$ denotes the coefficient of z^n in $[f(z)]^k$.) Formula (1.2) is of great importance in enumeration since many counting problems lead to equations of the form (1.1). Less often used is the "second form" of Lagrange's formula:

$$\langle z^n\rangle\frac{[f(z)]^k}{1-zg'[f(z)]} = \langle t^{n-k}\rangle[g(t)]^n, \tag{1.3}$$

whose combinatorial significance is less apparent.

In Part I we consider the equation

$$F = \sum_{n=-1}^{\infty} F^{n+1}x_n, \tag{1.4}$$

where the x_n are *noncommuting* variables. (A simple substitution of commuting variables reduces (1.4) to (1.1).) Then F is a formal sum of certain "words" in the x_n. These words have been studied by various authors (e.g. Gouyou-Beauchamps [13], Kemeny and Snell [16], Knuth [17; exercise 32, pp. 398–399; solution, pp. 587–588]) and may be interpreted as "reverse-Polish" codes for rooted plane trees. The noncommutative analogs of (1.2) and (1.3) are obtained by studying the relationship between the set of words counted by F and the set of all words. Formula (1.2) comes from the logarithm of a multiplicative decomposition of the set of all words, and formula (1.3) comes from an additive decomposition of the set

Presented to the Society August 10, 1978; received by the editors February 5, 1979.

AMS (MOS) subject classifications (1970). Primary 05A15, 05A19; Secondary 10A45, 20M05, 68A30.

Key words and phrases. Lagrange inversion, q-analog, factorization in free monoids, lattice path, q-Catalan numbers, continued fractions, labeled trees.

[1]Research partially supported by ONR Contract #N00014-76-C-0366.

of all words. A different combinatorial interpretation of (1.2) has been given by Raney [**22**], but there seems to be no combinatorial interpretation of (1.3) in the literature.

A formula in noncommuting variables contains much more information than the corresponding commutative formula, but it is usually difficult to extract this additional information in a "usable" form. In Part II we introduce a general method for obtaining q-analogs from noncommutative formulas, which can be applied to other combinatorial problems. We apply a linear operator to the noncommutative analog of (1.3) which "weights" each word by a power of q before substituting commuting variables. The formula we obtain enables us to solve equations of the form $f(z) = qz\sum_{n=0}^{\infty} g_n f(z)f(qz) \dots f(q^{n-1}z)$. (There does not seem to be a simple q-analog of (1.2).) The formula is used to obtain new results on Ramanujan's and related continued fractions [**14**, p. 294], Pólya's q-Catalan numbers [**21**], and the inversion enumerator for trees [**19**].

A different q-analog of the Lagrange inversion formula has been obtained by Andrews [**1**], but it does not seem to be closely related to the one discussed here.

Part I

2. Preliminaries. We take as our ring of scalars a commutative ring **R** containing the formal Laurent series ring $\mathbf{Q}((t))[[z]]$, where **Q** is the rationals. Thus **R** contains all series of the form $\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty} a_{ij}t^i z^j$ with $a_{ij} \in \mathbf{Q}$ and for fixed j, a_{ij} is nonzero for only finitely many negative values of i.

Let **A** be the ring of formal power series in the noncommuting variables $x_{-1}, x_0, x_1, x_2, \dots$ with coefficients in **R**. A *word* is an element of **A** of the form $w = x_{i_1}x_{i_2}\dots x_{i_n}$. The *length* of w is $l(w) = n$, the *rank* of w is $r(w) = -(i_1 + i_2 + \cdots + i_n)$, and the *reverse* of w is $\bar{w} = x_{i_n}x_{i_{n-1}}\dots x_{i_1}$. If v and w are words, then $l(vw) = l(v) + l(w)$, $r(vw) = r(v) + r(w)$, and $\overline{vw} = \bar{w}\bar{v}$. The "empty word" 1 is the identity of **A**; it has length zero and rank zero. We extend the map $w \mapsto \bar{w}$ by linearity to all of **A**. (By "linear" we always mean **R**-linear and continuous; thus if Λ is linear, $\Lambda(\sum r_w w) = \sum r_w \Lambda(w)$, where the sum is over all words w, and $r_w \in \mathbf{R}$.)

The following conventions will be used in Part I: lower-case letters represent words (except in §3), script capitals represent sets of words, and Roman capitals represent elements of **A** (usually the sum of the elements of the corresponding script capital).

3. The noncommutative formulas. In this section we state the noncommutative formulas and derive from them formulas (1.2) and (1.3).

We consider the equation in **A**

$$F = \sum_{n=-1}^{\infty} F^{n+1}x_n \tag{3.1}$$

for the unknown series F. We leave it to the reader to verify that (3.1) determines F uniquely; the first few terms are $F = x_{-1} + x_{-1}x_0 + x_{-1}x_0^2 + x_{-1}^2x_1 + x_{-1}x_0^3 + x_{-1}^2x_1x_0 + x_{-1}^2x_0x_1 + x_{-1}x_0x_{-1}x_1 + x_{-1}^3x_2 + \dots$. It is not difficult to show that the number of words in F of length n is the Catalan number $\binom{2n-2}{n-1}/n$. (We will see later that each word in F appears just once.) An easy induction shows that every word in F has rank one.

Since every word in F begins with x_{-1}, we may define $B \in \mathbf{A}$ by

$$F = x_{-1}B. \tag{3.2}$$

Then every word in B has rank zero. We now state without proof the basic noncommutative Lagrange inversion formula. (A stronger theorem will be proved in the next section.)

THEOREM A. *Let F and B be defined by* (3.1) *and* (3.2), *and let* $X = \sum_{n=-1}^{\infty} x_n$. *Then*

$$(1 - X)^{-1} = (1 + R)B(1 - F)^{-1}, \tag{3.3}$$

$$(1 - X)^{-1} = (1 - D)^{-1}(1 - F)^{-1} + S, \tag{3.4}$$

where R and S are sums of words of negative rank, and

$$D = \sum_{j,k=0}^{\infty} F^j x_{j+k} \bar{F}^k. \tag{3.5}$$

To justify our claim that Theorem A is a generalized Lagrange inversion formula, we derive from it the classical formulas.

Let g_n for $n \geqslant 0$ be elements of $\mathbf{R}$, and let $g(t) = \sum_{n=0}^{\infty} g_n t^n$. In Theorem A substitute $zg_{n+1}t^n$ for x_n and let $f(z)/t$ be the image of F under this substitution. Then the image of B is $f(z)/(zg_0)$ and the image of X is $(z/t)g(t)$. The image of D is

$$\sum_{j,k=0}^{\infty} \left[\frac{f(z)}{t}\right]^{j+k} zg_{j+k+1}t^{j+k} = z\sum_{n=0}^{\infty}(n+1)g_{n+1}[f(z)]^n = zg'[f(z)].$$

A word of length l and rank r is transformed into a term $z^l t^{-r}$ times a monomial in the g_n. Then (3.1) becomes $f(z) = zg[f(z)]$, which is (1.1), and (3.3) and (3.4) become

$$\left[1 - \frac{z}{t}g(t)\right]^{-1} = (1 + r)\frac{f(z)}{zg_0}\left[1 - \frac{f(z)}{t}\right]^{-1} \tag{3.6}$$

and

$$\left[1 - \frac{z}{t}g(t)\right]^{-1} = \{1 - zg'[f(z)]\}^{-1}\left[1 - \frac{f(z)}{t}\right]^{-1} + s, \tag{3.7}$$

where every term in r and s contains a positive power of t.

Equating coefficients of $z^n t^{-k}$ in (3.7) for $n, k \geqslant 0$ yields

$$\langle z^n\rangle\{1 - zg'[f(z)]\}^{-1}[f(z)]^k = \langle t^{-k}\rangle[g(t)/t]^n = \langle t^{n-k}\rangle[g(t)]^n,$$

which is (1.3). Taking logarithms in (3.6) we obtain

$$\log(1 + r) + \log\frac{f(z)}{zg_0} + \sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{f(z)}{t}\right]^k = \sum_{n=1}^{\infty}\frac{z^n}{n}\left[\frac{g(t)}{t}\right]^n.$$

Equating coefficients of $z^n t^{-k}$ for $n, k > 0$ yields (1.2); equating coefficients of $z^n t^0$ for $n > 0$ yields as a bonus,

$$\langle z^n \rangle \log \frac{f(z)}{zg_0} = \langle t^0 \rangle \frac{1}{n}\left[\frac{g(t)}{t}\right]^n = \frac{1}{n}\langle t^n \rangle [g(t)]^n. \tag{3.8}$$

Equation (3.8) was discovered by Schur [25], but is probably much older. It can also be derived directly from (1.3) for $k = 0$, using the fact that $f(z) = zg[f(z)]$ implies

$$\{1 - zg'[f(z)]\}^{-1} = zf'(z)/f(z) = 1 + z(d/dz)\log[f(z)/(zg_0)].$$

4. The algebraic proof. In this section we prove a stronger result than Theorem A: we give explicit formulas for R and S which allow a proof by straightforward algebraic manipulation. The proof has the disadvantage that the formulas are pulled out of a hat; in the next section we sketch a more satisfying (but longer) combinatorial proof.

THEOREM 4.1. *Let F be the unique series satisfying $F = \sum_{n=-1}^{\infty} F^{n+1}x_n$. Then*

$$(1 - X)^{-1} = (1 - BE)^{-1}B(1 - F)^{-1}, \tag{4.1}$$

where $F = x_{-1}B$ and $E = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} F^k x_n$. Moreover, every word in F has rank one, every word in B has rank zero, and every word in E (and hence in $(1 - BE)^{-1} - 1$) has negative rank.

We prove Theorem 4.1 with the help of two lemmas.

LEMMA 4.2. $\sum_{n=0}^{\infty} F^n x_n = 1 - B^{-1}$.

PROOF. $F\sum_{n=0}^{\infty} F^n x_n = \sum_{n=0}^{\infty} F^{n+1}x_n = F - x_{-1} = F - FB^{-1} = F(1 - B^{-1})$. □

LEMMA 4.3. $(1 - F)E = B^{-1} - 1 + \sum_{n=0}^{\infty} x_n$.

PROOF. $(1 - F)E = \sum_{n=1}^{\infty}[\sum_{k=0}^{n-1}(1 - F)F^k]x_n = \sum_{n=1}^{\infty}(1 - F^n)x_n = \sum_{n=0}^{\infty} x_n - \sum_{n=0}^{\infty} F^n x_n$. By Lemma 4.2, this is equal to $\sum_{n=0}^{\infty} x_n - 1 + B^{-1}$. □

PROOF OF THEOREM 4.1. We prove that $1 - X = (1 - F)B^{-1}(1 - BE)$, which is the reciprocal of (4.1). We have

$$\begin{aligned}(1 - F)B^{-1}(1 - BE) &= (1 - F)(B^{-1} - E)\\ &= B^{-1} - FB^{-1} - (1 - F)E.\end{aligned}$$

By Lemma 4.3 and the identity $FB^{-1} = x_{-1}$, this is $1 - x_{-1} - \sum_{n=0}^{\infty} x_n = 1 - X$. The verification that every word in E has negative rank is straightforward. □

THEOREM 4.4. *Under the assumptions of Theorem* 4.1,

$$(1 - X)^{-1} = (1 - D)^{-1}\left[(1 - F)^{-1} + H(1 - \bar{E}\bar{B})^{-1} - 1\right],$$

where

$$D = \sum_{j,k=0}^{\infty} F^j x_{j+k}\bar{F}^k$$

and

$$H = 1 + \sum_{j,k=1}^{\infty} \sum_{l=0}^{k-1} F^j x_{j+k} \bar{F}^l \bar{B}.$$

Moreover, every word in D (and hence in $(1-D)^{-1}$) has rank zero, and every word in $H(1-\bar{E}\bar{B})^{-1}-1$ has negative rank.

Again we need some preliminary lemmas.

LEMMA 4.5. $(1-\bar{E}\bar{B})(\bar{B}^{-1}-x_{-1}) = 1 - X$.

PROOF. By Theorem 4.1 we have

$$\begin{aligned} 1 - X &= (1-F)B^{-1}(1-BE) \\ &= (B^{-1} - FB^{-1})(1-BE) = (B^{-1}-x_{-1})(1-BE). \end{aligned}$$

The lemma follows by reversing both sides. □

LEMMA 4.6. $(1-F)H(\bar{B}^{-1}-x_{-1}) = (1-F)(1-D) - F(1-X)$.

PROOF. Using $\bar{B}x_{-1} = \bar{F}$, we have

$$\begin{aligned} H(\bar{B}^{-1}-x_{-1}) &= \bar{B}^{-1} - x_{-1} + \sum_{j,k=1}^{\infty} \sum_{l=0}^{k-1} F^j x_{j+k} \bar{F}^l (1-\bar{F}) \\ &= \bar{B}^{-1} - x_{-1} + \sum_{j,k=1}^{\infty} F^j x_{j+k} (1-\bar{F}^k) \\ &= \bar{B}^{-1} - x_{-1} + \sum_{\substack{j>0 \\ k>0}} F^j x_{j+k} (1-\bar{F}^k). \end{aligned}$$

From $\bar{B}^{-1} = 1 - \sum_{k=0}^{\infty} x_k \bar{F}^k$ (Lemma 4.2), we obtain

$$\begin{aligned} H(\bar{B}^{-1}-x_{-1}) &= 1 - x_{-1} - \sum_{j,k=0}^{\infty} F^j x_{j+k} \bar{F}^k + \sum_{\substack{j>0 \\ k>0}} F^j x_{j+k} \\ &= 1 - D - x_{-1} + \sum_{\substack{j>0 \\ k>0}} F^j x_{j+k}. \end{aligned}$$

Thus

$$\begin{aligned} &(1-F)H(\bar{B}^{-1}-x_{-1}) \\ &\quad = (1-F)(1-D) - (1-F)x_{-1} + (1-F) \sum_{\substack{j>0 \\ k>0}} F^j x_{j+k}. \qquad (4.2) \end{aligned}$$

Substituting $k = l - j$ in the sum in (4.2) we obtain

$$\begin{aligned} (1-F)\sum_{\substack{j>0 \\ k>0}} F^j x_{j+k} &= \sum_{l=1}^{\infty} \sum_{j=1}^{l} (1-F)F^j x_l \\ &= \sum_{l=1}^{\infty} (F - F^{l+1})x_l = \sum_{l=-1}^{\infty} (F - F^{l+1})x_l - (F-1)x_{-1} \\ &= FX - F + (1-F)x_{-1}. \end{aligned}$$

Returning to (4.2), we find that

$$(1-F)H(\bar{B}^{-1}-x_{-1}) = (1-F)(1-D) + FX - F$$
$$= (1-F)(1-D) - F(1-X). \quad \square$$

Proof of Theorem 4.4. First we have

$$(1-F)H(1-\bar{E}\bar{B})^{-1}$$
$$= (1-F)H(\bar{B}^{-1}-x_{-1})\left[(1-\bar{E}\bar{B})(\bar{B}^{-1}-x_{-1})\right]^{-1}.$$

By Lemmas 4.5 and 4.6, this is equal to

$$\left[(1-F)(1-D)-F(1-X)\right](1-X)^{-1}$$
$$= (1-F)(1-D)(1-X)^{-1} - F.$$

Thus

$$(1-D)^{-1}\left[(1-F)^{-1} + H(1-\bar{E}\bar{B})^{-1} - 1\right]$$
$$= (1-D)^{-1}(1-F)^{-1}\left[F + (1-F)H(1-\bar{E}\bar{B})^{-1}\right]$$
$$= (1-D)^{-1}(1-F)^{-1}\left[(1-F)(1-D)(1-X)^{-1}\right] = (1-X)^{-1}.$$

The assertions about ranks are straightforward. $\square$

5. The combinatorial proof. The most natural way of proving Theorems 4.1 and 4.4 is to interpret them as describing factorizations of words. In addition to explaining the genesis of these equations, this approach enables us to stop at Theorem A, which contains the most useful parts of Theorems 4.1 and 4.4.

Unfortunately, the large number of terms in Theorems 4.1 and 4.4, each of which must be individually explained, necessitates a long proof. Therefore we omit many of the details, which the reader should have no trouble filling in.

The reader may find the following lattice-path interpretation helpful in understanding the proof: we represent the word $w = x_{i_1}x_{i_2}\ldots x_{i_n}$ as a path in the plane from the origin to the point $(n, -r(w))$, with vertices $(0,0)$, $(1, i_1)$, $(2, i_1 + i_2), \ldots, (n, i_1 + \cdots + i_n)$. Thus the letter x_i corresponds to the step $(1, i)$. All the classes of words we consider have simple geometric interpretations.

For any set $\mathcal{U}$ of words, the *counting series* for $\mathcal{U}$ is $\Gamma(\mathcal{U}) = \sum_{u\in\mathcal{U}} u$. Now let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be sets of words. If every $u \in \mathcal{U}$ has a unique expression vw with $v \in \mathcal{V}$ and $w \in \mathcal{W}$, and every product vw, with $v \in \mathcal{V}$ and $w \in \mathcal{W}$, is in $\mathcal{U}$, then we say that $\mathcal{U}$ is the *product* of $\mathcal{V}$ and $\mathcal{W}$, and we write $\mathcal{U} = \mathcal{V}\mathcal{W}$. (Alternatively, $\mathcal{U} = \mathcal{V}\mathcal{W}$ if and only if the map $(v, w) \mapsto vw$ is a bijection from $\mathcal{V} \times \mathcal{W}$ to $\mathcal{U}$.) If $\mathcal{W} = \{w\}$, we write $\mathcal{V}w$ for $\mathcal{V}\mathcal{W}$. It is clear that $\mathcal{U} = \mathcal{V}\mathcal{W}$ if and only if $\Gamma(\mathcal{U}) = \Gamma(\mathcal{V})\Gamma(\mathcal{W})$.

Let $\mathcal{U}$ be a set of words containing the empty word. Let $\mathcal{P}$ be the set of nonempty words in $\mathcal{U}$ which cannot be expressed in the form w_1w_2 for $w_1, w_2 \in \mathcal{U} - \{1\}$. We say that $\mathcal{P}$ is the set of *primes* of $\mathcal{U}$. If every nonempty word in $\mathcal{U}$ has a unique expression in the form $p_1p_2\cdots p_m$, with $p_i \in \mathcal{P}$, and every such product is in $\mathcal{U}$, then $\mathcal{U}$ is the *free monoid* generated by $\mathcal{P}$, and we write $\mathcal{U} = \mathcal{P}^*$. It is clear that $\mathcal{U} = \mathcal{P}^*$ if and only if $\Gamma(\mathcal{U}) = [1 - \Gamma(\mathcal{P})]^{-1}$.

Given sets of words $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$, to prove that $\mathcal{U} = \mathcal{V}\mathcal{W}$ we need to prove three things: first, that any $u \in \mathcal{U}$ has a factorization $u = vw$ with $v \in \mathcal{V}$ and $w \in \mathcal{W}$; second, that this factorization is unique; and third, that every such product is in $\mathcal{U}$. In general we shall prove only the first, and leave the other two to the reader. A similar remark applies to proving that $\mathcal{U} = \mathcal{P}^*$.

Now let $\mathcal{W}$ be the set of all words, and let $\mathcal{X}$ be the set $\{x_{-1}, x_0, x_1, \ldots\}$. Then $\mathcal{W} = \mathcal{X}^*$, so $W = (1 - X)^{-1}$, where $W = \Gamma(\mathcal{W})$ and $X = \Gamma(\mathcal{X})$. (Note that here, as in the rest of this section, our notation is consistent with that of §4. This consistency will not, in general, be obvious from the definitions.)

We introduce a partial ordering on the set of words: $u \leqslant v$ if for some word w, $uw = v$. If $u \leqslant v$, we say that u is a *head* of v; if $1 < u \leqslant v$, then u is a *proper head* of v.

We now define three sets of words:

$\mathcal{A} = \{a \mid w < a \text{ implies } r(w) > r(a)\}$,

$\mathcal{B} = \{b \mid r(b) = 0 \text{ and } w \leqslant b \text{ implies } r(w) \geqslant 0\}$,

$\mathcal{C} = \{c \mid 1 < w \leqslant c \text{ implies } r(w) > 0\}$.

In the lattice-path interpretation, a word in $\mathcal{A}$ is a path which stays below the horizontal line through its endpoint (until the end), a word in $\mathcal{B}$ is a path which ends on the horizontal axis and never passes above it, and a word in $\mathcal{C}$ is a path which stays below the horizontal axis. (Keep in mind that positive rank = negative vertical coordinate.)

Observe that $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = \{1\}$, and that $a \in \mathcal{A} - \{1\}$ implies $r(a) < 0$ and $c \in \mathcal{C} - \{1\}$ implies $r(c) > 0$.

PROPOSITION 5.1. $\mathcal{W} = \mathcal{A}\mathcal{B}\mathcal{C}$.

PROOF. Let w be a word. Of the heads of w of minimal rank, let a be the shortest, and let $w = av$.

Let b be the longest head of v of rank zero, and define c by $v = bc$. It is easily verified that $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c \in \mathcal{C}$, and that the decomposition is unique. □

PROPOSITION 5.2. *$\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are free monoids.* □

Propositions 5.1 and 5.2 are valid in the more general setting of words in the letters x_n for *all* integers n. Factorizations related to these are described in Feller [7, p. 383] and Foata and Schützenberger [9]. An application of Proposition 5.1 to lattice path enumeration is given in [10].

PROPOSITION 5.3. *A head of a word in $\mathcal{C}$ is in $\mathcal{C}$.* □

PROPOSITION 5.4. *If $r(u) < r(uv)$ then v has a head of rank one.*

PROOF. We may assume that $u = 1$. Let w be the shortest head of v of positive rank. Since $w \neq 1$, we may write $w = yx_n$. Then $r(y) \leqslant 0$, so $r(w) = r(y) - n \leqslant -n \leqslant 1$, since $n \geqslant -1$. Thus $r(w) = 1$. □

PROPOSITION 5.5. *A word is in $\mathcal{C} - \{1\}$ if and only if it has positive rank and has no proper head of rank zero.*

PROOF. Suppose that w has positive rank, but is not in $\mathcal{C}$. Let u be the longest head of w of nonpositive rank. Then $u \neq 1$ since $w \notin \mathcal{C}$, and by Proposition 5.4, w has a head of rank $r(u) + 1$. Thus $r(u) = 0$. The converse is immediate. □

Now let $\mathfrak{F}$ be the set of paths in $\mathcal{C}$ of rank one.

PROPOSITION 5.6. $\mathcal{C} = \mathfrak{F}^*$.

PROOF. Give $c \in \mathcal{C} - \{1\}$, let f_1 be the longest head of c of rank one (by Proposition 5.4). Then $f_1 \in \mathfrak{F}$ by Proposition 5.3. Let $c = f_1 c'$. Then by Proposition 5.5, $c' \in \mathcal{C}$, so by induction, $c = f_1 f_2 \dots f_k$, where $f_i \in \mathfrak{F}$. □

PROPOSITION 5.7. *$\mathfrak{F}^k$ is the set of words in $\mathcal{C}$ of rank k.* □

PROPOSITION 5.8. *$\mathfrak{F}$ is the disjoint union $\bigcup_{n=-1}^{\infty} \mathfrak{F}^{n+1} x_n$.*

PROOF. For $f \in \mathfrak{F}$, let $f = cx_n$. Then by Proposition 5.3, $c \in \mathcal{C}$. Then $r(c) = r(cx_n) - r(x_n) = 1 + n$, so by Proposition 5.7, $c \in \mathfrak{F}^{n+1}$. □

PROPOSITION 5.9. $\mathfrak{F} = x_{-1}\mathfrak{B}$.

PROOF. If $f = x_n b \in \mathfrak{F}$, then $r(x_n) > 0$, so $x_n = x_{-1}$. Then $r(b) = r(x_{-1}b) - r(x_{-1}) = 0$, so $b \in \mathfrak{B}$ by the definitions of $\mathfrak{B}$ and $\mathcal{C}$. □

With $A = \Gamma(\mathfrak{A})$, $B = \Gamma(\mathfrak{B})$, and, $F = \Gamma(\mathfrak{F})$, Propositions 5.1, 5.6, 5.8, and 5.9 yield

$$(1 - X)^{-1} = AB(1 - F)^{-1}, \qquad F = \sum_{n=-1}^{\infty} F^{n+1} x_n, \qquad F = x_{-1}B,$$

which is the first part of Theorem A.

Although we will not need it, we give a description of the prime set of $\mathfrak{B}$ (cf. Lemma 4.2).

PROPOSITION 5.10. *The prime set of $\mathfrak{B}$ is the disjoint union $\bigcup_{n=0}^{\infty} \mathfrak{F}^n x_n$.*

PROOF. Let $p = cx_n$ be a prime of $\mathfrak{B}$. Then $r(p) = 0$ but every proper head of c has positive rank, so $c \in \mathcal{C}$. Moreover, $r(c) = r(cx_n) - r(x_n) = n$, so $p \in \mathfrak{F}^n x_n$. □

To complete the combinatorial proof of Theorem 4.1, we must determine the primes of $\mathfrak{A}$.

PROPOSITION 5.11. *If $a = uv$ is in $\mathfrak{A}$, then v is in $\mathfrak{A}$.* □

PROPOSITION 5.12. *The primes of $\mathfrak{A}$ are those words of negative rank of the form bcx_n for $b \in \mathfrak{B}$ and $c \in \mathcal{C}$.*

PROOF. By Proposition 5.11, a word in $\mathfrak{A}$ is prime if and only if no proper head other than itself is in $\mathfrak{A}$. Let $p = wx_n$ be a prime of $\mathfrak{A}$. Then w has no head of negative rank. Let $w = bc$ where b is the longest head of w of rank zero. It follows that $b \in \mathfrak{B}$ and $c \in \mathcal{C}$. Conversely, if $b \in \mathfrak{B}$, $c \in \mathcal{C}$, and $r(bcx_n)$ is negative, then bc has no head of negative rank, so bcx_n is a prime of $\mathfrak{A}$. □

By Proposition 5.12, the prime set of $\mathcal{A}$ is $\mathcal{B}[\bigcup_{0<k<n} \mathcal{F}^k x_n]$, so its counting series is $B\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} F^k x_n$. This completes the combinatorial proof of Theorem 4.1.

We now move onto Theorem 4.4. Let $\mathcal{Z}$ be the set of words of rank zero.

PROPOSITION 5.13. *$\mathcal{Z}$ is a free monoid.* □

Let $\mathcal{D}$ be the set of primes of $\mathcal{Z}$. Then $d \in \mathcal{D}$ if and only if $r(d) = 0$ but $1 < w < d$ implies $r(w) \neq 0$.

PROPOSITION 5.14. *$\mathcal{D}$ is the set of words of rank zero of the form $c_1 x_n \bar{c}_2$ for $c_1, c_2 \in \mathcal{C}$ and $n > 0$.*

PROOF. Let d be a word in $\mathcal{D}$. Let c_1 be the longest head of d which is in $\mathcal{C}$. Then we may write $d = c_1 x_n \bar{c}_2$ for some x_n and c_2, since c_1 cannot equal d. Since $c_1 x_n \notin \mathcal{C}$, we must have $r(c_1 x_n) \leq 0$. Thus $n > 0$, and if $c_2 \neq 1$, $r(c_2) > 0$. If $c_2 = uv$ where $r(u) = 0$, then $0 = r(d) = r(c_1 x_n \bar{v}\bar{u}) = r(c_1 x_n \bar{v})$, so $u = 1$. Then by Proposition 5.5, $c_2 \in \mathcal{C}$. □

From Proposition 5.14 it follows that $D = \Gamma(\mathcal{D}) = \sum_{j,k=0}^{\infty} F^j x_{j+k} \bar{F}^k$.

Now let $\mathcal{W}_+$ and $\mathcal{W}_-$ be respectively the sets of words of nonnegative and nonpositive rank. (Thus $\mathcal{W}_+ \cap \mathcal{W}_- = \mathcal{Z}$.)

PROPOSITION 5.15. $\mathcal{W}_+ = \mathcal{Z}\mathcal{C}$.

PROOF. For $w \in \mathcal{W}_+$, let $w = zc$ where z is the longest head of w of rank zero. Then $z \in \mathcal{Z}$ and $c \in \mathcal{C}$. □

Propositions 5.14 and 5.15 yield the second part of Theorem A.

Before completing the proof of Theorem 4.4, we give an example of a power-series proof of a theorem about free monoids.

PROPOSITION 5.16. *$\mathcal{W}_+$ is not a free monoid.*

PROOF. By Proposition 5.15, $\Gamma(\mathcal{W}_+)^{-1} = (1 - F)(1 - D) = 1 - (F + D - FD)$. If $\mathcal{W}_+$ is a free monoid, then $F + D - FD$ must be a sum of words. But $F + D - FD$ contains negative terms, since FD counts some words not in $\mathcal{F}$. □

We now finish the proof of Theorem 4.4.

PROPOSITION 5.17. *Let $\mathcal{H}$ be the set consisting of the empty word and all words of negative rank of the form $h = c_1 x_n \bar{c}_2 \bar{b}$ with $c_1 \in \mathcal{C} - \{1\}$, $c_2 \in \mathcal{C}$, and $b \in \mathcal{B}$. Then $\mathcal{W}_- = \mathcal{Z}\mathcal{H}\bar{\mathcal{A}}$, where $\bar{\mathcal{A}} = \{\bar{a} | a \in \mathcal{A}\}$.*

PROOF. We give only a rough sketch. Let w be a word of nonpositive rank. We may write $w = zw_1$ where z is the longest head of w of rank zero. We now consider two cases. In the first case, assume that every proper head of w_1 has negative rank. Thus if $\bar{w}_1 = \bar{u}\bar{v}$ and $\bar{v} \neq 0$, $r(\bar{u}) = r(\bar{w}_1) - r(\bar{v}) > r(\bar{w}_1)$. Thus $\bar{w}_1 \in \mathcal{A}$, so $w_1 \in \bar{\mathcal{A}}$.

In the second case, let c_1 be the longest head of w_1 of positive rank. By construction, w_1 has no proper head of rank zero, so by Proposition 5.5, $c_1 \in \mathcal{C} - \{1\}$. Now let $w_1 = c_1 x_n w_2$. Then $r(c_1 x_n) < 0$. By Proposition 5.1 we may write $w_2 = \bar{c}_2 \bar{b} \bar{a}$, where $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c_2 \in \mathcal{C}$, so $w_1 = c_1 x_n \bar{c}_2 \bar{b} \bar{a}$.

Given that $w_1 = c_1 x_n \bar{c}_2 \bar{b} \bar{a}$, with $c_1, c_2 \in \mathcal{C}$, $b \in \mathcal{B}$, and $a \in \mathcal{Q}$, with $r(c_1 x_n) < 0$, the condition that w_1 has no proper head of rank zero is equivalent to the assertion that $r(c_1 x_n \bar{c}_2 \bar{b}) < 0$, since $\bar{c}_2 \bar{b}$ is the head of $\bar{c}_2 \bar{b} \bar{a}$ of maximal rank. □

It follows from Proposition 5.17 that $H = \Gamma(\mathcal{K}) = 1 + \Sigma F^j x_n \bar{F}^k \bar{B}$, the sum being over $j \geqslant 1$, $n \geqslant 2$, and $n > j + k$. This completes the combinatorial proof of Theorem 4.4.

Raney [22] (see also Gouyou-Beauchamps [13], Schützenberger [27], and Wendel [29]) has given a different combinatorial proof of formula (1.2):

A *cyclic permutation* of a word w is a word w' of the form vu where $uv = w$. Raney showed that a word of rank $k > 0$ has exactly k cyclic permutations which are in $\mathcal{C}$ (counting multiplicities), from which formula (1.2) follows easily. For a connection between cyclic permutations and factorizations of words, see Schützenberger [26].

PART II

6. The q-analog. We now derive a q-analog of the second part of the Lagrange inversion formula by applying a linear operator to formula (3.4) of Theorem A.

We first define an integer-valued function α on words by

$$\alpha(x_{i_1} x_{i_2} \ldots x_{i_n}) = -\left[n i_1 + (n-1) i_2 + \cdots + i_n \right].$$

This definition is suggested by geometric considerations. In the lattice path interpretation of a word w, $\alpha(w)$ is the number of lattice points in the region bounded above by the horizontal axis and below by the path, minus the number of lattice points in the region bounded above by the path and below by the horizontal axis. (We count lattice points on the path, but not on the horizontal axis.)

We now assume that our ring of scalars $\mathbf{R}$ contains $\mathbf{Q}((q))((t))[[z]]$, where q is an indeterminate.

If w is a word, we define $\Theta(w)$ to be $q^{\alpha(w)} w$, and we extend Θ to all of $\mathbf{A}$ by linearity. To obtain the formula we want, we apply Θ to every term except S in the following formulas from Theorem A:

$$F = \sum_{n=-1}^{\infty} F^{n+1} x_n, \tag{6.1}$$

$$(1 - X)^{-1} = (1 - D)^{-1} (1 - F)^{-1} + S, \tag{6.2}$$

$$D = \sum_{j,k=0}^{\infty} F^j x_{j+k} \bar{F}^k. \tag{6.3}$$

LEMMA 6.1. *For words u and v, $\alpha(uv) = \alpha(u) + \alpha(v) + r(u) l(v)$. (Recall that $l(v)$ is the length of v.)*

PROOF. Let $u = x_{i_1} x_{i_2} \ldots x_{i_m}$, $v = x_{j_1} x_{j_2} \ldots x_{j_n}$. Then

$$\begin{aligned} -\alpha(uv) &= (m+n) i_1 + (m+n-1) i_2 + \cdots + (n+1) i_m \\ &\quad + n j_1 + (n-1) j_2 + \cdots + j_n \\ &= n(i_1 + i_2 + \cdots + i_m) + \left[m i_1 + (m-1) i_2 + \cdots + i_m \right] \\ &\quad + \left[n j_1 + (n-1) j_2 + \cdots + j_n \right] \\ &= -\left[l(v) r(u) + \alpha(u) + \alpha(v) \right]. \quad \square \end{aligned}$$

COROLLARY 6.2. *If $r(u) = 0$, then $\alpha(uv) = \alpha(u) + \alpha(v)$. If U is a sum of words of rank zero and $V \in \mathbf{A}$ then $\Theta(UV) = \Theta(U)\Theta(V)$.* □

LEMMA 6.3. *For words $w_1, w_2, \ldots, w_n$,*

$$\alpha(w_1 w_2 \ldots w_n) = \sum_{i=1}^{n} \alpha(w_i) + \sum_{1 < i < j < n} r(w_i) l(w_j).$$

PROOF. We use induction on n. The case $n = 1$ is trivial. For $n > 1$, Lemma 6.1 gives $\alpha(w_1 w_2 \ldots w_n) = \alpha(w_1 w_2 \ldots w_{n-1}) + \alpha(w_n) + r(w_1 \ldots w_{n-1}) l(w_n)$. By the induction hypothesis this is

$$\sum_{i=1}^{n} \alpha(w_i) + \sum_{1 < i < j < n-1} r(w_i) l(w_j) + \sum_{1 < i < n} r(w_i) l(w_n)$$
$$= \sum_{i=1}^{n} \alpha(w_i) + \sum_{1 < i < j < n} r(w_i) l(w_j). \quad \square$$

We now define for each integer n a linear operator Λ_n: $\mathbf{A} \to \mathbf{A}$; if w is a word we set $\Lambda_n(w) = q^{nl(w)}w$. For $U \in \mathbf{A}$ we write U^* for $\Theta(U)$ and U_n for $\Lambda_n(U)$. Since Θ and Λ_n commute, U_n^* is unambiguous. (We use this subscript convention on capital letters only, so w_1 and w_2 are still different variables.)

PROPOSITION 6.4. $\Theta(F^n) = F^* F_1^* \ldots F_{n-1}^*$.

PROOF. If $f_1, f_2, \ldots, f_n$ are in $\mathcal{F}$, then by Lemma 6.3,

$$\alpha(f_1 f_2 \ldots f_n) = \sum_{j=1}^{n} \alpha(f_j) + \sum_{1 < i < j < n} r(f_i) l(f_j).$$

Since $r(f_i) = 1$, this is $\sum_{j=1}^{n} [\alpha(f_j) + (j-1) l(f_j)]$. Thus

$$\Theta(f_1 f_2 \ldots f_n) = \prod_{j=1}^{n} q^{\alpha(f_j) + (j-1) l(f_j)} f_j = \prod_{j=1}^{n} \Lambda_{j-1}(f_j^*). \tag{6.4}$$

The proposition follows by summing (6.4) over all n-tuples $f_1, f_2, \ldots, f_n$ in $\mathcal{F}$. □

PROPOSITION 6.5. $\Theta(F^{n+1} x_n) = q F^* F_1^* \ldots F_n^* x_n$.

PROOF.

$$\begin{aligned} \Theta(F^{n+1} x_n) &= q^{n+1} \Theta(F^{n+1}) \Theta(x_n) \quad \text{(by Lemma 6.1)} \\ &= q \Theta(F^{n+1}) x_n \\ &= q F^* F_1^* \ldots F_n^* x_n \quad \text{(by Proposition 6.4).} \quad \square \end{aligned}$$

We now turn to the terms involving D. Since every word in D has rank zero, by Corollary 6.2, $\Theta[(1 - D)^{-1}] = [1 - \Theta(D)]^{-1}$ and $\Theta[(1 - D)^{-1}(1 - F)^{-1}] = [1 - \Theta(D)]^{-1} \Theta[(1 - F)^{-1}]$.

LEMMA 6.6. *For all words w, $\alpha(w) + \alpha(\bar{w}) = r(w)[1 + l(w)]$.*

PROOF. Let $w = x_{i_1} x_{i_2} \ldots x_{i_n}$. Then $\alpha(w) = -[n i_1 + (n-1) i_2 + \cdots + i_n]$ and $\alpha(\bar{w}) = -[n i_n + (n-1) i_{n-1} + \cdots + i_1]$. Thus $\alpha(w) + \alpha(\bar{w}) = -(1 + n)(i_1 + i_2 + \cdots + i_n) = [1 + l(w)] r(w)$.

LEMMA 6.7. *If* $r(u) = i$ *and* $r(v) = j$, *then* $\alpha(ux_{i+j}\bar{v}) = \alpha(u) - \alpha(v)$.

PROOF. By Lemma 6.2,

$$\begin{aligned}\alpha(ux_{i+j}\bar{v}) &= \alpha(u) + \alpha(x_{i+j}) + \alpha(\bar{v}) \\ &\quad + i[1 + l(v)] - (i+j)l(v) \\ &= \alpha(u) + \alpha(\bar{v}) - j[1 + l(v)].\end{aligned}$$

By Lemma 6.6 this is $\alpha(u) + \alpha(\bar{v}) - [\alpha(v) + \alpha(\bar{v})] = \alpha(u) - \alpha(v)$. □

PROPOSITION 6.8.

$$\Theta(X^n) = \left[\sum_{i_1=-1}^{\infty} q^{-ni_1}x_{i_1}\right]\left[\sum_{i_2=-1}^{\infty} q^{-(n-1)i_2}x_{i_2}\right]\cdots\left[\sum_{i_n=-1}^{\infty} q^{-i_n}x_{i_n}\right].$$

PROOF.

$$X^n = \sum_{i_1,i_2,\ldots,i_n=-1}^{\infty} x_{i_1}x_{i_2}\ldots x_{i_n},$$

so

$$\Theta(X^n) = \sum_{i_1,i_2,\ldots,i_n=-1}^{\infty} q^{-[ni_1+(n-1)i_2+\cdots+i_n]}x_{i_1}x_{i_2}\ldots x_{i_n}. \quad \square$$

Before stating the main result of this section, we introduce some notation. For any $h(z) = h(z; q)$ in $\mathbf{R}$, we set $h^{[k]}(z) = h(z)h(qz)\ldots h(q^{k-1}z)$ and $\tilde{h}^{[k]}(z) = h(z; q^{-1})h(q^{-1}z; q^{-1})\ldots h(q^{-(k-1)}z; q^{-1})$, with $\tilde{h}(z) = \tilde{h}^{[1]}(z)$. The reader should be warned that this interpretation of $\tilde{h}^{[k]}(z)$ is somewhat illogical; thus if $e(z) = \tilde{h}(z)$, $e^{[k]}(z) = h(z;\ q^{-1})\ldots h(q^{k-1}z;\ q^{-1}) \neq \tilde{h}^{[k]}(z)$. Note also that $h(q^{-1}z; q^{-1})\ldots h(q^{-k}z; q^{-1})$ is $\tilde{h}^{[k]}(q^{-1}z)$, not $\tilde{h}^{[k]}(qz)$.

THEOREM 6.9. *Let* $f(z) = f(z; q)$ *satisfy*

$$f(z) = qz\sum_{n=0}^{\infty} g_n f^{[n]}(z), \tag{6.5}$$

where the g_n *are indeterminates. Let* $g(t) = \sum_{n=0}^{\infty} g_n t^n$. *Then for* $n, k > 0$,

$$\langle z^n\rangle[1 - d(z)]^{-1}f^{[k]}(z) = q^{(n+1)n/2}\langle t^{n-k}\rangle\tilde{g}^{[n]}(q^{-1}t), \tag{6.6}$$

where

$$d(z) = z\sum_{i,j=0}^{\infty} g_{i+j+1}f^{[i]}(z)\tilde{f}^{[j]}(z). \tag{6.7}$$

PROOF. We note first that if we set $q = 1$, the theorem reduces to formulas (1.1) and (1.3).

Although Theorem 6.9 does not give a simple formula for the coefficients of $f(z)$ in terms of those of $g(t)$ (such a formula seems unlikely to exist), it does give an expression for $f(z)$ as a quotient of two series whose coefficients are explicitly described.

We define a homomorphism Ω: $\mathbf{A} \to \mathbf{R}$ by $\Omega(x_n) = zt^n g_{n+1}$ and we define $f(z) \in \mathbf{R}$ by

$$f(z) = t\Omega(F^*). \tag{6.8}$$

We now apply the composed operator $\Omega\Theta$ to formulas (6.1), (6.2) and (6.3).

If w is a word of length l, then $\Omega(w) = az^l$, where a does not contain z, so $\Omega[\Lambda_n(w)] = a(q^n z)^l$. Thus

$$\Omega(F_n^*) = t^{-1}f(q^n z). \tag{6.9}$$

Now by Proposition 6.5 we have

$$\begin{aligned}\Omega\Theta\big(F^{n+1}x_n\big) &= q\frac{f(z)}{t}\,\frac{f(qz)}{t}\cdots\frac{f(q^n z)}{t}\,zt^n g_{n+1}\\ &= \frac{qz}{t}\,g_{n+1}f^{[n+1]}(z),\end{aligned}$$

so from (6.1) we obtain

$$\frac{f(z)}{t} = \frac{qz}{t}\sum_{n=-1}^{\infty} g_{n+1}f^{[n+1]}(z),$$

so

$$f(z) = qz\sum_{n=0}^{\infty} g_n f^{[n]}(z).$$

We have thus shown that the definitions (6.5) and (6.8) for $f(z)$ are equivalent. (As in the case $q = 1$, formula (6.5) determines $f(z)$ uniquely.)

Now by Proposition 6.8,

$$\begin{aligned}\Omega\Theta(X^n) &= \left[\sum_{i_1=-1}^{\infty} q^{-ni_1}zt^{i_1}g_{i_1+1}\right]\cdots\left[\sum_{i_n=-1}^{\infty} q^{-i_n}zt^{i_n}g_{i_n+1}\right]\\ &= \left[z\sum_{j_1>0} g_{j_1}q^{-n(j_1-1)}t^{j_1-1}\right]\cdots\left[z\sum_{j_n>0} g_{j_n}q^{-(j_n-1)}t^{j_n-1}\right]\\ &= \left[q^n\frac{z}{t}\,g(q^{-n}t)\right]\left[q^{n-1}\frac{z}{t}\,g(q^{-(n-1)}t)\right]\cdots\left[q\frac{z}{t}\,g(q^{-1}t)\right]\\ &= q^{(n+1)n/2}(z/t)^n\tilde{g}^{[n]}(q^{-1}t).\end{aligned}$$

Thus

$$\Omega\Theta\big[(1-X)^{-1}\big] = \sum_{n=0}^{\infty} q^{(n+1)n/2}(z/t)^n\tilde{g}^{[n]}(q^{-1}t). \tag{6.10}$$

By Lemma 6.7,

$$\begin{aligned}\Omega\Theta\big(F^i x_{i+j}\bar{F}^j\big) &= \frac{f(z)}{t}\,\frac{f(qz)}{t}\cdots\\ &\qquad \frac{f(q^{i-1}z)}{t}\,zt^{i+j}g_{i+j+1}\frac{\tilde{f}(z)}{t}\,\frac{\tilde{f}(q^{-1}z)}{t}\cdots\frac{\tilde{f}(q^{-(j-1)})}{t}\\ &= zg_{i+j+1}f^{[i]}(z)\tilde{f}^{[j]}(z),\end{aligned}$$

so

$$\Omega\Theta(D) = z\sum_{i,j=0}^{\infty} g_{i+j+1}f^{[i]}(z)\tilde{f}^{[j]}(z) = d(z),$$

as defined in equation (6.7). Now using Corollary 6.2 we have

$$\Omega\Theta\left[(1-D)^{-1}(1-F)^{-1}\right]=\left[1-\Omega\Theta(D)\right]^{-1}\Omega\Theta\left[(1-F)^{-1}\right]$$
$$=\left[1-d(z)\right]^{-1}\sum_{k=0}^{\infty} t^{-k}f^{[k]}(z), \tag{6.11}$$

by Proposition 6.4 and equation (6.9).

From (6.2), (6.10) and (6.11) we obtain

$$\sum_{n=0}^{\infty} q^{(n+1)n/2}(z/t)^n\tilde{g}^{[n]}(q^{-1}t) = \left[1-d(z)\right]^{-1}\sum_{k=0}^{\infty} t^{-k}f^{[k]}(z) + ts(t, z), \tag{6.12}$$

where $s(t, z)$ contains no negative powers of t.

Equating coefficients of $z^n t^{-k}$ for $n, k \geqslant 0$ in (6.12) yields

$$\langle t^{-k}\rangle q^{(n+1)n/2}t^{-n}\tilde{g}^{[n]}(q^{-1}t) = \langle z^n\rangle\left[1-d(z)\right]^{-1}f^{[k]}(z),$$

which is equivalent to (6.6). □

7. A simpler proof. Once we know the "right powers of q" we should be able to give a more direct proof of Theorem 6.9. I have found such a proof for most of the theorem, but the method does not yield equation (6.7).

THEOREM 7.1. *Let $f(z)$ satisfy*

$$f(z) = qz\sum_{i=0}^{\infty} g_i f^{[i]}(z) \tag{7.1}$$

and let $g(t) = \sum_{n=0}^{\infty} g_n t^n$. Then for some formal power series $e(z)$, independent of n and k,

$$\langle z^n\rangle e(z)f^{[k]}(z) = q^{(n+1)n/2}\langle t^{n-k}\rangle\tilde{g}^{[n]}(q^{-1}t), \tag{7.2}$$

for $n, k \geqslant 0$.

PROOF. Define $A(n, k)$ by

$$A(n, k) = q^{(n+1)n/2}\langle t^{n-k}\rangle\tilde{g}^{[n]}(q^{-1}t). \tag{7.3}$$

Let $e(z) = \sum_{n=0}^{\infty} A(n, 0)z^n$, and define $B(n, k)$ by

$$e(z)f^{[k]}(z) = \sum_{n=0}^{\infty} B(n, k)z^n. \tag{7.4}$$

We shall show that $A(n, k)$ and $B(n, k)$ are equal for all $n, k \geqslant 0$ by showing that they satisfy the same recurrence and initial conditions.

By definition, $A(n, 0) = B(n, 0)$; for $k > 0$, $A(0, k) = B(0, k) = 0$. Substituting $q^k z$ for z in (7.1) and multiplying by $e(z)f^{[k]}(z)$ yields

$$e(z)f^{[k+1]}(z) = q^{k+1}z\sum_{i=0}^{\infty} g_i e(z)f^{[k+i]}(z). \tag{7.5}$$

Equating coefficients of z^{n+1} in (7.5) we obtain

$$B(n+1, k+1) = q^{k+1}\sum_{i=0}^{\infty} g_i B(n, k+i). \tag{7.6}$$

Now

$$\begin{aligned} A(n+1,k+1) &= q^{(n+2)(n+1)/2}\langle t^{n-k}\rangle \tilde{g}^{[n+1]}(q^{-1}t) \\ &= \langle t^{n-k}\rangle\big[q^{n+1}g(q^{-1}t)\big]\big[q^{(n+1)n/2}\tilde{g}^{[n]}(q^{-2}t)\big] \\ &= \langle t^{n-k}\rangle q^{n+1}\sum_{i=0}^{\infty} g_i(q^{-1}t)^i \sum_{j=0}^{n} A(n,j)(q^{-1}t)^{n-j} \\ &= q^{n+1}\sum_{i=0}^{\infty} g_i q^{-(n-k)}A(n,k+i) \\ &= q^{k+1}\sum_{i=0}^{\infty} g_i A(n,k+i). \end{aligned}$$

Thus by comparison with (7.6), $A(n,k) = B(n,k)$ for all $n, k \geqslant 0$. □

8. Another form. By making a change of variables in Theorem 6.9, we find an equivalent form which is easier to apply.

THEOREM 8.1. *Define $h(z)$ and $g(t)$ by*

$$h(z) = \sum_{n=0}^{\infty} q^{(n+1)n/2} g_n z^n h^{[n]}(z) \tag{8.1}$$

and $g(t) = \sum_{n=0}^{\infty} g_n t^n$. Then

$$\langle z^n\rangle\big[1-d(z)\big]^{-1}h^{[k]}(z) = q^{n(n-1)/2+nk}\langle t^n\rangle \tilde{g}^{[n+k]}(t), \tag{8.2}$$

where

$$d(z) = \sum_{i,j=0}^{\infty} g_{i+j+1} q^{(i+j+1)(i-j)/2} z^{i+j+1} h^{[i]}(z)\tilde{h}^{[j]}(z). \tag{8.3}$$

PROOF. In Theorem 6.9 set $f(z) = qzh(z)$. Then

$$f^{[n]}(z) = \big[qzh(z)\big]\big[q^2zh(qz)\big]\dots\big[q^nzh(q^{n-1}z)\big] = q^{(n+1)n/2}z^nh^{[n]}(z).$$

(Therefore $\tilde{f}^{[n]}(z) = q^{-(n+1)n/2}z^n\tilde{h}^{[n]}(z)$.) Then (6.5) becomes (8.1). With the identity $\binom{i+1}{2} - \binom{j+1}{2} = (i+j+1)(i-j)/2$, (6.7) becomes (8.3). With $n+k$ substituted for n, (6.6) becomes

$$\langle z^n\rangle\big[1-d(z)\big]^{-1}q^{(k+1)k/2}h^{[k]}(z) = q^{(n+k+1)(n+k)/2}\langle t^n\rangle\tilde{g}^{[n+k]}(q^{-1}t).$$

Thus

$$\begin{aligned} \langle z^n\rangle\big[1-d(z)\big]^{-1}h^{[k]}(z) &= q^{(n+k+1)(n+k)/2-(k+1)k/2}\langle t^n\rangle\tilde{g}^{[n+k]}(q^{-1}t) \\ &= q^{(n+k+1)(n+k)/2-(k+1)k/2-n}\langle t^n\rangle\tilde{g}^{[n+k]}(t) \\ &= q^{n(n-1)/2+nk}\langle t^n\rangle\tilde{g}^{[n+k]}(t). \end{aligned}$$

9. Some notation. We introduce some of the standard q-series notation. We write $(t)_n = (t;q)_n$ for $(1-t)(1-tq)\dots(1-tq^{n-1})$. (All empty products are taken to be one.) In particular, $(q)_n = (q;q)_n = (1-q)(1-q^2)\dots(1-q^n)$, and $(t)_\infty = \prod_{i=0}^{\infty}(1-tq^i)$. The *$q$-binomial coefficient* or *Gaussian polynomial* is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k(q)_{n-k}}.$$

We use the notation $\left[{n \atop k}\right]_r$ to denote $\left[{n \atop k}\right]$ with q replaced by q^r. Since $(q^{-1}; q^{-1})_n = (-1)^n q^{-n(n-1)/2}(q)_n$, and $\binom{m+n}{2} - \binom{m}{2} - \binom{n}{2} = mn$, we have

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_{-1} = q^{-mn}\begin{bmatrix} m+n \\ m \end{bmatrix}. \tag{9.1}$$

We note also that

$$\lim_{k\to\infty}\begin{bmatrix} m+k \\ n \end{bmatrix} = 1/(q)_n. \tag{9.2}$$

An important formula is the q-binomial theorem [2, p. 17]:

$$\frac{(ut)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty} \frac{(u)_n}{(q)_n} t^n, \tag{9.3}$$

and its special cases [2, p. 36]

$$(1+t)(1+tq)\dots(1+tq^{n-1}) = \sum_{j=0}^{n} q^{j(j-1)/2}\begin{bmatrix} n \\ j \end{bmatrix} t^j, \tag{9.4}$$

and [2, p. 19]

$$1/(t)_\infty = \sum_{n=0}^{\infty} \frac{t^n}{(q)_n}. \tag{9.5}$$

10. Continued fractions. We now consider a simple case of Theorem 8.1, that in which $g(t) = 1 + g_1 t + g_2 t^2$. For comparison, we first take $q = 1$. Then $h(z) = 1 + g_1 z h(z) + g_2 z^2 [h(z)]^2$, so

$$h(z) = \left\{1 - g_1 z - \left[1 - 2g_1 z + \left(g_1^2 - 4g_2\right)z^2\right]^{1/2}\right\}/2g_2 z^2, \tag{10.1}$$

and by formulas (1.2) and (1.3),

$$\langle z^n\rangle[h(z)]^k = k/(n+k)\langle t^n\rangle\left(1 + g_1 t + g_2 t^2\right)^{n+k} \tag{10.2}$$

and

$$\langle z^n\rangle[h(z)]^k/a(z) = \langle t^n\rangle\left(1 + g_1 t + g_2 t^2\right)^{n+k} \tag{10.3}$$

where $a(z) = [1 - 2g_1 z + (g_1^2 - 4g_2)z^2]^{1/2}$. Here we easily obtain

$$\langle t^n\rangle\left(1 + g_1 t + g_2 t^2\right)^{n+k} = \sum_{n/2 < i < n} \binom{n+k}{i}\binom{i}{2i-n} g_1^{2i-n} g_2^{n-i}. \tag{10.4}$$

In the special case $g_1 = 0$, $g_2 = 1$, we have

$$h(z) = \left[1 - (1-4z^2)^{1/2}\right]/2z^2 = \sum_{n=0}^{\infty} \frac{1}{2n+1}\binom{2n+1}{n} z^{2n}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} z^{2n},$$

in which the coefficients are Catalan numbers [6], [23], and more generally,

$$\left[h(z)\right]^k = \sum_{n=0}^{\infty} \frac{k}{2n+k}\binom{2n+k}{n} z^{2n},$$

in which the coefficients are ballot numbers [6], [23]. Also,

$$h(z)^k/(1-4z^2)^{1/2} = \sum_{n=0}^{\infty} \binom{2n+k}{n} z^{2n}.$$

Now let us consider the case $g(t) = (1+t)(1+st)$, so that $g_1 = 1+s$ and $g_2 = s$. Then from (10.1),

$$h(z) = \left\{1-(1+s)z - \left[1-2(1+s)z+(1-s)^2z^2\right]^{1/2}\right\}/2sz^2, \quad (10.5)$$

and from (10.2) and (10.3) we obtain

$$[h(z)]^k = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{k}{n+k} \binom{n+k}{j} \binom{n+k}{j+k} s^j z^n \quad (10.6)$$

and

$$[h(z)]^k/\left[1-2(1+s)z+(1-s)^2z^2\right]^{1/2} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n+k}{j} \binom{n+k}{j+k} s^j z^n.$$

The numbers $(n+1)^{-1}\binom{n+1}{j}\binom{n+1}{j+1}$ ($k=1$ in (10.6)), are the *Runyon numbers* considered by Riordan [23, p. 17].

We note that setting $s=1$ in (10.5) yields

$$\begin{aligned} h(z) &= \left[1-2z-(1-4z)^{1/2}\right]/2z^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{n+1}\binom{2n}{n} z^{n-1} = \left\{\left[1-(1-4z)^{1/2}\right]/2z\right\}^2, \end{aligned}$$

so we obtain again the Catalan and ballot numbers.

We now turn to the q-analogs.

PROPOSITION 10.1. *Let $h(z)$ satisfy*

$$h(z) = 1 + qg_1zh(z) + q^3g_2z^2h(z)h(qz). \quad (10.7)$$

Then

$$h(z) = \frac{1}{1-qg_1z-}\;\frac{q^3g_2z^2}{1-q^2g_1z-}\;\frac{q^5g_2z^2}{1-q^3g_1z-}\cdots \quad (10.8)$$

and

$$\begin{aligned} &\langle z^n\rangle\left[1-d(z)\right]^{-1}h^{[k]}(z) \\ &\quad = q^{n(n-1)/2+nk}\langle t^n\rangle \prod_{i=0}^{n+k-1}\left(1+q^{-i}g_1t+q^{-2i}g_2t^2\right), \quad (10.9) \end{aligned}$$

where

$$d(z) = g_1z + g_2z^2\left[qh(z) + q^{-1}\tilde{h}(z)\right]. \quad (10.10)$$

PROOF. From (10.7) we have $h(z)[1-qg_1z-q^3g_2h(qz)] = 1$, so $h(z) = 1/[1-qg_1z-q^3g_2h(qz)]$. Iterating this relation yields (10.8). (Convergence in the formal power series sense is straightforward.) Equations (10.9) and (10.10) are immediate from Theorem 8.1. □

I have not found a simple formula for the right side of (10.9) analogous to (10.4). A continued fraction closely related to (10.8) was studied by Hodel **[15]** in connection with a weighted lattice-path problem.

PROPOSITION 10.2. *Let $h(z)$ satisfy*

$$h(z) = 1 + q^3z^2h(z)h(qz). \tag{10.11}$$

Then

$$h(z) = \frac{1}{1-}\,\frac{q^3z^2}{1-}\,\frac{q^5z^2}{1-}\cdots \tag{10.12}$$

and

$$[1 - d(z)]^{-1}h^{[k]}(z) = \sum_{n=0}^{\infty} q^{-n^2}\begin{bmatrix} 2n+k \\ n \end{bmatrix}_2 z^{2n}, \tag{10.13}$$

where

$$d(z) = z^2[qh(z) + q^{-1}\tilde{h}(z)]. \tag{10.14}$$

PROOF. In Proposition 10.1, set $g_1 = 0$, $g_2 = 1$. Then (10.11), (10.12), and (10.14) are immediate. By (9.4),

$$\prod_{i=0}^{n+k-1}(1 + q^{-2i}t^2) = \sum_{i=0}^{n+k} q^{-i(i-1)}\begin{bmatrix} n+k \\ i \end{bmatrix}_{-2} t^{2i}. \tag{10.15}$$

The coefficient of t^n in (10.15) is zero for n odd; for $n = 2m$ it is $q^{-m(m-1)}\left[\begin{smallmatrix} 2m+k \\ m \end{smallmatrix}\right]_{-2}$. Thus from (10.9) and (9.1),

$$\begin{aligned}\langle z^{2m}\rangle[1 - d(z)]^{-1}h^{[k]}(z) &= q^{m(2m-1)+2mk-m(m-1)-2m(m+k)}\begin{bmatrix} 2m+k \\ m \end{bmatrix}_2 \\ &= q^{-m^2}\begin{bmatrix} 2m+k \\ m \end{bmatrix}_2,\end{aligned}$$

which is (10.13). □

We note in particular that for $k = 0$,

$$\{1 - z^2[qh(z) + q^{-1}\tilde{h}(z)]\}^{-1} = \sum_{n=0}^{\infty} q^{-n^2}\begin{bmatrix} 2n \\ n \end{bmatrix}_2 z^{2n},$$

and for $k = \infty$,

$$[1 - d(z)]^{-1}h^{[\infty]}(z) = \sum_{n=0}^{\infty} q^{-n^2}\frac{z^{2n}}{(q^2;q^2)_n}.$$

The continued fraction (10.12) was studied by Ramanujan in connection with the Rogers-Ramanujan identities **[14**, p. 294**]**. The coefficients of $h^{[k]}(z)$ are q-ballot numbers and have been studied by Carlitz **[3]**, Carlitz and Riordan **[4]**, Carlitz and Scoville **[5]**, and Riordan **[24]**.

We will return to these series from a more general perspective in §13.

PROPOSITION 10.3. *Let* $h(z)$ *satisfy*

$$h(z) = 1 + q(1+s)zh(z) + q^3sz^2h(z)h(qz). \tag{10.16}$$

Then

$$h(z) = \frac{1}{1-q(1+s)z-}\;\frac{q^3sz^2}{1-q^2(1+s)z-}\;\frac{q^5sz^2}{1-q^3(1+s)z-} \tag{10.17}$$

and

$$[1-d(z)]^{-1}h^{[k]}(z) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} q^{-j(n-j)}\begin{bmatrix} n+k \\ j \end{bmatrix}\begin{bmatrix} n+k \\ j+k \end{bmatrix} s^j z^n, \tag{10.18}$$

where

$$d(z) = (1+s)z + sz^2[qh(z) + q^{-1}\tilde{h}(z)]. \tag{10.19}$$

PROOF. We take $g(t) = (1+t)(1+st)$ in Proposition 10.1. Then (10.16), (10.17), and (10.19) are immediate. By (9.4),

$$\prod_{i=0}^{n+k-1}(1+q^{-i}t)(1+q^{-i}st)$$
$$= \sum_{i,j=0}^{n+k} q^{-i(i-1)/2-j(j-1)/2}\begin{bmatrix} n+k \\ i \end{bmatrix}_{-1}\begin{bmatrix} n+k \\ j \end{bmatrix}_{-1} s^j t^{i+j}.$$

Thus from (10.9) and (9.1) we have

$$\langle z^n\rangle[1-d(z)]^{-1}h^{[k]}(z) = \sum_{i+j=n} q^e\begin{bmatrix} n+k \\ i \end{bmatrix}\begin{bmatrix} n+k \\ j \end{bmatrix} s^j,$$

where

$$e = \binom{n}{2} - \binom{i}{2} - \binom{j}{2} + nk - i(j+k) - j(i+k)$$
$$= ij + nk - 2ij - (i+j)k = -ij + nk - nk = -j(n-j),$$

and (10.18) follows. □

We note the special cases $k = 0$:

$$\{1-(1+s)z + sz^2[qh(z) + q^{-1}\tilde{h}(z)]\}^{-1} = \sum_{n=0}^{\infty}\sum_{j=0}^{n} q^{-j(n-j)}\begin{bmatrix} n \\ j \end{bmatrix}^2 s^j z^n,$$

and $k = \infty$:

$$[1-d(z)]^{-1}h^{[\infty]}(z) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} q^{-j(n-j)}\begin{bmatrix} n \\ j \end{bmatrix} s^j \frac{z^n}{(q)_n}.$$

A few special values of s in Proposition 10.3 are of interest. For $s = -1$, we have the q-Catalan numbers of Proposition 10.2, and for $s = 1$, we have Pólya's q-Catalan numbers [21], discussed in the next section. The case $s = q^{1/2}$, with a different normalization, is contained in Proposition 13.1.

11. Pólya's q-Catalan numbers. We consider pairs (π, σ) of lattice paths in the plane of the same length, each path starting at the origin and consisting of unit horizontal and vertical steps in the positive direction.

Now let S be the set of such path-pairs with the following properties:

(i) π and σ end at the same point;

(ii) π begins with a vertical step and σ with a horizontal;

(iii) π and σ do not meet between the origin and their common endpoint.

We call elements of S *polygons*.

Now let $c(m, j, n)$ be the number of polygons (π, σ) which end at the point $(j, n-j)$ and which enclose an area m, and let

$$P_n(s; q) = \sum_{j=0}^{n} \sum_{m=0}^{\infty} c(m, j, n) q^m s^j.$$

PROPOSITION 11.1. *Let* $P(z; q) = \sum_{n=2}^{\infty} P_n(s; q) z^n$. *Then* $P(z; q) = sqz^2 h(z)$, *where* $h(z)$ *is as in Proposition* 10.3. *In particular,*

$$(1 + s)z + P(z; q) + P(z; q^{-1}) = 1 - \left\{ \sum_{n=0}^{\infty} \sum_{j=0}^{n} q^{-j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix}^2 s^j z^n \right\}^{-1}. \tag{11.1}$$

PROOF. First we represent a pair of paths of the same length as a sequence of pairs of steps. Let v be the vertical step and h the horizontal step. Then we represent the pair (π, σ), where $\pi = a_1 a_2 \ldots a_n$ and $\sigma = b_1 b_2 \ldots b_n$, and each a_i and b_i is a v or h, as the sequence of step-pairs $(a_1, b_1)(a_2, b_2) \ldots (a_n, b_n)$. Recalling the terminology of Part I, we code such a sequence of step-pairs as a word in the letters $\{x_{-1}, x_0, x_1\}$ as follows:

$$(v, h) \mapsto x_{-1}, \qquad (v, v) \mapsto x_0,$$
$$(h, h) \mapsto x_0, \qquad (h, v) \mapsto x_1.$$

We write $c(\pi, \sigma)$ for the image of (π, σ) under this encoding. Note that a word with n x_0's corresponds to 2^n path-pairs.

Let (π, σ) be a polygon. Then we define $A(\pi, \sigma)$ to be the area enclosed by (π, σ). We call (σ, π) a *reversed polygon* and we define $A(\sigma, \pi)$ to be $-A(\pi, \sigma)$. We also define $A(v, v) = A(h, h) = 0$. (The step-pairs (v, v) and (h, h) are in a sense "degenerate polygons", although they are not, according to our definition, polygons.)

If $c(\pi, \sigma) = w$, then π and σ end at the same point if and only if w has rank zero. Thus with the set $\mathcal{D}$ defined as in §5, but restricted to words in the letters $\{x_{-1}, x_0, x_1\}$, $c^{-1}(\mathcal{D})$ consists of

(a) polygons,

(b) reversed polygons, and

(c) the path-pairs (v, v) and (h, h).

We next observe that with α defined as in §6, if (π, σ) is a polygon then $\alpha(c(\pi, \sigma)) = A(\pi, \sigma)$ and $\alpha(c(\sigma, \pi)) = A(\sigma, \pi) = -A(\pi, \sigma)$. Also $\alpha(c(v, v)) = \alpha(c(h, h)) = 0$. This follows easily from the lattice-point interpretation of α; we leave the details to the reader.

Now in the proof of Theorem 6.9, let us take $g(t) = 1 + (1 + s)t + st^2$. Let w be a word which codes a polygon, and assume that w contains i x_0's and j x_1's. Then

$$\Omega\Theta(w) = q^{\alpha(w)}(1+s)^i s^j z^{l(w)}$$
$$= z^{l(w)} \sum_{(\pi,\sigma)\in c^{-1}(w)} q^{A(\pi,\sigma)} s^{e(\pi,\sigma)}, \tag{11.2}$$

where $e(\pi, \sigma)$ is the horizontal coordinate of the common endpoint of π and σ.

We note that by symmetry, (11.2) also holds for words w that code reversed polygons. If w' is obtained from w by interchanging x_{-1}'s and x_1's, then $\alpha(w') = -\alpha(w)$ and w' codes a reversed polygon if and only if w codes a polygon. Thus with $D = \Gamma(\mathfrak{D})$, and using $\Omega\Theta(x_0) = (1+s)z$, we have $d(z) = \Omega\Theta(D) = (1+s)z + P(z; q) + P(z; q^{-1})$, and the proposition follows from the proof of Theorem 6.9. □

We remark that a shorter proof of (11.1) be given without using the coding of path-pairs by words, but this proof would not explain the connection between Propositions 11.1 and 10.3.

Pólya [21] considered the case $s = 1$ of Proposition 11.1 and found formula (11.1) for this case. For $q = 1$, equation (10.6) gives

$$P_{n+2}(s, 1) = \sum_{j=0}^{n} \frac{1}{n+1}\binom{n+1}{j}\binom{n+1}{j+1} s^{j+1},$$

in which the coefficients are Runyon numbers. This formula was found by Narayana [20] and Levine [18]. Finally,

$$P_{n+2}(1,1) = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j}\binom{n+1}{j+1}$$
$$= \frac{1}{n+1}\binom{2n+2}{n} = \frac{1}{n+2}\binom{2n+2}{n+1},$$

a Catalan number. (See Shapiro [28].)

12. A "generalization". Another change of variables enables us to use the q-Lagrange inversion formula in its full generality more easily. For convenience, we make the substitutions in two steps.

LEMMA 12.1. *Let m be a positive integer, let $\gamma(t) = \sum_{n=0}^{\infty} g_n t^n$, and define $f(z)$ by*

$$f(z) = \sum_{n=0}^{\infty} g_n q^{n(mn+1)/2} z^n f^{[mn]}(z). \tag{12.1}$$

Then

$$\langle z^n \rangle [1 - e(z)]^{-1} f^{[k]}(z) = q^{n(mn+1)/2 + nk} \langle t^n \rangle \tilde{\gamma}^{[mn+k]}(q^{-1}t), \tag{12.2}$$

where

$$e(z) = \sum_{n=0}^{\infty} \sum_{i+j+1=mn} g_n q^{n(i-j)/2} z^n f^{[i]}(z) \tilde{f}^{[j]}(z). \tag{12.3}$$

PROOF. In Theorem 8.1, set $g(t) = \gamma(t^m)$ and $h(z) = f(z^m)$. Then replacing z^m by z in (8.1) we have

$$f(z) = \sum_{n=0}^{\infty} q^{mn(mn+1)/2} g_n \prod_{i=0}^{mn-1} f(q^{mi}z). \tag{12.4}$$

Let $d(z) = e(z^m)$. Then from (8.3), with z^m replaced by z,

$$e(z) = \sum_{n=0}^{\infty} \sum_{i+j+1=mn} g_n q^{mn(i-j)/2} z^n \prod_{k=0}^{i-1} f(q^{mk}z) \prod_{l=0}^{j-1} \tilde{f}(q^{-ml}z). \tag{12.5}$$

From (8.2),

$$\begin{aligned}\langle z^n\rangle[1 - e(z)]^{-1} \prod_{i=0}^{k-1} f(q^{mi}z) &= \langle z^{mn}\rangle[1 - d(z)]^{-1} h^{[k]}(z) \\ &= q^{mn(mn-1)/2+mnk}\langle t^{mn}\rangle \prod_{i=0}^{mn+k-1} \gamma[(q^{-i}t)^m] \\ &= q^{mn(mn-1)/2+mnk}\langle t^n\rangle \prod_{i=0}^{mn+k-1} \gamma(q^{-mi}t). \end{aligned} \tag{12.6}$$

Replacing q^m by q everywhere, we obtain (12.1) and (12.3) from (12.4) and (12.5), and (12.6) becomes

$$\begin{aligned}\langle z^n\rangle[1 - e(z)]^{-1} f^{[k]}(z) &= q^{n(mn-1)/2+nk}\langle t^n\rangle \tilde{\gamma}^{[mn+k]}(t) \\ &= q^{n(mn+1)/2+nk}\langle t^n\rangle \tilde{\gamma}^{[mn+k]}(q^{-1}t),\end{aligned}$$

which is (12.2). □

THEOREM 12.2. *Let m be a positive integer and let $p = q^m$. Define $h(z)$ by*

$$h(z) = \sum_{n=0}^{\infty} g_n q^{mn^2/2} z^n h^{[mn]}(z), \tag{12.7}$$

with $g_0 = 1$.

Let $H(z) = [h^{[\infty]}(z)]^{-1}$. Then

$$h^{[k]}(z) = H(q^k z)/H(z) \tag{12.8}$$

and

$$H(qz) = \sum_{n=0}^{\infty} g_n p^{n^2/2} z^n H(p^n z). \tag{12.9}$$

Let $g(t) = \sum_{n=0}^{\infty} g_n t^n$ and let $G(t) = g^{[\infty]}(t)$. Then

$$\langle z^n\rangle H(z) = p^{n^2/2}\langle t^n\rangle G(t)^{-1}, \tag{12.10}$$

$$\langle z^n\rangle[1 - d(z)]^{-1} h^{[k]}(z) = q^{nk+mn^2/2}\langle t^n\rangle \tilde{g}^{[mn+k]}(q^{-1}t), \tag{12.11}$$

and if we set $u = q^k$,

$$\langle z^n\rangle[1 - d(z)]^{-1} H(uz)/H(z) = u^n p^{n^2/2}\langle t^n\rangle G(p^{-n}u^{-1}t)/G(t), \tag{12.12}$$

where

$$d(z) = \sum_{n=1}^{\infty} \sum_{i+j+1=mn} g_n z^n q^{n(i-j-1)/2} h^{[i]}(z) \tilde{h}^{[j]}(q^{-1}z). \tag{12.13}$$

Moreover, if we take p, q, and u to be independent indeterminates, and define $H(z)$ by (12.9), *then* (12.10) *and* (12.12) *are valid* (*where $d(z)$ is now defined by* (12.12) *for* $u = 1$).

PROOF. In Lemma 12.1, set $\gamma(t) = g(t)$, $f(z) = h(q^{1/2}z)$, and $e(z) = d(q^{1/2}z)$. Then substitute $q^{-1/2}z$ for z. Formulas (12.7), (12.11), and (12.13) follow from (12.1), (12.2), and (12.3), where the powers of q in (12.13) are computed as follows:

We have $f(z;\ q) = h(q^{1/2}z;\ q)$ so $f(z;\ q^{-1}) = h(q^{-1/2}z;\ q^{-1})$ and $\tilde{f}^{[j]}(z) = \tilde{h}^{[j]}(q^{-1/2}z)$. Thus

$$d(q^{1/2}z) = e(z) = \sum g_n q^{n(i-j)/2} z^n h^{[i]}(q^{1/2}z)\tilde{h}^{[j]}(q^{-1/2}z)$$

so

$$d(z) = \sum g_n q^{n(i-j-1)/2} z^n h^{[i]}(z)\tilde{h}^{[j]}(q^{-1}z).$$

With (12.8), which is immediate, (12.9) follows from (12.7), and (12.12) follows from (12.11). It remains to prove (12.10).

Let $a_n = \langle z^n\rangle H(z)$. Then from (12.9),

$$\sum_{n=0}^{\infty} q^n a_n z^n = \sum_{j=0}^{\infty} g_j p^{j^2/2} z^j \sum_{k=0}^{\infty} a_k p^{jk} z^k,$$

so

$$q^n a_n = \sum_{k=0}^{n} g_{n-k} p^{(n-k)^2/2+(n-k)k} a_k$$
$$= p^{n^2/2} \sum_{k=0}^{n} g_{n-k} p^{-k^2/2} a_k.$$

Thus

$$p^{-n^2/2} q^n a_n = \sum_{k=0}^{n} g_{n-k} p^{-k^2/2} a_k. \tag{12.14}$$

Now let $J(t) = \sum_{n=0}^{\infty} p^{-n^2/2} a_n t^n$. From (12.14), $J(qt) = g(t)J(t)$, so $g(t) = J(qt)/J(t)$. Then

$$G(t) = g^{[\infty]}(t) = \prod_{n=0}^{\infty} \frac{J(q^{n+1}t)}{J(q^n t)} = J(t)^{-1}.$$

Thus $a_n = p^{n^2/2}\langle t^n\rangle J(t) = p^{n^2/2}\langle t^n\rangle G(t)^{-1}$.

13. An example. The simplest example of Theorem 12.2 is the case $g(t) = 1 - t$.

PROPOSITION 13.1. *Let*

$$H(z) = \sum_{n=0}^{\infty} p^{n^2/2} \frac{z^n}{(q)_n}. \tag{13.1}$$

Then for all u,

$$H(uz) = \sum_{n=0}^{\infty} (u-1)(u-q)\dots(u-q^{n-1}) p^{n^2/2} \frac{z^n}{(q)_n} H(p^n z), \tag{13.2}$$

and in particular (for $u = q$),

$$H(qz) = H(z) - p^{1/2}zH(pz). \tag{13.3}$$

Also,

$$s(z)H(uz)/H(z) = \sum_{n=0}^{\infty} (up^n - 1)(up^n - q)\dots(up^n - q^{n-1})p^{-n^2/2}\frac{z^n}{(q)_n}, \tag{13.4}$$

where

$$s(z) = \sum_{n=0}^{\infty} (p^n - 1)(p^n - q)\dots(p^n - q^{n-1})p^{-n^2/2}\frac{z^n}{(q)_n}. \tag{13.5}$$

In the special case $p = q^m$, m a positive integer, let $h(z) = H(qz)/H(z)$. Then for $k > 0$,

$$h^{[k]}(z) = \sum_{n=0}^{k} (-1)^n q^{n[n(m+1)-1]/2}\begin{bmatrix} k \\ n \end{bmatrix} z^n h^{[mn]}(z), \tag{13.6}$$

and in particular,

$$h(z) = 1 - q^{m/2}zh^{[m]}(z). \tag{13.7}$$

Also,

$$s(z)h^{[k]}(z) = \sum_{n=0}^{\infty} (-1)^n q^{-n[n(m-1)+1]/2}\begin{bmatrix} mn + k \\ n \end{bmatrix} z^n, \tag{13.8}$$

where

$$s(z) = \left\{1 - z \sum_{i+j=m-1} q^{(i-j-1)/2}h^{[i]}(z)\tilde{h}^{[j]}(q^{-1}z)\right\}^{-1}. \tag{13.9}$$

PROOF. In Theorem 12.2 take $g(t) = 1 - t$, so $G(t) = (t)_\infty$. Then (13.1), (13.3), (13.4), (13.8) and (13.9) follow from (12.10), (12.9), (12.12), (12.11), and (12.13) respectively, and (13.5) follows from (13.4) by setting $u = 1$.

Now take $G(t) = (t; r)_\infty$ in Theorem 12.2. Then

$$H(z) = \sum_{n=0}^{\infty} p^{n^2/2}\frac{z^n}{(r;r)_n} \tag{13.10}$$

by (12.10), and

$$g(t) = G(t)/G(qt) = \sum_{n=0}^{\infty} \frac{(q^{-1};r)_n}{(r;r)_n}(qt)^n$$
$$= \sum_{n=0}^{\infty} (q-1)(q-r)\dots(q-r^{n-1})\frac{t^n}{(r;r)_n}.$$

Thus by (12.9),

$$H(qz) = \sum_{n=0}^{\infty} (q-1)(q-r)\dots(q-r^{n-1})p^{n^2/2}\frac{z^n}{(r;r)_n} H(p^n z). \tag{13.11}$$

If we replace q by u and r by q, then (13.10) becomes (13.1) and (13.11) becomes (13.2). Finally, (13.6) follows from (13.2) on setting $u = q^k$ and dividing both sides by $H(z)$.

We give an application of Proposition 13.1 to Ramanujan's continued fraction.

PROPOSITION 13.2. *Let*

$$f(z) = \frac{z}{1+}\,\frac{qz}{1+}\,\frac{q^2z}{1+}\,\frac{q^3z}{1+}\cdots. \tag{13.12}$$

Then

$$f(z)f(qz) = \frac{qz^2}{1+(1+q)qz-}\,\frac{q^5z^2}{1+(1+q)q^3z-}\,\frac{q^9z^2}{1+(1+q)q^5z-}\cdots. \tag{13.13}$$

(*The general term is*

$$\left.\frac{q^{4n+1}z^2}{1+(1+q)q^{2n+1}z-}.\right)$$

PROOF. Take $p = q^2$ in Proposition 13.1. Then $h(z) = 1 - qzh(z)h(qz)$ by (13.7), so

$$h(z) = \frac{1}{1+}\,\frac{qz}{1+}\,\frac{q^2z}{1+}\cdots.$$

Setting $f(z) = zh(z)$, we obtain (13.12).

By (13.6), $h^{[2]}(z) = 1 - (1+q)qzh^{[2]}(z) + q^5z^2h^{[4]}(z)$, so with $e(z) = h(z)h(qz)$, $e(z) = 1 - (1+q)qze(z) + q^5z^2e(z)e(q^2z)$. Thus $e(z) = 1/[1 + (1+q)qz - q^5z^2e(q^2z)]$, and (13.13) follows by iteration ($f(z)f(qz) = qz^2e(z)$). □

Once we have found the formula, we can, of course, show by direct algebraic manipulation that $f(z) = z/[1 + f(qz)]$ implies $f(z)f(qz) = qz^2/[1 + (1+q)qz - f(q^2z)f(q^3z)]$.

14. The inversion enumerator for trees. Let T be a tree on the vertices $\{1, 2, \ldots, n\}$. An *inversion* of T is a pair of vertices i, j, with $1 < i < j$, such that j lies on the unique path from i to 1. Let $J_n(p) = \sum_T p^{I(T)}$, where the sum is over all trees T on $\{1, 2, \ldots, n\}$, and $I(T)$ is the number of inversions of T. Then $J_n(p)$ is a polynomial in p of degree $\binom{n-1}{2}$ with constant term $(n-1)!$, $J_n(1) = n^{n-2}$, $J_n(2)$ is the number of connected labeled graphs on n vertices, and $\sum_{n=0}^{\infty} J_{n+1}(-1)(z^n/n!) = \sec z + \tan z$.

Mallows and Riordan [19] showed that

$$\sum_{n=1}^{\infty}(p-1)^{n-1}J_n(p)\frac{z^n}{n!} = \log\left[\sum_{n=0}^{\infty} p^{n(n-1)/2}\frac{z^n}{n!}\right]. \tag{14.1}$$

(See also Foata [8, pp. 144–147] and Gessel and Wang [11].) We derive some identities for $J_n(p)$ from Proposition 13.1.

PROPOSITION 14.1. *Let* $J(z) = \sum_{n=0}^{\infty} J_{n+1}(p)(z^n/n!)$. *Then*

$$J(z)J(pz)\cdots J(p^{k-1}z) = \sum_{n=0}^{\infty}(1+p+\cdots+p^{k-1})^n p^{n(n-1)/2}J(z)J(pz)\cdots J(p^{n-1}z)\frac{z^n}{n!}, \tag{14.2}$$

and in particular,

$$J(z) = \sum_{n=0}^{\infty} p^{n(n-1)/2} J(z)J(pz)\ldots J(p^{n-1}z)\frac{z^n}{n!}. \tag{14.3}$$

Also,

$$a(z)J(z)\ldots J(p^{k-1}z) = \sum_{n=0}^{\infty} p^{-(n+1)n/2}\left(\frac{p^{n+k}-1}{p-1}\right)^n \frac{z^n}{n!}, \tag{14.4}$$

where

$$a(z) = \sum_{n=0}^{\infty} p^{-(n+1)n/2}\left(\frac{p^n-1}{p-1}\right)^n \frac{z^n}{n!}. \tag{14.5}$$

In particular,

$$J(z) = \frac{\sum_{n=0}^{\infty} p^{-(n+1)n/2}(1+p+\cdots+p^n)^n \dfrac{z^n}{n!}}{\sum_{n=0}^{\infty} p^{-(n+1)n/2}(1+p+\cdots+p^{n-1})^n \dfrac{z^n}{n!}}. \tag{14.6}$$

PROOF. Let $E(z) = \sum_{n=0}^{\infty} p^{n(n-1)/2} z^n/n!$. Then differentiating (14.1) with respect to t we have

$$\begin{aligned} J[(p-1)z] &= \sum_{n=0}^{\infty} (p-1)^n J_{n+1}(p)\frac{z^n}{n!} \\ &= E'(z)/E(z) = E(pz)/E(z). \end{aligned} \tag{14.7}$$

Now with $H(z)$ as in Proposition 13.1,

$$E(z) = H\left[(1-q)z/p^{1/2}\right]|_{q=1},$$

so from (13.2) we have

$$E(uz) = \sum_{n=0}^{\infty} (u-1)^n p^{n(n-1)/2} \frac{z^n}{n!} E(p^n z), \tag{14.8}$$

and from (13.4) and (13.5) we obtain

$$s(z)E(uz)/E(z) = \sum_{n=0}^{\infty} (up^n - 1)^n p^{-(n+1)n/2} \frac{z^n}{n!} \tag{14.9}$$

and

$$s(z) = \sum_{n=0}^{\infty} (p^n-1)^n p^{-(n+1)n/2} \frac{z^n}{n!}. \tag{14.10}$$

Let $a(z) = s[z/(p-1)]$. Then substituting $z/(p-1)$ for z and p^k for u in (14.8), (14.9), and (14.10); and using (14.7), we obtain (14.2), (14.4), and (14.5). □

In the special case $p = 1$, (14.3) becomes $J(z) = e^{zJ(z)}$, which by classical Lagrange inversion has the solution $[J(z)]^k = \sum_{n=0}^{\infty} k(n + k)^{n-1} z^n/n!$ and

$$\frac{[J(z)]^k}{1 - zJ(z)} = \sum_{n=0}^{\infty} (n + k)^n \frac{z^n}{n!}.$$

(See Riordan [23, p. 147].)

NOTE ADDED IN PROOF. Formula (3.8) is actually due to J. L. Lagrange, *Nouvelle méthode pour résoudre les équations littérales par le moyen de séries*, Mém. Acad. Roy. Sci. Belles-Lettres Berlin **24** (1770); *Oeuvres*, Vol. 3, Gauthier-Villars, Paris, 1869, pp. 3–73.

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